Accepted Manuscript

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PII: S0377-0427(15)00192-2
DOI: http://dx.doi.org/10.1016/j.cam.2015.03.039
Reference: CAM 10094

To appear in: *Journal of Computational and Applied Mathematics*

Received date: 24 December 2014
Revised date: 10 March 2015

Numerical methods for a one-dimensional non-linear Biot's model

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Abstract

Staggered finite difference methods for a one-dimensional Biot's problem are considered. The permeability tensor of the porous medium is assumed to depend on the strain, thus yielding a non-linear model. Some strong two-side estimates for displacements and for pressure are provided and convergence results in the discrete L^2-norm are proved. Numerical examples are given to illustrate the good performance of the proposed numerical approach.

Keywords: Finite-difference scheme, Maximum principle, Non-linear Biot's model

1. Introduction

Biot's model addresses the time-dependent coupling between the deformation of a porous matrix and the fluid flow inside. The porous matrix is supposed to be saturated by the fluid phase and the flow is governed by Darcy's law. The state of the continuous medium is characterized by the knowledge of the displacements and the fluid pressure at each point of the

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domain. The one-dimensional theory of isothermal consolidation was first formulated by Terzaghi [24] which was extended to a general 3D consolidation theory by Biot [2, 3]. Existence and uniqueness of the solution of the problem are analyzed by Showalter in [22] and by Ženišek in [26]. Nowadays, Biot's models are frequently used in a great variety of disciplines as in geomechanics, petrol engineering, hydrogeology, biomechanics and food processing.

Analytical solution of the Biot's model is available only in very special cases, and therefore, numerical methods are commonly used for solving this problem. In general, the solution of complex poromechanics problems is usually approximated by finite elements, see for instance the monograph by Lewis and Schrefler [11]. Problems where the solution is smooth are satisfactorily solved by standard finite element discretizations. Nevertheless, when strong pressure gradients appear, solutions generated by finite element methods exhibit non-physical oscillations. These oscillations can be minimized if stable finite element methods are used. As for Stokes problems, approximation spaces for the vector and the scalar fields satisfying the LBB stability condition [6] can be used. This approach has been analyzed, for example in [18], for the classical quasi–static Biot's model. Nevertheless, these methods still present small oscillations in the pressure approximation when very sharp boundary layers occur.

Naturally, as for finite elements, standard finite-difference schemes may suffer the same unstable behaviour in the pressure approximation. In [8], a reason for this non-monotone behaviour for one-dimensional consolidation problems has been identified, and to avoid this effect, the use of staggered grid discretizations was suggested, theoretically analyzed and tested in two-dimensions [9]. Notice that the use of staggered grids is the way to incorporate a discrete infsup condition in the finite-difference framework, see for example [21]. An extension of this method to the case of discontinuous coefficients through harmonic averaging has been presented in [7]. For other Biot's models, such as the secondary consolidation model [10], the double porosity model [5] and the fully dynamic problem [4], staggered grids have also been successfully applied. In this work, we also apply this technique to non-linear poroelasticity problems in order to avoid the pressure oscillations.

Most of the works in this area treat the linear case. However, the hydraulic permeability of hydrogels and other hydrated soft tissues (e.g., cartilage and intervertebral disc) is deformation dependent [17]. Also, in simulations of hydraulic fracturing of rocks, it is often considered a dependence of the permeability tensor on the stress in exponential form [25]. All these models give rise to non-linear problems. Barucq et al. proved in [1] the existence and uniqueness of the solution of a non-linear fully dynamic poroelastic model, where the nonlinearity appeared in the first equation. Tavakoli and Ferronato [23] used the Galerkin's method to prove the existence and uniqueness of solution of the variational problem associated to a non-linear Biot's model where the permeability of the material depended on the strain. The aim of this research is to provide results of stability and convergence of a finite difference scheme on staggered grids for this non-linear Biot's model.

For solving numerically non-linear Biot's models, it is important to develop monotone schemes. Notice that monotone schemes (schemes that satisfy the discrete maximum principle) have remarkable properties providing physically correct solutions, see [13, 15, 16]. The study of these schemes will be carried out on a class of one-dimensional problems uncoupled due to the considered boundary conditions. This fact let us to simplify the problem to the case of non-linear parabolic equations with boundary conditions of the second type. For linear parabolic problems, an approach for the construction of second-order monotone finite difference schemes with boundary conditions of the second and third kind without using the differential equation on the boundary of the domain was suggested in [12]. The main idea was based on the extension of the solution of the problem in some small neighborhood of the domain and the use of half-integer grid points. Later, such approach was applied to develop monotone finite difference schemes for non-linear parabolic equations with boundary conditions of the first and third type [14]. In this work, this approach will be extended to the non-linear parabolic problem in the case of boundary conditions of the second type, and as a consequence to the one-dimensional non-linear Biot's model.

The rest of the manuscript is organized as follows. In Section 2, we propose the discretization by finite-difference schemes of non-linear parabolic equations with boundary conditions of the second type. Two-side estimates of the numerical solution and convergence results in the discrete L^2-norm are provided in Section 3 and Section 4 respectively. These results will be used to prove the corresponding estimates of the pressure and of the displacements as well as convergence results for the non-linear Biot's model. Numerical results are presented in Section 6, and some conclusions are drawn at the end in Section 7.

