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Improving the domain of parameters for Newton's method with applications☆

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Abstract

We present a new technique to improve the convergence domain for Newton's method both in the semilocal and local case. It turns out that with the new technique the sufficient convergence conditions for Newton's method are weaker, the error bounds are tighter and the information on the location of the solution is at least as precise as in earlier studies. Numerical examples are given showing that our results apply to solve nonlinear equations in cases where the old results cannot apply.

Keywords: Banach space, majorizing sequence, local/semilocal convergence, domain of parameters.

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution x^* of equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet-differentiable operator defined on a convex subset D of a Banach space X with values in a Banach space Y.

Many problems in Applied Sciences including engineering can be solved by means of finding the solutions of equations in a form like (1.1) using Mathematical Modelling [2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 14, 16]. For example, dynamic

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systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. Except in special cases, the solutions of these equations can rarely be found in closed form. This is the main reason why the most commonly used solution methods are usually iterative. The convergence analysis of iterative methods is usually divided into two categories: semilocal and local convergence analysis. The semilocal convergence matter is, based on the information around an initial point, to give criteria ensuring the convergence of iteration procedures; while the local one is, based on the information about a solution, to find estimates of the radii of the convergence balls. A very important problem in the study of iterative procedures is the convergence domain. In general the convergence domain is small. Therefore, it is important to enlarge the convergence domain without additional hypotheses. Another important problem is to find more precise error estimates on the distances $\|x_{n+1} - x_n\|$, $\|x_n - x^*\|$.

Newton's method defined for each $n = 0, 1, 2, \dots$ by

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \tag{1.2}$$

where x_0 is an initial point,is undoubtedly the most popular method for generating a sequence $\{x_n\}$ approximating x^*.

Let $U(z, \varrho)$, $\bar{U}(z, \varrho)$ stand, respectively for the open and closed ball in X with center $z \in X$ and of radius $\varrho > 0$. Let also $L(X, Y)$ stand for the space of bounded linear operators from X into Y.

The best known semilocal convergence result for Newton's method is the Newton-Kantorovich theorem [3, 4, 5, 9] (see Theorem 1 that follows) which is based on the hypotheses (given in affine invariant form) by:

(H_1) There exists $x_0 \in D$ such that $F'(x_0)^{-1} \in L(Y, X)$ and a parameter $\eta \geq 0$ such that

$$\|F'(x_0)^{-1}F(x_0)\| \leq \eta$$

(H_2) There exists a parameter $L > 0$ such that for each $x, y \in D$

$$\|F'(x_0)^{-1}(F'(x) - F'(y))\| \leq L\|x - y\|.$$

and

(H_3) $\bar{U}(x_0, R) \subseteq D$ for some $R > 0$.

The sufficient semilocal convergence condition of Newton's method is given by the famous for its simplicity and clarity Kantorovich hypothesis

$$h = 2L\eta \leq 1. \tag{1.3}$$

There are simple examples in the literature to show that hypothesis (1.3) is not satisfied but Newton's method converges starting at x_0 (See Example 2.2). Moreover, the convergence domain of Newton's method depending on the parameters L and η is in general small. Therefore, it is important to enlarge the convergence domain by using the same constants L and η using techniques as the ones that appeared in [8, 9, 13, 14]. Argyros et all in a series of papers [3, 4, 5, 6] presented weaker sufficient convergence conditions for Newton's method by using more precise majorizing sequences than before [9]. These conditions are

$$h_1 = 2A_1\eta \le 1, \tag{1.4}$$
$$h_2 = 2A_2\eta \le 1, \tag{1.5}$$
$$h_3 = 2A_3\eta \le 1, \tag{1.6}$$
$$h_4 = 2A_4\eta_0 \le 1, \tag{1.7}$$

where

$$A_1 = \frac{L_0 + L}{2}, \quad A_2 = \frac{1}{8}\left(L + 4L_0 + \sqrt{L^2 + 8L_0L}\right),$$
$$A_3 = \frac{1}{8}\left(4L_0 + \sqrt{L_0L + 8L_0^2} + \sqrt{L_0L}\right), \quad A_4 = \frac{1}{\eta_0},$$

η_0 is the small positive root of a quadratic polynomial (see Theorem in [3, 4, 5, 6] or Theorem 5 or Theorem 6 that follows) and $L_0 > 0$, is the center-Lipschitz constant such that

$$\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \le L_0\|x - x_0\| \quad \text{for each } x \in D \tag{1.8}$$

The existence of L_0 is always implied by (H_2).

