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Convergence of the variants of the Chebyshev–Halley iteration family under the Hölder condition of the first derivative☆

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Abstract

The present paper is concerned with the convergence problem of the variants of the Chebyshev–Halley iteration family with parameters for solving nonlinear operator equations in Banach spaces. Under the assumption that the first derivative of the operator satisfies the Hölder condition of order p, a convergence criterion of order $1+p$ for the iteration family is established. An application to a nonlinear Hammerstein integral equation of the second kind is provided.

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1. Introduction

Let X and Y be (real or complex) Banach spaces, $\Omega \subseteq X$ be an open subset and let $F : \Omega \subseteq X \to Y$ be a nonlinear operator on Ω. Finding solutions of the nonlinear operator equation

$$F(x) = 0 \tag{1.1}$$

in Banach spaces is a very general subject which is widely used in both theoretical and applied areas of mathematics. When F is Fréchet differentiable, the most important method to find the approximation solution is Newton's method. One of the famous results on Newton's method is the well-known Kantorovich theorem (cf. [21]) which guarantees convergence of Newton's sequence to a solution under very mild conditions. Further researches on Newton's method are referred to [24–26].

As it is well known, in the case when F has the second continuous Fréchet derivative on Ω, there are several kinds of cubic generalizations for Newton's method. The most important two are the Chebyshev method and the Halley method, see e.g., [1–4,17,20,28], respectively. Another more general family of the cubic extensions is the family of Chebyshev–Halley-type methods, which was proposed in [13] by Gutiérrez and Hernández. This family includes the

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Chebyshev method ($\alpha = 0$) and the Halley method ($\alpha = \frac{1}{2}$) as well as the convex acceleration of Newton's method (or the super-Halley method) ($\alpha = 1$, cf. [5,15,14]) as its special cases and has been explored extensively in [13,16,27]. Let $\alpha \in [0, 1]$. Then the family of Chebyshev–Halley-type methods is defined by

$$x_{\alpha,n+1} = x_{\alpha,n} - \left[\mathbf{I} + \tfrac{1}{2}L_F(x_{\alpha,n})[\mathbf{I} - \alpha L_F(x_{\alpha,n})]^{-1}\right] F'(x_{\alpha,n})^{-1}F(x_{\alpha,n}), \quad n = 0, 1, \ldots, \tag{1.2}$$

where $\mathbf{I}$ is the identity and, for each $x \in X$, $L_F(x)$ is a bounded linear operator from X to Y defined by

$$L_F(x) = F'(x)^{-1}F''(x)F'(x)^{-1}F(x), \quad x \in X. \tag{1.3}$$

Recent interests are focused on the study of the variants of the Chebyshev iteration and the Halley iteration as well as the convex acceleration of Newton's method, which are obtained by replacing the second derivative in (1.3) with the difference of the first derivatives at x and z:

$$F''(x)(z - x) \approx F'(z) - F'(x),$$

where $z = x + \lambda(-F'(x)^{-1}F(x))$ while $\lambda \in (0, 1]$ is a parameter. This is equivalent that $L_F(x)$ is replaced by the bounded linear operator $H(x, y) : X \to Y$ defined by

$$H(x, y) = \frac{1}{\lambda}F'(x)^{-1}[F'(x + \lambda(y - x)) - F'(x)], \tag{1.4}$$

where $y = x - F'(x)^{-1}F(x)$.

Such a variant has the advantage that avoids the computation of the second derivatives (so works for operators with the first derivatives only) but keeps the higher orders of convergence. The variant of the convex acceleration of Newton's method was first presented by Ezquerro and Hernández in [7], where a cubical convergence criterion based on the Lipschitz constant and the boundary of the second derivative was established under the assumption that the second derivative of F satisfies the Lipschitz condition. The same variant was done in [18] for the Chebyshev method, and cubical convergence criterions for this variant were studied in [18,19]. Convergence criterions under the Lipschitz condition of the first derivative were discussed for the variants of the convex acceleration of Newton's method, the Chebyshev method and the Halley method, respectively, in [8,18,30].