2. Difference schemes for non-linear parabolic problems with mixed boundary conditions

We consider a finite difference scheme for the solution of the non-linear parabolic differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right) = f(x,t), \quad x \in \Omega = (0,l), \quad t \in (0,T], \tag{1}$$

with initial and boundary conditions given by

$$\begin{aligned} u(x,0) &= u_0(x), \quad x \in \bar{\Omega}, \\ u(0,t) &= \mu_1(t), \quad k(u)\frac{\partial u}{\partial x}(l,t) = 0, \quad t \in (0,T]. \end{aligned} \tag{2}$$

We assume that the functions $k(u)$ and $f(x,t)$ are sufficiently smooth in such a way that the solution $u(x,t) \in C^{4,2}(Q_T)$, with $Q_T = [0,l] \times [0,T]$. Moreover, we suppose that there exist values k_1 and k_2 such that

$$0 < k_1 \leq k(u) \leq k_2, \quad \forall u \in [m_1, m_2],$$

where m_1 and m_2 are two constants such that

$$m_1 = \min\left\{\min_{t\in[0,T]} \mu_1(t), \min_{x\in[0,l]} u_0(x)\right\} + \int_0^T \min_{x\in[0,l]} f(x,\xi)\,\mathrm{d}\xi,$$
$$m_2 = \max\left\{\max_{t\in[0,T]} \mu_1(t), \max_{x\in[0,l]} u_0(x)\right\} + \int_0^T \max_{x\in[0,l]} f(x,\xi)\,\mathrm{d}\xi.$$

Let N and N_0 be positive integers and let $h = 2l/(2N+1)$ and $\tau = T/N_0$ be the space and time discretization parameters, respectively. Then, we introduce the uniform grids

$$\bar{\omega}_h = \left\{x_i = ih, i = 0,\ldots,N+1\right\}, \tag{3}$$
$$\bar{\omega}_t = \left\{t_n = n\tau, n = 0,\ldots,N_0,\ \tau N_0 = T\right\}. \tag{4}$$

On the grid $\bar{\omega}_h \times \bar{\omega}_t$, we approximate the differential problem by the implicit difference scheme

$$y^n_{t,i} = \left(a(y^n)y^{n+1}_{\bar{x}}\right)_{x,i} + \varphi^{n+1}_i, \quad i = 1,\ldots,N, \tag{5}$$
$$y^0_i = u_0(x_i), \quad i = 0,\ldots,N+1, \tag{6}$$
$$y^{n+1}_0 = \mu^{n+1}_1, \quad a^n_{N+1}y^{n+1}_{\bar{x},N+1} = 0, \tag{7}$$

where

$$a(y_i^n) = \frac{1}{2}\left(k(y_{i-1}^n) + k(y_i^n)\right), \quad \varphi_i^{n+1} = \frac{1}{\tau}\int_{t_n}^{t_{n+1}} f(x_i,\xi)\,\mathrm{d}\xi, \quad i = 1,\dots,N.$$

Here, we have used the standard notation of the theory of difference schemes [20]:

$$y_{t,i}^n = \frac{y_i^{n+1} - y_i^n}{\tau}, \quad u_{x,i} = \frac{u_{i+1} - u_i}{h}, \quad u_{\bar{x},i} = \frac{u_i - u_{i-1}}{h}.$$

Notice that due to the assumed regularity, the solution of problem (1)-(2) can be continuously extended to the domain $[0, l + h/2] \times [0, T]$.

3. Two-side estimates of the numerical solution

We start this section establishing a maximum principle for three-point difference schemes written in the form (see [20])

$$A_i^n y_{i-1}^{n+1} - C_i^n y_i^{n+1} + B_i^n y_{i+1}^{n+1} = -F_i^{n+1}, \quad i = 1,\dots,N-1, \tag{8}$$

$$y_0^{n+1} = \mu_1^{n+1}, \quad C_N^n y_N^{n+1} = A_N^n y_{N-1}^{n+1} + F_N^{n+1}. \tag{9}$$

Theorem 1. *Suppose that the following conditions are fulfilled*

$$A_i^n > 0, \quad B_i^n > 0, \quad C_i^n - A_i^n - B_i^n = 1, \quad i = 1,\dots,N-1, \tag{10}$$

$$A_N^n > 0, \quad B_N^n = 0, \quad C_N^n - A_N^n = 1, \tag{11}$$

then the following bounds for the solution hold

$$\min\{\mu_1^{n+1}, \min_{1\le i\le N} F_i^{n+1}\} \le y_i^{n+1} \le \max\{\mu_1^{n+1}, \max_{1\le i\le N} F_i^{n+1}\}, \quad i = 0,\dots,N. \tag{12}$$

Proof. We prove the upper bound. In a similar way, the lower bound can be proved. If the grid function $y(x)$ reaches its maximum on the boundary point $x = 0$, then for $i = 0,\dots,N$

$$y_i^{n+1} \le \max_{0\le i\le N} y_i^{n+1} = \mu_1^{n+1}.$$

If the grid function y reaches its maximum at an interior grid-point $x_{i^*}, 1 \le i^* \le N-1$. Then, as

$$C_{i^*}^n y_{i^*}^{n+1} = A_{i^*}^n y_{i^*-1}^{n+1} + B_{i^*}^n y_{i^*+1}^{n+1} + F_{i^*}^{n+1} \le (A_{i^*}^n + B_{i^*}^n) y_{i^*}^{n+1} + F_{i^*}^{n+1}.$$

from (10) we have

$$\max_{0\le i\le N} y_i^{n+1} = y_{i^*}^{n+1} \le F_{i^*}^{n+1}.$$

Finally, if the maximum is reached at $i = N$, then, the following inequality holds

$$C_N y_N^{n+1} = A_N^n y_{N-1}^{n+1} + F_N^{n+1} \le A_N^n y_N^{n+1} + F_N^{n+1},$$

and from (11) we obtain

$$\max_{0\le i\le N} y_i^{n+1} = y_N^{n+1} \le F_N^{n+1}.$$

□

Next, we apply the previous result to obtain estimates of the numerical solution of problem (5)-(7).

