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A note on the preconditioner $P_m = (I + S_m)$

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ABSTRACT

Kotakemori et al. [H. Kotakemori, K. Harada, M. Morimoto, H. Niki, A comparison theorem for the iterative method with the preconditioner $(I + S_{\max})$, Journal of Computational and Applied Mathematics 145 (2002) 373–378] have reported that the convergence rate of the iterative method with a preconditioner $P_m = (I + S_m)$ was superior to one of the modified Gauss–Seidel method under the condition. These authors derived a theorem comparing the Gauss–Seidel method with the proposed method. However, through application of a counter example, Wen Li [Wen Li, A note on the preconditioned GaussSeidel (GS) method for linear systems, Journal of Computational and Applied Mathematics 182 (2005) 81–91] pointed out that there exists a special matrix that does not satisfy this comparison theorem. In this note, we analyze the reason why such a to counter example may be produced, and propose a preconditioner to overcome this problem.

1. Introduction

We herein consider the following preconditioned linear system:

$$PA\boldsymbol{x} = P\boldsymbol{b}, \tag{1.1}$$

where $A = (a_{ij}) \in R^{n\times n}$ is a nonsingular $\boldsymbol{M}$-matrix, $P \in R^{n\times n}$ is a preconditioner, and $\boldsymbol{x}, \boldsymbol{b} \in R^n$ are vectors. Without loss of generality, we assume that A has a splitting of the form $A = I - L - U$, where I denotes the $n \times n$ identity, and $-L$ and $-U$ are the strictly lower, and upper triangular parts of A, respectively.

In 1991, Gunawardena et al. [1] proposed the modified Gauss–Seidel method in which $P = (I + S)$, with

$$S = (s_{ij}) = \begin{cases} -a_{ii+1} & \text{for } i = 1, 2, \ldots, n-1, j = i+1, \\ 0 & \text{otherwise.} \end{cases}$$

These authors proved that if $0 < a_{ii+1}a_{i+1i} < 1$ then the inequality:

$$\rho(T_S) \le \rho(T) < 1,$$

is satisfied, where $\rho(T_S)$ and $\rho(T)$ denote the spectral radius of the GaussSeidel iterative matrices T_S and T associated with $A_S = (I + S)A$ and A, respectively.

In 2002, Kotakemori et al. [2] proposed to use $P_m = (I + S_m)$, where S_m is defined by

$$S_m = (s_{ij}^{(m)}) = \begin{cases} -a_{ik_i} & \text{for } 1 \le i < n, i+1 < j \le n, \\ 0 & \text{otherwise,} \end{cases}$$

where $k_i = minI_i$, $I_i = \{j : |a_{ij}| \text{ is maximal for } i+1 \le j \le n\}$, for $1 \le i < n$.

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Then $A_m = (I + S_m)A$ can be written as follows:

$$A_m = I - L - U + S_m - S_mL - S_mU = M_m - N_m,$$

where $M_m = (I - D_m) - (L + E_m)$ and $N_m = U - S_m + F_m + S_mU$ and D_m, E_m and F_m are the diagonal and strictly lower and strictly upper triangular parts of S_mL, respectively. Under the condition $0 < a_{ik_i}a_{k_ii} < 1$, Kotakemori et al. derived the following result.

Lemma 1 (*Kotakemori et al. [2], Lemma 3.4*). *Let A be an $\boldsymbol{M}$-matrix. Suppose that*

$$a_{ii+1}a_{i+1j} \le a_{ik_i}a_{k_ij}, \quad 1 \le i \le n-2, j \le i. \tag{1.2}$$

Then the following inequality holds:

$$M_m^{-1} \ge M_s^{-1}.$$

Theorem 2 (*Kotakemori et al. [2], Theorem 3.5*). *Let A be an $\boldsymbol{M}$-matrix. Then the Gauss–Seidel splittings $A = M - N$and $A_s = M_s - N_s$ are convergent and the following inequality holds:*

$$\rho(T_S) \le \rho(T) < 1. \tag{1.3}$$

Proof. It easily follows from the assumption of the present theorem together with Theorem of [4] that $\rho(M^{-1}N) < 1$. Since A_S is also an $\boldsymbol{M}$-matrix, A_S admits the convergent splitting $A_S = M_S - N_S$. By putting $A = P_S^{-1}(M_S - N_S)$, we have $A = M - N = P_S^{-1}(M_S - N_S)$. Since $A = M - N$ is the Gauss–Seidel convergent splitting, there exists a positive vector $\boldsymbol{x}$ satisfing $\rho(M^{-1}N)\boldsymbol{x} = M^{-1}N\boldsymbol{x}$. We then have the following relation:

$$A\boldsymbol{x} = (M - N) = M(I - M^{-1}N)\boldsymbol{x} = \frac{1 - \rho(M^{-1}N)}{\rho(M^{-1}N)}N\boldsymbol{x} \ge O.$$

