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JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

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Journal of Computational and Applied Mathematics 186 (2006) 300–323

www.elsevier.com/locate/cam

Iterates of the infinitesimal generator and space–time harmonic polynomials of a Markov process

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Received 28 September 2004; received in revised form 23 March 2005

Abstract

We relate iterates of the infinitesimal generator of a Markov process to space–time harmonic functions. First, we develop the theory for a general Markov process and create a family a space–time martingales. Next, we investigate the special class of subordinators. Combinatorics results on space–time harmonic polynomials and generalized Stirling numbers are developed and interpreted from a probabilistic point of view. Finally, we introduce the notion of pairs of subordinators in duality, investigate the implications on the associated martingales and consider some explicit examples.

Keywords: Markov process; Lévy processes; Subordinators; Space–time harmonic polynomials; Martingales; Infinitesimal generator; Stirling numbers

1. Introduction

Fundamental martingale-additive functionals can be associated to a nice Markov process X_t. There are of the type

$$M_t(f) \triangleq f(X_t) - f(X_0) - \int_0^t L_{\mathrm{e}}(f)(X_s)\,\mathrm{d}s,$$

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0377-0427/$ - see front matter
doi:10.1016/j.cam.2005.04.014

where L_e is the (extended) infinitesimal generator of X and f is any measurable function belonging to the domain of L_e. These martingales generate, in the Kunita–Watanabe sense, the set of all the martingales of the Markov process.

In this paper, from the martingales $M_t(f)$, we create a family of similar space–time martingales obtained by using some formulae involving the iterates of the generator. We illustrate this construction in the case of the Brownian motion and the Poisson process in Section 2 of the paper. Section 3 is devoted to the case of subordinators. Results from combinatorics (see e.g. [26]) involving space–time harmonic polynomials and generalized Stirling numbers are developed and interpreted from a probabilistic point of view. Many connections between stochastic processes and combinatorics can be found in Pitman's Saint–Flour course [20]. Relations between stochastic processes and orthogonal polynomials are described in [10], [11] and [23].

2. Iterates of the infinitesimal generator and associated martingales

2.1. *Definition of the extended infinitesimal generator* L_e

In this section, we consider a general Markov process $X=(X_t, t\geqslant 0)$ taking values in the measurable state space $(E, \mathscr{E})$ and endowed with the laws $(P_x, x \in E)$ such that

$$P_x(X_0=x)=1 \quad \text{for each } x.$$

The notion of extended (infinitesimal) generator L_e associated with the Markov process X was first in Kunita [14,15] and is quite convenient to exhibit important sets of martingales (under all P_x's) associated to X. More precisely,

Definition 2.1. Let f be a measurable function on E such that there exists a function $g: E \to \mathbb{R}$ and

$$M_t(f) \triangleq f(X_t) - f(X_0) - \int_0^t g(X_s)\,\mathrm{d}s$$

is a (P_x)-martingale for all x, then f is said to belong to D_e, the domain of L_e, the operator defined on D_e as

$$L_e(f)=g.$$

Some assumptions are needed regarding the function g. In particular, g may be assumed to be bounded, but the weaker assumption

$$\int_0^t |g(X_s)|\,\mathrm{d}s < \infty \quad P_x \text{ a.s.} \quad \text{for all } x \text{ and } t.$$

is sufficient.

This definition extends that of any "stronger" infinitesimal generator L (for more details, please refer for instance, to [17–19] or [9] in Chapter XV and its errata in the last two pages of [8]). In particular, the martingale $M_t(f)$ is introduced in formula (2) p. 130 in [17].

We then define iterates of L_{e} in the obvious manner, i.e.,

$$L_{\mathrm{e}}^n = L_{\mathrm{e}}(L_{\mathrm{e}}^{n-1}), \quad n \in \mathbb{N}^* \quad \text{and} \quad L_{\mathrm{e}}^0 f = f. \tag{1}$$

As a particular example, we consider the case

$$f(\cdot) = U^p(h)(\cdot) \equiv E_{(\cdot)}\left[\int_0^\infty \mathrm{d}t\ \exp(-pt)h(X_t)\right]$$

for h bounded and Borel in $D_{\mathrm{e}}(L^{n-1})$ and p a positive real number.

It is a well-known result that

$$L_{\mathrm{e}} f = pU^p(h) - h \equiv pf - h \tag{2}$$

from which, more generally, the iterates of the infinitesimal generator L_{e} computed on f can easily be derived as

$$L_{\mathrm{e}}^n f = p^n f - \left(\sum_{k=0}^{n-1} p^k L_{\mathrm{e}}^{n-(k+1)} h\right) \quad \text{for } n \in \mathbb{N}^*; \quad L_{\mathrm{e}}^0 f = f.$$

2.2. General result

The first result of this note is the following:

Proposition 2.2. *For every $T \in \mathbb{R}$, for every $N \in \mathbb{N}$ and $f \in D(L_{\mathrm{e}}^{N+1})$, the process*

$$M_t^{N,T}(f) = \sum_{n=0}^{N} \frac{(T-t)^n}{n!} L_{\mathrm{e}}^n f(X_t) - \int_0^t \frac{(T-s)^N}{N!} L_{\mathrm{e}}^{N+1} f(X_s)\,\mathrm{d}s \tag{3}$$

is a martingale. More precisely, it satisfies

$$M_t^{N,T}(f) - M_0^{N,T}(f) = \sum_{n=0}^{N} \int_0^t \frac{(T-s)^n}{n!}\,\mathrm{d}M_s(L_{\mathrm{e}}^n f).$$

For a general semi-martingale version of this result, see Exercise 6.17 in [7].

2.3. Some comments

2.3.1. Some general remarks on formula (3)

We might call formula (3) a stochastic Taylor formula for the following reason. Indeed, if $X_t = x + \alpha t$ is the deterministic constant velocity (Markov) process, starting from x, then

$$\text{For any } C^1 \text{ function } f : \mathbb{R} \to \mathbb{R}, \quad L_{\mathrm{e}} f(x) = \alpha f'(x).$$

So that formula (3) becomes

$$f(x) = \sum_{n=0}^{N} \frac{(-\alpha t)^n}{n!} f^{(n)}(x + \alpha t) + (-\alpha)^{N+1} \int_0^t \mathrm{d}s \frac{s^N}{N!} f^{(N+1)}(x + \alpha s) \tag{4}$$

since, for this degenerate Markov process, the martingale given in Eq. (3) is identically equal to its value for $t=0$. And identity (4) is nothing else but Taylor's expansion at order N, around $(x+\alpha t)$, for f considered between x and $x+\alpha t$.

However, we shall refrain from using the terminology "stochastic Taylor formula", as this term is already used in a number of different contexts. In particular, Azencott [3] considers Taylor expansion with respect to a parameter ε, near $\varepsilon=0$, of a family of solutions of SDE's depending on ε. Another variant has also been recently developed in [24], who iterate Itô's formula for the Poisson process.