We have that

$$L_0 \le L \tag{1.9}$$

holds in general and $\frac{L}{L_0}$ can be arbitrarily large [3]. Notice also (1.8) is not an additional to (H_2) hypothesis, since in practice the computation of parameter L involves the computation of L_0 as a special case. Notice that if $L_0 = L$ conditions (1.4)–(1.7) reduce to condition(1.3). However, if $L_0 < L$, then we have [4, 6]

$$h \le 1 \Rightarrow h_1 \le 1 \Rightarrow h_2 \le 1 \Rightarrow h_3 \le 1 \Rightarrow h_4 \le 1, \tag{1.10}$$

$$\begin{aligned} &\frac{h_1}{h} \to \frac{1}{2}, \quad \frac{h_2}{h} \to \frac{1}{4}, \quad \frac{h_2}{h_1} \to \frac{1}{2}, \quad \frac{h_3}{h} \to 0, \\ &\text{and} \\ &\frac{h_3}{h_2} \to 0, \quad \frac{h_3}{h_1} \to 0, \quad \text{as } \frac{L_0}{L} \to 0. \end{aligned} \tag{1.11}$$

Estimates (1.11) show by how many times (at most) a condition is improving the previous one.

Notice also that the error bounds on the distances involved as well as the location on the solution x^* are also improved under these weaker conditions [3, 4, 5, 6]. In the present study, the main goal is to improve further conditions (1.3)–(1.7) by using smaller than L_0 and L parameters and by restricting the domain D. Similar ideas are used to improve the error bounds and enlarge convergence radii in the local convergence case.

The rest of the paper is organized as follows: The semilocal and local convergence analysis is presented in Section 2. The numerical examples are given in the concluding Section 3.

2. Convergence Analysis

We present first the semilocal convergence analysis of Newton's method. Next, we state the following version of the Newton Kantorovich theorem [3, 4, 5, 9].

Theorem 1. *Let $F : \mathbb{D} \subset X \to Y$ be a Fréchet-differentiable operator. Suppose that* (1.3) *and conditions* (H_1)–(H_3) *hold, where*

$$R = \frac{1-\sqrt{1-h}}{L}.$$

Then, the sequence $\{x_n\}$ generated by Newton's method is well defined, remains in $U(x_0, R)$ for each $n = 0, 1, 2, \dots$ and converges to a unique solution $x^* \in \bar{U}(x_0, R)$ of equation (1.1).

Let us consider an academic example, where the Newton-Kantorovich hypothesis (1.3) is not satisfied.

Example 1. *Let $X = Y = \mathbb{R}$, $x_0 = 1$, $D = U(1, 1-p)$ for $p \in (0, \frac{1}{2})$ and define function F on D by*

$$F(x) = x^3 - p.$$

We have that $\eta = \frac{1-p}{3}$ and $L = 2(2-p)$. Then, hypothesis (1.3) *is not satisfied, since*

$$h = \frac{4}{3}(2-p)(1-p) > 1 \quad \text{for each } p \in (0, \frac{1}{2}).$$

Hence, there is no guarantee under the hypotheses of Theorem 1 that sequence $\{x_n\}$starting from $x_0 = 1$ converges to $x^* = \sqrt[3]{p}$.

Next, we present a semilocal convergence result that extends the applicability of Theorem 1.

Theorem 2. *Let $F : \mathbb{D} \subset X \to Y$ be a Fréchet-differentiable operator. Suppose that there exist $x_0 \in D$, $\eta \geq 0$, $\gamma > 1$, $L_\gamma > 0$ such that*

$$F'(x_0) \in L(Y, X),$$

$$\|F'(x_0)^{-1}F(x_0)\| \leq \eta,$$

$$D_\gamma = U(x_0, \gamma\eta) \subseteq D,$$

$$\|F'(x_0)^{-1}(F'(x) - F'(y))\| \leq L_\gamma \|x-y\| \quad \text{for each } x, y \in D_\gamma,$$

$$h_\gamma = 2L_\gamma \eta \leq 1$$

and

$$R_\gamma \leq \gamma\eta,$$

where

$$R_\gamma = \frac{1-\sqrt{1-h_\gamma}}{L_\gamma}.$$

Then, the sequence $\{x_n\}$ generated by Newton's method is well defined, remains in $U(x_0, R_\gamma)$ for each $n = 0, 1, 2, \dots$ and converges to a unique solution $x^* \in \bar{U}(x_0, R_\gamma)$ of equation (1.1).

Proof. The hypotheses of Theorem 1 on D_γ are satisfied. ■

Example 2. *Returning back to the Example 1 let* $p = 0.49$ *and* $\gamma = 1.9$. *Then, we have that* $\eta = 0.17$, $\gamma\eta = 0.323$ *and* $R_\gamma = 0.258202394 < 0.323 < 1 - p = 0.51$. *Hence, the hypotheses of Theorem 2 are satisfied.*

Next, we present a semilocal result given in [6] involving condition (1.6).

Theorem 3. *Let* $F : \mathbb{D} \subset X \to Y$ *be a continuously Fréchet-differentiable operator. Suppose that* (1.6) *and conditions* $(H_1) - (H_3)$ *hold, where*

$$r_3 = \eta + \frac{L_0\eta^2}{2(1-\alpha)(1-L_0\eta)},$$

and

$$\alpha = \frac{2L}{L + \sqrt{L^2 + 8L_0L}}.$$

Then, the sequence $\{x_n\}$ *generated by Newton's method is well defined, remains in* $U(x_0, r_3)$ *for each* $n = 0, 1, 2, \ldots$ *and converges to a unique solution* $x^* \in \bar{U}(x_0, r_3)$ *of equation* (1.1).

