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Convergence rates of moving mesh methods for moving boundary partial integro-differential equations from regime-switching jump-diffusion Asian option pricing *

Jingtang Ma† and Han Wang‡

Abstract

This paper studies the convergence rates of moving mesh methods for a system of moving boundary partial integro-differential equations (PIDEs) which arise in the Asian option pricing under the state-dependent regime-switching jump-diffusion models. The value function of the Asian option under the model is governed by a system of two-dimensional PIDEs. In this paper, the two-dimensional PIDEs are recast into a one-dimensional moving boundary problem of the PIDEs. A moving finite difference method (FDM) is proposed to solve the one-dimensional moving boundary problem and the convergence rates are proved. Numerical examples are provided to confirm the theoretical results.

2010 MSC: 65M06, 65M12, 91G20, 91G60, 91G80

Keywords: Partial integro-differential equations, moving boundary problems, Asian option pricing, regime-switching, jump diffusion models, moving mesh methods, convergence rates

1 Introduction

Let $(\Omega, \mathcal{F}, \mathcal{Q})$ be a complete probability space with risk-neutral measure. Assume that the price of the underlying asset $S(t)$ follows the state-dependent regime-switching jump-diffusion model under risk-neutral measure:

$$\frac{dS(t)}{S(t)} = \left[r\left(B(t)\right) - \delta\left(B(t)\right) - \lambda\left(B(t)\right)\kappa\left(B(t)\right)\right]dt + \sigma\left(B(t)\right)dW_t + \left[\eta\left(B(t)\right) - 1\right]d\aleph_t, \quad (1)$$

where W_t is a standard Brownian motion, $B(t)$ is a continuous-time Markov chain with the state $B(t) \in \{b_i : i = 1, 2, \ldots, d\}$. Assume that at each state, the interest rates

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$r(b_i) = r_i$, dividend yields $\delta(b_i) = \delta_i$ and volatilities $\sigma(b_i) = \sigma_i$ for $i \in \mathcal{D} \equiv \{1, 2, \ldots, d\}$ are nonnegative constants. $\aleph_t$ is a Poisson process with intensity $\lambda(B(t)) = \lambda_i \geq 0$, the amplitude $\eta(B(t)) - 1 = \eta_i - 1$, and the expectation of the random amplitude $\kappa(B(t)) = \kappa_i = E(\eta_i - 1)$, where $\eta_i = e^{Y(B(t))} = e^{Y_i}$, and the jump sizes Y_i for $i \in \mathcal{D}$ are independent random variables with density functions

$$f_i(y) = \frac{1}{\varrho_i\sqrt{2\pi}} \exp\left\{-\frac{(y-\mu_i)^2}{2\varrho_i^2}\right\}, \tag{2}$$

where $\varrho_i > 0$ and $\mu_i \geq 0$ are the constants depending solely on the regime state b_i for $i \in \mathcal{D}$. Let $\mathrm{Q} = (q_{il})_{i,l\in\mathcal{D}}$ be the generator matrix of the Markov chain process whose elements are constants satisfying $q_{il} \geq 0$ for $i \neq l$ and $\sum_{l=1}^{d} q_{il} = 0$ for $i \in \mathcal{D}$. Assume that $B(t)$, W_t and $\aleph_t$ are conditionally independent.

The study of the Black-Scholes (BS) model is popular, but it has several deficiencies. One of the important extensions to the BS model is the model of regime-switching and jump-diffusions. The regime-switching model for the dynamics of stock price is first introduced by Hamilton [11, 12]. The pricing of Vanilla-type options under regime-switching models has been well studied in the literature (see e.g., Bollen [2], Duan et al. [9], Khaliq and Liu [13], Yuen and Yang [23, 24], Liu [15, 16], Liu and Zhao [17], Ma et al. [21]). The pricing of Asian options under the regime-switching models is studied by Boyle and Draviam [3], Ma and Zhou [20].

Dang et al. [8] study the arithmetic Asian option under the state-dependent regime-switching jump-diffusion models. The governing equation is a system of PIDEs. Dang et al. [8] construct a monotonic sequence to decouple the PIDE system and prove the limit is a strong solution of the PIDE system. The sequence contains d single PIDEs, which are solved by the numerical methods – finite difference methods for discretization of the time variable and finite element methods for the space variable. The convergence results of the numerical methods are not given in their paper.

For pricing Asian options using the PDE methods, one of the difficulties is to set up the boundary conditions. For the Asian options under geometric Brownian motion models, the two dimensional PDEs are recast into one dimensional PDEs and the boundaries condition are derived (see e.g., Zvan, Forsyth and Vetzal [25], Večeř [22], Dubois and Lelièvre [10]). Following the similar idea, we derive the one-dimensional moving boundary PIDEs. To be more specific, the problem is described as follows. Denote

$$I(t) = \int_0^t S(u)\,du.$$

Then the value of the continuous arithmetic average Asian options with payoff: $\max(I(T)/T - K, 0)$, where K is the fixed strike and T is the expiry time of the option, is given by

$$V(S(t), I(t), t, i) = e^{-r_i(T-t)} E_t\left[\max(I(T)/T - K, 0)\right], \quad i \in \mathcal{D},$$

where E_t denotes the conditional expectation at t (see e.g., [8]). Also from [8], the value function of the Asian option is given by the following system of PIDEs:

$$\begin{aligned}
&\frac{\partial V(S,I,t,i)}{\partial t} + \frac{1}{2}\sigma_i^2 S^2 \frac{\partial^2 V(S,I,t,i)}{\partial S^2} + (r_i - \delta_i - \lambda_i\kappa_i) S \frac{\partial V(S,I,t,i)}{\partial S} \\
&+ S\frac{\partial V(S,I,t,i)}{\partial I} + \lambda_i \int_{-\infty}^{+\infty} V(e^y S, I, t, i)\, f_i(y) dy + \sum_{l=1}^{d} q_{il} V(S,I,t,l) \\
&-(r_i + \lambda_i) V(S,I,t,i) = 0, \quad i \in \mathcal{D},
\end{aligned} \tag{3}$$

with terminal condition $V(S,I,T,i) = \max(I/T - K, 0)$, boundary condition $V(S,-\infty,t,i) = 0$, $i \in \mathcal{D}$, where S and I are dummy variables. Each equation in (3) is a two-dimensional problem and there is no diffusion in the I direction. These facts cause many difficulties in the numerical solutions and analysis with the standard finite difference methods.

Motivated by Dubois and Lelièvre [10], we recast (3) into a moving boundary problem of one-dimensional PIDEs. To this end, we first construct an explicit solution to (3) in the region $I \geq KT$ for all $t \leq T$ as

$$V(S,I,t,i) = \left(\frac{I}{T} - K\right) e^{-r_i(T-t)} + \frac{S}{(\delta_i - r_i)T}\left(e^{-r_i(T-t)} - e^{-\delta_i(T-t)}\right), \quad \text{for } i \in \mathcal{D}, \tag{4}$$

which can be immediately verified by substituting it into (3).