The variant of the family of Chebyshev–Halley-type methods was presented in [29]. Under the assumption that the second derivative F'' satisfies the Hölder condition on some suitable closed ball $\mathbf{B}(x_0, R)$:

$$\|F''(x) - F''(y)\| \leqslant K\|x - y\|^p \quad \text{for all } x, y \in \mathbf{B}(x_0, R), \tag{1.5}$$

a unified convergence criterion depending on the values of the operator, its first derivative and second derivative at the initial point x_0 as well as the Hölder constant K was established for the variant.

The present paper is a continuation of the paper [29]. More precisely, just assuming that the first derivative F' satisfies the Hölder condition on some suitable closed ball $\mathbf{B}(x_0, R)$:

$$\|F'(x) - F'(y)\| \leqslant K\|x - y\|^p \quad \text{for all } x, y \in \mathbf{B}(x_0, R) \tag{1.6}$$

(its second derivative is not necessary), we establish a unified convergence criterion only depending on the values of the operator and its first derivative at the initial point x_0 as well as the Hölder constant for the variant of the family of Chebyshev–Halley-type methods. The main theorem is stated in Section 3, which includes the corresponding results for the variant of the convex acceleration of Newton's method and the variant of the Chebyshev method as well as the variant of the Halley method obtained in [8,18,30] as special examples. An application to a nonlinear Hammerstein integral equation of the second kind (cf. [22]) is given in the final section.

We should compare the convergence criterion in the present paper with that in [29]. The main difference between them is that we use condition (1.6) here instead of (1.5). Clearly, (1.5) implies (1.6) (with different constant K). In particular, in the case when F'' does not exist or F'' is unbounded, condition (1.5) is not satisfied. Section 4 of the present paper provides such an example (cf. Example 4.1), where the operator F has the first derivative F' satisfying (1.6) (so the convergence criterion in the present paper may be applicable) but does not have the second derivative on any closed ball containing x_0 and the convergence criterion in [29] is not applicable.

We end this section by introducing some notations and basic assumptions. Let $\alpha \in [0,1]$, $\lambda \in (0,1]$ and $x_0 \in \Omega$. Recall that H is defined by (1.4) and define

$$Q(x,y) = -\tfrac{1}{2}H(x,y)[\mathbf{I} + \alpha H(x,y)]^{-1}. \tag{1.7}$$

Note that, for each $x, y \in X$, $Q(x,y)$ is a bounded linear operator from X to Y. Then the variant of the Chebyshev–Halley iteration family with parameters λ and initial point $x_{\alpha,0} = x_0$ can be represented as

$$y_{\alpha,n} = x_{\alpha,n} - F'(x_{\alpha,n})^{-1}F(x_{\alpha,n}) \quad \text{for each } n = 0,1,2,\ldots, \tag{1.8}$$

$$x_{\alpha,n+1} = y_{\alpha,n} + Q(x_{\alpha,n}, y_{\alpha,n})(y_{\alpha,n} - x_{\alpha,n}) \quad \text{for each } n = 0,1,2,\ldots. \tag{1.9}$$

Throughout the whole paper, we shall always assume that $\alpha \in [0,1]$, $\lambda \in (0,1]$, $p \in (0,1]$ and that $x_0 \in \Omega$ such that the inverse $F'(x_0)^{-1}$ of F' at x_0 exists. For $r > 0$, we use $\mathbf{B}(x_0, r)$ to denote the closed ball with radius r and center x_0. Moreover, for convenience, we shall delete the subscript α in (1.8) and (1.9); that is, we write $y_n = y_{\alpha,n}$ and $x_n = x_{\alpha,n}$ for each n.

2. Iteration sequence $\{a_n\}$

We begin with two real-valued functions defined below, which will play an important role in analyzing the convergence order of the variant of the Chebyshev–Halley iteration family. The technique used here was developed by Ezquerro, Gutiérrez and Hernández in [6,12] and has been also used in papers [7–10,18,19,29], etc. For a real-valued function h and a point $t_0 \in \mathbb{R}$, we shall use $h(t_0-)$ to denote the left limit of h at t_0, i.e., $h(t_0-) := \lim_{t \to t_0-} h(t)$. Consider the function s defined by

$$s(t) := 1 - \lambda^{1-p} t \left(1 + \frac{t}{2(1-\alpha t)}\right)^p \quad \text{for each } t \in \left[0, \frac{1}{\alpha}\right).$$