2.3.2. *Proper functions*

Moreover, applying formula (3) to the particular case of proper functions, i.e. functions f such that there exists a parameter λ and $L_{\mathrm{e}}(f)=\lambda f$, we obtain that

$$\sum_{n=0}^{N} \frac{(\lambda(T-t))^n}{n!} f(X_t) - \int_0^t \frac{(T-s)^N}{N!} \lambda^{N+1} f(X_s)\,\mathrm{d}s$$

is a martingale. Then, the well-known result of $\exp(-\lambda t) f(X_t)$ being a martingale is immediate, letting N tend to ∞ and considering the case $T=0$.

2.3.3. *Polynomial case*

Finally, a very particular and interesting situation is that of polynomials. Let $E=\mathbb{R}$ and consider a Markov process $(X_t;\, t\geqslant 0)$ such that L_{e} has the following stability property with respect to polynomials:

$$L_{\mathrm{e}}(\mathscr{P}_n) \subseteq \mathscr{P}_{n-1}, \tag{5}$$

where $\mathscr{P}_n$ is the space of polynomials of degree less than or equal to n.

Then, applying Proposition 2.2 to polynomials leads to an interesting family of space–time harmonic polynomials in the sense that we obtain a sequence of polynomials in both variables x and t. Moreover, denoting by Π this polynomial, $\Pi(X_t, t)$ is a martingale so that

$$\left(L_x + \frac{\partial}{\partial t}\right)\Pi = 0.$$

More precisely,

Proposition 2.3. *Let* $\mathbf{p} \in \mathscr{P}_N$. *Then*

$$\sum_{n=0}^{N} \frac{(-t)^n}{n!} (L_{\mathrm{e}}^n \mathbf{p})(x) \tag{6}$$

is a space–time harmonic polynomial, in other terms

$$M_t^{N,0}(\mathbf{p}) = \sum_{n=0}^{N} \frac{(-t)^n}{n!} (L_{\mathrm{e}}^n \mathbf{p})(X_t)$$

is a martingale.

We indicate two directions towards which we shall go in the sequel of our paper:

1. We shall look for a sequence $(p_N)_{N=0,1,2,\ldots}$ of polynomials in x with respective degrees N, such that

$$L_{\mathrm{e}}^n(p_N) = p_{N-n}, \quad n \leqslant N.$$

In this case, the formula (6) reduces to a space–time harmonic polynomial

$$\sum_{n=0}^{N} \frac{(-t)^n}{n!} p_{N-n}(x). \tag{7}$$

We shall consider especially this notion of space–time harmonic polynomials when working with subordinators.

2. However, it would be a pity to limit ourselves to the hypothesis (5) as there are many interesting cases of real-valued diffusions for which

$$L_{\mathrm{e}}(\mathscr{P}_n) \subseteq \mathscr{P}_n \tag{8}$$

holds and (5) is not satisfied, as we will see later in Section 2.5.3. In fact, Mazet [16] obtains a characterization of diffusion semigroups which is related to the property (8). Mazet's definition of a diffusion is that its infinitesimal generator L_{e} acts on polynomials with the "Itô rule"

$$L_{\mathrm{e}}(\Phi(f)) = \Phi'(f) L_{\mathrm{e}} f + \tfrac{1}{2}\Phi''(f)[L_{\mathrm{e}}(f^2) - 2 f L_{\mathrm{e}} f],$$

where Φ and f are polynomials. He then shows that, again on the space of polynomials, L is equal to

$$(Ax^2 + Bx + C)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (ax + b)\frac{\mathrm{d}}{\mathrm{d}x}$$

thus, obtaining a five-parameter family of Markov processes which satisfy (8). This discussion is also held in various references among which the S.M.F. publication in [2].

2.3.4. *Some heuristic considerations*

Before proving Proposition 2.2, we begin with the following heuristic consideration: for any bounded measurable function $f : E \to \mathbb{R}$ and for $0 \leqslant t \leqslant T$, if $\mathscr{F}_t = \sigma(X_s; 0 \leqslant s \leqslant t)$, the process

$$E_x(f(X_T)/\mathscr{F}_t) = \mathscr{P}_{T-t} f(X_t) \tag{9}$$

is a martingale.

Now $\mathscr{P}_u$ is "generated" by L_{e}, the extended infinitesimal generator, i.e.,

$$\mathscr{P}_u = \exp(u L_{\mathrm{e}}),$$

so that formula (9) may be written informally as

$$E_x(f(X_T)/\mathscr{F}_t) = \sum_{n=0}^{\infty} \frac{(T-t)^n}{n!} L_{\mathrm{e}}^n f(X_t). \tag{10}$$

For nice f's, the right-hand side of this equation is meaningful for any $T \in \mathbb{R}$ and $t \geqslant 0$. This defines a martingale.

2.4. Proof of Proposition 2.2

In order to prove Proposition 2.2, we proceed in several steps:

(i) For $N=0$, we have

$$M_t^{0,T}(f)=M_t(f)+f(X_0).$$

Hence, this is a martingale.

(ii) For $N\in\mathbb{N}^*$, consider

$$M_t^{N,T}(f)=\sum_{n=0}^{N-1}\frac{(T-t)^n}{n!}L_{\mathrm{e}}^n f(X_t)+\Delta_t^{N,T}(f),$$

where

$$\Delta_t^{N,T}(f)=\frac{(T-t)^N}{N!}(L_{\mathrm{e}}^N f)(X_t)-\int_0^t\frac{(T-s)^N}{N!}L_{\mathrm{e}}^{N+1}f(X_s)\,\mathrm{d}s,$$

which, using integration by parts, is equal to

$$\Delta_t^{N,T}(f)=\frac{T^N}{N!}L_{\mathrm{e}}^N f(X_0)+\int_0^t\frac{(T-s)^N}{N!}\,\mathrm{d}M_s(L_{\mathrm{e}}^N f)-\int_0^t\frac{(T-s)^{N-1}}{(N-1)!}L_{\mathrm{e}}^N f(X_s)\,\mathrm{d}s.$$

Consequently, we have obtained

$$M_t^{N,T}(f)=M_t^{N-1,T}(f)+\frac{T^N}{N!}L_{\mathrm{e}}^N f(X_0)+\int_0^t\frac{(T-s)^N}{N!}\,\mathrm{d}M_s(L_{\mathrm{e}}^N f).$$

Iterating this formula, we get

$$M_t^{N,T}(f)=\sum_{n=0}^{N}\frac{T^n}{n!}L_{\mathrm{e}}^n f(X_0)+\sum_{n=0}^{N}\int_0^t\frac{(T-s)^n}{n!}\,\mathrm{d}M_s(L_{\mathrm{e}}^n f).$$

Or equivalently

$$M_t^{N,T}(f)=M_0^{N,T}(f)+\sum_{n=0}^{N}\int_0^t\frac{(T-s)^n}{n!}\,\mathrm{d}M_s(L_{\mathrm{e}}^n f).$$