Corollary 2. *Let y^{n+1} be the numerical approximation of the non-linear parabolic problem (1)-(2) generated by scheme (5)-(7), then we have the following a priori estimate of the solution with respect to the right-hand-side, boundary and initial conditions*

$$\max_{0\le i\le N} y_i^{n+1} \le \max\left\{\max_{0\le n\le N_0-1} \mu_1^{n+1}, \max_{0\le i\le N} u_i^0\right\} + \int_0^T \max_{0\le i\le N} f(x_i,\xi)\,\mathrm{d}\xi = m_2. \tag{13}$$

Besides, we have the lower estimate

$$\min_{0\le i\le N} y_i^{n+1} \ge \min\left\{\min_{0\le n\le N_0-1} \mu_1^{n+1}, \min_{0\le i\le N} u_i^0\right\} + \int_0^T \min_{0\le i\le N} f(x_i,\xi)\,\mathrm{d}\xi = m_1. \tag{14}$$

Proof. First, by removing the fictitious point x_{N+1} we can write difference scheme (5)-(7) in the canonical form (8)-(9), with

$$A_i^n = \frac{\tau}{h^2} a_i^n, \quad B_i^n = A_{i+1}^n, \quad C_i^n = 1 + A_i^n + B_i^n, \quad i = 1,\dots,N-1, \tag{15}$$

$$F_i^{n+1} = y_i^n + \tau\varphi_i^{n+1}, \quad \varphi_i^{n+1} = \frac{1}{\tau}\int_{t_n}^{t_{n+1}} f(x_i,\xi)\,\mathrm{d}\xi, \quad i = 1,\dots,N-1, \tag{16}$$

$$A_N^n = \frac{\tau}{h^2} a_N^n, \quad C_N = 1 + A_N^n, \quad F_N^{n+1} = y_N^n + \tau\varphi_N^{n+1}. \tag{17}$$

The following upper and lower bounds follow directly from Theorem 1.

$$y_i^{n+1} \leq \max\left\{\mu_1^{n+1}, \max_{1\leq i\leq N}\left(y_i^n + \int_{t_n}^{t_{n+1}} f(x_i,\xi)\,\mathrm{d}\xi\right)\right\}, \quad i=0,\dots,N, \tag{18}$$

$$y_i^{n+1} \geq \min\left\{\mu_1^{n+1}, \min_{1\leq i\leq N}\left(y_i^n + \int_{t_n}^{t_{n+1}} f(x_i,\xi)\,\mathrm{d}\xi\right)\right\}, \quad i=0,\dots,N, \tag{19}$$

and therefore estimates (13) and (14) are obtained. □

4. Convergence in L^2

In this section, we use the energy inequality method to obtain estimates of the error and convergence results in the discrete L^2-norm. Defining the approximation error ψ^{n+1} in the interior nodes as

$$\psi_i^{n+1} = \varphi_i^{n+1} + \big(a(u^n)u_{\bar{x}}^{n+1}\big)_{x,i} - u_{t,i}^n, \quad i=1,\dots,N,$$

and at the right boundary point as

$$\psi_{N+1}^{n+1} = -a(u_{N+1}^n)u_{\bar{x},N+1}^{n+1},$$

the grid-function error $z = y - u$ is the solution of the discrete problem

$$z_{t,i}^n = \big(a(y^n)y_{\bar{x}}^{n+1} - a(u^n)u_{\bar{x}}^{n+1}\big)_{x,i} + \psi_i^{n+1}, \quad i=1,\dots,N, \tag{20}$$

$$a(y_{N+1}^n)y_{\bar{x},N+1}^{n+1} - a(u_{N+1}^n)u_{\bar{x},N+1}^{n+1} = \psi_{N+1}^{n+1}, \tag{21}$$

with initial and boundary conditions,

$$z_i^0 = 0, \quad i=0,\dots,N+1, \tag{22}$$

$$z_0^n = 0, \quad n=1,\dots,N_0. \tag{23}$$

It is easy to see that the approximation errors have order $O(h^2+\tau)$ in all the nodes, including the boundary point $x=l$, i.e. $\psi_i^{n+1} = O(h^2+\tau)$, $i = 1,\dots,N+1$.

We now define

$$(y,v)_{\omega_h^-} = \sum_{i=1}^{N-1} h y_i v_i, \quad \|y\|_{\omega_h^-} = \sqrt{(y,y)_{\omega_h^-}}, \tag{24}$$

$$(y,v)_{\omega_h} = \sum_{i=1}^{N} h y_i v_i, \quad \|y\|_{\omega_h} = \sqrt{(y,y)_{\omega_h}}, \tag{25}$$

where $w_h = \{x_i = ih, i = 0, \ldots, N\}$. The following results will be useful to prove the convergence of the scheme. With this purpose, we will use the formula of summation by parts [20], and the Gronwall's inequality.

Lemma 3. (Summation by parts). *For any grid functions y, v defined in ω_h vanishing on the boundary point $x_0 = 0$, the following identity holds*

$$(y, v_x)_{\omega_h^-} = -(y_{\bar{x}}, v)_{\omega_h} + y_N v_N. \tag{26}$$

Lemma 4. (Gronwall's inequality). *Let ε_n and f_n be non-negative discrete functions defined on the grid $\omega_t = \{t_n = n\tau, n = 0, 1, \ldots\}$ and $\rho > 0$ a constant such that the following inequalities are satisfied*

$$\varepsilon_{n+1} \leq \rho\, \varepsilon_n + f_n, \quad n = 0, 1, \ldots$$

Then, the following estimate holds

$$\varepsilon_{n+1} \leq \rho^{n+1} \varepsilon_0 + \sum_{k=0}^{n} \rho^{n-k} f_k.$$

Proof. See, for example, page 159 in [19].

Theorem 5. *The solution of scheme (20)-(23) satisfies the estimate*

$$\|z^{n+1}\|_{\omega_h} \leq C(\tau + h^2),$$

being C a positive constant independent of the discretization parameters.