Clearly, s is a strictly monotonic decreasing, continuous function with $s(0) = 1$ and $s(1/\alpha -) < 0$. Hence, there exists $s_\alpha \in [0, 1/\alpha)$ such that $s(s_\alpha) = 0$. Now define

$$f(s) := \frac{[2(1-\alpha s)]^p}{[2(1-\alpha s)]^p - \lambda^{1-p} s[(1-2\alpha)s + 2]^p} \quad \text{for each } s \in [0, s_\alpha) \tag{2.1}$$

and

$$g(s) := \frac{(p+1)s[2(1-\alpha s)]^p + \lambda^{1-p} s[(1-2\alpha)s+2]^{p+1}}{(p+1)[2(1-\alpha s)]^{p+1}} \quad \text{for each } s \in [0, s_\alpha). \tag{2.2}$$

Note that f and g can be, respectively, rewritten as

$$f(s) = \frac{1}{1 - \lambda^{1-p} s(1 + (s/2(1-\alpha s)))^p} \quad \text{for each } s \in [0, s_\alpha)$$

and

$$g(s) = \frac{s}{2(1-\alpha s)} + \frac{\lambda^{1-p} s}{p+1}\left(1 + \frac{s}{2(1-\alpha s)}\right)^{p+1} \quad \text{for each } s \in [0, s_\alpha).$$

The following lemma, which describes some properties of the functions f and g, is direct.

Lemma 2.1. *Let f and g be defined as above. Then*

(i) *f is a strictly monotonic increasing and continuous function on $[0, s_\alpha)$ with $f(0) = 1$,*
(ii) *g is a strictly monotonic increasing and continuous function on $[0, s_\alpha)$ satisfying*

$$g(\gamma s) < \gamma g(s) \quad \textit{for each } \gamma \in (0,1), \quad s \in (0, s_\alpha). \tag{2.3}$$

Define

$$G_\alpha(s) := f^{1+p}(s)g^p(s) = \frac{s^p\{[2(1-\alpha s)]^p + \lambda^{1-p}/(p+1)[(1-2\alpha)s+2]^{p+1}\}^p}{\{[2(1-\alpha s)]^p - \lambda^{1-p}s[(1-2\alpha)s+2]^p\}^{1+p}} \quad \text{for each } s \in [0, s_\alpha). \tag{2.4}$$

Then G_α is strictly monotonic increasing and continuous on $[0, s_\alpha)$ thanks to Lemma 2.1. Since $G_\alpha(0) = 0$ and $G_\alpha(s_\alpha -) > 1$, there exists a unique $r_\alpha \in (0, s_\alpha)$ such that $G_\alpha(r_\alpha) = 1$.

Let $a_0 \in [0, r_\alpha)$ and define an iteration $\{a_n\}$ by

$$a_n = a_{n-1}G_\alpha(a_{n-1}) = a_{n-1}f^{1+p}(a_{n-1})g^p(a_{n-1}) \quad \text{for each } n = 1, 2, \ldots . \tag{2.5}$$

Then we have the following obvious lemmas.

Lemma 2.2. *Let $a_0 \in [0, r_\alpha)$. Then*

(i) $f^{1+p}(a_0)g^p(a_0) < 1$,
(ii) $\{a_n\}$ *is a strictly monotonic decreasing sequence*,
(iii) *for each $n = 1, 2, \ldots$,*

$$\lambda^{1-p}a_n\left(1 + \frac{a_n}{2(1-\alpha a_n)}\right)^p < 1. \tag{2.6}$$

Lemma 2.3. *Let $a_0 \in (0, r_\alpha)$ and set $\gamma := f^{1+p}(a_0)g^p(a_0)$. Then $\gamma = a_1/a_0 < 1$ and the following assertions hold for all $n = 1, 2, \ldots$*

$$a_n \leqslant \gamma^{((1+p)^n-1)/p}a_0 \tag{2.7}$$

and

$$\prod_{j=0}^{n-1} f(a_j)g(a_j) \leqslant \gamma^{((1+p)^n-1)/p^2}(f(a_0)^{-1/p})^n. \tag{2.8}$$