Next, we present an improvement of Theorem 3.

Theorem 4. *Let* $F : \mathbb{D} \subset X \to Y$ *be a Fréchet-differentiable operator. Suppose that there exist* $x_0 \in D$, $\eta \geq 0$, $\gamma > 1$, $L_\gamma > 0$, $L_{0,\gamma} \geq 0$ *such that*

$$F'(x_0) \in L(Y, X),$$
$$\|F'(x_0)^{-1}F(x_0)\| \leq \eta,$$
$$D_\gamma \subseteq D,$$
$$\|F'(x_0)^{-1}(F'(x) - F'(x_0)) \leq L_{0,\gamma}\|x - x_0\| \quad \textit{for each } x \in D_\gamma,$$
$$\|F'(x_0)^{-1}(F'(x) - F'(y)) \leq L_\gamma\|x - y\| \quad \textit{for each } x, y \in D_\gamma,$$
$$h_{3,\gamma} = 2L_{3,\gamma}\eta \leq 1$$

and

$$r_{3,\gamma} \leq \gamma\eta,$$

where the set D_γ *is given in Theorem 2*

$$L_{3,\gamma} = \frac{1}{4}(4L_{0,\gamma} + \sqrt{L_\gamma L_{0,\gamma} + 8L_{0,\gamma}^2} + \sqrt{L_{0,\gamma}L_\gamma})$$

$$r_{3,\gamma} = \eta + \frac{L_{0,\gamma}\eta^2}{2(1-\alpha_\gamma)(1-L_{0,\gamma}\eta)},$$

and

$$\alpha_\gamma = \frac{2L_\gamma}{L_\gamma + \sqrt{L_\gamma^2 + 8L_{0,\gamma}L_\gamma}}.$$

Then, the sequence $\{x_n\}$ *generated by Newton's method is well defined, remains in* $U(x_0, r_{3,\gamma})$ *for each* $n = 0, 1, 2, \ldots$ *and converges to a unique solution* $x^* \in \bar{U}(x_0, r_{3,\gamma})$ *of equation* (1.1).

Proof. The hypotheses of Theorem 3 on D_γ are satisfied. ■

We present a semilocal convergence result of Newton's method involving condition (1.7) [6].

Theorem 5. *Let $F : \mathbb{D} \subset X \to Y$ be a Fréchet-differentiable operator. Suppose that conditions $(H_1) - (H_3)$ and $D_\gamma \subset D$ holds. Moreover, suppose that there exist $K_0 > 0$, $K_1 > 0$ such that*

$$\|F'(x_0)^{-1}(F'(x_1) - F'(x_0))\| \leq K_0 \|x_1 - x_0\|,$$

$$\|F'(x_0)^{-1}(F'(x_0 + \theta(x_1 - x_0)) - F'(x_0))\| \leq K_1 \theta \|x_1 - x_0\| \quad \text{for each } \theta \in [0, 1],$$

$$h_4 = 2L_4 \eta_0 \leq 1,$$

where

$$x_1 = x_0 - F'(x_0)^{-1} F(x_0),$$

$$L_4 = \frac{1}{2\eta_0} \leq 1,$$

$$\alpha_0 = \frac{L(t_2 - t_1)}{2(1 - L_0 t_2)},$$

$$t_1 = \eta, \quad t_2 = \eta + \frac{K_1 \eta^2}{2(1 - K_0 \eta)},$$

$$r_4 = \eta + \left(1 + \frac{\alpha_0}{1 - \alpha_\gamma}\right) \frac{K\eta^2}{2(1 - K_0 \eta)},$$

η_0 is defined by

$$\eta_0 = \begin{cases} \dfrac{1}{L_0 + K_0}, & \text{if } B = LK + 2\alpha_\gamma(L_0(K - 2K_0)) = 0, \\ \text{positive root of } p, & \text{if } B > 0, \\ \text{small positive root of } p, & \text{if } B < 0, \end{cases}$$

and

$$p(t) = (LK + 2\alpha_\gamma L_0(K - 2K_0))t^2 + 4\alpha_\gamma(L_0 + K_0)t - 4\alpha_\gamma.$$

Then, the sequence $\{x_n\}$ generated by Newton's method is well defined, remains in $U(x_0, r_4)$ for each $n = 0, 1, 2, \ldots$ and converges to a unique solution $x^* \in \bar{U}(x_0, r_4)$ of equation (1.1).

Notice that if $B = 0$, then $p(t) = 0$, if $t = \eta_0$. The discriminant of the quadratic polynomial p is positive, since

$$\begin{aligned} &16\alpha_\gamma^2(L_0 + K_0)^2 + 16\alpha_\gamma(LK + 2\alpha_\gamma L_0(K - 2K_0)) \\ &= 16\alpha_\gamma[\alpha_\gamma(L_0 - K_0)^2 + 2\alpha_\gamma L_0(K_0 + K) + KL]. \end{aligned}$$

If $B > 0$, then p has a unique positive root by the Descarte's rule of signs. But if $B < 0$, by the Vietta relations of the roots of the quadratic polynomial, the multiple of the roots equals $-\frac{4\alpha_\gamma}{B} > 0$ and the sum of the roots equals $\frac{-4\alpha_\gamma(L_0+K_0)}{B} > 0$. Therefore, p has two positive roots.

