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A note on constrained degree reduction of polynomials in Bernstein–Bézier form over simplex domain☆

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ABSTRACT

In the paper [H.S. Kim, Y.J. Ahn, Constrained degree reduction of polynomials in Bernstein–Bézier form over simplex domain, J. Comput. Appl. Math. 216 (2008) 14–19], Kim and Ahn proved that the best constrained degree reduction of a polynomial over d-dimensional simplex domain in L_2-norm equals the best approximation of weighted Euclidean norm of the Bernstein–Bézier coefficients of the given polynomial. In this paper, we presented a counterexample to show that the approximating polynomial of lower degree to a polynomial is virtually non-existent when $d \geq 2$. Furthermore, we provide an assumption to guarantee the existence of solution for the constrained degree reduction.

1. Introduction

Following the notations in [2], the Bernstein basis of degree n over the d-dimensional ($d \geq 1$) simplex domain Δ is defined by

$$B_\alpha^n(x) = \binom{n}{\alpha} x^\alpha \left(1 - \sum_{i=1}^{d} x_i\right)^{n-|\alpha|}, \quad |\alpha| \leq n,$$

where $x = (x_1, \ldots, x_d) \in \Delta$ and $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$. We write polynomials in Bernstein–Bézier form as

$$B^n b := [B_\alpha^n]_{|\alpha|\leq n} \cdot [b_\alpha]_{|\alpha|\leq n} = \sum_{|\alpha|\leq n} B_\alpha^n(x) b_\alpha.$$

Let $\mathbb{P}_n$ be the linear space of polynomials of degree less than or equal to n. For the nonnegative integer $a \leq n/(d+1)$, let

$$I_n^a = \{|\alpha| \leq n: \alpha_1 \geq a, \ldots, \alpha_d \geq a, |\alpha| \leq n - a\},$$
$$J_n^a = \left\{|\alpha| \leq n: \alpha \notin I_n^a\right\}.$$

In the paper [2], Kim and Ahn have considered the best constrained degree reduction over simplex domain:

Theorem 1 *([2, Theorem 3]). Given a polynomial $B^n b$ of degree n, the approximation problem*

$$\min_{p\in\mathbb{P}_m} \left\{\left\|B^n b - p\right\| : p = B^n c \in \mathbb{P}_m, b_\alpha = c_\alpha \quad \text{for } \alpha \in J_n^a\right\}$$

has the same minimizer for the norm induced either by the L_2-inner product or the weighted Euclidean inner product.

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When $a = 0$, J_n^a becomes an empty set, which means that the degree reduction is unconstrained. So, we assume from now on that $a \geq 1$. To solve the constrained approximation problem in Theorem 1, the following assumption must hold; otherwise, the degree reduction may be unable to be carried out since we cannot guarantee the existence of a degree m polynomial satisfying the constraint conditions described in Theorem 1.

Assumption 1. For a given polynomial $B^n b$ of degree n, there exists a polynomial p of lower degree m such that

$$p = B^n c \in \mathbb{P}_m, \qquad b_\alpha = c_\alpha \quad \text{for } \alpha \in J_n^a.$$

For the one-dimensional case, i.e., $d = 1$, Assumption 1 is verified to be always satisfied, see [1]. However, for $d \geq 2$, the situation is different. We can find many polynomials that satisfy it, for example, the polynomial $B^n b$ with $b_\alpha = 0$ for all $\alpha \in J_n^a$. On the other hand, we give a simple counterexample in the following.

Example 1. For $d \geq 2$, let $a = 1$, and let $b_\alpha = 0$ for all $\alpha \in J_n^a$ except $b_{(n,0,\ldots,0)} = 1$. Then the polynomial $p = B^n c$, satisfying $c_\alpha = b_\alpha$ for $\alpha \in J_n^a$, has a degree of n whatever the chosen values of $[c_\alpha]_{\alpha \in I_n^a}$.

Proof. For any $\alpha \in I_n^a$, noting that $\alpha_1 + n - |\alpha| \leq n - 1$ since $\alpha_2 \geq 1$, we have

$$\frac{\partial^n}{\partial x_1^n} B_\alpha^n(x) = \binom{n}{\alpha} \frac{\partial^n}{\partial x_1^n} x_1^{\alpha_1} \cdots x_d^{\alpha_d} \left(1 - \sum_{i=1}^d x_i\right)^{n-|\alpha|} = 0.$$

Then, we express p as

$$p = \sum_{|\alpha| \leq n} B_\alpha^n(x) c_\alpha = x_1^n + \sum_{\alpha \in I_n^a} B_\alpha^n(x) c_\alpha.$$

Finally, from

$$\frac{\partial^n p}{\partial x_1^n} = \frac{\partial^n x_1^n}{\partial x_1^n} + \sum_{\alpha \in I_n^a} c_\alpha \frac{\partial^n}{\partial x_1^n} B_\alpha^n(x) = n!,$$

we can conclude that the polynomial p has a degree of n. $\square$

Assumption 2. For a given polynomial $B^n b$ of degree n, the constrained coefficients $[b_\alpha]_{\alpha \in J_n^a}$ are the parts of the coefficients of a degree n polynomial which is raised from a certain polynomial $B^m c$ of degree m, i.e., there exists a matrix T (that depends only on n and m) such that

$$[b_\alpha]_{\alpha \in J_n^a} = T \cdot [c_\beta]_{\beta \in J_m^a}. \tag{1}$$

Also, $[c_\beta]_{\beta \in J_m^a}$ are the parts of the coefficients of the degree m polynomial after degree reduction.