3. Convergence criterion of the iterations

Throughout this section, we shall always assume that $R > 0$ and

$$\|F'(x_0)^{-1}F(x_0)\| \leqslant \eta, \tag{3.1}$$

$$\|F'(x_0)^{-1}(F'(x) - F'(y))\| \leqslant K\|x-y\|^p \quad \text{for all } x, y \in \mathbf{B}(x_0, R), \tag{3.2}$$

where $a_0 := K\lambda^{p-1}\eta^p$ and

$$R = \left(1 + \frac{a_0}{2(1-\alpha a_0)}\right)\frac{\eta}{1 - [\gamma f(a_0)^{-1}]^{1/p}}. \tag{3.3}$$

Recall that r_α is the unique solution of the equation $G_\alpha(s) = 1$ and $\gamma = f^{1+p}(a_0)g^p(a_0)$. Then $R > 0$ is well-defined if $a_0 < r_\alpha$. Before giving our main theorem, we first need several lemmas, the proofs of which are standard, see for example [29].

Lemma 3.1. *Suppose that $a_0 < r_\alpha$. Then $KR^p < 1$. Consequently, $\forall x \in \mathbf{B}(x_0, R)$, $F'(x)^{-1}$ exists and*

$$\|F'(x)^{-1}F'(x_0)\| \leqslant \frac{1}{1 - K\|x - x_0\|^p}. \tag{3.4}$$

Let $\{y_n\}$ and $\{x_n\}$ be the sequences generated by (1.8) and (1.9), respectively. Recall that the functions H and Q are defined by (1.4) and (1.7), respectively. In the remainder, we always assume that $a_0 < r_\alpha$.

Lemma 3.2. *Let $n = 1, 2, \dots$ and let $x_n, y_n \in \mathbf{B}(x_0, R)$ satisfy*

$$\alpha K \lambda^{p-1} \|F'(x_n)^{-1} F'(x_0)\| \|y_n - x_n\|^p < 1. \tag{3.5}$$

Then

$$\|Q(x_n, y_n)\| \leqslant \frac{K\lambda^{p-1}\|F'(x_n)^{-1}F'(x_0)\|\|y_n - x_n\|^p}{2(1 - \alpha K\lambda^{p-1}\|F'(x_n)^{-1}F'(x_0)\|\|y_n - x_n\|^p)} \tag{3.6}$$

and

$$\|x_{n+1} - x_n\| \leqslant \left(1 + \frac{K\lambda^{p-1}\|F'(x_n)^{-1}F'(x_0)\|\|y_n - x_n\|^p}{2(1 - \alpha K\lambda^{p-1}\|F'(x_n)^{-1}F'(x_0)\|\|y_n - x_n\|^p)}\right)\|y_n - x_n\|. \tag{3.7}$$

The following lemma is a direct application of the Taylor's expression.

Lemma 3.3. *Let $n = 1, 2, \dots$ and let $x_n, y_n \in \mathbf{B}(x_0, R)$ satisfy* (3.5). *Then*

$$\begin{aligned} \|F'(x_n)^{-1}F(x_{n+1})\| \leqslant & \frac{K\lambda^{p-1}\|F'(x_n)^{-1}F'(x_0)\|\|y_n - x_n\|^{1+p}}{2(1 - \alpha K\lambda^{p-1}\|F'(x_n)^{-1}F'(x_0)\|\|y_n - x_n\|^p)} \\ & + \frac{K}{p+1}\|F'(x_n)^{-1}F'(x_0)\|\|x_{n+1} - x_n\|^{1+p}. \end{aligned} \tag{3.8}$$

The following lemma, which plays a key role in the proof of the main theorem, can be verified by mathematical induction similar to the proof of [29, Lemma 3.3].

Lemma 3.4. *The following inequalities hold for each $n = 0, 1, 2, \dots$*

$$\|y_n - x_n\| \leqslant \gamma^{((1+p)^n - 1)/p^2}(f(a_0)^{-1/p})^n \eta, \tag{3.9}$$

$$K\lambda^{p-1}\|F'(x_n)^{-1}F'(x_0)\|\|y_n - x_n\|^p \leqslant a_n, \tag{3.10}$$

$$\|x_{n+1} - x_n\| \leqslant \left(1 + \frac{a_n}{2(1 - \alpha a_n)}\right)\|y_n - x_n\|, \tag{3.11}$$

$$\|x_{n+1} - x_0\| \leqslant \eta\left(1 + \frac{a_0}{2(1 - \alpha a_0)}\right)\frac{1 - (\gamma f(a_0)^{-1})^{(n+1)/p}}{1 - (\gamma f(a_0)^{-1})^{1/p}} < R, \tag{3.12}$$

$$\|F'(x_{n+1})^{-1}F'(x_0)\| \leqslant f(a_n)\|F'(x_n)^{-1}F'(x_0)\|. \tag{3.13}$$

Now we are ready to prove the main theorem of the present paper. Recall that G is defined by (2.4).