Using transformation of variables, for $i \in \mathcal{D}$,

$$x = \frac{T-\tau}{T} + \frac{K - I/T}{S}, \quad G(x,\tau,i) = \frac{V(S,I,T-\tau,i)}{S}, \quad \tau = T - t, \tag{5}$$

formula (4) becomes

$$\begin{aligned} G(x,\tau,i) &= -\left(x - \frac{T-\tau}{T}\right) e^{-r_i\tau} \\ &+ \frac{1}{(\delta_i - r_i)T}\left(e^{-r_i\tau} - e^{-\delta_i\tau}\right), \quad \text{for } x \in \left(-\infty, \frac{T-\tau}{T}\right], \end{aligned}$$

and the PIDEs (3) are re-written into, for $i \in \mathcal{D}$,

$$\begin{aligned} &\frac{\partial G(x,\tau,i)}{\partial \tau} - \frac{1}{2}\sigma_i^2\left(x - \frac{T-\tau}{T}\right)^2 \frac{\partial^2 G(x,\tau,i)}{\partial x^2} \\ &+(r_i - \delta_i - \lambda_i\kappa_i)\left(x - \frac{T-\tau}{T}\right)\frac{\partial G(x,\tau,i)}{\partial x} \\ &-\lambda_i \int_{-\infty}^{+\infty} e^y G\left(\frac{x - \frac{T-\tau}{T}}{e^y} + \frac{T-\tau}{T}, \tau, i\right) f_i(y)dy + (\lambda_i + \delta_i + \lambda_i\kappa_i)G(x,\tau,i) \\ &-\sum_{l=1}^{d} q_{il}G(x,\tau,l) = 0, \quad \text{for } x \in \left(\frac{T-\tau}{T}, +\infty\right), \ \tau \in (0,T], \end{aligned} \tag{6}$$

with initial and boundary conditions,

$$G(x,0,i) = 0, \quad \text{for } x \in [1,+\infty), \tag{7}$$

$$G\left(\frac{T-\tau}{T}, \tau, i\right) = \frac{1}{(\delta_i - r_i)T}\left(e^{-r_i\tau} - e^{-\delta_i\tau}\right), \quad \text{for } \tau \in (0,T], \tag{8}$$

$$G(+\infty,\tau,i) = 0, \quad \text{for } \tau \in (0,T]. \tag{9}$$

Since the problems (6) - (9) contain a moving boundary, it is natural to develop the moving mesh methods to solve the problem. In this paper, we analyze the convergence rates of the moving FDMs. Although we derive the one dimensional problem similarly to the work by Zvan, Forsyth and Vetzal [25], Večeř [22], Dubois and Lelièvre [10], the convergence analysis is not given in these papers. Ma and Zhou [19], [20] study the convergence rates of the moving FDMs for the Asian option pricing under geometric Brownian motion models and regime-switching models. In their papers, the governing equations do not contain the integral terms. The moving FDMs and convergence analysis in this paper will incorporate the discretization and analysis for the integral terms. Moreover since the PIDEs (6) contain a Fredholm type integral term, the analysis of moving FDMs is essentially different from the paper [18] for the partial Volterra integro-differential equations.

The remainder of this paper is arranged as follows: In Section 2, we construct the moving FDMs and prove the convergence rates. In Section 3, we use numerical examples to confirm the theoretical results. In the final section, we give the conclusions.

2 Moving FDMs and the convergence rates

For the aim of computation, the semi-infinite domain $\left(\frac{T-\tau}{T}, +\infty\right)$ is truncated into a finite one $\Omega_\tau \equiv \left(\frac{T-\tau}{T}, X\right)$ with an appropriate value of X such that $G(X,\tau,i) \approx 0$. Denote $\overline{\Omega}_\tau$ as the closure of Ω_τ. We define the uniform mesh for time:

$$\tau_n = \frac{n}{M}T, \quad n = 0, 1, \ldots, M, \tag{10}$$

and moving mesh for space:

$$x_j^n = \frac{T-\tau_n}{T} + \frac{X - \frac{T-\tau_n}{T}}{N} j, \quad j = 0, 1, \ldots, N, \tag{11}$$

and denote the meshsize by $\Delta\tau_n = \tau_{n+1} - \tau_n$, $h_j^{n+1} = x_j^{n+1} - x_{j-1}^{n+1}$, for $n = 0, 1, \ldots, M-1$; $j = 1, \ldots, N$. It is easy to see that

$$\Delta\tau_n = TM^{-1}, \quad \text{for } n = 0, 1, \ldots, M-1, \tag{12}$$

$$h_j^{n+1} = \frac{X - \frac{T-\tau_{n+1}}{T}}{N} \le CN^{-1}. \tag{13}$$

Obviously the moving mesh in (11) is driven by the time-dependent boundary and there is no need to use the monitor function. This is often used for the moving boundary problems, although for some physics problems the moving mesh is generated by the equidistribution using the monitor function, see e.g., [6] and [7]. Since the locations of the spatial mesh points x_j^n depend on the time level τ_n, the standard FDMs to (6) - (9) need to be modified, which are called moving FDMs.

We first discretize the integral term in (6). To this end, we denote $\xi = \frac{x_j^{n+1} - x_0^{n+1}}{e^y} + x_0^{n+1}$. Then we have

$$\begin{aligned}
&\int_{-\infty}^{+\infty} e^y G\left(\frac{x_j^{n+1} - x_0^{n+1}}{e^y} + x_0^{n+1}, \tau_{n+1}, i\right) f_i(y)dy \qquad (14)\\
=\ & \int_{\frac{T-\tau_{n+1}}{T}}^{+\infty} \frac{x_j^{n+1} - x_0^{n+1}}{(\xi - x_0^{n+1})^2} G(\xi, \tau_{n+1}, i) f_i\left(\ln\left(\frac{x_j^{n+1} - x_0^{n+1}}{\xi - x_0^{n+1}}\right)\right) d\xi\\
\approx\ & \sum_{k=1}^{N} \int_{x_{k-1}^{n+1}}^{x_k^{n+1}} \left\{ \frac{x_j^{n+1} - x_0^{n+1}}{(\xi - x_0^{n+1})^2} \frac{\xi - x_k^{n+1}}{x_{k-1}^{n+1} - x_k^{n+1}} G(x_{k-1}^{n+1}, \tau_{n+1}, i) f_i\left(\ln\left(\frac{x_j^{n+1} - x_0^{n+1}}{\xi - x_0^{n+1}}\right)\right)\right.\\
&\left. + \frac{x_j^{n+1} - x_0^{n+1}}{(\xi - x_0^{n+1})^2} \frac{\xi - x_{k-1}^{n+1}}{x_k^{n+1} - x_{k-1}^{n+1}} G(x_k^{n+1}, \tau_{n+1}, i) f_i\left(\ln\left(\frac{x_j^{n+1} - x_0^{n+1}}{\xi - x_0^{n+1}}\right)\right)\right\} d\xi,
\end{aligned}$$

where we have used piecewise linear interpolation for function G:

$$G(\xi,\tau_{n+1},i)\approx\frac{\xi-x_k^{n+1}}{x_{k-1}^{n+1}-x_k^{n+1}}G(x_{k-1}^{n+1},\tau_{n+1},i)+\frac{\xi-x_{k-1}^{n+1}}{x_k^{n+1}-x_{k-1}^{n+1}}G(x_k^{n+1},\tau_{n+1},i).$$

We further derive that

$$\begin{aligned}
&\int_{x_{k-1}^{n+1}}^{x_k^{n+1}}\frac{x_j^{n+1}-x_0^{n+1}}{(\xi-x_0^{n+1})^2}\frac{\xi-x_k^{n+1}}{x_{k-1}^{n+1}-x_k^{n+1}}f_i\left(\ln\left(\frac{x_j^{n+1}-x_0^{n+1}}{\xi-x_0^{n+1}}\right)\right)d\xi \qquad (15)\\
&=-\int_{x_{k-1}^{n+1}}^{x_k^{n+1}}\frac{x_j^{n+1}-x_0^{n+1}}{(\xi-x_0^{n+1})}\frac{\xi-x_k^{n+1}}{x_{k-1}^{n+1}-x_k^{n+1}}f_i\left(\ln\left(\frac{x_j^{n+1}-x_0^{n+1}}{\xi-x_0^{n+1}}\right)\right)d\ln\left(\frac{x_j^{n+1}-x_0^{n+1}}{\xi-x_0^{n+1}}\right)\\
&=-\exp\left(\mu_i+\frac{\varrho_i^2}{2}\right)\int_{x_{k-1}^{n+1}}^{x_k^{n+1}}\frac{\xi-x_k^{n+1}}{x_{k-1}^{n+1}-x_k^{n+1}}dF_i\left(\ln\left(\frac{x_j^{n+1}-x_0^{n+1}}{\xi-x_0^{n+1}}\right)\right)\\
&=\exp\left(\mu_i+\frac{\varrho_i^2}{2}\right)\left\{F_i\left(\ln\left(\frac{x_j^{n+1}-x_0^{n+1}}{x_{k-1}^{n+1}-x_0^{n+1}}\right)\right)-\frac{\int_{x_{k-1}^{n+1}}^{x_k^{n+1}}F_i\left(\ln\left(\frac{x_j^{n+1}-x_0^{n+1}}{\xi-x_0^{n+1}}\right)\right)d\xi}{x_k^{n+1}-x_{k-1}^{n+1}}\right\},
\end{aligned}$$