The improvement of Theorem 5 is:

Theorem 6. *Let* $F : \mathbb{D} \subset X \to Y$ *be a Fréchet-differentiable operator. Suppose that there exist* $x_0 \in D$, $\eta \geq 0$, $\gamma > 1$, $K_{0,\gamma} > 0$, $K_\gamma > 0$, $L_\gamma > 0$, $L_{0,\gamma} \geq 0$ *such that*

$$F'(x_0) \in L(Y, X),$$

$$\|F'(x_0)^{-1}F(x_0)\| \leq \eta,$$

$$D_\gamma \subseteq D,$$

$$\|F'(x_0)^{-1}(F'(x_1) - F'(x_0)) \leq K_{0,\gamma}\|x_1 - x_0\|,$$

$$\|F'(x_0)^{-1}(F'(x_0 + \theta(x_1 - x_0)) - F'(x_0)) \leq K_\gamma \theta \|x_1 - x_0\| \quad \textit{for each } \theta \in [0, 1],$$

$$\|F'(x_0)^{-1}(F'(x) - F'(x_0)) \leq L_{0,\gamma}\|x - x_0\| \quad \textit{for each } x \in D_\gamma,$$

$$\|F'(x_0)^{-1}(F'(x) - F'(y)) \leq L_\gamma \|x - y\| \quad \textit{for each } x, y \in D_\gamma,$$

$$h_{4,\gamma} = 2L_{4,\gamma}\eta_{0,\gamma} \leq 1$$

and

$$r_{4,\gamma} \leq \gamma\eta,$$

where

$$x_1 = x_0 - F'(x_0)^{-1}F(x_0),$$

$$\alpha_{0,\gamma} = \frac{L_\gamma(t_2 - t_1)}{2(1 - L_{0,\gamma}t_2)},$$

$$t_1 = \eta, \quad t_2 = \eta + \frac{K_\gamma \eta^2}{2(1 - K_{0,\gamma}\eta)},$$

$$r_{4,\gamma} = \eta + \left(1 + \frac{\alpha_{0,\gamma}}{1 - \alpha_\gamma}\right)\frac{K_\gamma \eta^2}{2(1 - K_{0,\gamma}\eta)},$$

$\eta_{0,\gamma}$ *is defined by*

$$\eta_{0,\gamma} = \begin{cases} \dfrac{1}{L_{0,\gamma} + K_{0,\gamma}}, & \textit{if } B = L_\gamma K_\gamma + 2\alpha_\gamma(L_{0,\gamma}(K_\gamma - 2K_{0,\gamma})) = 0, \\ \textit{positive root of } p_\gamma, & \textit{if } L_\gamma K_\gamma + 2\alpha_\gamma L_{0,\gamma}(K_\gamma - 2K_{0,\gamma} > 0, \\ \textit{small positive root of } p_\gamma, & \textit{if } L_\gamma K_\gamma + 2\alpha_\gamma L_{0,\gamma}(K_\gamma - 2K_{0,\gamma}) < 0, \end{cases}$$

and

$$p_\gamma(t) = (L_\gamma K_\gamma + 2\alpha_\gamma L_{0,\gamma}(K_\gamma - 2K_{0,\gamma}))t^2 + 4\alpha_\gamma(L_{0,\gamma} + K_{0,\gamma})t - 4\alpha_\gamma.$$

Then, the sequence $\{x_n\}$ *generated by Newton's method is well defined, remains in* $U(x_0, r_{4,\gamma})$ *for each* $n = 0, 1, 2, \ldots$ *and converges to a unique solution* $x^* \in \bar{U}(x_0, r_{4,\gamma})$ *of equation* (1.1).

Proof. The hypotheses of Theorem 6 on D_γ are satisfied. ■

Next, we present the local convergence results.

Theorem 7. *[5, 6] Let $F : \mathbb{D} \subset X \to Y$ be a Fréchet-differentiable operator. Suppose that there exist $x^* \in D$, $l_0 > 0$, $l > 0$ such that*

$$F(x^*) = 0,$$

$$F'(x^*) \in L(Y, X),$$

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*)) \le l_0 \|x - x^*\|, \quad \textit{for each} \quad x \in D$$

$$\|F'(x^*)^{-1}(F'(x) - F'(y)) \le l \|x - y\|, \quad \textit{for each} \quad x, y \in D$$

and

$$\bar{U}(x^*, \varrho) \subset D,$$

where

$$\varrho = \frac{2}{2l_0 + l}.$$

Then, the sequence $\{x_n\}$ generated for $x_0 \in U(x^*, \varrho) \setminus \{x^*\}$ by Newton's method is well defined, remains in $U(x^*, \varrho)$ for each $n = 0, 1, 2, \ldots$ and converges to a x^*. Moreover, for $T \in [\varrho, \frac{2}{l_0})$, the limit point x^* is the only solution of equation (1.1) in $\bar{U}(x^*, T) \cap D$.