Theorem 3.1. *Let $F : \Omega \subseteq X \to Y$ be a nonlinear operator with continuous first derivative F' and let $x_0 \in \Omega$ be such that $F'(x_0)^{-1}$ exists. Let r_α be the unique positive solution of the equation $G_\alpha(t) = 1$. Suppose that conditions* (3.1) *and* (3.2) *are satisfied. Let $a_0 := K\lambda^{p-1}\eta^p$ be such that $a_0 < r_\alpha$ and $\mathbf{B}(x_0, R) \subseteq \Omega$ where R is defined by* (3.3). *Then the sequence $\{x_n\}$ generated by* (1.8) *and* (1.9) *with initial point x_0 converges at a rate of order $1 + p$ to a unique solution x^* of the equation $F(x) = 0$ on $\mathbf{B}(x_0, R)$.*

Proof. We first apply Lemma 3.4 to verify that $\{x_n\}$ is a Cauchy sequence. For this purpose, note that, by (3.9) and (3.11),

$$\begin{aligned} \|x_{i+1} - x_i\| & \leqslant \eta\left(1 + \frac{a_i}{2(1 - \alpha a_i)}\right)\gamma^{((1+p)^i - 1)/p^2}(f(a_0)^{-1/p})^i \\ & \leqslant \eta\left(1 + \frac{a_0}{2(1 - \alpha a_0)}\right)\gamma^{((1+p)^i - 1)/p^2}(f(a_0)^{-1/p})^i \end{aligned} \tag{3.14}$$

holds as $a_i < a_0$ for each $i = 0, 1, \ldots$. This implies that, for each $m, n = 0, 1, \ldots$ with $m > n$,

$$\begin{aligned}\|x_{m+n} - x_n\| &\leqslant \sum_{i=n}^{n+m-1} \|x_{i+1} - x_i\| \\ &\leqslant \sum_{i=n}^{n+m-1} \eta\left(1 + \frac{a_0}{2(1-\alpha a_0)}\right)\gamma^{((1+p)^i-1)/p^2}(f(a_0)^{-1/p})^i \\ &\leqslant \left(1 + \frac{a_0}{2(1-\alpha a_0)}\right)\eta\frac{\left(\gamma^{((1+p)^n-1)/p^2}(f(a_0)^{-1/p})^n\right)\left(1-((\gamma f(a_0)^{-1})^{1/p})^m\right)}{1-(\gamma f(a_0)^{-1})^{1/p}} \\ &\leqslant R\left(\gamma^{((1+p)^n-1)/p^2}(f(a_0)^{-1/p})^n\right)\left(1-((\gamma f(a_0)^{-1})^{1/p})^m\right),\end{aligned} \tag{3.15}$$

since $((1+p)^{i+1}-1)/p^2 \geqslant i+1$ for each $i \geqslant n$. Hence $\{x_n\}$ is a Cauchy sequence because $(\gamma f(a_0)^{-1})^{1/p} < 1$ by Lemma 2.1. Consequently, $\{x_n\}$ converges to, say x^*. Letting $m \to \infty$ on the two-side hands of (3.15) yields that

$$\|x^* - x_n\| \leqslant R\left(\gamma^{((1+p)^n-1)/p^2}\right)\left(f(a_0)^{-1/p}\right)^n.$$

This shows that $\{x_n\}$ converges to x^* at a rate of order $1+p$. In particular, letting $n=0$, we have that

$$\|x^* - x_0\| \leqslant R.$$

Since $F'(x_n)^{-1}F(x_n) \to 0$ and the function $F'^{-1}F$ is continuous, $F'(x^*)^{-1}F(x^*) = 0$; hence, $F(x^*) = 0$.