where $F_i(\cdot)$ is the distribution function for a normal random variable with expectation $\mu_i+\varrho_i^2$ and variance ϱ_i^2. Similarly, we have

$$\begin{aligned}
&\int_{x_{k-1}^{n+1}}^{x_k^{n+1}}\frac{x_j^{n+1}-x_0^{n+1}}{(\xi-x_0^{n+1})^2}\frac{\xi-x_{k-1}^{n+1}}{x_k^{n+1}-x_{k-1}^{n+1}}f_i\left(\ln\left(\frac{x_j^{n+1}-x_0^{n+1}}{\xi-x_0^{n+1}}\right)\right)d\xi \qquad (16)\\
&=-\exp\left(\mu_i+\frac{\varrho_i^2}{2}\right)\left\{F_i\left(\ln\left(\frac{x_j^{n+1}-x_0^{n+1}}{x_k^{n+1}-x_0^{n+1}}\right)\right)-\frac{\int_{x_{k-1}^{n+1}}^{x_k^{n+1}}F_i\left(\ln\left(\frac{x_j^{n+1}-x_0^{n+1}}{\xi-x_0^{n+1}}\right)\right)d\xi}{x_k^{n+1}-x_{k-1}^{n+1}}\right\}.
\end{aligned}$$

Combining (15) - (16) into (14) gives that

$$\begin{aligned}
&\int_{-\infty}^{+\infty}e^yG\left(\frac{x_j^{n+1}-x_0^{n+1}}{e^y}+x_0^{n+1},\tau_{n+1},i\right)f_i(y)dy\\
&\approx\exp\left(\mu_i+\frac{\varrho_i^2}{2}\right)\sum_{k=1}^{N}G(x_{k-1}^{n+1},\tau_{n+1},i)\\
&\qquad\cdot\left\{F_i\left(\ln\left(\frac{x_j^{n+1}-x_0^{n+1}}{x_{k-1}^{n+1}-x_0^{n+1}}\right)\right)-\frac{\int_{x_{k-1}^{n+1}}^{x_k^{n+1}}F_i\left(\ln\left(\frac{x_j^{n+1}-x_0^{n+1}}{\xi-x_0^{n+1}}\right)\right)d\xi}{x_k^{n+1}-x_{k-1}^{n+1}}\right\}\\
&-\exp\left(\mu_i+\frac{\varrho_i^2}{2}\right)\sum_{k=1}^{N}G(x_k^{n+1},\tau_{n+1},i)\\
&\qquad\cdot\left\{F_i\left(\ln\left(\frac{x_j^{n+1}-x_0^{n+1}}{x_k^{n+1}-x_0^{n+1}}\right)\right)-\frac{\int_{x_{k-1}^{n+1}}^{x_k^{n+1}}F_i\left(\ln\left(\frac{x_j^{n+1}-x_0^{n+1}}{\xi-x_0^{n+1}}\right)\right)d\xi}{x_k^{n+1}-x_{k-1}^{n+1}}\right\}\\
&\equiv\mathcal{I}_j^{n+1}G(x_j^{n+1},\tau_{n+1},i). \qquad (17)
\end{aligned}$$

Let $G_j^n(i)$ be the approximation of $G(x,\tau,i)$ at point $x = x_j^n$, $\tau = \tau_n$, i.e., $G_j^n(i) \approx G(x_j^n, \tau_n, i)$, for $i \in \mathcal{D}$. Then the moving mesh FDM is defined by, for $i \in \mathcal{D}$, $n = 0, 1, \ldots, M-1$; $j = 1, \ldots, N-1$,

$$\begin{aligned}
&\frac{G_j^{n+1}(i) - \hat{G}_j^n(i)}{\Delta\tau_n} + (r_i - \delta_i - \lambda_i\kappa_i)\left(x_j^{n+1} - \frac{T-\tau_{n+1}}{T}\right)\frac{G_{j+1}^{n+1}(i) - G_{j-1}^{n+1}(i)}{h_j^{n+1} + h_{j+1}^{n+1}} \\
&-\frac{1}{2}\sigma_i^2\left(x_j^{n+1} - \frac{T-\tau_{n+1}}{T}\right)^2 \frac{2}{h_j^{n+1} + h_{j+1}^{n+1}}\left[\frac{G_{j+1}^{n+1}(i) - G_j^{n+1}(i)}{h_{j+1}^{n+1}} - \frac{G_j^{n+1}(i) - G_{j-1}^{n+1}(i)}{h_j^{n+1}}\right] \\
&= \lambda_i \mathcal{I}_j^{n+1} G(i) + \sum_{l=1}^{d} q_{il} G_j^{n+1}(l) - (\lambda_i + \delta_i + \lambda_i\kappa_i) G_j^{n+1}(i),
\end{aligned} \tag{18}$$

with

$$G_0^n(i) = G\left(\frac{T-\tau_n}{T}, \tau_n, i\right) = \frac{1}{(\delta_i - r_i)T}\left(e^{-r_i\tau_n} - e^{-\delta_i\tau_n}\right), \quad n = 0, 1, \ldots, M, \tag{19}$$

$$G_N^n(i) = G(X, \tau_n, i) \approx 0, \quad n = 0, 1, \ldots, M, \tag{20}$$

$$G_j^0(i) = 0, \quad j = 0, 1, \ldots, N, \tag{21}$$

where, $\hat{G}_j^n(i)$, $i \in \mathcal{D}$, is the quadratic interpolation of the computational solutions at time-level τ_n:

$$\begin{aligned}
\hat{G}_j^n(i) \;=\; &\frac{(x_j^{n+1} - x_j^n)(x_j^{n+1} - x_{j+1}^n)}{(x_{j-1}^n - x_j^n)(x_{j-1}^n - x_{j+1}^n)} G_{j-1}^n(i) + \frac{(x_j^{n+1} - x_{j-1}^n)(x_j^{n+1} - x_{j+1}^n)}{(x_j^n - x_{j-1}^n)(x_j^n - x_{j+1}^n)} G_j^n(i) \\
+ &\frac{(x_j^{n+1} - x_j^n)(x_j^{n+1} - x_{j-1}^n)}{(x_{j+1}^n - x_j^n)(x_{j+1}^n - x_{j-1}^n)} G_{j+1}^n(i),
\end{aligned} \tag{22}$$

and

$$\begin{aligned}
\mathcal{I}_j^{n+1} G(i) \;\equiv\; &\exp\left(\mu_i + \frac{\varrho_i^2}{2}\right) \sum_{k=1}^{N} G_{k-1}^{n+1}(i) \\
&\cdot \left\{ F_i\left(\ln\left(\frac{x_j^{n+1} - x_0^{n+1}}{x_{k-1}^{n+1} - x_0^{n+1}}\right)\right) - \frac{\int_{x_{k-1}^{n+1}}^{x_k^{n+1}} F_i\left(\ln\left(\frac{x_j^{n+1} - x_0^{n+1}}{\xi - x_0^{n+1}}\right)\right) d\xi}{x_k^{n+1} - x_{k-1}^{n+1}} \right\} \\
&- \exp\left(\mu_i + \frac{\varrho_i^2}{2}\right) \sum_{k=1}^{N} G_k^{n+1}(i) \\
&\cdot \left\{ F_i\left(\ln\left(\frac{x_j^{n+1} - x_0^{n+1}}{x_k^{n+1} - x_0^{n+1}}\right)\right) - \frac{\int_{x_{k-1}^{n+1}}^{x_k^{n+1}} F_i\left(\ln\left(\frac{x_j^{n+1} - x_0^{n+1}}{\xi - x_0^{n+1}}\right)\right) d\xi}{x_k^{n+1} - x_{k-1}^{n+1}} \right\}.
\end{aligned} \tag{23}$$