Hence $(M_t^{N,T}(f),t\geqslant 0)$ is a martingale. □

2.5. Some examples

2.5.1. Brownian motion

Let us consider on a probability space $(\Omega,\mathfrak{I},\mathbb{P})$, a standard $\mathbb{P}$-Brownian motion $(W_t;t\geqslant 0)$ and a function f which belongs to $C^{2(N+1)}\subset D(L_{\mathrm{e}}^{N+1})$. Applying Itô's lemma to $f(W_t)$, we obtain the characterization of the generator L_{e} as

$$L_{\mathrm{e}}(f)(x)=\tfrac{1}{2}f''(x).$$

This case corresponds to

$$L_{\mathrm{e}}(\mathscr{P}_n) \subseteq \mathscr{P}_{n-2}.$$

Hence, by a simple recurrence argument, we get the iterates of the generator as

$$\forall 0 \leqslant n \leqslant N, \quad L_{\mathrm{e}}^n(f)(x) = \frac{1}{2^n} \frac{\mathrm{d}^{2n} f}{\mathrm{d}x^{2n}}(x).$$

According to Proposition 2.2

$$\sum_{n=0}^{N} \frac{1}{2^n} \frac{(-t)^n}{n!} \frac{\mathrm{d}^{2n} f}{\mathrm{d}x^{2n}}(W_t) + (-1)^{N+1} \int_0^t \mathrm{d}s \frac{1}{2^{N+1}} \frac{s^N}{N!} \frac{\mathrm{d}^{2(N+1)} f}{\mathrm{d}x^{2(N+1)}}(W_s) \tag{11}$$

is a martingale.

Let now f be a polynomial of degree N. For instance

$$f(x) = x^N$$

whose derivatives are obviously computed. Denoting by $[N/2]$ the integer part of $N/2$, formula (11) simply becomes

$$M_t^{N,0} \triangleq \sum_{n=0}^{\left[\frac{N}{2}\right]} \frac{1}{2^n} \frac{(-t)^n}{n!} N(N-1)\ldots(N-2n+1)(W_t)^{N-2n} = N!(\sqrt{2t})^N \sum_{n=0}^{\left[\frac{N}{2}\right]} \frac{(-1)^n}{n!} \frac{(\frac{W_t}{\sqrt{2t}})^{N-2n}}{(N-2n)!}.$$

Therefore, the process $(M_t^{N,0}; t \geqslant 0)$ is a martingale.

Moreover, this process can be expressed in terms of the Nth Hermite polynomials. More precisely, using the following definition of the Hermite polynomials, found in [13]

$$H_N(x) = N! \sum_{n=0}^{\lfloor N/2 \rfloor} \frac{(-1)^n}{n!} \frac{(2x)^{N-2n}}{(N-2n)!},$$

we directly obtain the following equality:

$$M_t^{N,0} = (\sqrt{2t})^N H_N\left(\frac{W_t}{\sqrt{2t}}\right).$$

Hence, the standard result that the above version of the Hermite polynomials are martingales is easily found in this framework (see for instance [23,25]). See also [28].

Remark 2.4. Another way to obtain some relationship between Brownian motion and Hermite polynomials is to compute conditional moments

$$E(W_T^N | \mathscr{F}_t) = H_N\left(\frac{W_t}{\sqrt{2(T-t)}}\right)(\sqrt{2(T-t)})^N, \quad t < T.$$

2.5.2. Poisson process

Let us consider a probability space $(\Omega, \mathfrak{I}, \mathbb{P})$ and $(N_t;\, t \geqslant 0)$ a $\mathbb{P}$-Poisson process with intensity parameter 1. Moreover, let f be a function which belongs to $D(L_{\mathrm{e}}^{K+1})$ where $K \in \mathbb{N}^*$. Applying Itô's lemma to $f(N_t)$, we obtain the characterization of the generator L_{e} as

$$L_{\mathrm{e}}(f)(x) = f(x+1) - f(x).$$

Let now f be defined as

$$f(x) = p_K(x) = \frac{[x]_K}{K!},$$

where

$$[x]_n = x(x-1)\ldots(x-n+1) \quad \text{for } x \in \mathbb{R}.$$

Applying Itô's lemma to $f(N_t)$, we obtain the explicit characterization of the generator L_{e} and the following recurrence relation:

$$L_{\mathrm{e}}(p_K) = p_{K-1}.$$

Note that we are precisely in the case

$$L_{\mathrm{e}}(\mathscr{P}_n) \subseteq \mathscr{P}_{n-1}.$$

Hence, using Proposition 2.2, the following process

$$M_t^{K,0} \triangleq \sum_{n=0}^{K} \frac{(-t)^n}{n!} L_{\mathrm{e}}^n(p_K)(N_t) = \sum_{n=0}^{K} \frac{(-t)^n}{n!} p_{K-n}(N_t) = \sum_{n=0}^{K} \frac{(-t)^n}{n!} \frac{[N_t]_{K-n}}{(K-n)!}$$

is a martingale.

Moreover, this process $(M_t^{K,0};\, t \geqslant 0)$ can be related to Charlier polynomials. More precisely, using the following definition of the Charlier polynomials, found in [13]:

$$C_K(x,t) = K! \sum_{n=0}^{K} \frac{[x]_{K-n}}{(K-n)!} \frac{(-t)^n}{n!}$$

we directly obtain the following equality:

$$M_t^{K,0} = \frac{C_K(N_t, t)}{K!}.$$

Hence, the close relationship existing between Poisson process and Charlier polynomials is underlined in this equality (see also [25]).

Remark 2.5. Tsilevich and Vershik develop in [27] some isomorphism between the Fock spaces of Brownian motion and the Poisson process which relates the Hermite and Charlier polynomials.

2.5.3. Exponential of Brownian motion

Let us consider a probability space $(\Omega, \Im, \mathbb{P})$, a standard $\mathbb{P}$-Brownian motion $(W_t; t \geqslant 0)$, the following process $(X_t; t \geqslant 0)$ defined as

$$X_t = \exp(W_t)$$

and a function *f* which belongs to $D(L_{\mathrm{e}}^{N+1})$.

Applying Itô's lemma to $f(X_t)$, we obtain the expression of the generator L_{e} on C^2 functions as

$$L_{\mathrm{e}}(f)(x) = \tfrac{1}{2}(x^2 f''(x) + x f'(x)).$$

Hence, by a simple recurrence argument, we get its iterates as

$$\forall 0 \leqslant n \leqslant N, \quad L_{\mathrm{e}}^n(f)(x) = \frac{1}{2^n} \sum_{k=1}^{n} \phi_{2k}(x) \frac{\mathrm{d}^k f}{\mathrm{d}x^k}(x),$$

where $\phi_{2k}(x)$ is the Bell polynomial of degree $2k$. More precisely, the Bell polynomials of degree *n* are generated by

$$\phi_n(x) = x \sum_{k=1}^{n} \binom{n-1}{k-1} \phi_{k-1}(x), \quad \phi_0(x) = 1 \quad \text{and} \quad \phi_n(1) = B_n$$

where $\binom{n}{k}$ are the binomial coefficients and B_n are Bell numbers given by $B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}$.