Proof. Multiplying scalarly (20) by $2\tau z^{n+1}$, we obtain

$$2\tau (z_t^n, z^{n+1})_{\omega_h^-} = 2\tau \left(z^{n+1}, \left(a(y^n) y_{\bar{x}}^{n+1} - a(u^n) u_{\bar{x}}^{n+1} \right)_x \right)_{\omega_h^-} + 2\tau (z^{n+1}, \psi^{n+1})_{\omega_h^-}. \tag{27}$$

Using the identity $z^{n+1} = \dfrac{1}{2}\left(z^{n+1} + z^n\right) + \dfrac{\tau}{2} z_t^n$, the left-hand side in equation (27) satisfies the equality

$$2\tau (z_t^n, z^{n+1})_{\omega_h^-} = \|z^{n+1}\|^2_{\omega_h^-} - \|z^n\|^2_{\omega_h^-} + \tau^2 \|z_t^n\|^2_{\omega_h^-}. \tag{28}$$

On the other hand, using the formula of summation by parts (26), we have that

$$\begin{aligned} 2\tau \left(z^{n+1}, \left(a(y^n) y_{\bar{x}}^{n+1} - a(u^n) u_{\bar{x}}^{n+1} \right)_x \right)_{\omega_h^-} &= -2\tau \left(z_{\bar{x}}^{n+1}, a(y^n) y_{\bar{x}}^{n+1} - a(u^n) u_{\bar{x}}^{n+1} \right)_{\omega_h} \\ &\quad + 2\tau z_N^{n+1} \left(a(y_N^n) y_{\bar{x},N}^{n+1} - a(u_N^n) u_{\bar{x},N}^{n+1} \right). \end{aligned} \tag{29}$$

Next, we are going to bound the two different terms appearing in the right-hand side of the above equation. Considering equation (20) for $i = N$ and (21) we obtain

$$2\tau z_N^{n+1}\left(a(y_N^n)y_{\bar{x},N}^{n+1} - a(u_N^n)u_{\bar{x},N}^{n+1}\right) = -2h\tau z_N^{n+1} z_{t,N}^n + 2\tau z_N^{n+1}\left(\psi_{N+1}^{n+1} + h\psi_N^{n+1}\right).$$

Combining the embedding theorem

$$|z_i^{n+1}| \le \max_{0\le i\le N} |z_i^{n+1}| \le \sqrt{l}\|z_{\bar{x}}^{n+1}\|_{\omega_h}, \tag{30}$$

and the generalized Cauchy-Schwarz inequality

$$ab \le \varepsilon_1 a^2 + \frac{1}{4\varepsilon_1}b^2, \quad a, b \in R,$$

for $a = \|z_{\bar{x}}^{n+1}\|_{\omega_h}$ and $b = \sqrt{l}\left(\left|\psi_N^{n+1}\right| + \left|\psi_{N+1}^{n+1}\right|\right)$, we immediately get the following inequality

$$\begin{aligned}2\tau z_N^{n+1}\left(\left|\psi_{N+1}^{n+1}\right| + h\left|\psi_N^{n+1}\right|\right) &\le 2\tau\sqrt{l}\|z_{\bar{x}}^{n+1}\|_{\omega_h}\left(\left|\psi_N^{n+1}\right| + \left|\psi_{N+1}^{n+1}\right|\right) \le 2\tau\varepsilon_1\|z_{\bar{x}}^{n+1}\|_{\omega_h}^2 \\ &+\frac{\tau l}{\varepsilon_1}\max\left\{\left|\psi_{N+1}^{n+1}\right|^2, \left|\psi_N^{n+1}\right|^2\right\} \le 2\tau\varepsilon_1\|z_{\bar{x}}^{n+1}\|_{\omega_h}^2 + \tau c(h^2+\tau)^2,\end{aligned}$$

where c is a constant independent of h and τ and we have assumed that $h \le 1$. Using that

$$z_N^{n+1} = \frac{z_N^{n+1} + z_N^n}{2} + \frac{\tau}{2}z_{t,N}^n,$$

we obtain the following bound for the second term of the right-hand side of equality (29)

$$\begin{aligned}2\tau z_N^{n+1}\left(a(y_N^n)y_{\bar{x},N}^{n+1} - a(u_N^n)u_{\bar{x},N}^{n+1}\right) &\le -h(z_N^{n+1})^2 + h(z_N^n)^2 - \tau^2 h(z_{t,N}^n)^2 \\ &+2\tau\varepsilon_1\|z_{\bar{x}}^{n+1}\|_{\omega_h}^2 + \tau c(h^2+\tau)^2.\end{aligned} \tag{31}$$

Next, we focus on the first term of the right-hand side in (29). Taking into account that

$$a(y^n)y_{\bar{x}}^{n+1} - a(u^n)u_{\bar{x}}^{n+1} = a(y^n)z_{\bar{x}}^{n+1} + (a(y^n) - a(u^n))u_{\bar{x}}^{n+1},$$

we have the equality

$$\begin{aligned}-2\tau\left(z_{\bar{x}}^{n+1}, a(y^n)y_{\bar{x}}^{n+1} - a(u^n)u_{\bar{x}}^{n+1}\right)_{\omega_h} &= -2\tau\left(z_{\bar{x}}^{n+1}, a(y^n)z_{\bar{x}}^{n+1}\right)_{\omega_h} - \\ &\quad 2\tau\left(z_{\bar{x}}^{n+1}, (a(y^n) - a(u^n))u_{\bar{x}}^{n+1}\right)_{\omega_h}.\end{aligned}$$

Since $a(y) \geq k_1$ for $y \in [m_1, m_2]$, and taking into account that due to the assumed regularity of function k, there exists a positive constant L such that for $i = 1, \dots, N$

$$|a(y_i^n) - a(u_i^n)| \leq L|z_i^n|_{(0.5)},$$

where the grid-function $z_{(0.5)}$ is defined as $(z_i)_{(0.5)} = (z_i + z_{i-1})/2$, we have that

$$-2\tau \left(z_{\bar{x}}^{n+1}, a(y^n)y_{\bar{x}}^{n+1} - a(u^n)u_{\bar{x}}^{n+1}\right)_{\omega_h} \leq -2\tau k_1 \|z_{\bar{x}}^{n+1}\|_{\omega_h}^2 + 2\tau L \left(|z^n|_{(0.5)}|u_{\bar{x}}^{n+1}|, |z_{\bar{x}}^{n+1}|\right)_{\omega_h}$$