In (18), we use the central difference to discretize the first-order term. However it is noted that it can be done for the upwind difference, see e.g., [5]. To investigate the error analysis, we need to define and estimate the local truncation error for the scheme. Define the local truncation error for the scheme (18) for $i \in \mathcal{D}$; $n = 0, 1, \ldots, M-1$; $j = 1, \ldots, N-1$, as

$$
\begin{aligned}
\zeta_j^{n+1}(i) \quad &\equiv \quad \frac{G(x_j^{n+1},\tau_{n+1},i)-G(x_j^{n+1},\tau_n,i)}{\Delta\tau_n} \\
&+ \quad (r_i-\delta_i-\lambda_i\kappa_i)\left(x_j^{n+1}-\frac{T-\tau_{n+1}}{T}\right)\frac{G(x_{j+1}^{n+1},\tau_{n+1},i)-G(x_{j-1}^{n+1},\tau_{n+1},i)}{h_j^{n+1}+h_{j+1}^{n+1}} \\
&- \quad \frac{1}{2}\sigma_i^2\left(x_j^{n+1}-\frac{T-\tau_{n+1}}{T}\right)^2\frac{2}{h_j^{n+1}+h_{j+1}^{n+1}}\left[\frac{G(x_{j+1}^{n+1},\tau_{n+1},i)-G(x_j^{n+1},\tau_{n+1},i)}{h_{j+1}^{n+1}}\right. \\
&\qquad \left.-\frac{G(x_j^{n+1},\tau_{n+1},i)-G(x_{j-1}^{n+1},\tau_{n+1},i)}{h_j^{n+1}}\right]-\lambda_i\mathcal{I}_j^{n+1}G(x_j^{n+1},\tau_{n+1},i) \\
&- \quad \sum_{l=1}^{d}q_{il}G(x_j^{n+1},\tau_{n+1},l)+(\lambda_i+\delta_i+\lambda_i\kappa_i)G(x_j^{n+1},\tau_{n+1},i),
\end{aligned}
\tag{24}
$$

where the integral term $\mathcal{I}_j^{n+1}G(x_j^{n+1},\tau_{n+1},i)$ is defined by (17).

The local truncation error is estimated as follows.

Lemma 2.1 *The local truncation error* (24) *is estimated by*

$$
|\zeta_j^{n+1}(i)| = O\left(N^{-2}\right)+O\left(M^{-1}\right), \quad \textit{for } i\in\mathcal{D};\ n=0,1,\ldots,M-1;\ j=1,\ldots,N-1. \tag{25}
$$

Proof We use Taylor's theorem to expand the finite difference terms in (24) at point (x_j^{n+1},τ_{n+1}) and obtain the following results (The details of the derivation can be referred to Ma and Zhou [19]):

$$
\frac{G(x_j^{n+1},\tau_{n+1},i)-G(x_j^{n+1},\tau_n,i)}{\Delta\tau_n} \quad = \quad \frac{\partial G}{\partial\tau}(x_j^{n+1},\tau_{n+1},i)+O\left(\Delta\tau_n\right), \tag{26}
$$

$$
\begin{aligned}
\frac{G(x_{j+1}^{n+1},\tau_{n+1},i)-G(x_{j-1}^{n+1},\tau_{n+1},i)}{h_j^{n+1}+h_{j+1}^{n+1}} \quad &= \quad \frac{\partial G}{\partial x}(x_j^{n+1},\tau_{n+1},i) \\
&+ \quad O\left((h_j^{n+1})^2\right)+O\left((h_{j+1}^{n+1})^2\right),
\end{aligned}
\tag{27}
$$

and

$$
\begin{aligned}
&\frac{2}{h_j^{n+1}+h_{j+1}^{n+1}}\left[\frac{G(x_{j+1}^{n+1},\tau_{n+1},i)-G(x_j^{n+1},\tau_{n+1},i)}{h_{j+1}^{n+1}}\right. \\
&\left.-\frac{G(x_j^{n+1},\tau_{n+1},i)-G(x_{j-1}^{n+1},\tau_{n+1},i)}{h_j^{n+1}}\right] \\
&=\frac{\partial^2 G}{\partial x^2}(x_j^{n+1},\tau_{n+1},i)+O\left((h_j^{n+1})^2\right)+O\left((h_{j+1}^{n+1})^2\right).
\end{aligned}
\tag{28}
$$

Now we derive the disretization of the term $\mathcal{I}_j^{n+1}G(x_j^{n+1},\tau_{n+1},i)$ in (24). To this end, using Taylor expansion gives that

$$
\begin{aligned}
G(\xi,\tau_{n+1},i) \quad &= \quad G(x_{k-1}^{n+1},\tau_{n+1},i)+\frac{\partial G(x_{k-1}^{n+1},\tau_{n+1},i)}{\partial x}(\xi-x_{k-1}^{n+1}) \\
&+ \quad \frac{1}{2}\frac{\partial^2 G(x_{k-1}^{n+1},\tau_{n+1},i)}{\partial x^2}(\xi-x_{k-1}^{n+1})^2+O\left((h_k^{n+1})^3\right),
\end{aligned}
\tag{29}
$$

and

$$
\begin{aligned}
G(\xi,\tau_{n+1},i) &= G(x_k^{n+1},\tau_{n+1},i)+\frac{\partial G(x_k^{n+1},\tau_{n+1},i)}{\partial x}(\xi-x_k^{n+1})\\
&+ \frac{1}{2}\frac{\partial^2 G(x_k^{n+1},\tau_{n+1},i)}{\partial x^2}(\xi-x_k^{n+1})^2+O\left((h_k^{n+1})^3\right). \qquad (30)
\end{aligned}
$$

Multiplying (29) and (30) by $\frac{\xi-x_k^{n+1}}{x_{k-1}^{n+1}-x_k^{n+1}}$ and $\frac{\xi-x_{k-1}^{n+1}}{x_k^{n+1}-x_{k-1}^{n+1}}$, respectively, and then adding them, we obtain that

$$
\begin{aligned}
G(\xi,\tau_{n+1},i) &= \frac{\xi-x_k^{n+1}}{x_{k-1}^{n+1}-x_k^{n+1}}\Bigg[G(x_{k-1}^{n+1},\tau_{n+1},i)+\frac{\partial G(x_{k-1}^{n+1},\tau_{n+1},i)}{\partial x}(\xi-x_{k-1}^{n+1})\\
&+ \frac{1}{2}\frac{\partial^2 G(x_{k-1}^{n+1},\tau_{n+1},i)}{\partial x^2}(\xi-x_{k-1}^{n+1})^2+O\left((h_k^{n+1})^3\right)\Bigg]\\
&+ \frac{\xi-x_{k-1}^{n+1}}{x_k^{n+1}-x_{k-1}^{n+1}}\Bigg[G(x_k^{n+1},\tau_{n+1},i)+\frac{\partial G(x_k^{n+1},\tau_{n+1},i)}{\partial x}(\xi-x_k^{n+1})\\
&+ \frac{1}{2}\frac{\partial^2 G(x_k^{n+1},\tau_{n+1},i)}{\partial x^2}(\xi-x_k^{n+1})^2+O\left((h_k^{n+1})^3\right)\Bigg]. \qquad (31)
\end{aligned}
$$