Therefore, according to Proposition 2.2

$$\sum_{n=0}^{N} \frac{1}{2^n} \frac{(-t)^n}{n!} \sum_{k=1}^{n} \phi_{2k}(X_t) \frac{\mathrm{d}^k f}{\mathrm{d}x^k}(X_t) + (-1)^{N+1} \int_0^t \mathrm{d}s \frac{1}{2^{N+1}} \frac{s^N}{N!} \sum_{k=1}^{N+1} \phi_{2k}(X_s) \frac{\mathrm{d}^k f}{\mathrm{d}x^k}(X_s) \tag{12}$$

is a martingale.

Let now *f* be a polynomial of degree *N*. For example

$$f(x) = x^N.$$

The different derivatives of the function *f* are explicitly known. In particular, one has

$$L_{\mathrm{e}}(f)(x) = \frac{N^2}{2} x^N$$

and

$$\forall 0 \leqslant n \leqslant N, \quad L_{\mathrm{e}}^n(f)(x) = \frac{N^{2n}}{2^n} x^N.$$

Note that we are precisely in the case

$$L_{\mathrm{e}}(\mathscr{P}_n) \subseteq \mathscr{P}_n.$$

Hence, formula (12) simply becomes

$$
\begin{aligned}
M_t^{N,0} &\triangleq \sum_{n=0}^{N} \frac{(-t)^n}{n!} L_{\mathrm{e}}^n(f)(X_t) + (-1)^{N+1} \int_0^t \mathrm{d}s \frac{s^N}{N!} L_{\mathrm{e}}^{N+1}(f)(X_s) \\
&= \left(\sum_{n=0}^{N} \frac{(-t)^n}{n!} \frac{N^{2n}}{2^n} \right) (X_t)^N + \frac{(-1)^{N+1}}{N!} \frac{N^{2(N+1)}}{2^{N+1}} \int_0^t \mathrm{d}s \; s^N (X_s)^N
\end{aligned}
$$

and $(M_t^{N,0}; t \geqslant 0)$ is a martingale.

3. Space–time harmonic polynomials associated with subordinators

An important class of Markov processes are Lévy processes, i.e. processes with independent and homogeneous increments. We shall even consider the more particular case when the process is increasing, that is a subordinator (for more details on Lévy processes, please refer for instance to [4,22]).

3.1. Framework

In the following, $(X_t; t \geqslant 0)$ denotes a general subordinator with no drift and Lévy measure $v(\mathrm{d}y)$ such that

$$
\text{for some } \varepsilon > 0, \quad \int_0^\infty v(\mathrm{d}y)(\exp(\varepsilon y) - 1) < \infty. \tag{13}
$$

This condition is equivalent to requiring that

$$
\text{for some } \varepsilon' > 0, \quad \int_0^\infty v(\mathrm{d}y) y \, \exp(\varepsilon' y) < \infty. \tag{14}
$$

In particular, stable subordinators do not satisfy condition (13), or (14) even for $\varepsilon' = 0$!

Denote, for any $\xi > 0$, the associated Lévy exponent

$$
\Psi(\xi) = \int_0^\infty v(\mathrm{d}y)(1 - \exp(-\xi y))
$$

so that $(\exp(-\xi X_t + \Psi(\xi)t), t \geqslant 0)$ is a martingale.

The expression of the infinitesimal generator L_{e} of X, acting on a function f, is written as

$$
L_{\mathrm{e}} f(x) = \int v(\mathrm{d}y)[f(x+y) - f(x)].
$$

Note that it is well defined for any polynomial f.

As previously, we denote by $\mathscr{P}_n$ the space of polynomials of degree less than or equal to n. In this framework, we always have

$$
L_{\mathrm{e}}(\mathscr{P}_n) = \mathscr{P}_{n-1}.
$$

Indeed, let us consider the following polynomial in $\mathscr{P}_n$:

$$f(x)=\sum_{k=0}^{n} f_k x^k.$$

Then

$$L_{\mathrm{e}}(f)(x)=\sum_{k=0}^{n-1}\widetilde{f}_k x^k,\ \text{ where }\ \widetilde{f}_k=\sum_{l=k}^{n-1}\binom{l+1}{k} f_{l+1}v_{(l+1-k)}\ \text{ and}$$
$$v_{(m)}=\int v(\mathrm{d}y)y^m \quad (<\infty \text{ for } m\geqslant 1), \tag{15}$$

where $\binom{l}{k}$ are the Binomial coefficients.

It follows from Eq. (15) that the coefficients $(f_1, f_2, \ldots, f_n)$ can be retrieved from $(\widetilde{f}_k; k=0,1,\ldots, n-1)$. Hence the result

$$L_{\mathrm{e}}(\mathscr{P}_n)=\mathscr{P}_{n-1}.$$

3.2. Space–time harmonic polynomials

First note that the associated Lévy exponent Ψ is strictly increasing from $[0;\infty)$ to $[0;\widetilde{v})$ where $\widetilde{v}=\int_0^\infty v(\mathrm{d}y)(\leqslant\infty)$. Thus Ψ^{-1} is well-defined on $[0;\widetilde{v})$.

Developing the following expressions:

$$\exp(-\xi x+t\Psi(\xi))=\sum_{j=0}^{\infty}\xi^j Q_j(x,t) \quad \text{and} \quad \exp(-\Psi^{-1}(a)x+ta)=\sum_{i=0}^{\infty}a^i P_i(x,t),$$

we obtain two sequences of polynomials in both variables x and t, with P_i and Q_j, respectively, of degree i and j in either variable x and t. Moreover, for $\Pi=P_i$ or Q_j, $\Pi(X_t,t)$ is a martingale so that

$$\left(L_x+\frac{\partial}{\partial t}\right)\Pi=0.$$

There has been a long standing interest in such functions Π, which are called *space–time harmonic* (relative to X).

3.3. Main results

The following theorems are related to results in the combinatorics literature (see e.g. [26, Exercise 5.37]) as was pointed out to us by Jim Pitman. Here, we make some connections with stochastic processes.