Taking into account that due to the smoothness of the solution the following inequality holds

$$\left|u_{\bar{x}}^{n+1}\right| \leq \frac{1}{h} \int_{x_{i-1}}^{x_i} \left|\frac{\partial u^{n+1}}{\partial x}\right| \mathrm{d}x \leq c_2,$$

and applying the generalized Cauchy-Schwarz inequality, the following estimate is obtained

$$2\tau L \left(|z^n|_{(0.5)}|u_{\bar{x}}^{n+1}|, |z_{\bar{x}}^{n+1}|\right)_{\omega_h} \leq 2\tau L c_2 \varepsilon_2 \|z_{\bar{x}}^{n+1}\|_{\omega_h}^2 + \frac{\tau L c_2}{2\varepsilon_2}\|z^n\|_{\omega_h}^2,$$

and therefore

$$-2\tau \left(z_{\bar{x}}^{n+1}, a(y^n)y_{\bar{x}}^{n+1} - a(u^n)u_{\bar{x}}^{n+1}\right)_{\omega_h} \leq -2\tau \left(k_1 - L c_2 \varepsilon_2\right) \|z_{\bar{x}}^{n+1}\|_{\omega_h}^2 + \frac{\tau L c_2}{2\varepsilon_2}\|z^n\|_{\omega_h}^2. \tag{32}$$

For the second term in the right-hand side in (27) we have,

$$2\tau(z^{n+1}, \psi^{n+1})_{\omega_h^-} \leq 2\tau\|z^{n+1}\|_{\omega_h^-}\|\psi^{n+1}\|_{\omega_h^-} \leq 2\tau l\|z_{\bar{x}}^{n+1}\|_{\omega_h}\|\psi^{n+1}\|_{\omega_h^-},$$

and applying again the generalized Cauchy-Schwarz inequality

$$2\tau(z^{n+1}, \psi^{n+1})_{\omega_h^-} \leq 2\tau\varepsilon_3 l\|z_{\bar{x}}^{n+1}\|_{\omega_h}^2 + \frac{\tau l}{2\varepsilon_3}\|\psi^{n+1}\|_{\omega_h^-}^2 \leq 2\tau\varepsilon_3 l\|z_{\bar{x}}^{n+1}\|_{\omega_h}^2 + \tau c(h^2+\tau)^2. \tag{33}$$

Taking into account (27), (28), (31), (32) and (33)

$$\|z^{n+1}\|_{\omega_h}^2 + \tau^2\|z_t^n\|_{\omega_h}^2 + 2\tau(k_1 - \varepsilon_1 - Lc_2\varepsilon_2 - l\varepsilon_3)\|z_{\bar{x}}^{n+1}\|_{\omega_h}^2 \leq (1+\tau c)\|z^n\|_{\omega_h}^2 + \tau c(h^2+\tau)^2.$$

Choosing $\varepsilon_1, \varepsilon_2, \varepsilon_3$ small enough in order to satisfy that $k_1 - \varepsilon_1 - cL\varepsilon_2 - l\varepsilon_3 > 0$, we obtain the recurrent estimate

$$\|z^{n+1}\|_{\omega_h}^2 \leq (1 + \tau c)\|z^n\|_{\omega_h}^2 + \tau c(h^2 + \tau)^2 \leq \mathrm{e}^{\tau c}\|z^n\|_{\omega_h}^2 + \tau c(h^2 + \tau)^2.$$

Finally, using Gronwall's inequality we arrive to the result. □

5. Finite difference schemes for the non-linear poroelasticity problem

In this section, we consider a non-linear one-dimensional Biot's model. Considering as unknowns the displacement of the solid $u(x,t)$ and the pore pressure of the fluid $p(x,t)$, the governing equations read

$$-\frac{\partial}{\partial x}\left((\lambda+2\mu)\frac{\partial u}{\partial x}\right)+\frac{\partial p}{\partial x}=0, \quad x\in(0,l), \tag{34}$$

$$\frac{\partial}{\partial t}\left(\phi\beta p+\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial x}\left(K\left(\frac{\partial u}{\partial x}\right)\frac{\partial p}{\partial x}\right)=q(x,t), \quad x\in(0,l),\ 0<t\leq T, \tag{35}$$

with initial and boundary conditions given by

$$\left(\phi\beta p+\frac{\partial u}{\partial x}\right)(x,0)=0, \quad x\in(0,l), \tag{36}$$

$$p(0,t)=\mu_0(t), \quad (\lambda+2\mu)\frac{\partial u}{\partial x}(0,t)=-s_0(t), \tag{37}$$

$$u(l,t)=\mu_1(t), \quad K\left(\frac{\partial u}{\partial x}\right)\frac{\partial p}{\partial x}(l,t)=0, \tag{38}$$

where λ and μ are the Lamé coefficients, ϕ is the porosity, β the compressibility coefficient of the fluid, K is the permeability of the porous medium which is assumed to be dependent on the strain $\partial u/\partial x$, and the source term $q(x,t)$ represents a forced extraction or injection process.

Due to the considered boundary conditions for the displacements and for the pressure, the above problem is decoupled. In effect, from equation (34) the following relation between the pressure and the strain holds

$$p(x,t)=(\lambda+2\mu)\frac{\partial u}{\partial x}(x,t)+\varphi(t). \tag{39}$$

Using the boundary conditions at $x=0$ we have that $\varphi(t)=\mu_0(t)+s_0(t)$, and therefore

$$\frac{\partial u}{\partial x}(x,t)=\frac{1}{\lambda+2\mu}\left(p(x,t)-\mu_0(t)-s_0(t)\right). \tag{40}$$