Then, the improvement of Theorem 7 is given by:

Theorem 8. *Let $F : \mathbb{D} \subset X \to Y$ be a Fréchet-differentiable operator. Suppose that there exist $x^* \in D$, $\delta \ge 1$, $l_{0,\delta} > 0$, $l_\delta > 0$ such that*

$$F(x^*) = 0, \quad F'(x^*)^{-1} \in L(Y, X),$$

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*)) \le l_{0,\delta} \|x - x^*\| \quad \textit{for each } x \in D_\delta$$

$$\|F'(x^*)^{-1}(F'(x) - F'(y)) \le l_\delta \|x - y\| \quad \textit{for each } x, y \in D_\delta$$

$$D_\delta = \bar{U}(x^*, \delta\|x_0 - x^*\|) \subseteq D \quad \textit{for } x_0 \in D$$

and

$$\varrho_\delta \le \delta \|x_0 - x^*\|,$$

where

$$\varrho_\delta = \frac{2}{2l_{0,\delta} + l_\delta}.$$

Then, the sequence $\{x_n\}$ generated for $x_0 \in U(x^*, \varrho_\delta) \setminus \{x^*\}$ by Newton's method is well defined, remains in $U(x^*, \varrho_\delta)$ for each $n = 0, 1, 2, \ldots$ and converges to a x^*. Moreover, for $T \in [\varrho_\delta, \frac{2}{l_{0,\delta}})$, the limit point x^* is the only solution of equation (1.1) in $\bar{U}(x^*, T) \cap D$.

Proof. The hypotheses of Theorem 8 are satisfied on the domain D_δ. ■

Remark 9. **(a)** *If $D = U(x_0, \xi)$ for some $\xi > \eta$, then $\gamma \in [1, \frac{\xi}{\eta}]$, if $\eta \ne 0$.*

(b) *If we set* $\gamma = \dfrac{2}{1+\sqrt{1-2L_\gamma\eta}}$, *then condition* $R_\gamma \leq \gamma\eta$ *is satisfied as equality. Another choice for* γ *is given by* $\gamma = 2$. *Then, again* $R_\gamma \leq \gamma\eta$, *since we have that* $R_\gamma \leq 2\eta = \gamma\eta$.

(c) *Clearly, we have that*

$$L_{0,\gamma} \leq L,$$

$$L_\gamma \leq L,$$

$$K_{0,\gamma} \leq K_0$$

and

$$K_\gamma \leq K.$$

Therefore, we get that

$$h \leq 1 \Rightarrow h_\gamma \leq 1$$

$$h_3 \leq 1 \Rightarrow h_{3,\gamma} \leq 1$$

and

$$h_4 \leq 1 \Rightarrow h_{4,\gamma} \leq 1$$

but not necessarily vice versa unless if $L_{0,\gamma} = L$, $L_\gamma = L$, $K_{0,\gamma} = K_0$ *and* $K_\gamma = K$.

Notice also that the new majorizing sequences are more precise that the corresponding older ones. As an example, the majorizing sequences $\{t_n\}$, $\{\bar{t}_n\}$ *for Newton's method corresponding to conditions* $h \leq 1$ *and* $h_\gamma \leq 1$ *are:*

$$t_0 = 0, \quad t_1 = \eta, \quad t_{n+1} = t_n + \frac{L(t_n - t_{n-1})^2}{2(1 - Lt_n)}$$

$$\bar{t}_0 = 0, \quad \bar{t}_1 = \eta, \quad \bar{t}_{n+1} = \bar{t}_n + \frac{L_\gamma(\bar{t}_n - \bar{t}_{n-1})^2}{2(1 - L_{0,\gamma}\bar{t}_n)}$$

Then, a simple induction argument shows that

$$\bar{t}_n \leq t_n$$

$$0 \leq \bar{t}_{n+1} - \bar{t}_n \leq t_{n+1} - t_n$$

and

$$R_\gamma \leq R.$$

If $L_\gamma < L$, *then the strict inequality holds for* $n \geq 2$ *in the first inequality and for* $n \geq 1$ *in the second inequality. Moreover, we have that* $R_\gamma < R$. *Hence, in this case the information on the location of the solution* x^* *is more precise under the new approach. Similar comments can be made for the majorizing sequences corresponding to the other "h" and corresponding "h_γ" conditions. Finally, notice that the majorizing sequences corresponding to conditions* (1.5)–(1.7) *have already been shown to be more precise than sequence* $\{t_n\}$ *which corresponds to condition* (1.4) *[4, 6].*

(d) *If* $D = U(x^*, \xi)$ *for some* $\xi > \|x_0 - x^*\|$, *then* δ *can b chosen so that* $\delta \in [1, \frac{\xi}{\|x_0 - x^*\|})$ *for* $x_0 \neq x^*$.