Theorem 3.1. *Consider a sequence of polynomials* $(p_N(x))_{N=0,1,2\ldots}$, *exactly of degree* N, *satisfying the condition*:

(c1): $p_0\equiv 1 \quad \text{and} \quad p_N(0)=0 \quad \forall N\geqslant 1.$

Then, the following properties are equivalent:

(i) $L(p_N)=p_{N-1} \quad \forall N\geqslant 1$;
(ii) *The polynomials* $P_N(x,t)$ *defined by*

$$P_N(x,t)=\sum_{k=0}^{N}\frac{(-1)^k t^{N-k}}{(N-k)!}p_k(x) \tag{16}$$

are space–time harmonic;
(iii) $(p_N(x))_{N=0,1,2\ldots}$ *is the sequence obtained from the development of*

$$\exp(-\Psi^{-1}(a)x)=\sum_{N=0}^{\infty}(-a)^N p_N(x). \tag{17}$$

More precisely, defining the double sequence $\sigma_n^{(m)}$ $(n\geqslant m)$ *as follows*:

$$(\Psi^{-1}(a))^m=m!\sum_{n=m}^{\infty}\sigma_n^{(m)}\frac{a^n}{n!}, \tag{18}$$

there is the formula for the polynomial p_N

$$p_N(x)=\frac{1}{N!}\sum_{m=0}^{N}(-1)^{N-m}\sigma_N^{(m)}x^m. \tag{19}$$

Looking at Eq. (7), it is also natural to exchange the role of x and t, i.e. to look for a sequence $(q_N)_{N=0,1,2,\ldots}$ of polynomials in t with respective degrees N, such that

$$\sum_{n=0}^{N}\frac{(-x)^n}{n!}q_{N-n}(t)$$

is a space–time harmonic polynomial.

Theorem 3.2. *Consider a sequence of polynomials* $(q_k(t))_{k=0,1,2\ldots}$, *exactly of degree k, satisfying the condition*

$$\text{(c2)}: q_0\equiv 1 \quad \text{and} \quad q_k(0)=0 \quad \forall k\geqslant 1.$$

Then, the following properties are equivalent:

(i) *The polynomials* $Q_j(x,t)$ *defined by*

$$Q_j(x,t)=\sum_{k=0}^{j}\frac{(-x)^{j-k}}{(j-k)!}q_k(t) \tag{20}$$

are space–time harmonic;

(ii) $(q_k(t))_{k=0,1,2...}$ *is the sequence obtained from the development of*

$$\exp(t\Psi(\xi)) = \sum_{n=0}^{\infty} \xi^n q_n(t). \tag{21}$$

More precisely, defining the double sequence $s_n^{(m)}$ $(n \geqslant m)$ *as follows*:

$$(\Psi(\xi))^m = m! \sum_{n=m}^{\infty} s_n^{(m)} \frac{\xi^n}{n!}, \tag{22}$$

there is the formula for the polynomial q_n

$$q_n(t) = \frac{1}{n!} \sum_{m=0}^{n} s_n^{(m)} t^m. \tag{23}$$

Note that some similar results are presented in Pitman's Saint Flour course [20] (Chapter 2) when he studies moments of Lévy processes and relates them to sequences of polynomials of Binomial type.

3.3.1. An introduction to certain generalized Stirling numbers

The classical Stirling numbers and the Gamma process: If the subordinator process is the Gamma process $(\Gamma_t;\, t \geqslant 0)$ i.e.

$$E[\exp(-\xi\Gamma_t)] = \frac{1}{(1+\xi)^t},$$

so that

$$\Psi_\Gamma(\xi) = \log(1+\xi), \quad \Psi_\Gamma^{-1}(a) = (\exp(a) - 1),$$

then $(\sigma_n^{(m)})$ are the classical Stirling numbers of the second kind, defined by (see e.g. [1, Chapter 24, p. 824])

$$(\exp(a) - 1)^m = m! \sum_{n=m}^{\infty} \sigma_n^{(m)} \frac{a^n}{n!}$$

and $(s_n^{(m)})$ are the classical Stirling numbers of the first kind, defined by (see for e.g. [1, Chapter 24, p. 824])

$$(\log(1+\xi))^m = m! \sum_{n=m}^{\infty} s_n^{(m)} \frac{\xi^n}{n!}.$$

The classical Stirling numbers and the Poisson process: If the subordinator process is the Poisson process $(\eta_t;\, t \geqslant 0)$ i.e.

$$E[\exp(-\xi\eta_t)] = \exp[-t(1-\exp(-\xi))],$$

so that

$$\Psi_\eta(\xi) = 1 - \exp(-\xi), \quad \Psi_\eta^{-1}(a) = -\log(1-a).$$

The classical Stirling numbers of the first and the second kind may be related to the Poisson process, with their role inverted in comparison with the Gamma process case.

A generalization of the Stirling numbers: From the two above examples, the coefficients $(\sigma_n^{(m)})$ and $(s_n^{(m)})$ deserve the name of generalized Stirling numbers of the second, resp., first, kind.

We are well aware that the terminology "generalized Stirling numbers" designates a vast class of generalizations of Stirling numbers (see, e.g. [12] or [6]).

The generalized Stirling numbers, which appear in, e.g. these two papers do not necessarily fit into our framework. We shall show later that the intersection between our generalized Stirling numbers and those for e.g. in [12] is, however reasonably large. In particular, our generalized Stirling numbers associated with the Esscher transformed stable subordinators, whose Lévy measure is

$$\nu(\mathrm{d}y) = C\frac{\exp(-ay)}{y^{1+b}}\,\mathrm{d}y \quad \text{for } b < 1,$$

appear in the generalized Stirling numbers' family proposed in [12].

These subordinators are well-known to play an essential role in the study of the Poisson–Dirichlet laws (see e.g. [21]).

3.4. Proofs of Theorems 3.1 and 3.2

3.4.1. Proof of Theorem 3.1

(ii) $\Rightarrow$ (i) Since $P_N(x,t)$ is a space–time harmonic polynomial, it satisfies

$$L_x(P_N) + \frac{\partial}{\partial t}P_N = 0,$$

where for clarity, L_x denotes the operator L acting on a function of x.

This yields

$$\sum_{k=0}^{N}\frac{t^{N-k}}{(N-k)!}L_x(p_k)(x) + \sum_{k=0}^{N-1}\frac{t^{N-k-1}}{(N-k-1)!}p_k(x) = 0.$$

Identifying the coefficients of t^k $(0 \leqslant k \leqslant N)$ gives (i).

(i) $\Leftrightarrow$ (iii) *Existence of the* p_N*'s satisfying* (c1) and (iii) (:(iii) $\Rightarrow$ (i))

Since the process

$$\{\exp(-\Psi^{-1}(a)X_t + ta), t \geqslant 0\}$$

is a martingale, we have

$$L_x(\exp(-\Psi^{-1}(a)x)) = -a\exp(-\Psi^{-1}(a)x). \tag{24}$$

Developing $\exp(-\Psi^{-1}(a)x)$ with the help of formula (17) on both sides of Eq. (24), we obtain (i). We note that the property (c1) follows directly from Eq. (17) by taking $x = 0$.

Uniqueness of the p_N*'s* (:(i) $\Rightarrow$ (iii))

For $N \geqslant 1$, we write

$$p_N(x) = \sum_{k=1}^{N} F_{k,N} x^k$$

and we need to show that the double sequence $(F_{k,N})_{k \leqslant N}$ is uniquely determined from (c1)) and (i).