On the other hand, from (39) and the initial condition (36), we have

$$p(x,0)=c_1(\mu_0(0)+s_0(0)),$$

where

$$c_1 = \frac{1}{(1 + \phi\beta(\lambda + 2\mu))}.$$

By substituting the strain (40) in (35), we obtain a non-linear parabolic problem for the pressure

$$\begin{aligned} \frac{\partial p}{\partial t} &= c_2 \frac{\partial}{\partial x}\left(k(p)\frac{\partial p}{\partial x}\right) = f(x,t), \quad x \in \Omega = (0,l), \quad t \in (0,T], &(41)\\ p(x,0) &= c_1(\mu_0(0) + s_0(0)), \quad x \in \bar{\Omega}, &(42)\\ p(0,t) &= \mu_0(t), \quad k(p)\frac{\partial p}{\partial x}(l,t) = 0, \quad t \in (0,T], &(43) \end{aligned}$$

where

$$f(x,t) = c_2 q(x,t) + c_2 c_3(\mu_0'(t) + s_0'(t)), \quad c_3 = \frac{1}{\lambda + 2\mu}, \quad c_2 = \frac{1}{\phi\beta + c_3}.$$

Relation (40) shows explicitly the dependence of the permeability on the pressure. We have introduced function k as $k(p) = K(\partial u/\partial x(p))$. Once the pressure is obtained solving problem (41)-(43), we can calculate the displacements by using (40). We assume that the input data are sufficiently smooth in such a way that the solutions $p(x,t) \in C^{4,2}(Q_T)$ and $u(x,t) \in C^{3,2}(Q_T)$. Moreover, we suppose that there exist values k_3 and k_4 such that

$$0 < k_3 \le k(p) \le k_4, \quad \forall p \in [m_3, m_4],$$

where m_3 and m_4 are two constants such that

$$\begin{aligned} m_3 &= \min\left\{\min_{t\in[0,T]} \mu_0(t), c_1(\mu_0(0) + s_0(0))\right\} + \int_0^T \min_{x\in[0,l]} f(x,\xi)\,\mathrm{d}\xi,\\ m_4 &= \max\left\{\max_{t\in[0,T]} \mu_0(t), c_1(\mu_0(0) + s_0(0))\right\} + \int_0^T \max_{x\in[0,l]} f(x,\xi)\,\mathrm{d}\xi. \end{aligned}$$

In order to obtain schemes providing solutions without oscillations, we will consider staggered grids which have been widely used for the stabilization of difference schemes in CFD. Given a positive integer N and $h = \frac{2l}{2N+1}$, let us define two different grids, $\overline{\omega}_p$ to discretize the pressure and $\overline{\omega}_u$ to discretize the displacement,

$$\begin{aligned} \overline{\omega}_p &= \{x_i \,|\, x_i = ih, \ i = 0, \ldots, N+1\},\\ \overline{\omega}_u &= \left\{x_{i+1/2} \,|\, x_{i+1/2} = (i + \tfrac{1}{2})h, i = 0, \ldots, N\right\}. \end{aligned}$$

The grid points for u are shown in Figure 1 by small circles, while the grid points for p are shown by filled circles. Notice that we have introduced the fictitious node $x_{N+1} = l + h/2$. Here, again we consider a uniform grid

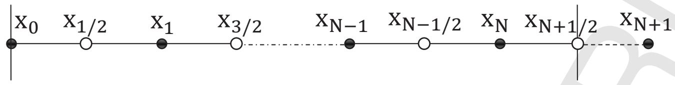


Figure 1: Staggered grid: Meshes for displacement $\circ$ and for pressure $\bullet$, including the fictitious point.

for time discretization $\overline{\omega}_t$ with step size $\tau > 0$, as defined in (4). Problem (34)-(38) is approximated by the finite difference scheme

$$-\left((\lambda + 2\mu)u_{\bar{x}}^{n+1}\right)_{x,i} + p_{\bar{x},i}^{n+1} = 0, \quad i = 1, \dots, N-1, \tag{44}$$

$$(\phi\beta p^n + u_{\bar{x}}^n)_{t,i} - \left(K(u_x^n)p_{\bar{x}}^{n+1}\right)_{x,i} = q_i^{n+1}, \quad i = 1, \dots, N, \tag{45}$$

$$\phi\beta p_i^0 + u_{\bar{x},i}^0 = 0, \quad i = 0, \dots, N, \tag{46}$$

$$p_0^{n+1} = \mu_0^{n+1}, \quad (\lambda + 2\mu)u_{\bar{x},0}^{n+1} = -s_0^{n+1}, \tag{47}$$

$$u_{N+1/2}^{n+1} = \mu_1^{n+1}, \quad K(u_x^n)p_{\bar{x},N+1}^{n+1} = 0. \tag{48}$$

It is trivial to see that the solutions of this problem satisfy the following relation

$$u_{\bar{x},i}^{n+1} = \frac{1}{\lambda + 2\mu}\left(p_i^{n+1} - \mu_0^{n+1} - s_0^{n+1}\right), \quad i = 0, \dots, N,$$

which corresponds to the approximation of (40). Therefore, we deduce that the approximate pressure is the solution of the finite difference scheme

$$p_{t,i}^n = c_2\left(a(p^n)p_{\bar{x}}^{n+1}\right)_{x,i} + \varphi_i^{n+1}, \quad i = 1, \dots, N, \tag{49}$$

$$p_i^0 = c_1(\mu_0^0 + s_0^0), \quad i = 0, \dots, N, \tag{50}$$

$$p_0^{n+1} = \mu_0^{n+1}, \quad a_{N+1}^n p_{\bar{x},N+1}^{n+1} = 0, \tag{51}$$

where again

$$a(p_i^n) = \frac{1}{2}\left(k(p_{i-1}^n) + k(p_i^n)\right), \quad i = 1, \dots, N.$$

Once the approximation of the pressure is obtained, the discrete displacements can be calculated as

$$\begin{aligned} u^{n+1}_{N+1/2} &= \mu^{n+1}_1, \\ u^{n+1}_{i+1/2} &= u^{n+1}_{N+1/2} - c_3 \left(\sum_{k=i+1}^{N} h p^{n+1}_k - (l - x_{i+1/2})(\mu^{n+1}_0 + s^{n+1}_0) \right), \quad i = N-1, \dots, 0. \end{aligned}$$