Thus, to complete the proof, it remains to show that the solution of the equation $F(x) = 0$ is unique in $\mathbf{B}(x_0, R)$. For this end, let $y^* \in \mathbf{B}(x_0, R)$ be such that $F(y^*) = 0$. Then

$$\int_0^1 F'(x_0)^{-1}F'(x^* + t(y^* - x^*))\,dt\,(y^* - x^*) = F'(x_0)^{-1}[F(y^*) - F(x^*)] = 0.$$

Since

$$\begin{aligned}\left\|\int_0^1 F'(x_0)^{-1}[F'(x^* + t(y^* - x^*)) - F'(x_0)]\,dt\right\| &\leqslant K\int_0^1 [(1-t)\|x^* - x_0\| + t\|y^* - x_0\|]^p\,dt \\ &< K\int_0^1 [(1-t)R + tR]^p\,dt \\ &< K\|x^* - x_0\|^p.\end{aligned}$$

It follows from Lemma 3.1 that

$$\left\|F'(x_0)^{-1}\int_0^1 [F'(x^* + t(y^* - x^*)) - F'(x_0)]\,dt\right\| < 1.$$

Thus by the Banach Lemma, one gets that $\int_0^1 F'(x^* + t(y^* - x^*))\,dt$ is invertible. Hence, $y^* = x^*$ and the proof is complete. □

In particular, taking $\alpha = 0, \frac{1}{2}, 1$, respectively, and $p = 1$, the corresponding G_α are, respectively, as follows:

$$G_0(s) = \frac{s(2 + \frac{1}{2}(s+2)^2)}{(2 - 2s - s^2)^2} \quad \text{for each } s \in [0, \sqrt{3} - 1),$$

$$G_{1/2}(s) = \frac{s(4-s)}{(2-3s)^2} \quad \text{for each } s \in \left[0, \frac{2}{3}\right)$$

and

$$G_1(s) = \frac{s(2-2s+\frac{1}{2}(2-s)^2)}{(2-4s+s^2)^2} \quad \text{for each } s \in [0, 2-\sqrt{2}).$$

Then r_0, $r_{1/2}$ and r_1 are, respectively, equal to $0.326664\ldots$, $(4-\sqrt{6})/5$ and $0.292246\ldots$. Hence, by Theorem 3.1, we immediately obtain the following results, which have been studied, respectively, in [18,30,8].

Corollary 3.1. *Let $F : \Omega \subseteq X \to Y$ be a nonlinear operator with continuous first derivative F' and let $x_0 \in \Omega$ be such that $F'(x_0)^{-1}$ exists. Suppose that conditions* (3.1) *and* (3.2) *are satisfied with $p = 1$. Let $a_0 := K\eta$ be such that $a_0 < r_0 = 0.326664\ldots$ and $\mathbf{B}(x_0, R_0) \subseteq \Omega$ where $R_0 = (1 + a_0/2)\eta/(1 - \gamma f(a_0)^{-1})$. Then the sequence $\{x_n\}$ generated by the variant of the Chebyshev method with initial point x_0 converges at a rate of order 2 to a unique solution x^* of the equation $F(x) = 0$ on $\mathbf{B}(x_0, R_0)$.*

Corollary 3.2. *Let $F : \Omega \subseteq X \to Y$ be a nonlinear operator with continuous first derivative F' and let $x_0 \in \Omega$ be such that $F'(x_0)^{-1}$ exists. Suppose that conditions* (3.1) *and* (3.2) *are satisfied with $p = 1$. Let $a_0 = K\eta$ be such that $a_0 < r_{1/2} = (4-\sqrt{6})/5$ and $\mathbf{B}(x_0, R_{1/2}) \subseteq \Omega$ where $R_{1/2} = (1 + a_0/(2-a_0))\eta/(1 - \gamma f(a_0)^{-1})$. Then the sequence $\{x_n\}$ generated by the variant of the Halley method with initial point x_0 converges at a rate of order 2 to a unique solution x^* of the equation $F(x) = 0$ on $\mathbf{B}(x_0, R_{1/2})$.*

Corollary 3.3. *Let $F : \Omega \subseteq X \to Y$ be a nonlinear operator with continuous first derivative F' and let $x_0 \in \Omega$ be such that $F'(x_0)^{-1}$ exists. Suppose that conditions* (3.1) *and* (3.2) *are satisfied with $p = 1$. Let $a_0 = K\eta$ be such that $a_0 < r_1 = 0.292246\ldots$ and $\mathbf{B}(x_0, R_1) \subseteq \Omega$ where $R_1 = (1 + a_0/(2(1-a_0)))\eta/(1 - \gamma f(a_0)^{-1})$. Then the sequence $\{x_n\}$ generated by the variant of the convex acceleration of Newton's method with initial point x_0 converges at a rate of order 2 to a unique solution x^* of the equation $F(x) = 0$ on $\mathbf{B}(x_0, R_1)$.*