From the formula

$$Lf(x) = \int \nu(\mathrm{d}y)[f(x+y) - f(x)],$$

which is valid at least for polynomials, we easily obtain

$$L(p_N)(x) = \sum_{j=0}^{N-1} x^j \left(\sum_{k=j+1}^{N} \binom{k}{j} F_{k,N} \right) \nu_{(k-j)},$$

where

$$\nu_{(m)} \stackrel{(def)}{=} \int \nu(\mathrm{d}y) y^m,$$

is the mth moment of ν.

Thus, the property (i)

$$L(p_N) = p_{N-1}$$

amounts to

$$\forall j \leqslant N-1, \quad \sum_{l=j}^{N-1} \left(\binom{l+1}{j} \nu_{(l+1-j)} \right) F_{l+1,N} = F_{j,N-1}. \tag{25}$$

But this linear system of equations with the unknowns $(F_{k,N})_{1 \leqslant k \leqslant N}$ admits one and only one solution since it may be written as

$$\begin{pmatrix} \binom{1}{0}\nu_{(1)} & \binom{2}{0}\nu_{(2)} & \binom{3}{0}\nu_{(3)} & \cdots & \binom{N}{0}\nu_{(N)} \\ 0 & \binom{2}{1}\nu_{(1)} & \binom{3}{1}\nu_{(2)} & \cdots & \binom{N}{1}\nu_{(N-1)} \\ 0 & 0 & \binom{3}{2}\nu_{(1)} & \cdots & \binom{N}{2}\nu_{(N-2)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \binom{N}{N-1}\nu_{(1)} \end{pmatrix} \begin{pmatrix} F_{1,N} \\ F_{2,N} \\ F_{3,N} \\ \cdots \\ F_{N,N} \end{pmatrix} = \begin{pmatrix} F_{0,N-1} \\ F_{1,N-1} \\ F_{2,N-1} \\ \cdots \\ F_{N-1,N-1} \end{pmatrix}.$$

Hence, the vector $(F_{k,N})_{1 \leqslant k \leqslant N}$ is determined uniquely.

(iii) $\Rightarrow$ (ii) As written previously, for any $a > 0$, the process $\{\exp(-\Psi^{-1}(a)X_t + ta), t \geqslant 0\}$ is a martingale. We recall the definition of the p_m's via (iii)

$$\exp(-\Psi^{-1}(a)x) = \sum_{m=0}^{\infty} (-a)^m p_m(x).$$

On the other hand

$$\exp(ta) = \sum_{n=0}^{\infty} \frac{(at)^n}{n!}.$$

Hence

$$\exp(-\Psi^{-1}(a)x + ta) = \sum_{m=0}^{\infty} (-a)^m p_m(x) \sum_{n=0}^{\infty} \frac{(at)^n}{n!}$$

and

$$\exp(-\Psi^{-1}(a)x + ta) = \sum_{i=0}^{\infty} a^i P_i(x, t), \tag{26}$$

where

$$P_i(x, t) = \sum_{k=0}^{i} \frac{(-1)^k t^{i-k}}{(i-k)!} p_k(x)$$

is a sequence of space–time harmonic polynomials. □

3.4.2. Proof of Theorem 3.2

(ii) ⇒ (i) Recall that for any $\xi > 0$, the process $\{\exp(-\xi X_t + t\Psi(\xi)), t \geqslant 0\}$ is a martingale.

It follows from the classical formula:

$$\exp(-\xi x) = \sum_{m=0}^{\infty} \frac{(-\xi x)^m}{m!}$$

together with (19) that

$$\exp(-\xi x + t\Psi(\xi)) = \sum_{m=0}^{\infty} \frac{(-\xi x)^m}{m!} \sum_{n=0}^{\infty} \xi^n q_n(t).$$

From which, if we define the sequence Q_j by

$$\exp(-\xi x + t\Psi(\xi)) = \sum_{j=0}^{\infty} \xi^j Q_j(x, t), \tag{27}$$

where

$$Q_j(x, t) = \sum_{k=0}^{j} \frac{(-x)^{j-k}}{(j-k)!} q_k(t).$$

Thus, from Eq. (27) the Q_j's are space–time harmonic polynomials.

(i) ⇒ (ii) Consider the family $(q_0, q_1, \ldots, q_j)$ to be known. We want to determine q_{j+1}.

According to (i), $Q_{j+1}(x,t)$ is space–time harmonic. Hence

$$L_x(Q_{j+1})(x,t) + \frac{\partial}{\partial t} Q_{j+1}(x,t) = 0.$$

Such an equation is valid for every x, especially for $x=0$

$$L_x(Q_{j+1})(0,t) + \frac{\partial}{\partial t} Q_{j+1}(0,t) = 0. \tag{28}$$

Let us denote $\widehat{Q}_{j+1}(x,t)$ the following polynomial:

$$\widehat{Q}_{j+1}(x,t) = Q_{j+1}(x,t) - q_{j+1}(t),$$

i.e. the polynomial composed by the $(j+1)$ first terms of $Q_{j+1}(x,t)$, as given in (20).

Hence, Eq. (28) may also be written as

$$L_x(\widehat{Q}_{j+1} + q_{j+1}(t))(0,t) + \frac{\partial}{\partial t}[\widehat{Q}_{j+1}(0,t) + q_{j+1}(t)] = 0,$$

which simplifies as

$$L_x(\widehat{Q}_{j+1})(0,t) + \frac{\partial}{\partial t} q_{j+1}(t) = 0,$$

since $L_x(q_{j+1}(t)) = 0$ and $(\partial/\partial t)\widehat{Q}_{j+1}(0,t) = 0$.

Moreover

$$\begin{aligned} L_x(\widehat{Q}_{j+1})(0,t) &= \int \nu(\mathrm{d}y)[\widehat{Q}_{j+1}(y,t) - \widehat{Q}_{j+1}(0,t)] \equiv \int \nu(\mathrm{d}y)\widehat{Q}_{j+1}(y,t) \\ &= \sum_{k=0}^{j} q_k(t) \int \nu(\mathrm{d}y)\left[\frac{(-y)^{j+1-k}}{(j+1-k)!}\right] = \sum_{k=0}^{j} q_k(t)\frac{(-1)^{j+1-k}}{(j+1-k)!}\nu_{(j+1-k)}, \end{aligned}$$

where $\nu_{(n)}$ denotes the nth moment of the Lévy measure ν associated with X.

Consequently

$$\frac{\partial}{\partial t} q_{j+1}(t) = \sum_{k=0}^{j} q_k(t)\frac{(-1)^{j-k}}{(j+1-k)!}\nu_{(j+1-k)}$$

or

$$q_{j+1}(t) = \sum_{k=0}^{j} \widehat{q}_k(t)\frac{(-1)^{j-k}}{(j+1-k)!}\nu_{(j+1-k)}, \tag{29}$$

where $\widehat{q}_k(t)$ is the primitive of $q_k(t)$ such that

$$\widehat{q}_k(0) = 0$$

as to satisfy condition (c2).