(e) *We have that*

$$l_{0,\delta} \leq l_0$$

and

$$l_\delta \leq l.$$

Therefore, we get that

$$\varrho \leq \varrho_\delta.$$

Moreover, if $l_{0,\delta} < l_0$ *or* $l_\delta < l$, *then* $\varrho < \varrho_\delta$. *The corresponding error bounds are also improved, since we have*

$$\|x_{n+1} - x^*\| \leq \frac{l\|x_n - x^*\|^2}{2(1 - l_0\|x_n - x^*\|)}.$$

Notice that, if $l_0 = l$, *then Theorem 2.9 reduces to the corresponding by Rheinboldt [15] and Traub [16]. The radius found independently by these authors is given by*

$$\bar{\varrho} = \frac{2}{3l}.$$

However, if $l_0 < l$, *then our radius is such that*

$$\bar{\varrho} < \varrho < \varrho_\delta$$

and

$$\frac{\bar{\varrho}}{\varrho} \to \frac{1}{3} \text{ as } \frac{l_0}{l} \to 0.$$

Hence, our radius of convergence ϱ *can be at most three times larger than* $\bar{\varrho}$.

3. Numerical examples

We present numerical examples in this section.
Example 3.1 Let $D = U(x_0, 1), x^* = \sqrt[3]{2}$ and define function F on D by

$$F(x) = x^3 - 2. \tag{3.1}$$

We are going to consider such initial points for which previous conditions cannot be satisfied but our new conditions are satisfied. That is, the improvement that we get is the new weaker conditions.
The sufficient condition in Theorem 1 is

$$h = 2L\eta \leq 1.$$

So, we want to get the values of x_0 and γ for which condition $h \leq 1$ is not satisfied but the conditions of Theorem 2 are satisfied.

We get that

$$\eta = \frac{1}{3}\left|\frac{-2+x_0^3}{x_0^2}\right|,$$

$$L = \left|\frac{2(1+x_0)}{x_0^2}\right|,$$

$$h = \frac{4|1+x_0|\left|\frac{-2+x_0^3}{x_0^2}\right|}{3x_0^2},$$

$$R = \frac{x_0^2\left(1-\sqrt{1-\frac{4|1+x_0|\left|\frac{-2+x_0^3}{x_0^2}\right|}{3x_0^2}}\right)}{2|1+x_0|}$$

$$L_\gamma = \frac{2\left(x_0+\frac{1}{3}\gamma\left|\frac{-2+x_0^3}{x_0^2}\right|\right)}{x_0^2}$$

$$h_\gamma = \frac{4\left|\frac{-2+x_0^3}{x_0^2}\right|\left(x_0+\frac{1}{3}\gamma\left|\frac{-2+x_0^3}{x_0^2}\right|\right)}{3x_0^2}$$

and

$$R_\gamma = \frac{x_0^2\left(1-\sqrt{1-\frac{4\left|\frac{-2+x_0^3}{x_0^2}\right|\left(x_0+\frac{1}{3}\gamma\left|\frac{-2+x_0^3}{x_0^2}\right|\right)}{3x_0^2}}\right)}{2\left(x_0+\frac{1}{3}\gamma\left|\frac{-2+x_0^3}{x_0^2}\right|\right)}$$

Imposing the following conditions, we recall (SL)-conditions:

- $h > 1$
- $h_\gamma \leq 1$
- $R_\gamma \leq \gamma\eta$
- $\gamma\eta \leq 1$

we obtain that the coloured zone corresponds to the cases for which previous conditions (Theorem 1) cannot guarantee the convergence to the solution but our new weaker criteria can. In order to compute the graphics we associate the pair (x_0,γ) of the $xy-$plane, where $x = x_0$ and $y = \gamma$. Moreover, if we consider the set of points

$$V = \{(x_0,\gamma) \in \mathbb{R}^2 : (SL) - \text{conditions are satisfied}\}$$

we can observe that every point x_0 chosen such that the pair associated (x_0,γ) belongs to V cannot be chosen as a starting point with the old condition but can be chosen with the conditions of Theorem 2. Two examples of these regions in which Theorem 2 can guarantee the convergence but previous results can't are shown in Figure 1 and Figure 2.

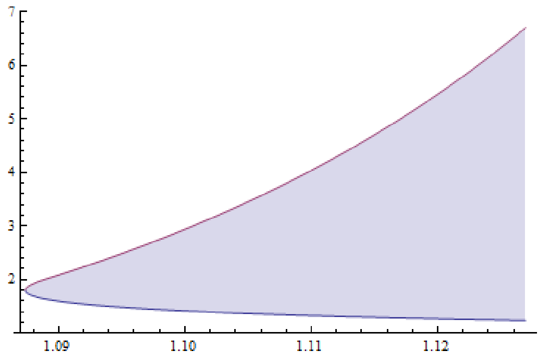

Figure 1: One of the cases in which previous conditions are not satisfied but conditions of Theorem are satisfied.