Using (31), we derive that

$$
\begin{aligned}
&\sum_{k=1}^{N}\int_{x_{k-1}^{n+1}}^{x_k^{n+1}}\frac{x_j^{n+1}-x_0^{n+1}}{(\xi-x_0^{n+1})^2}G(\xi,\tau_{n+1},i)f_i\left(\ln\left(\frac{x_j^{n+1}-x_0^{n+1}}{\xi-x_0^{n+1}}\right)\right)d\xi\\
&=\mathcal{I}_j^{n+1}G(x_j^{n+1},\tau_{n+1},i)+(\mathrm{I})+(\mathrm{II}), \qquad (32)
\end{aligned}
$$

with

$$
\begin{aligned}
(\mathrm{I}) &\equiv \sum_{k=1}^{N}\int_{x_{k-1}^{n+1}}^{x_k^{n+1}}\frac{x_j^{n+1}-x_0^{n+1}}{(\xi-x_0^{n+1})^2}\Bigg[\frac{(\xi-x_{k-1}^{n+1})(\xi-x_k^{n+1})}{x_{k-1}^{n+1}-x_k^{n+1}}\frac{\partial G(x_{k-1}^{n+1},\tau_{n+1},i)}{\partial x}\\
&+\frac{(\xi-x_k^{n+1})(\xi-x_{k-1}^{n+1})}{x_k^{n+1}-x_{k-1}^{n+1}}\frac{\partial G(x_k^{n+1},\tau_{n+1},i)}{\partial x}\Bigg]f_i\left(\ln\left(\frac{x_j^{n+1}-x_0^{n+1}}{\xi-x_0^{n+1}}\right)\right)d\xi, \qquad (33)
\end{aligned}
$$

and

$$
\begin{aligned}
(\mathrm{II}) &\equiv \sum_{k=1}^{N}\int_{x_{k-1}^{n+1}}^{x_k^{n+1}}\frac{x_j^{n+1}-x_0^{n+1}}{(\xi-x_0^{n+1})^2}\frac{\xi-x_k^{n+1}}{x_{k-1}^{n+1}-x_k^{n+1}}f_i\left(\ln\left(\frac{x_j^{n+1}-x_0^{n+1}}{\xi-x_0^{n+1}}\right)\right)\\
&\cdot\left[\frac{1}{2}\frac{\partial^2 G(x_{k-1}^{n+1},\tau_{n+1},i)}{\partial x^2}(\xi-x_{k-1}^{n+1})^2+O\left((h_k^{n+1})^3\right)\right]d\xi\\
&+\sum_{k=1}^{N}\int_{x_{k-1}^{n+1}}^{x_k^{n+1}}\frac{x_j^{n+1}-x_0^{n+1}}{(\xi-x_0^{n+1})^2}\frac{\xi-x_{k-1}^{n+1}}{x_k^{n+1}-x_{k-1}^{n+1}}f_i\left(\ln\left(\frac{x_j^{n+1}-x_0^{n+1}}{\xi-x_0^{n+1}}\right)\right)\\
&\cdot\left[\frac{1}{2}\frac{\partial^2 G(x_k^{n+1},\tau_{n+1},i)}{\partial x^2}(\xi-x_k^{n+1})^2+O\left((h_k^{n+1})^3\right)\right]d\xi. \qquad (34)
\end{aligned}
$$

Now we estimate term (I) as follows:

$$\begin{aligned}|(\mathrm{I})| &= \Bigg|\sum_{k=1}^{N}\int_{x_{k-1}^{n+1}}^{x_k^{n+1}}\frac{x_j^{n+1}-x_0^{n+1}}{(\xi-x_0^{n+1})^2}\frac{(\xi-x_{k-1}^{n+1})(\xi-x_k^{n+1})}{x_{k-1}^{n+1}-x_k^{n+1}}f_i\left(\ln\Big(\frac{x_j^{n+1}-x_0^{n+1}}{\xi-x_0^{n+1}}\Big)\right)\\
&\quad\cdot\left[\frac{\partial G(x_{k-1}^{n+1},\tau_{n+1},i)}{\partial x}-\frac{\partial G(x_k^{n+1},\tau_{n+1},i)}{\partial x}\right]d\xi\Bigg| \qquad (35)\\
&\le \Bigg|\sum_{k=1}^{N}\int_{x_{k-1}^{n+1}}^{x_k^{n+1}}\frac{x_j^{n+1}-x_0^{n+1}}{(\xi-x_0^{n+1})^2}\frac{(\xi-x_{k-1}^{n+1})(\xi-x_k^{n+1})}{x_{k-1}^{n+1}-x_k^{n+1}}f_i\left(\ln\Big(\frac{x_j^{n+1}-x_0^{n+1}}{\xi-x_0^{n+1}}\Big)\right)\\
&\quad(x_{k-1}^{n+1}-x_k^{n+1})d\xi\Bigg|\left\|\frac{\partial^2 G(x,\tau_{n+1},i)}{\partial x^2}\right\|_\infty\\
&\le \Bigg|\sum_{k=1}^{N}\int_{x_{k-1}^{n+1}}^{x_k^{n+1}}\frac{x_j^{n+1}-x_0^{n+1}}{(\xi-x_0^{n+1})^2}f_i\left(\ln\Big(\frac{x_j^{n+1}-x_0^{n+1}}{\xi-x_0^{n+1}}\Big)\right)d\xi\Bigg|\left\|\frac{\partial^2 G(x,\tau_{n+1},i)}{\partial x^2}\right\|_\infty (h_k^{n+1})^2.\end{aligned}$$

Now we derive that

$$\begin{aligned}&\sum_{k=1}^{N}\int_{x_{k-1}^{n+1}}^{x_k^{n+1}}\frac{x_j^{n+1}-x_0^{n+1}}{(\xi-x_0^{n+1})^2}f_i\left(\ln\Big(\frac{x_j^{n+1}-x_0^{n+1}}{\xi-x_0^{n+1}}\Big)\right)d\xi \qquad (36)\\
&=\int_{x_0^{n+1}}^{X}\frac{x_j^{n+1}-x_0^{n+1}}{(\xi-x_0^{n+1})^2}f_i\left(\ln\Big(\frac{x_j^{n+1}-x_0^{n+1}}{\xi-x_0^{n+1}}\Big)\right)d\xi\\
&=\int_{x_0^{n+1}}^{X}-\frac{x_j^{n+1}-x_0^{n+1}}{\xi-x_0^{n+1}}\frac{1}{\varrho_i\sqrt{2\pi}}\exp\left\{-\frac{\left(\ln\Big(\frac{x_j^{n+1}-x_0^{n+1}}{\xi-x_0^{n+1}}\Big)-\mu_i\right)^2}{2\varrho_i^2}\right\}d\ln\Big(\frac{x_j^{n+1}-x_0^{n+1}}{\xi-x_0^{n+1}}\Big).\end{aligned}$$

Using the transform of variables $y=\ln\Big(\frac{x_j^{n+1}-x_0^{n+1}}{\xi-x_0^{n+1}}\Big)$, $y_N^{n+1}=\ln\Big(\frac{x_j^{n+1}-x_0^{n+1}}{X-x_0^{n+1}}\Big)$ and $y_0^{n+1}=$

$\lim_{\xi\to x_0^{n+1}} \ln\left(\frac{x_j^{n+1}-x_0^{n+1}}{\xi-x_0^{n+1}}\right)$, we calculate that

$$\begin{aligned}
&\int_{x_0^{n+1}}^{X} -\frac{x_j^{n+1}-x_0^{n+1}}{\xi-x_0^{n+1}}\frac{1}{\varrho_i\sqrt{2\pi}}\exp\left\{-\frac{\left(\ln\left(\frac{x_j^{n+1}-x_0^{n+1}}{\xi-x_0^{n+1}}\right)-\mu_i\right)^2}{2\varrho_i^2}\right\} d\ln\left(\frac{x_j^{n+1}-x_0^{n+1}}{\xi-x_0^{n+1}}\right)\\
&= \int_{y_0^{n+1}}^{y_N^{n+1}} -\exp(y)\frac{1}{\varrho_i\sqrt{2\pi}}\exp\left\{-\frac{(y-\mu_i)^2}{2\varrho_i^2}\right\}dy\\
&= \int_{y_N^{n+1}}^{y_0^{n+1}} \frac{1}{\varrho_i\sqrt{2\pi}}\exp\left\{-\frac{(y-\mu_i)^2-2\varrho_i^2 y}{2\varrho_i^2}\right\}dy\\
&= \int_{y_N^{n+1}}^{y_0^{n+1}} \frac{1}{\varrho_i\sqrt{2\pi}}\exp\left\{-\frac{\left(y-\mu_i-\varrho_i^2\right)^2-\left(\mu_i+\varrho_i^2\right)^2+\mu_i^2}{2\varrho_i^2}\right\}dy\\
&= \exp\left(\mu_i+\frac{\varrho_i^2}{2}\right)\int_{y_N^{n+1}}^{y_0^{n+1}} \frac{1}{\varrho_i\sqrt{2\pi}}\exp\left\{-\frac{\left(y-\mu_i-\varrho_i^2\right)^2}{2\varrho_i^2}\right\}dy\\
&\leq \exp\left(\mu_i+\frac{\varrho_i^2}{2}\right). \qquad (37)
\end{aligned}$$