By recurrence, we obtain a unique sequence (q_k). Moreover, the sequence (q_n) presented in (ii) satisfies (i). Hence, it is the only sequence which satisfies (i). □

Remark 3.3. The relation (29) may simply be understood as a consequence of the martingale property of

$$X_t^{j+1} - \int_0^t \mathrm{d}s\, L_{\mathrm{e}}(x^{j+1})(X_s),$$

since, by formula (34), $q_n(-t)$ is a multiple of $E[X_t^n]$.

3.5. Various comments

3.5.1. Relations between both theorems

(a) Comparing formulae (26) and (27), we easily obtain the following relationship between the two sequences (P_n) and (Q_j) of space–time harmonic polynomials:

$$P_n(x,t) = \frac{1}{n!}\sum_{j=0}^{n} j!\sigma_n^{(j)} Q_j(x,t). \tag{30}$$

Moreover, when looking deeper at formulae (16) and (20), we get

$$\frac{(-1)^{n-m}}{m!} p_{n-m}(x) = \frac{1}{n!}\sum_{j=m}^{n} j!\sigma_n^{(j)} \sum_{l=0}^{j-m} \frac{(-1)^l s_{j-l}^{(m)}}{l!(j-l)!} x^l = \frac{1}{n!}\sum_{l=0}^{n-m} \frac{(-1)^l}{l!} x^l \sum_{j=l+m}^{n} \frac{j!\sigma_n^{(j)} s_{j-l}^{(m)}}{(j-l)!}.$$

Comparing this result with formula (19), i.e.

$$p_{n-m}(x) = \frac{1}{(n-m)!}\sum_{h=0}^{n-m} (-1)^{n-m-h} \sigma_{n-m}^{h} x^h,$$

we deduce

$$\sigma_{n-m}^{(h)} = \frac{1}{\binom{n}{m}} \sum_{l=m}^{n-h} \binom{l+h}{h} \sigma_n^{(l+h)} s_l^{(m)}. \tag{31}$$

In particular, when $h=0$, one has the well-known formula

$$\sum_{n=m}^{p} \sigma_p^{(n)} s_n^{(m)} = \delta_{p,m}.$$

(b) Symmetrically, there exist formulae expressing the sequence (Q_j) in terms of the (P_n)'s

$$Q_j(x,t) = \frac{1}{j!}\sum_{n=0}^{j} n! s_j^{(n)} P_n(x,t). \tag{32}$$

The relation (32) follows from Eq. (26) where we have taken $a = \Psi(\xi)$, and Eq. (22).

3.5.2. Some remarks on the generalized Stirling numbers

Recurrence relations between the classical Stirling numbers: We have shown the following results for the Stirling numbers of the first kind ($s_n^{(m)}$) and of the second kind ($\sigma_n^{(m)}$):

$$\sigma_{n-m}^{(h)} = \frac{1}{\binom{n}{m}} \sum_{l=m}^{n-h} \binom{l+h}{h} \sigma_n^{(l+h)} s_l^{(m)}.$$

Moreover, according to formula (25), applied to the Gamma process, we obtain that the Stirling numbers of the second kind have to satisfy the following ascending recurrence relation, with respect to N:

$$\forall j \leqslant N-1, \quad \frac{1}{N} \sum_{l=j}^{N-1} \frac{(l+1)!}{(l+1-j)} (-1)^l \sigma_N^{(l+1)} = (-1)^j j! \sigma_{N-1}^{(j)}. \tag{33}$$

Relation between the generalized Stirling numbers and the moments of X_t and of v: As written previously

$$E[\exp(-\xi X_t)] = \exp(-t\Psi(\xi)).$$

Developing both sides, we obtain, using Eq. (21)

$$\sum_{n=0}^{\infty} \frac{(-\xi)^n}{n!} E[X_t^n] = \sum_{n=0}^{\infty} \xi^n q_n(-t).$$

Hence

$$q_n(-t) = \frac{(-1)^n}{n!} E[X_t^n]. \tag{34}$$

Comparing with Eq. (23),

$$q_n(-t) = \frac{1}{n!} \sum_{m=0}^{n} s_n^{(m)} (-t)^m,$$

we obtain

$$E[X_t^n] = \sum_{m=0}^{n} (-1)^{n+m} s_n^{(m)} t^m.$$

On the other hand, the moments of v are related to those of the variables (X_t) with t varying, since from Remark 3.3

$$E[X_t^{j+1}] = \sum_{k=0}^{j} \binom{j+1}{k} \int_0^t \mathrm{d}s\, E(X_s^k) v_{(j+1-k)}.$$

Generalized Stirling numbers associated to the generalized stable subordinators: In this section, we show that some particular class of particular generalized Stirling numbers, as defined in [12], are in fact “our” generalized Stirling numbers for the generalized stable subordinators, whose Lévy measure is:

$$v(\mathrm{d}y) = C\, \frac{\exp(-ay)}{y^{1+b}}\, \mathrm{d}y \quad \text{for } b < 1. \tag{35}$$

In Theorem 2 in [12, p. 372], we take the particular case $r=0$, which gives

$$\left(\frac{(1+\alpha\xi)^{\beta/\alpha}-1}{\beta}\right)^k = k!\sum_{n=0}^{\infty} S_{\alpha,\beta}(n,k)\frac{\xi^n}{n!}. \tag{36}$$

We note that the left-hand side of Eq. (36) may be obtained as

$$\int v_{\alpha,\beta}(\mathrm{d}y)(1-\exp(-\xi y)),$$

where

$$v_{\alpha,\beta}(\mathrm{d}y) = \frac{\exp(-y/\alpha)}{\alpha^{1+(\beta/\alpha)}} y^{1+(\beta/\alpha)}\,\mathrm{d}y.$$

We recognize the Lévy measure in Eq. (35), where

$$C = \frac{1}{\alpha^{1+(\beta/\alpha)}};$$
$$a = \frac{1}{\alpha};$$
$$b = \frac{\beta}{\alpha}.$$

3.6. *On pairs of subordinators in duality*

3.6.1. *Introduction: the Poisson and Gamma processes*

Let us denote, respectively, by $(\eta_t; t \geqslant 0)$ and $(G_t, t \geqslant 0)$ the standard Poisson and Gamma processes, whose laws are characterized by

$$E[\exp(-a\eta_t)] = \exp(-t(1-\exp(-a))),$$
$$E[\exp(-\xi G_t)] = \exp(-t\log(1+\xi)).$$

Hence, with obvious notation, we have

$$\Psi_\eta(a) = 1-\exp(-a) \quad \text{and} \quad \Psi_G(\xi) = \log(1+\xi),$$

so that

$$-\Psi_G^{-1}(-a) = \Psi_\eta(a) \quad \text{or equivalently,} \quad -a = \Psi_G(-\Psi_\eta(a)).$$