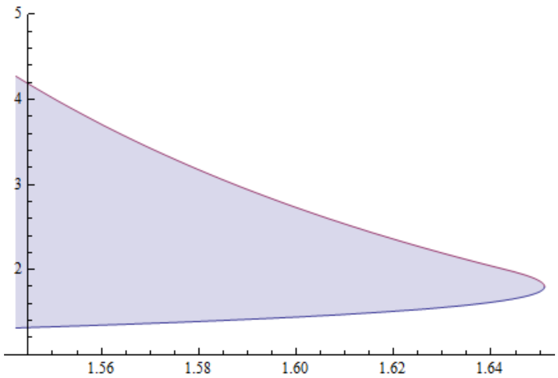

Figure 2: One of the cases in which previous conditions are not satisfied but conditions of Theorem are satisfied.

Example 3.2 Let $X = Y = \mathcal{C}[0,1]$, be the space of continuous functions defined in $[0,1]$ be equipped with the max-norm. Let $\Omega = \{x \in \mathcal{C}[0,1]; \|x\| \leq R\}$, for $R > 0$. Define operator F on Ω [2, 7, 8, 9, 14] by

$$F(x)(s) = x(s) - f(s) - \lambda \int_0^1 G(s,t)x(t)^3 \, dt, \quad x \in C[0,1], \; s \in [0,1],$$

where $f \in \mathcal{C}[0,1]$ is a given function, λ is a real constant and the kernel G is the Green's function defined by

$$G(s,t) = \begin{cases} (1-s)t, & t \leq s, \\ s(1-t), & s \leq t. \end{cases}$$

In this case, for each $x \in \Omega$, $F'(x)$ is a linear operator defined on Ω by the following expression:

$$[F'(x)(v)](s) = v(s) - 3\lambda \int_0^1 G(s,t)x(t)^2 v(t)\ dt, \quad v \in C[0,1],\ s \in [0,1].$$

If we choose $x_0(s) = f(s) = 1$, it follows $\|I - F'(x_0)\| \leq 3|\lambda|/8$. Thus, if $|\lambda| < 8/3$, $F'(x_0)^{-1}$ is defined and

$$\|F'(x_0)^{-1}\| \leq \frac{8}{8 - 3|\lambda|}.$$

Moreover,

$$\|F(x_0)\| \leq \frac{|\lambda|}{8},$$

$$\eta = \|F'(x_0)^{-1}F(x_0)\| \leq \frac{|\lambda|}{8 - 3|\lambda|}.$$

On the other hand, for each $x, y \in \Omega$, we have

$$\|F'(x) - F'(y)\| \leq \|x - y\| \frac{1 + 3|\lambda|(\|x + y\|)}{8} \leq \|x - y\| \frac{1 + 6R|\lambda|}{8}.$$

and

$$\|F'(x) - F'(1)\| \leq \|x - 1\| \frac{1 + 3|\lambda|(\|x\| + 1)}{8} \leq \|x - 1\| \frac{1 + 3(1 + R)|\lambda|}{8}.$$

Choosing $\lambda = 1$ and $R = 2$, we have

$$\eta = 0.2,$$
$$L = 2.6,$$
$$L_0 = 2,$$
$$L_3 = 22.8624$$
$$\alpha = 0.544267\ldots$$

and

$$r_3 = 0.346284\ldots.$$

Condition (1.3) is not satisfied, since

$$h = 1.04 > 1.$$

However, choosing $\gamma = 6$ conditions of Theorem 4 are satisfied, since

$$\eta = 0.2,$$

$$L_\gamma = 1.64,$$

$$L_{0,\gamma} = 1.52,$$

$$L_{3,\gamma} = 12.3667\ldots,$$

$$\alpha_\gamma = 0.512715\ldots,$$

$$r_{3,\gamma} = 0.289636\ldots \leq \gamma\eta = 1.2$$

and

$$2L_{3,\gamma}\eta = 0.656 \leq 1.$$

The convergence of Newton's method is ensured by Theorem 4.

Example 3.3 Let $\mathbb{X} = \mathbb{Y} = \mathcal{C}[0,1]$, equipped with the max-norm. Consider the following nonlinear boundary value problem

$$\begin{cases} u'' = -u^3 - \alpha\ u^2 \\ u(0) = 0, \quad u(1) = 1. \end{cases}$$

It is well known that this problem can be formulated as the integral equation

$$u(s) = s + \int_0^1 G(s,t)\ (u^3(t) + \alpha\ u^2(t))\ dt \tag{3.2}$$

where, G is the Green's function defined by

$$G(s,t) = \begin{cases} t\ (1-s), & t \leq s \\ s\ (1-t), & s < t. \end{cases}$$

We observe that

$$\max_{0 \leq s \leq 1} \int_0^1 |G(s,t)|\, dt = \frac{1}{8}.$$

Then problem (3.2) is in the form (1.1), where, $F : \mathbb{D} \longrightarrow \mathbb{Y}$ is defined as

$$[F(x)]\,(s) = x(s) - s - \int_0^1 G(s,t)\ (x^3(t) + \alpha\ x^2(t))\ dt.$$

Set $u_0(s) = s$ and $\mathbb{D} = U(u_0, 1)$. It is easy to verify that $\|u_0\| = 1$. If $2\ \alpha < 5$, the operator F' satisfies the invertibility conditions. Choosing $\alpha = 0.5$, and $\gamma = 1.5$ we obtain that

$$\eta = 0.375,$$

$$L_\gamma = 0.59375,$$

$$h_\gamma = 0.429688\ldots$$

and

$$R_\gamma = 0.411354 \leq \gamma\eta = 0.5625$$

So, we can ensure the convergence of $\{x_n\}$ by Theorem 2.