So we have

$$|(\mathrm{I})| \leq \exp\left(\mu_i+\frac{\varrho_i^2}{2}\right)\left\|\frac{\partial^2 G(x,\tau_{n+1},i)}{\partial x^2}\right\|_\infty (h_k^{n+1})^2 = O\left((h_k^{n+1})^2\right). \qquad (38)$$

Similarly we estimate that

$$\begin{aligned}
|(\mathrm{II})| &\leq \exp\left(\mu_i+\frac{\varrho_i^2}{2}\right)\left\|\frac{\partial^2 G(x,\tau_{n+1},i)}{\partial x^2}\right\|_\infty (h_k^{n+1})^2 + O\left((h_k^{n+1})^3\right)\\
&= O\left((h_k^{n+1})^2\right), \qquad (39)
\end{aligned}$$

where

$$\left\|\frac{\partial^2 G(x,\tau_{n+1},i)}{\partial x^2}\right\|_\infty = \max_{\varpi\in\bar{\Omega}_{\tau_{n+1}}}\left|\frac{\partial^2 G(\varpi,\tau_{n+1},i)}{\partial x^2}\right|.$$

Since the equation is parabolic, from PDE theory [14], the solution G is sufficiently smooth, i.e., the second derivatives of G are bounded. Incorporating (26) - (39) into (24) gives that

$$\begin{aligned}
\zeta_j^{n+1}(i) &= \frac{\partial G}{\partial \tau}(x_j^{n+1},\tau_{n+1},i) - \frac{1}{2}\sigma_i^2(x_j^{n+1}-\frac{T-\tau_{n+1}}{T})^2\frac{\partial^2 G}{\partial x^2}(x_j^{n+1},\tau_{n+1},i)\\
&+ (r_i-\delta_i-\lambda_i\kappa_i)(x_j^{n+1}-\frac{T-\tau_{n+1}}{T})\frac{\partial G}{\partial x}(x_j^{n+1},\tau_{n+1},i)\\
&+ (\lambda_i+\delta_i+\lambda_i\kappa_i)G(x_j^{n+1},\tau_{n+1},i) - \sum_{l=1}^{d} q_{il}G(x_j^{n+1},\tau_{n+1},l)\\
&- \lambda_i\sum_{k=1}^{N}\int_{x_{k-1}^{n+1}}^{x_k^{n+1}} \frac{x_j^{n+1}-x_0^{n+1}}{(\xi-x_0^{n+1})^2}G(\xi,\tau_{n+1},i)f_i\left(\ln\left(\frac{x_j^{n+1}-x_0^{n+1}}{\xi-x_0^{n+1}}\right)\right)d\xi\\
&+ O\left((h_i^{n+1})^2\right)+O\left((h_{i+1}^{n+1})^2\right)+O\left(\Delta\tau_n\right). \qquad (40)
\end{aligned}$$

From (6) with $x = x_j^{n+1}$, $\tau = \tau_{n+1}$, (40) becomes

$$\zeta_j^{n+1}(i) = O\left((h_j^{n+1})^2\right) + O\left((h_{j+1}^{n+1})^2\right) + O\left(\Delta\tau_n\right). \tag{41}$$

Further using (12) and (13) for (41), we complete the proof of this lemma. □

In the following, we shall use the following mesh-dependent norm

$$\|\psi\|_n \equiv \left(\sum_j \left(\psi(x_j^n)\right)^2 \frac{x_{j+1}^n - x_{j-1}^n}{2}\right)^{1/2}, \tag{42}$$

and denote C as a generic positive constant which is independent of the grids for space and time.

To prove the convergence rates, we need the following Lemma 2.2 and Lemma 2.3.

Lemma 2.2 *Denote, for $i \in \mathcal{D}$; $j = 0, 1, \ldots, N$; $n = 0, 1, \ldots, M-1$,*

$$e_j^{n+1}(i) \equiv G(x_j^{n+1}, \tau_{n+1}, i) - G_j^{n+1}(i), \quad \hat{e}_j^n(i) \equiv G(x_j^{n+1}, \tau_n, i) - \hat{G}_j^n(i).$$

and assume that $N \le \widetilde{C}M$ where $\widetilde{C}$ is a positive constant satisfying $\widetilde{C} \le \frac{1-\sqrt{\frac{1}{2}}}{2}(X-1)$. Then we have

$$\begin{aligned}\left[\sum_j \left(\hat{e}_j^n(i)\right)^2 \frac{x_{j+1}^{n+1} - x_{j-1}^{n+1}}{2}\right]^{1/2} \quad &\le \quad (1 + C\Delta\tau_n)\left[\sum_j \left(e_j^n(i)\right)^2 \frac{x_{j+1}^n - x_{j-1}^n}{2}\right]^{1/2} \\ &+ \quad C\Delta\tau_n(M^{-2} + N^{-2}).\end{aligned} \tag{43}$$

Proof The proof of this lemma follows from Ma and Zhou [19, Lemma 4.1]. □

Lemma 2.3 *Let $\Delta\tau_n > 0$ and α_n, β_n, $q_n \ge 0$, for $1 \le n \le m$, with $\beta_n\Delta\tau_n \le 1/2$ and $\beta = \max_n \beta_n$. Then, if*

$$(1 - \beta_n\Delta\tau_n)q_n \le \Delta\tau_n\alpha_n + (1 + \beta_n\Delta\tau_n)q_{n-1},$$

there exists a positive constant C_m such that

$$\max_{0\le n\le m} q_n \le C_m \left\{q_0 + \sum_{n=1}^m \alpha_n\Delta\tau_n\right\},$$

where

$$C_m = \prod_{n=1}^m \frac{1 + \beta_n\Delta\tau_n}{1 - \beta_n\Delta\tau_n} \le C\exp\left(c\sum_{n=1}^m \beta_n\Delta\tau_n\right) \le C\exp\left(c\beta T\right),$$

where c and C are some positive constants.

Proof The proof is provided by Bank and Santos[1, Lemma 2.1]. □

Now it is ready to present the main convergence theorem.

Theorem 2.1 *The error of the moving FDM (18) is estimated by*

$$\begin{aligned}\max_{1\le n\le m}\|e^n(i)\|_n &\equiv \max_{1\le n\le m}\left[\sum_j\left(e_j^n(i)\right)^2\frac{x_{j+1}^n-x_{j-1}^n}{2}\right]^{1/2}\\ &\le C(M^{-1}+N^{-2}),\quad \textit{for } i\in\mathcal{D};\ m=1,\ldots,M.\end{aligned}\tag{44}$$

Proof Subtracting (18) by (24), for $i\in\mathcal{D}$; $n=0,1,\ldots,M-1$; $j=1,\ldots,N-1$, gives that

$$\begin{aligned}&\frac{e_j^{n+1}(i)-\hat{e}_j^n(i)}{\Delta\tau_n}+(r_i-\delta_i-\lambda_i\kappa_i)(x_j^{n+1}-\frac{T-\tau_{n+1}}{T})\frac{e_{j+1}^{n+1}(i)-e_{j-1}^{n+1}(i)}{h_j^{n+1}+h_{j+1}^{n+1}}\\ &-\frac{1}{2}\sigma_i^2(x_j^{n+1}-\frac{T-\tau_{n+1}}{T})^2\frac{2}{h_j^{n+1}+h_{j+1}^{n+1}}\left[\frac{e_{j+1}^{n+1}(i)-e_j^{n+1}(i)}{h_{j+1}^{n+1}}-\frac{e_j^{n+1}(i)-e_{j-1}^{n+1}(i)}{h_j^{n+1}}\right]\\ &-\lambda_i\mathcal{I}_j^{n+1}e(i)-\sum_{l=1}^d q_{il}e_j^{n+1}(l)+(\lambda_i+\delta_i+\lambda_i\kappa_i)e_j^{n+1}(i)=\zeta_j^{n+1}(i).\end{aligned}\tag{45}$$