4. Application to a nonlinear integral equation of Hammerstein type

In this section, we provide an application of the main result to a special nonlinear Hammerstein integral equation of the second kind (cf. [22]). Letting $\mu \in \mathbb{R}$ and $p \in (0, 1]$, we consider

$$x(s) = l(s) + \int_a^b G(s,t)[x(t)^{1+p} + \mu x(t)]\,\mathrm{d}t, \quad s \in [a,b], \tag{4.1}$$

where l is a continuous function such that $l(s) > 0$ for all $s \in [a, b]$ and the kernel G is a non-negative continuous function on $[a, b] \times [a, b]$. This kind of nonlinear Hammerstein integral equation has been already studied by many authors, see for example [9,10,19,29], etc.

Note that if G is the Green function defined by

$$G(s,t) = \begin{cases} \dfrac{(b-s)(t-a)}{b-a}, & t \leqslant s, \\ \dfrac{(s-a)(b-t)}{b-a}, & s \leqslant t, \end{cases} \tag{4.2}$$

Eq. (4.1) is equivalent to the following boundary value problem (cf. [23]):

$$\begin{cases} x'' = -x^{1+p} - \mu x \\ x(a) = v(a), \quad x(b) = v(b). \end{cases}$$

To apply Theorem 3.1, let $X = Y = C[a, b]$, the Banach space of real-valued continuous functions on $[a, b]$ with the uniform norm. Let $\mathbf{Q}$ denote the set of all rational numbers $p \in (0, 1]$ such that $p = u/q$ for some odd number q and positive integer u. Let

$$\Omega_p = \begin{cases} \{x \in C[a,b] : x(s) > 0,\ s \in [a,b]\}, & p \in (0,1]\backslash\mathbf{Q}, \\ C[a,b], & p \in \mathbf{Q}. \end{cases}$$

Define $F : \Omega_p \to C[a, b]$ by

$$[F(x)](s) = x(s) - l(s) - \int_a^b G(s,t)[x(t)^{1+p} + \mu x(t)]\,\mathrm{d}t, \quad s \in [a, b]. \tag{4.3}$$

Then solving Eq. (4.1) is equivalent to solving Eq. (1.1) with F being defined by (4.3).

We start by calculating the parameter η in the study. Firstly, we have

$$[F'(x)u](s) = u(s) - \int_a^b G(s,t)[(1+p)x(t)^p + \mu]u(t)\,\mathrm{d}t, \quad s \in [a, b].$$

Let $x_0 \in \Omega_p$ be fixed. Then

$$\|\mathbf{I} - F'(x_0)\| \leqslant M((1+p)\|x_0\|^p + \mu),$$

where

$$M = \max_{s\in[a,b]} \int_a^b |G(s,t)|\,\mathrm{d}t.$$

By the Banach Lemma, if $M((1+p)\|x_0\|^p + \mu) < 1$, one has

$$\|F'(x_0)^{-1}\| \leqslant \frac{1}{1 - M((1+p)\|x_0\|^p + \mu)}.$$

Since

$$\|F(x_0)\| \leqslant \|x_0 - l\| + M(\|x_0\|^{1+p} + \mu\|x_0\|),$$

it follows that

$$\|F'(x_0)^{-1}F(x_0)\| \leqslant \frac{\|x_0 - l\| + M(\|x_0\|^{1+p} + \mu\|x_0\|)}{1 - M((1+p)\|x_0\|^p + \mu)}. \tag{4.4}$$

Therefore, η is estimated.

On the other hand, for $x, y \in \Omega_p$,

$$[(F'(x) - F'(y))u](s) = -\int_a^b G(s,t)[(1+p)(x(t)^p - y(t)^p)]u(t)\,\mathrm{d}t, \quad s \in [a, b]$$

and consequently,

$$\|F'(x) - F'(y)\| \leqslant M(1+p)\|x - y\|^p \quad \text{for all}\, x, y \in \Omega_p. \tag{4.5}$$

This means that $K = M(1+p)$. Thus, we can establish the following result from Theorem 3.1.