Since $M_S^{-1} \ge O$ and $P_S \ge O$, then we further have that $M_S^{-1}P_S \ge M_S^{-1} \ge M^{-1}$. It follows that

$$\begin{aligned}(M_S^{-1}P_S - M^{-1})A\boldsymbol{x} &= M_S^{-1}P_S\{P_S^{-1}(M_S - N_S)\}\boldsymbol{x} - (I - M^{-1}N)\boldsymbol{x}\\ &= (I - M_S^{-1}N_S)\boldsymbol{x} - (I - M^{-1}N)\boldsymbol{x}\\ &= M^{-1}N\boldsymbol{x} - M_S^{-1}N_S\boldsymbol{x} = \rho(M^{-1}N)\boldsymbol{x} - M_S^{-1}N_S\boldsymbol{x} \ge O,\end{aligned}$$

which by Theorem (2.2) of [1] implies (1.3). ■

Theorem 3. *Let A be an $\boldsymbol{M}$-matrix. Let $A_S = M_S - N_S$ and $A_m = M_m - N_m$ be Gauss–Seidel convergent splittings of A_S and A_m, respectively. Assume that only one of the inequalities $A_m\boldsymbol{x} \ge A_S\boldsymbol{x}$ or $A_m\boldsymbol{y} \ge A_s\boldsymbol{y}$, where $\boldsymbol{x}$ and $\boldsymbol{y}$ are positive eigenvectors associated with T_S and T_m, respectively. Under the assumptions in Lemma 1, the following inequality holds:*

$$\rho(T_m) \le \rho(T_S).$$

Proof. By putting $A = A_S$ and $A_S = A_m$, the proof follows in a similar manner to that of Theorem 2. ■

2. The case $\rho(T_S) \le \rho(T_m)$ for $M_m^{-1} \ge M_S^{-1}$

In 2005, Wen Li showed the following counter example [5]:

$$A = \begin{pmatrix} 1.0 & -0.1 & -0.1 & -0.1 & -0.2\\ -0.1 & 1 & -0.1 & -0.1 & -0.2\\ -0.1 & -0.1 & 1 & -0.1 & -0.2\\ -0.1 & -0.1 & -0.1 & 1 & -0.2\\ -0.1 & -0.1 & -0.1 & -0.1 & 1 \end{pmatrix}.$$

For this matrix, we have the following results:

$$A_S = \begin{pmatrix} 0.99 & 0 & -0.11 & -0.11 & -0.22\\ -0.11 & 0.99 & 0 & -0.11 & -0.22\\ -0.11 & -0.11 & 0.99 & 0 & -0.22\\ -0.12 & -0.12 & -0.12 & 0.98 & 0\\ -0.1 & -0.1 & -0.1 & -0.1 & 1 \end{pmatrix},$$

$$M_S^{-1} = \begin{pmatrix} 1.010 & 0 & 0 & 0 & 0\\ 0.112 & 1.010 & 0 & 0 & 0\\ 0.125 & 0.112 & 1.010 & 0 & 0\\ 0.153 & 0.137 & 0.124 & 1.020 & 0\\ 0.140 & 0.126 & 0.113 & 0.102 & 1 \end{pmatrix}, \qquad T_S = \begin{pmatrix} 0 & 0 & 0.111 & 0.111 & 0.111\\ 0 & 0 & 0.012 & 0.123 & 0.247\\ 0 & 0 & 0.014 & 0.026 & 0.274\\ 0 & 0 & 0.017 & 0.032 & 0.091\\ 0 & 0 & 0.015 & 0.029 & 0.083 \end{pmatrix},$$

and $\rho(T_S) = 0.1497$. On the other hand, for P_m we have:

$$A_m = \begin{pmatrix} 0.98 & -0.12 & -0.12 & -0.12 & 0 \\ -0.12 & 0.98 & -0.12 & -0.12 & 0 \\ -0.12 & -0.12 & 0.98 & -0.12 & 0 \\ -0.12 & -0.12 & -0.12 & 0.98 & 0 \\ -0.1 & -0.1 & -0.1 & -0.1 & 1 \end{pmatrix},$$