By multiplying (45) by $e_j^{n+1}(i)\left(x_{j+1}^{n+1}-x_{j-1}^{n+1}\right)/2$ and summing up for all the allowable index i, we derive that, for $i\in\mathcal{D}$,

$$\begin{aligned}&\sum_j\left[\left(e_j^{n+1}(i)\right)^2\frac{x_{j+1}^{n+1}-x_{j-1}^{n+1}}{2}\right]\\ &=\Delta\tau_n\sum_j\left[\zeta_j^{n+1}(i)e_j^{n+1}(i)\frac{x_{j+1}^{n+1}-x_{j-1}^{n+1}}{2}\right]+\sum_j\left[\hat{e}_j^n(i)e_j^{n+1}(i)\frac{x_{j+1}^{n+1}-x_{j-1}^{n+1}}{2}\right]\\ &+\frac{1}{2}\Delta\tau_n\sigma_i^2\sum_j\left[\left(x_j^{n+1}-\frac{T-\tau_{n+1}}{T}\right)^2\left(\frac{e_{j+1}^{n+1}(i)-e_j^{n+1}(i)}{h_{j+1}^{n+1}}-\frac{e_j^{n+1}(i)-e_{j-1}^{n+1}(i)}{h_j^{n+1}}\right)e_j^{n+1}(i)\right]\\ &-\frac{1}{2}\Delta\tau_n(r_i-\delta_i-\lambda_i\kappa_i)\sum_j\left[\left(x_j^{n+1}-\frac{T-\tau_{n+1}}{T}\right)\left(e_{j+1}^{n+1}(i)-e_{j-1}^{n+1}(i)\right)e_j^{n+1}(i)\right]\\ &-\Delta\tau_n(\lambda_i+\delta_i+\lambda_i\kappa_i)\sum_j\left[\left(e_j^{n+1}(i)\right)^2\frac{x_{j+1}^{n+1}-x_{j-1}^{n+1}}{2}\right]\\ &+\Delta\tau_n\sum_{l=1}^d q_{il}\sum_j\left[e_j^{n+1}(l)e_j^{n+1}(i)\frac{x_{j+1}^{n+1}-x_{j-1}^{n+1}}{2}\right]\\ &+\Delta\tau_n\lambda_i\sum_j\left[\mathcal{I}_j^{n+1}e(i)e_j^{n+1}(i)\frac{x_{j+1}^{n+1}-x_{j-1}^{n+1}}{2}\right]\\ &\equiv \text{(i)}+\text{(ii)},\end{aligned}\tag{46}$$

where

$$\text{(ii)}\equiv\Delta\tau_n\lambda_i\sum_j\left[\mathcal{I}_j^{n+1}e(i)e_j^{n+1}(i)\frac{x_{j+1}^{n+1}-x_{j-1}^{n+1}}{2}\right],$$

and (i) the remaining terms on the right-hand side of (46). The estimation of (i) can obtained by following Ma and Zhou [20] as

$$
\begin{aligned}
|(\mathrm{i})| \leq\ & \Delta\tau_n \left[\sum_j \left(\zeta_j^{n+1}(i)\right)^2 \frac{x_{j+1}^{n+1}-x_{j-1}^{n+1}}{2}\right]^{1/2} \left[\sum_j \left(e_j^{n+1}(i)\right)^2 \frac{x_{j+1}^{n+1}-x_{j-1}^{n+1}}{2}\right]^{1/2} \\
+\ & \left[\sum_j \left(\hat{e}_j^{n}(i)\right)^2 \frac{x_{j+1}^{n+1}-x_{j-1}^{n+1}}{2}\right]^{1/2} \left[\sum_j \left(e_j^{n+1}(i)\right)^2 \frac{x_{j+1}^{n+1}-x_{j-1}^{n+1}}{2}\right]^{1/2} \\
+\ & \frac{1}{2}\Delta\tau_n \left(|r_i-\delta_i-\lambda_i\kappa_i| + \sigma_i^2\right) \left[\sum_j \left(e_j^{n+1}(i)\right)^2 \frac{x_{j+1}^{n+1}-x_{j-1}^{n+1}}{2}\right] \\
+\ & \Delta\tau_n |\lambda_i+\delta_i+\lambda_i\kappa_i| \sum_j \left[\left(e_j^{n+1}(i)\right)^2 \frac{x_{j+1}^{n+1}-x_{j-1}^{n+1}}{2}\right] \\
+\ & \Delta\tau_n \sum_{l=1}^{d} |q_{il}| \left[\sum_j \left(e_j^{n+1}(l)\right)^2 \frac{x_{j+1}^{n+1}-x_{j-1}^{n+1}}{2}\right]^{1/2} \\
& \cdot \left[\sum_j \left(e_j^{n+1}(i)\right)^2 \frac{x_{j+1}^{n+1}-x_{j-1}^{n+1}}{2}\right]^{1/2}. \qquad (47)
\end{aligned}
$$

Now we estimate (ii) as follows:

$$
\begin{aligned}
|(\mathrm{ii})| =\ & \Bigg|\Delta\tau_n\lambda_i \sum_j \sum_k \Bigg[\int_{x_{k-1}^{n+1}}^{x_k^{n+1}} \frac{x_j^{n+1}-x_0^{n+1}}{(\xi-x_0^{n+1})^2} \frac{\xi-x_{k-1}^{n+1}}{x_k^{n+1}-x_{k-1}^{n+1}} f_i\left(\ln\left(\frac{x_j^{n+1}-x_0^{n+1}}{\xi-x_0^{n+1}}\right)\right) d\xi \\
& \cdot e_k^{n+1}(i) e_j^{n+1}(i) \frac{x_{j+1}^{n+1}-x_{j-1}^{n+1}}{4}\Bigg]\Bigg| \\
+\ & \Bigg|\Delta\tau_n\lambda_i \sum_j \sum_k \Bigg[\int_{x_{k-1}^{n+1}}^{x_k^{n+1}} \frac{x_j^{n+1}-x_0^{n+1}}{(\xi-x_0^{n+1})^2} \frac{\xi-x_{k}^{n+1}}{x_{k-1}^{n+1}-x_{k}^{n+1}} f_i\left(\ln\left(\frac{x_j^{n+1}-x_0^{n+1}}{\xi-x_0^{n+1}}\right)\right) d\xi \\
& \cdot e_{k-1}^{n+1}(i) e_j^{n+1}(i) \frac{x_{j+1}^{n+1}-x_{j-1}^{n+1}}{4}\Bigg]\Bigg| \\
\leq\ & \Bigg|\Delta\tau_n\lambda_i \sum_j \sum_k \Bigg[\int_{x_{k-1}^{n+1}}^{x_k^{n+1}} \frac{x_j^{n+1}-x_0^{n+1}}{(\xi-x_0^{n+1})^2} f_i\left(\ln\left(\frac{x_j^{n+1}-x_0^{n+1}}{\xi-x_0^{n+1}}\right)\right) d\xi \\
& \cdot e_k^{n+1}(i) e_j^{n+1}(i) \frac{x_{j+1}^{n+1}-x_{j-1}^{n+1}}{4}\Bigg]\Bigg| \\
+\ & \Bigg|\Delta\tau_n\lambda_i \sum_j \sum_k \Bigg[\int_{x_{k-1}^{n+1}}^{x_k^{n+1}} \frac{x_j^{n+1}-x_0^{n+1}}{(\xi-x_0^{n+1})^2} f_i\left(\ln\left(\frac{x_j^{n+1}-x_0^{n+1}}{\xi-x_0^{n+1}}\right)\right) d\xi \\
& \cdot e_{k-1}^{n+1}(i) e_j^{n+1}(i) \frac{x_{j+1}^{n+1}-x_{j-1}^{n+1}}{4}\Bigg]\Bigg| \\
\equiv\ & \Delta\tau_n\lambda_i \left[(\text{Term } 1) + (\text{Term } 2)\right], \qquad (48)
\end{aligned}
$$

where the meaning of (Term 1) and (Term 2) is obvious. To continue the estimation, we denote

$$g(\gamma;\xi) = \frac{\gamma - x_0^{n+1}}{(\xi - x_0^{n+1})^2} f_i\left(\ln\Big(\frac{\gamma - x_0^{n+1}}{\xi - x_0^{n+1}}\Big)\right), \quad \gamma \in \Omega_{\tau_{n+1}} = \Big(\frac{T-\tau_{n+1}}{T}, X\Big). \tag{49}$$