Theorem 4.1. *Let F be the nonlinear operator defined in* (4.3) *and $x_0 \in \Omega_p$ a point such that* $M((1+p)\|x_0\|^{1+p} + \mu) < 1$. *Let r_α be the unique solution of the equation $G_\alpha(s) = 1$ on $(0, s_\alpha)$. Let $a_0 := K\lambda^{p-1}\eta^p$ be such that $a_0 < r_\alpha$ and $\mathbf{B}(x_0, R) \subseteq \Omega$ where R is defined by* (3.3). *Then the sequence $\{x_n\}$ generated by* (1.8) *and* (1.9) *with initial point x_0 converges at a rate of order* $1 + p$ *to a unique solution x^* of Eq.* (4.1) *on* $\mathbf{B}(x_0, R)$.

The following example provides an operator F which has the first derivative satisfying (1.6) but does not have the second derivative.

Example 4.1. Let G be Green's function on $[0, 1] \times [0, 1]$ defined by (4.2). Consider the following particular case of (4.1):

$$x(s) = \frac{1}{32} + \int_0^1 G(s,t)(x(t)^{4/3} + x(t))\,\mathrm{d}t, \quad s \in [a, b]. \tag{4.6}$$

Table 1
The values of r_α and a_0

$\alpha\backslash\lambda$	0.005	0.009	0.01	0.1	0.2	0.5
0	1.4794	1.3347	1.3070	0.6744	0.5113	0.4280
0.1	1.3249	1.2161	1.1947	0.6544	0.5017	0.3335
0.2	1.1941	1.1117	1.0951	0.6344	0.4920	0.3303
0.3	1.0833	1.0202	1.0072	0.6145	0.4822	0.3271
0.5	0.9081	0.8698	0.8616	0.5754	0.4622	0.3204
1	0.6373	0.6238	0.6208	0.4853	0.4120	0.3025
a_0	1.8771	1.2685	1.1825	0.2548	0.1605	0.1225

Table 2
TF values of $a_0 < r_\alpha$

$\alpha\backslash\lambda$	0.005	0.009	0.01	0.1	0.2	0.5
0	*F*	*T*	*T*	*T*	*T*	*T*
0.1	*F*	*F*	*T*	*T*	*T*	*T*
0.2	*F*	*F*	*F*	*T*	*T*	*T*
0.3	*F*	*F*	*F*	*T*	*T*	*T*
0.5	*F*	*F*	*F*	*T*	*T*	*T*
1	*F*	*F*	*F*	*T*	*T*	*T*

The corresponding operator $F : \Omega_p \to C[a, b]$ is equal to

$$[F(x)](s) = x(s) - \frac{1}{32} - \int_0^1 G(s,t)(x(t)^{4/3} + x(t))\,\mathrm{d}t, \quad s \in [a, b]. \tag{4.7}$$

Clearly $p = \frac{1}{3}$ and $\Omega_p = C[0, 1]$. Choose $x_0 = 0$ for Theorem 4.1. Since $M = \frac{1}{8}$ and $p = \frac{1}{3}$, we have $K = \frac{1}{6}$. By (4.4), we can take $\eta = \frac{1}{28}$. Hence

$$a_0 = K\lambda^{-2/3}\eta^{1/3} = \frac{1}{6\sqrt[3]{28\lambda^2}}.$$

Note that $\mathbf{B}(x_0, R) \subseteq \Omega$ holds for each R. It follows from Theorem 4.1 that the sequence $\{x_n\}$ generated by (1.8) and (1.9) with initial point $x_0 = 0$ converges at a rate of order $\frac{4}{3}$ provided $a_0 < r_\alpha$.

For some special values of λ and α, the corresponding values of r_α, a_0 and TF values of "$a_0 < r_\alpha$" are given in the following Tables 1 and 2, respectively.

Note that the operator F defined by (4.7) does not have the second derivative at any point $x \in C[0, 1]$ with the Lebesgue measure $\mu\{t \in [0, 1] : x(t) = 0\} > 0$. Hence, condition (1.5) is not satisfied and the convergence criterion in [29] is not applicable. Note also that the equation given in this example cannot be solved by Halley's method and the convergence theorems in [11] is not applicable too.

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