$$M_m^{-1} = \begin{pmatrix} 1.020 & 0 & 0 & 0 & 0 \\ 0.125 & 1.020 & 0 & 0 & 0 \\ 0.140 & 0.125 & 1.020 & 0 & 0 \\ 0.157 & 0.140 & 0.125 & 1.020 & 0 \\ 0.144 & 0.129 & 0.115 & 0.102 & 1 \end{pmatrix}, \quad T_m = \begin{pmatrix} 0 & 0.122 & 0.122 & 0.122 & 0 \\ 0 & 0.015 & 0.137 & 0.137 & 0 \\ 0 & 0.017 & 0.032 & 0.154 & 0 \\ 0 & 0.019 & 0.036 & 0.051 & 0 \\ 0 & 0.017 & 0.033 & 0.046 & 0 \end{pmatrix},$$

and $\rho(T_m) = 0.1555$. While this matrix satisfies condition equation (1.2), the inequality $\rho(T_m) \le \rho(T_S)$ does not hold. We test the following matrix:

$$B = \begin{pmatrix} 1.0 & -0.1 & -0.1 & -0.1 & -0.3 \\ -0.1 & 1 & -0.1 & -0.1 & -0.3 \\ -0.1 & -0.1 & 1 & -0.1 & -0.3 \\ -0.1 & -0.1 & -0.1 & 1 & -0.1 \\ -0.1 & -0.1 & -0.1 & -0.1 & 1 \end{pmatrix}.$$

For this matrix, we have the following results:

$$M_S^{-1} = \begin{pmatrix} 1.010 & 0 & 0 & 0 & 0 \\ 0.112 & 1.010 & 0 & 0 & 0 \\ 0.125 & 0.112 & 1.010 & 0 & 0 \\ 0.139 & 0.125 & 0.112 & 1.010 & 0 \\ 0.139 & 0.125 & 0.112 & 0.101 & 1 \end{pmatrix}, \quad T_S = \begin{pmatrix} 0 & 0 & 0.111 & 0.111 & 0.333 \\ 0 & 0 & 0.012 & 0.123 & 0.370 \\ 0 & 0 & 0.014 & 0.026 & 0.391 \\ 0 & 0 & 0.015 & 0.029 & 0.122 \\ 0 & 0 & 0.015 & 0.029 & 0.122 \end{pmatrix},$$

and $\rho(T_S) = 0.1873$. On the other hand, for P_m we have:

$$M_m^{-1} = \begin{pmatrix} 1.031 & 0 & 0 & 0 & 0 \\ 0.138 & 1.031 & 0 & 0 & 0 \\ 0.157 & 0.138 & 1.031 & 0 & 0 \\ 0.147 & 0.130 & 0.115 & 1.010 & 0 \\ 0.147 & 0.130 & 0.115 & 0.101 & 1 \end{pmatrix}, \quad T_m = \begin{pmatrix} 0 & 0.134 & 0.134 & 0.134 & 0 \\ 0 & 0.018 & 0.152 & 0.152 & 0 \\ 0 & 0.020 & 0.038 & 0.172 & 0 \\ 0 & 0.019 & 0.036 & 0.051 & 0 \\ 0 & 0.019 & 0.036 & 0.051 & 0 \end{pmatrix}.$$

Then $\rho(T_m) = 0.1682$. Thus for this case the inequality $\rho(T_m) \le \rho(T_S)$ holds. From the results above, we know that there exist $|a_{in}|$ $(1 \le i \le n-2)$ such that $\rho(T_S) \ge \rho(T_m)$. It is, however, in general difficut to determine $|a_{in}|$ such that the relation $\rho(T_S) \ge \rho(T_m)$ holds a proori. Motivated by this problem, in next section we propose a preconditioner that satisfies $\rho(T_S) \ge \rho(T_m)$ unconditionaly.

3. A preconditioner satisfying $\rho(T_S) \ge \rho(T_m)$

Method 1. We propose the preconditioner $P_{m1} = (I + S_{m1})$, where S_{m1} is defined by

$$S_{m1} = (s_{ij}^{(m1)}) = \begin{cases} -a_{12} & \\ -a_{ik_i} & \text{for } 2 \le i < n, i < j \le n, \\ 0 & \text{otherwise,} \end{cases}$$

where $k_i = minI_i$, $I_i = \{j : |a_{ij}| \text{ is maximal for } i \le j \le n\}$, for $2 \le i < n$.

By using this preconditioner P_{m1} for A, we obtain

$$A_{m1} = \begin{pmatrix} .97 & 0 & -0.11 & -0.11 & -0.22 \\ -0.12 & .98 & -0.12 & -0.12 & 0 \\ -0.12 & -0.12 & .98 & -0.12 & 0 \\ -0.12 & -0.12 & -0.12 & .98 & 0 \\ -0.1 & -0.1 & -0.1 & -0.1 & 1 \end{pmatrix}.$$

Then $\rho(T_{m1}) = 0.1376 < \rho(T_S)$.