Example 3.4 Let $X = Y = \mathbb{R}^3$, $D = \overline{U}(0,1)$. Define F on D for $v = (x,y,z)^T$ by

$$F(v) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T. \tag{3.3}$$

Then, the Fréchet-derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that $x^* = (0,0,0)^T$, $F'(x^*) = F'(x^*)^{-1} = diag\{1,1,1\}$, $l_0 = e-1 < l = e$. Choosing $x_0 = (0.2, 0.2, 0.2)^T$, we see in the following Table the radius found by Traub, our old one and the new one presented in this paper. Notice that our radius ϱ_δ is larger than the older one ϱ and the one given by Traub r_{TR}.

Table 1: Radius of convergence

δ	R_{TR}	ϱ	ϱ_δ
3	0.245253	0.324947	0.448563
4	0.245253	0.324947	0.418223
5	0.245253	0.324947	0.324947

Table 2: Corresponding error bounds for Traub's condition, i.e. $l = l_0 = e$

n	$\|x_n - x^*\|$
0	0.000955501
1	3.94473×10^{-7}
2	8.41053×10^{-14}
3	3.84158×10^{-27}

Table 3: Corresponding error bounds for our old condition, i.e. $l = e$, $l_0 = e-1$

n	$\|\|x_n - x^*\|\|$
0	0.00092994
1	3.9426×10^{-7}
2	8.41053×10^{-14}
3	3.84158×10^{-27}

Example 3.5 We consider the following Planck's radiation law [1] problem which calculates the energy density within an isothermal blackbody and is given by:

$$\vartheta(\lambda) = \frac{8\pi c P \lambda^{-5}}{\mathrm{e}^{\frac{cP}{\lambda BT}} - 1}, \tag{3.4}$$

where λ is the wavelength of the radiation, T is the absolute temperature of the blackbody, B is the Boltzmann constant, P is the Planck constant and c

Table 4: Corresponding error bounds for our new condition with $\delta = 4$

n	$\|\|x_n - x^*\|\|$
0	0.000584918
1	2.49195×10^{-7}
2	5.31647×10^{-14}
3	2.42834×10^{-27}

Table 5: Corresponding error bounds for our new condition with $\delta = 3$

n	$\|\|x_n - x^*\|\|$
0	0.000582411
1	2.49173×10^{-7}
2	5.31647×10^{-14}
3	2.42834×10^{-27}

is the speed of light. We are interested in determining wave length λ which corresponds to maximum energy density $\vartheta(\lambda)$.

From (3.4), we obtain

$$\vartheta'(\lambda) = \left(\frac{8\pi cP\lambda^{-6}}{\mathrm{e}^{\frac{cP}{\lambda BT}} - 1}\right)\left(\frac{\frac{cP}{\lambda Bt}\mathrm{e}^{\frac{cP}{\lambda BT}}}{\mathrm{e}^{\frac{cP}{\lambda BT}} - 1} - 5\right), \tag{3.5}$$

so that the maxima of ϑ occurs when

$$\frac{\frac{cP}{\lambda Bt}\mathrm{e}^{\frac{cP}{\lambda BT}}}{\mathrm{e}^{\frac{cP}{\lambda BT}} - 1} = 5. \tag{3.6}$$

After that, if $x = \frac{cP}{\lambda BT}$, then (3.6) is satisfied if

$$F(x) = \mathrm{e}^{-x} + \frac{x}{5} - 1 = 0. \tag{3.7}$$

Therefore, the solutions of $F(x) = 0$ give the maximum wave length of radiation λ by means of the following formula:

$$\lambda \approx \frac{cP}{x^* BT}, \tag{3.8}$$

where x^* is a solution of (3.7).

Function (3.7) is continuous and such that $F(1) = -0.432121\ldots$ and $F(9) = 0.800123\ldots$. By the intermediate value theorem function $F(x)$ has zeros in the interval $[1, 9]$. So, we consider $\mathcal{D} = U(5, 4)$ and $x^* = 4.965114\ldots$.

We choose $x_0 = 3$, $\delta = 1.5$ and we obtain:

$$l_{0,\delta} = 0.2214851\ldots,$$

$$l_\delta = 0.6890122\ldots$$

and

$$\rho_\delta = 1.76681\ldots \leq 2.94767\ldots = \delta \|x_0 - x^*\|.$$

As a consequence, conditions of Theorem 8 are satisfied as a consequence we can ensure the convergence of Newton's method (1.2).

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