By removing the fictitious point x_{N+1}, difference scheme (49)-(51) can be written in the canonical form

$$A^n_i p^{n+1}_{i-1} - C^n_i p^{n+1}_i + B^n_i p^{n+1}_{i+1} = -F^{n+1}_i, \quad i = 1, \dots, N-1, \tag{52}$$

$$p^0_i = c_1(\mu^0_0 + s^0_0), \tag{53}$$

$$p^{n+1}_0 = \mu^{n+1}_0, \quad C^n_N p^{n+1}_N = A^n_N p^{n+1}_{N-1} + F^{n+1}_N, \tag{54}$$

where

$$A^n_i = \frac{c_2 \tau}{h^2} a^n_i, \quad B^n_i = A^n_{i+1}, \quad C^n_i = 1 + A^n_i + B^n_i, \quad i = 1, \dots, N-1, \tag{55}$$

$$F^{n+1}_i = p^n_i + \tau \varphi^{n+1}_i, \; \varphi^{n+1}_i = \frac{1}{\tau} \int_{t_n}^{t_{n+1}} f(x_i, \xi)\, \mathrm{d}\xi,$$

$$f(x_i, t_{n+1}) = c_2 q^{n+1}_i + c_2 c_3 (\mu^n_{0,t} + s^n_{0,t}), \; i = 1, \dots, N-1, \tag{56}$$

$$A^n_N = \frac{c_2 \tau}{h^2} a^n_N, \quad C_N = 1 + A^n_N, \quad F^{n+1}_N = p^n_N + \tau \varphi^{n+1}_N. \tag{57}$$

Next, two-side estimates for the solution of the non-linear poroelasticity problem are given.

Theorem 6. *The solution of finite difference scheme (44)-(48) satisfies the following bounds*

$$m_3 \le p^{n+1}_i \le m_4,$$

$$u^{n+1}_{i+1/2} \ge \mu^{n+1}_1 - c_3(l - x_{i+1/2}) \left(m_4 - \mu^{n+1}_0 - s^{n+1}_0 \right), \tag{58}$$

$$u^{n+1}_{i+1/2} \le \mu^{n+1}_1 - c_3(l - x_{i+1/2}) \left(m_3 - \mu^{n+1}_0 - s^{n+1}_0 \right), \quad i = 0, \dots, N, \tag{59}$$

where constants m_3 and m_4 only depend on the input data of the problem in the following way

$$m_3 = \min \left\{ \min_{0 \le n \le N_0 - 1} \mu^{n+1}_0, c_1(\mu^0_0 + s^0_0) \right\} + \int_0^T \min_{1 \le i \le N} f(x_i, \xi)\, \mathrm{d}\xi,$$

$$m_4 = \max \left\{ \max_{0 \le n \le N_0 - 1} \mu^{n+1}_0, c_1(\mu^0_0 + s^0_0) \right\} + \int_0^T \max_{1 \le i \le N} f(x_i, \xi)\, \mathrm{d}\xi.$$

Proof. The result follows immediately applying Corollary 2.

The error grid-functions for the pressure and for the displacements z and w respectively are the solution of

$$z_{t,i}^n = \left(a(p^n) z_{\bar{x}}^{n+1}\right)_{x,i} + \left((a(p^n) - a(p(x_i, t_{n+1}))) p_{\bar{x}}^{n+1}\right)_{x,i} + \psi_{p,i}^{n+1}, \ i = 1, \ldots, N, \tag{60}$$

$$a(p_{N+1}^n) z_{\bar{x},N+1}^{n+1} = \psi_{p,N+1}^{n+1} - (a(p_{N+1}^n) - a(p(x_{N+1}, t_{n+1}))) p_{\bar{x},N+1}^{n+1}, \tag{61}$$

$$z_i^0 = 0, \quad i = 1, \ldots, N, \qquad\qquad z_0^{n+1} = 0, \quad n = 0, 1, \ldots, \tag{62}$$

$$w_{\bar{x},i}^{n+1} = z_i^{n+1} + \psi_{u,i}^{n+1}, \quad i = 0, \ldots, N, \tag{63}$$

where $\psi_{p,i}^{n+1} = O(h^2 + \tau)$ and $\psi_{u,i}^{n+1} = O(h^2)$ are the approximation errors for the pressure and the displacements, respectively. Finally, the following convergence result holds

Theorem 7. *The solution of scheme (60)-(63) satisfies the estimates*

$$\|z^{n+1}\|_{\omega_h} \le C_1(h^2 + \tau), \quad \max_{1 \le i \le N} |w_i^{n+1}| \le C_2(h^2 + \tau),$$

where C_1 and C_2 are positive constants independent of the discretization parameters.

Proof. The convergence result for the pressure follows from Theorem 3. Regarding the displacements, we have that

$$\max_{0 \le i \le N} |w_i^{n+1}| \le \sqrt{l} \|w_{\bar{x}}^{n+1}\|_{\omega_h} \le \sqrt{l} \left(\|z^{n+1}\|_{\omega_h} + \|\psi_{u,i}^{n+1}\|_{\omega_h}\right) \le C(h^2 + \tau).$$

$\square$

6. Numerical experiments

In this section we consider two numerical experiments of the non-linear Biot's model. In both cases, we fix $\lambda + 2\mu = 1$ and $\beta = 0$. The results obtained are similar for either set of values.