Method 2. We propose the precondioner $P_{m2} = (I + S_{m2})$, where S_{m2} is defined by

$$S_{m2} = (s_{ij}^{(m2)}) = \begin{cases} -a_{12} & \\ -a_{1k_i} & \text{for } 1 < j \le n, \\ -a_{ik_i} & \text{for } 2 \le i < n, i < j \le n, \\ 0 & \text{otherwise,} \end{cases}$$

where $k_i = minI_i$, $I_i = \{j : |a_{ij}| \text{ is maximal for } i+1 \le j \le n\}$, for $1 \le i < n$.

By using this preconditioner P_{m2} for A, we obtain

$$A_{m2} = \begin{pmatrix} .97 & -0.02 & -0.13 & -0.13 & -0.02 \\ -0.12 & .98 & -0.12 & -0.12 & 0 \\ -0.12 & -0.12 & .98 & -0.12 & 0 \\ -0.12 & -0.12 & -0.12 & .98 & 0 \\ -0.1 & -0.1 & -0.1 & -0.1 & 1 \end{pmatrix},$$

and $\rho(T_{m2}) = 0.1235$. As descrived in the next section, since $|a_{21}^{(m2)}| < 0.073$, $\rho(T_{m2}) < \rho(T_{m1})$ holds

Remark 4. For the preconditioner $P_{m1} = (I + S_{m1})$, Eq. (1.2) is weakened as follows:

$$a_{ii+1}a_{i+1j} \le a_{ik_i}a_{k_ij}, \quad 2 \le i \le n-2, j \le i.$$

Method 3. To ensure the inequality $\rho(T_S) \ge \rho(T_m)$ is satisfied, condition Eq. (1.2) must be satisfied. In order to overcome with this drawback, Morimoto et al. [3] proposed the preconditioner $P_{sm} = (I + S + S_m)$. For this preconditione, S_m is

$$S_m = (s_{ij}^{(m)}) = \begin{cases} -a_{il_i} & \text{for } 1 \le i < n, j > n-1, \\ 0 & \text{otherwise,} \end{cases}$$

where $l_i = minI_i$, $I_i = \{j : |a_{ij}| \text{ is maximal for } i+2 \le j \le n\}$, for $1 \le i < n-1$.

By using this preconditioner to A, we have the following results:

$$A_{sm} = \begin{pmatrix} 0.979 & -0.02 & -0.13 & -0.13 & -0.021 \\ -0.13 & 0.97 & -0.02 & -0.13 & -0.02 \\ -0.13 & -0.13 & 0.97 & -0.02 & -0.02 \\ -0.12 & -0.12 & -0.12 & 0.98 & 0 \\ -0.1 & -0.1 & -0.1 & -0.1 & 1 \end{pmatrix},$$

$$T_{sm} = \begin{pmatrix} 0 & 0.021 & 0.134 & 0.134 & 0.021 \\ 0 & 0.003 & 0.039 & 0.152 & 0.023 \\ 0 & 0.003 & 0.023 & 0.059 & 0.027 \\ 0 & 0.003 & 0.024 & 0.042 & 0.009 \\ 0 & 0003 & 0.022 & 0.039 & 0.008 \end{pmatrix},$$

and so $\rho(T_{sm}) = 0.0908 < \rho(T_{m2})$.

4. Concluding remarks

1. For matrix A, if $|a_{12}| < 0.073$, the preconditioning effect of P_S is insufficent, and so $\rho(T_S) < \rho(T_m)$. In cotrast, for $|a_{15}| > 0.296$, the preconditioning effect of P_m is sufficent, hence $\rho(T_S) > \rho(T_m)$ holds. If $|a_{51}| \ge 0.2$, then a sufficiently large value of M_m^{-1} is obtained such that $\rho(T_S) > \rho(T_m)$ holds.
2. Assume that $L^T \ge U$ and in particular that $|a_{ik_i}| < |a_{k_ii}|$ $(1 \le i \le n-2)$, then $\rho(T_S) > \rho(T_m)$ holds.
3. For a matrix with structuer of example A, either of the inequalities $\rho(T_S) \ge \rho(T_m)$ or $\rho(T_S) \le \rho(T_m)$ may be satisfied. To ensure that $\rho(T_S) \ge \rho(T_m)$ holds, A must satisfy an appropriate condition. However, the form of this condition remains an open question.
4. We have shown that the preconditioner P_{sm} is an effective precondioner.

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