The linear system (1) shows that the constrained coefficients $[c_\beta]_{\beta \in J_m^a}$ after degree reduction are closely related to the constrained coefficients $[b_\alpha]_{\alpha \in J_n^a}$ of the given polynomial. Actually, it can be used to calculate the constrained coefficients. This is because if the approximate polynomial is degree raised to degree n, its corresponding constrained coefficients must agree with $[b_\alpha]_{\alpha \in J_n^a}$. The derivation of the matrix T will be discussed in Section 2 and can be obtained from (3).

If the given polynomial satisfies Assumption 2, we can guarantee the existence of solution for the constrained degree reduction. In essence, Assumptions 1 and 2 are equivalent. But actually the latter one is more computationally feasible. In Theorem 3, we will show the reason why Assumption 2 is always satisfied when $d = 1$. And in Theorem 4, we will show the reason why Assumption 2 is not always satisfied when $d \geq 2$.

2. Discussions and remarks

For raising the degree of the polynomial $p = B^m c$ by one without changing its shape, i.e.,

$$p = B^m c = B^{m+1} c^{(1)},$$

we can show that the new coefficients $c_\alpha^{(1)}$ are obtained from linear combinations of the old ones

$$c_\alpha^{(1)} = \frac{\alpha_1}{m+1} c_{\alpha - \mathbf{e}_1} + \cdots + \frac{\alpha_d}{m+1} c_{\alpha - \mathbf{e}_d} + \frac{m+1-|\alpha|}{m+1} c_\alpha, \quad |\alpha| \leq m+1, \tag{2}$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_d$ are the standard unit vectors of $\mathbb{R}^d$. We can rewrite (2) in matrix form $c^{(1)} = T_{m+1,m} c$.

After repeating this process $r = n - m$ times, the coefficients $c_\alpha^{(r)}$ of the polynomial $p = B^n c^{(r)}$ are given by

$$c_\alpha^{(r)} = \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \leq m,\, |\gamma| \leq r}} \frac{\binom{m}{\beta}\binom{r}{\gamma}}{\binom{n}{\alpha}} c_\beta, \quad |\alpha| \leq n. \tag{3}$$

Similarly, we rewrite (3) as $c^{(r)} = T_{n,m}c$, where $T_{n,m}$ can also be decomposed into elementary degree-raising steps as

$$T_{n,m} = T_{n,n-1}T_{n-1,n-2}\cdots T_{m+1,m}.$$

Theorem 2. *Let $p = B^m c = B^n c^{(r)}$, $n = m + r$ be a polynomial of degree m. Then it satisfies Assumption 2.*

Proof. For any coefficient $c_\alpha^{(r)}$, $\alpha \in J_n^a$, we can see from (3) that $c_\alpha^{(r)}$ is a linear combination of the coefficients c_β, $|\beta| \le m$. Hence, to prove the theorem, it is equivalent to showing that all $\beta \in J_m^a$ for any $\alpha \in J_n^a$. Otherwise, suppose that $\beta \in I_m^a$. We then have

$$\beta_1 \ge a, \ldots, \beta_d \ge a, \quad m - |\beta| \ge a.$$

Noting that $\alpha = \beta + \gamma = (\beta_1 + \gamma_1, \ldots, \beta_d + \gamma_d)$, we obtain

$$\alpha_1 \ge a, \ldots, \alpha_d \ge a$$

and

$$n - |\alpha| = n - |\beta| - |\gamma| \ge n - |\beta| - r = m - |\beta| \ge a,$$

which means $\alpha \in I_n^a$. This is a contradiction. $\square$

We indicate the cardinalities of the sets I_n^a and J_n^a as follows.

Lemma 1. *For the integer n, let $a \le n/(d+1)$. Then,*

(1) $\operatorname{card}(I_n^a) = \binom{n-(d+1)a+d}{d}$,

(2) $\operatorname{card}(J_n^a) = \binom{n+d}{d} - \binom{n-(d+1)a+d}{d}$.

Theorem 3. *When $d = 1$, linear system (1) is well-determined.*

Proof. From Lemma 1, we obtain

$$\operatorname{card}(J_m^a) = \operatorname{card}(J_n^a) = 2a.$$

In this case, (3) becomes ($\alpha = \alpha_1$, $\beta = \beta_1$)

$$c_\alpha^{(r)} = \sum_{\beta=\max(0,\alpha-r)}^{\min(m,\alpha)} \frac{\binom{m}{\beta}\binom{r}{\alpha-\beta}}{\binom{n}{\alpha}} c_\beta, \quad \alpha = 0, 1, \ldots, n.$$

Thus, the matrix T in (1) is a square matrix and is nonsingular. $\square$

Theorem 4. *When $d \ge 2$, the linear system (1) is over-determined.*

Proof. Denote

$$J_m^a = J_m^{a,1} \cup J_m^{a,2} := \left\{\alpha \in J_m^a : \exists\, i,\ \alpha_i < a\right\} \cup \left\{\alpha \in J_m^a : \alpha_1 \ge a, \ldots, \alpha_d \ge a, |\alpha| > m - a\right\}.$$

If $\alpha \in J_m^{a,1}$, it is obvious that $\alpha \in J_n^a$. If $\alpha \in J_m^{a,2}$, we can find the corresponding element $(\alpha_1 + r, \alpha_2, \ldots, \alpha_d) \in J_n^a$. Furthermore, we can easily get an index $\alpha = (0, \ldots, 0, n)$ such that $\alpha \in J_n^a$ and $\alpha \notin J_m^a$. Therefore,

$$\operatorname{card}(J_m^a) < \operatorname{card}(J_n^a). \quad \square$$

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