6.1. Analytical solution of a model problem

We start this section by setting a simple analytic solution that satisfies the system of poroelasticity equations. We consider the non-linear system of partial differential equations

$$
\begin{aligned}
&-\frac{\partial^2 u}{\partial x^2} + \frac{\partial p}{\partial x} = 0, \quad x \in (0, \pi/2),\\
&\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial x}\left(K\left(\frac{\partial u}{\partial x}\right)\frac{\partial p}{\partial x}\right) = q(x,t), \quad x \in (0,\pi/2),\ 0 < t \leq 1,\\
&\left(\frac{\partial u}{\partial x}\right)(x,0) = \sin(x), \quad x \in (0,\pi/2),\\
&p(0,t) = 0, \quad \frac{\partial u}{\partial x}(0,t) = 0,\\
&u(\pi/2,t) = 0, \quad K\left(\frac{\partial u}{\partial x}\right)\frac{\partial p}{\partial x}(\pi/2,t) = 0,
\end{aligned}
\tag{64}
$$

where $K(\partial u/\partial x) = \exp(\partial u/\partial x)$ and $q(x,t)$ are such that the solutions of the problem are $u(x,t) = -\exp(-t)\cos(x)$ and $p(x,t) = \exp(-t)\sin(x)$. To

	h	$h/2$	$h/4$	$h/8$
τ	0.007017	0.006880	0.006845	0.006836
$\tau/4$	0.001794	0.001650	0.001613	0.001604
$\tau/16$	0.000588	0.000441	0.000403	0.000394
$\tau/64$	0.000296	0.000147	0.000109	0.000100

Table 1: L^2-norm errors for pressure with different time and space discretization parameters.

check the accuracy of the proposed scheme, in Table 1 and Table 2 we show the L^2-norm of the pressure error and the maximum norm for the displacement error respectively for different values of space and time discretization parameters. In particular $h = \pi/(2N+1)$ with $N = 20$, and $\tau = 0.05$ have been initially chosen, and from them, $h/2^k$ and $\tau/4^k$, are considered with $k = 1, 2, 3$. As we can observe, in both cases first order convergence is achieved as expected from the previous theoretical results.

6.2. A second model problem

We consider an idealized problem consisting of a column of fluid-saturated porous media of height $l = 1$. The column is bounded by rigid, imperme-

	h	$h/2$	$h/4$	$h/8$
τ	0.008408	0.0081192	0.008136	0.008122
$\tau/4$	0.002208	0.001980	0.001922	0.001907
$\tau/16$	0.000775	0.000544	0.000485	0.000470
$\tau/64$	0.000426	0.000195	0.000135	0.000120

Table 2: Maximum norm errors for displacement with different time and space discretization parameters.

able bottom and walls. The fluid flows freely through the top surface at atmospheric pressure and therefore a null pressure is imposed. A unit load is applied on the top of the column. This problem is an extension of the classical one-dimensional problem of Terzaghi [24]. Under these conditions, we have to solve the following problem

$$\begin{aligned} &-\frac{\partial^2 u}{\partial x^2} + \frac{\partial p}{\partial x} = 0, \quad x \in (0,1), \\ &\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial x}\left(K\left(\frac{\partial u}{\partial x}\right)\frac{\partial p}{\partial x}\right) = 0, \quad x \in (0,1),\ 0 < t \leq T, \end{aligned} \tag{65}$$

with initial an boundary conditions

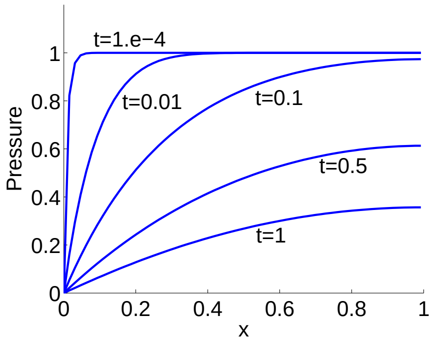
$$\begin{aligned} &\left(\frac{\partial u}{\partial x}\right)(x,0) = 0, \quad x \in (0,1), \\ &p(0,t) = 0, \quad \frac{\partial u}{\partial x}(0,t) = -1, \\ &u(1,t) = 0, \quad K\left(\frac{\partial u}{\partial x}\right)\frac{\partial p}{\partial x}(1,t) = 0, \end{aligned}$$

where $K\left(\frac{\partial u}{\partial x}\right) = \exp\left(\frac{\partial u}{\partial x}\right)$.

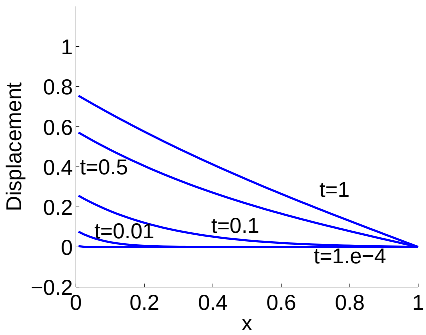
Applying Theorem 6 to this numerical experiment, we can obtain bounds for the pressure and displacement solutions. Taking into account that in this case $m_1 = 0$ and $m_2 = 1$, we have

$$0 \leq p_i^{n+1} \leq 1, \quad 0 \leq u_{i+1/2}^{n+1} \leq 1 - x_{i+1/2}, \quad i = 0, \ldots, N.$$

In order to illustrate the previous bounds, in Figure 2, we show the numerical approximation for the pressure and the displacement at different times. For these results we have chosen $h = 2/(2N+1)$ with $N = 64$, and $\tau = 10^{-4}$. We can observe that indeed the obtained pressure solutions are between 0 and 1 and the displacement approximations are between 0 and $1-x$.



(a)



(b)

Figure 2: Numerical approximation for (a) the pressure and (b) the displacement at different times: $t = 1.e-4$, $t = 0.01$, $t = 0.1$, $t = 0.5$ and $t = 1$.

7. Conclusions

We have considered staggered finite difference methods for a one-dimensional Biot's model. The permeability tensor depends on the strain, thus yielding a non-linear model. Strong estimates for the displacement and pressure solutions are provided, and convergence results are proved in the discrete L^2-norm. Planned future work includes the extension of these results to two and three dimensional non-linear poroelasticity problems.

Acknowledgements

This research has been partially supported by Spanish Project MTM2013-40842-P, by a grant from Iceland, Liechtenstein and Norway through the EEA Financial Mechanism (Operated by Universidad Complutense de Madrid) and the DGA (Grupo consolidado PDIE).

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