3.6.2. *Towards a generalization*

Existence of pairs of subordinators: In this section, we look for pairs of subordinators X and Y which play the roles of G and η in that

$$-a = \Psi_X(-\Psi_Y(a)) \tag{37}$$

or equivalently

$$-\Psi_X^{-1}(-a) = \Psi_Y(a). \tag{38}$$

In terms of the Lévy measures of X and Y, this may be written

$$\begin{aligned}-a &= \int \nu_X(\mathrm{d}x)[1 - \exp\{-x(-\Psi_Y(a))\}] \\ &= \int \nu_X(\mathrm{d}x)[1 - \exp\{x\Psi_Y(a)\}] \\ &= \int \nu_X(\mathrm{d}x)\left(1 - \frac{1}{E[\exp(-aY_x)]}\right),\end{aligned}$$

so that finally

$$\int \nu_X(\mathrm{d}x)\left(\frac{1}{E[\exp(-aY_x)]} - 1\right) = a. \tag{39}$$

Given X, the existence of a subordinator Y satisfying (37) is not obvious at all.

Implications for polynomials: We now discuss the symmetric roles which are played by X and Y, assumed to satisfy Eq. (38). Indeed, from this equation, we see that

$$\Psi_Y^{-1}(-\xi) = -\Psi_X(\xi),$$

so that the pairs (X, Y) and (Y, X) entertain the same relationship.

Raising both sides of Eq. (38) to the power m, we easily obtain

$$(-1)^{n+m}\sigma_n^{(m)}(X) = s_n^{(m)}(Y)$$

with obvious notation.

As a consequence, from both Eqs. (19) and (23), we obtain

$$p_n(X; x) = q_n(Y; x).$$

Hence, from both Eqs. (16) and 20

$$P_n(X; x, t) = (-1)^n Q_n(Y; t, x).$$

3.6.3. A compound Poisson process example

Let us take for (X_t) the compound Poisson process with

$$\nu_X(\mathrm{d}x) = C\exp(-\alpha x)\,\mathrm{d}x,$$

so that

$$\Psi_X(\xi) = C\int_0^\infty \mathrm{d}x\exp(-\alpha x)(1 - \exp(-x\xi)) = \frac{C\xi/\alpha^2}{1 + \xi/\alpha}. \tag{40}$$

We then show the existence of Y satisfying (39), which in this case, takes the form

$$C\int_0^\infty \mathrm{d}x\exp(-\alpha x)\{\exp(x\Psi_Y(a)) - 1\} = a,$$

that is

$$C\left\{\frac{1}{\alpha - \Psi_Y(a)} - \frac{1}{\alpha}\right\} = a.$$

It follows that

$$\Psi_Y(a) = \frac{\alpha^2 a/C}{1+\alpha a/C}$$

which, when compared with Eq. (40), yields

$$\nu_Y(\mathrm{d}y) = C' \exp(-\alpha' y)\,\mathrm{d}y, \quad \text{where} \quad C' = C \text{ and } \alpha' = \frac{C}{\alpha}.$$

3.6.4. On generalized stable subordinators

In this paragraph, we consider the three parameters family of subordinators (X_t) given by

$$\nu_X(\mathrm{d}x) = Cx^{\beta-1}\exp(-\alpha x)\,\mathrm{d}x \quad \text{for } \beta > -1. \tag{41}$$

This subordinator is related to the CGMY process introduced in [5] where in the bounded variations case, the CGMY process is the difference between two processes of the family (41). Moreover, we should like to call X a generalized stable subordinator, since in case $\alpha = 0$ and $\beta = -\gamma$, $0 < \gamma < 1$, X is a stable (γ) subordinator. But this family (41) also contains the Gamma case (for $\beta = 0$) and the compound Poisson case (for $\beta = 1$) with exponential jumps.

Case $\beta > 0$: We now compute $\Psi_X(\xi)$

$$\Psi_X(\xi) = C\int_0^\infty \mathrm{d}x\ x^{\beta-1}\exp(-\alpha x)(1-\exp(-x\xi)) = C\Gamma(\beta)\left(\frac{1}{\alpha^\beta} - \frac{1}{(\alpha+\xi)^\beta}\right). \tag{42}$$

Then, we look for Y such that

$$C\int_0^\infty \mathrm{d}x\ x^{\beta-1}\exp(-\alpha x)\{\exp(x\Psi_Y(a)) - 1\} = a,$$

which yields, after some elementary computation

$$\Psi_Y(a) = \alpha - \frac{1}{(\frac{1}{\alpha^\beta} + \frac{a}{C\Gamma(\beta)})^{1/\beta}} = (C\Gamma(\beta))^{1/\beta}\left(\frac{1}{(\frac{C\Gamma(\beta)}{\alpha^\beta})^{1/\beta}} - \frac{1}{\left(\frac{C\Gamma(\beta)}{\alpha^\beta}+a\right)^{1/\beta}}\right).$$

Hence, by comparison with Eq. (42), we find that Y belongs to the same family, with parameters (C', α', β') given by

$$\beta' = \frac{1}{\beta}, \quad \alpha' = \frac{C\Gamma(\beta)}{\alpha^\beta}, \quad C'\Gamma\left(\frac{1}{\beta}\right) = [C\Gamma(\beta)]^{1/\beta}.$$

Case $-1 < \beta < 0$: We now compute $\Psi_X(\xi)$

$$\Psi_X(\xi) = C\int_0^\infty \mathrm{d}x\ x^{\beta-1}\exp(-\alpha x)(1-\exp(-x\xi)),$$

i.e. when setting $\beta = -\gamma$,

$$\Psi_X(\xi) = C\Gamma(-\gamma)(\alpha^\gamma - (\alpha+\xi)^\gamma).$$

Since

$$\Gamma(-\gamma) = \frac{-1}{\gamma}\Gamma(1-\gamma),$$

we finally obtain

$$\Psi_X(\xi) = C\frac{\Gamma(1-\gamma)}{\gamma}((\alpha+\xi)^\gamma - \alpha^\gamma)$$

$$\Psi_X(\xi) = C\frac{\Gamma(1+\beta)}{\beta}\left(\frac{1}{\alpha^\beta} - \frac{1}{(\alpha+\xi)^\beta}\right) = C\Gamma(\beta)\left(\frac{1}{\alpha^\beta} - \frac{1}{(\alpha+\xi)^\beta}\right).$$

Hence, formula (42) may be extended to the case $-1<\beta<0$. The results of the previous case are also extended.

Acknowledgements

We thank Marc Yor for his generosity, his constant availability and patience and for his helpful insights on this work. We also thank Jim Pitman for mentioning Stanley’s book and some relevant literature. W. Schoutens dedicates this paper to his scientific mentor J.L. Teugels and would like to thank him for his continuous guidance through the scientific world.

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