The maximum of $g(\gamma;\xi)$ on $\overline{\Omega}_{\tau_{n+1}}$ is achieved at $\gamma^* = x_0^{n+1} + (\xi - x_0^{n+1})\exp(\mu_i + \varrho_i^2)$. Now we define

$$\widetilde{g}(\gamma;\xi) = \begin{cases} g(\gamma^*;\xi), & \gamma \in [\gamma^*, \gamma^* + h_k^{n+1}), \\ g(\gamma - h_k^{n+1};\xi), & \gamma \in [\gamma^* + h_k^{n+1}, X + h_k^{n+1}]. \end{cases} \tag{50}$$

We then derive that

$$\begin{aligned}
&\sum_j \frac{x_j^{n+1} - x_0^{n+1}}{(\xi - x_0^{n+1})^2} f_i\left(\ln\Big(\frac{x_j^{n+1} - x_0^{n+1}}{\xi - x_0^{n+1}}\Big)\right)(x_{j+1}^{n+1} - x_j^{n+1}) \\
&= \sum_j g(x_j^{n+1};\xi)(x_{j+1}^{n+1} - x_j^{n+1}) \\
&\le \int_{x_0^{n+1}}^{\gamma^*} g(\gamma;\xi)d\gamma + \int_{\gamma^*}^{X} \widetilde{g}(\gamma;\xi)d\gamma \\
&= \int_{x_0^{n+1}}^{\gamma^*} g(\gamma;\xi)d\gamma + \int_{\gamma^*}^{\gamma^*+h_k^{n+1}} \widetilde{g}(\gamma;\xi)d\gamma + \int_{\gamma^*+h_k^{n+1}}^{X+h_k^{n+1}} \widetilde{g}(\gamma;\xi)d\gamma \\
&= \int_{x_0^{n+1}}^{\gamma^*} g(\gamma;\xi)d\gamma + \int_{\gamma^*}^{\gamma^*+h_k^{n+1}} \widetilde{g}(\gamma;\xi)d\gamma + \int_{\gamma^*}^{X} g(\gamma;\xi)d\gamma \\
&= \int_{x_0^{n+1}}^{X} g(\gamma;\xi)d\gamma + g(\gamma^*;\xi)h_k^{n+1} \\
&\le \int_{x_0^{n+1}}^{X} g(\gamma;\xi)d\gamma + \frac{\exp(\mu_i + \frac{\varrho_i^2}{2})}{\varrho_i\sqrt{2\pi}}.
\end{aligned} \tag{51}$$

Moreover we calculate

$$\begin{aligned}
&\int_{x_0^{n+1}}^{X} g(\gamma;\xi)d\gamma \\
&= \int_{x_0^{n+1}}^{X} \frac{\gamma - x_0^{n+1}}{(\xi - x_0^{n+1})^2} \frac{1}{\varrho_i\sqrt{2\pi}} \exp\left\{-\frac{\left(\ln\Big(\frac{\gamma - x_0^{n+1}}{\xi - x_0^{n+1}}\Big) - \mu_i\right)^2}{2\varrho_i^2}\right\} d\gamma \\
&= \int_{x_0^{n+1}}^{X} \frac{(\gamma - x_0^{n+1})^2}{(\xi - x_0^{n+1})^2} \frac{1}{\varrho_i\sqrt{2\pi}} \exp\left\{-\frac{\left(\ln\Big(\frac{\gamma - x_0^{n+1}}{\xi - x_0^{n+1}}\Big) - \mu_i\right)^2}{2\varrho_i^2}\right\} d\ln\Big(\frac{\gamma - x_0^{n+1}}{\xi - x_0^{n+1}}\Big).
\end{aligned} \tag{52}$$

Denote $z = \ln\Big(\frac{\gamma - x_0^{n+1}}{\xi - x_0^{n+1}}\Big)$, $z_N^{n+1} = \ln\Big(\frac{X - x_0^{n+1}}{\xi - x_0^{n+1}}\Big)$ and $z_0^{n+1} = \lim_{\gamma\to x_0^{n+1}} \ln\Big(\frac{\gamma - x_0^{n+1}}{\xi - x_0^{n+1}}\Big)$. Then

(52) is further estimated as

$$
\begin{aligned}
&\int_{x_0^{n+1}}^{X} \frac{(\gamma - x_0^{n+1})^2}{(\xi - x_0^{n+1})^2} \frac{1}{\varrho_i\sqrt{2\pi}} \exp\left\{-\frac{\left(\ln\left(\frac{\gamma - x_0^{n+1}}{\xi - x_0^{n+1}}\right) - \mu_i\right)^2}{2\varrho_i^2}\right\} d\ln\left(\frac{\gamma - x_0^{n+1}}{\xi - x_0^{n+1}}\right)\\
&= \int_{z_0^{n+1}}^{z_N^{n+1}} \exp(2z)\frac{1}{\varrho_i\sqrt{2\pi}} \exp\left\{-\frac{(z-\mu_i)^2}{2\varrho_i^2}\right\} dz\\
&= \int_{z_0^{n+1}}^{z_N^{n+1}} \frac{1}{\varrho_i\sqrt{2\pi}} \exp\left\{-\frac{(z-\mu_i)^2 - 4\varrho_i^2 z}{2\varrho_i^2}\right\} dy\\
&= \int_{z_0^{n+1}}^{z_N^{n+1}} \frac{1}{\varrho_i\sqrt{2\pi}} \exp\left\{-\frac{\left(z-\mu_i-2\varrho_i^2\right)^2 - \left(\mu_i + 2\varrho_i^2\right)^2 + \mu_i^2}{2\varrho_i^2}\right\} dz\\
&= \exp\left(2\mu_i + 2\varrho_i^2\right) \int_{z_0^{n+1}}^{z_N^{n+1}} \frac{1}{\varrho_i\sqrt{2\pi}} \exp\left\{-\frac{\left(z-\mu_i-2\varrho_i^2\right)^2}{2\varrho_i^2}\right\} dz\\
&\leq \exp\left(2\mu_i + 2\varrho_i^2\right). \qquad (53)
\end{aligned}
$$

Using (36), (37), (51), (52) and (53), we derive that

$$
\begin{aligned}
&(\text{Term 1})\\
&\leq \sum_j \sum_k \left[\left(e_k^{n+1}(i)\right)^2 \frac{x_{j+1}^{n+1} - x_{j-1}^{n+1}}{8} \int_{x_{k-1}^{n+1}}^{x_k^{n+1}} \frac{x_j^{n+1} - x_0^{n+1}}{(\xi - x_0^{n+1})^2} f_i\left(\ln\left(\frac{x_j^{n+1} - x_0^{n+1}}{\xi - x_0^{n+1}}\right)\right) d\xi\right]\\
&+ \sum_j \sum_k \left[\left(e_j^{n+1}(i)\right)^2 \frac{x_{j+1}^{n+1} - x_{j-1}^{n+1}}{8} \int_{x_{k-1}^{n+1}}^{x_k^{n+1}} \frac{x_j^{n+1} - x_0^{n+1}}{(\xi - x_0^{n+1})^2} f_i\left(\ln\left(\frac{x_j^{n+1} - x_0^{n+1}}{\xi - x_0^{n+1}}\right)\right) d\xi\right]\\
&= \frac{1}{4}\sum_k \left[\left(e_k^{n+1}(i)\right)^2 \int_{x_{k-1}^{n+1}}^{x_k^{n+1}} \sum_j \frac{x_j^{n+1} - x_0^{n+1}}{(\xi - x_0^{n+1})^2} f_i\left(\ln\left(\frac{x_j^{n+1} - x_0^{n+1}}{\xi - x_0^{n+1}}\right)\right) (x_{j+1}^{n+1} - x_j^{n+1}) d\xi\right]\\
&+ \frac{1}{4}\sum_j \left[\left(e_j^{n+1}(i)\right)^2 \frac{x_{j+1}^{n+1} - x_{j-1}^{n+1}}{2} \sum_k \int_{x_{k-1}^{n+1}}^{x_k^{n+1}} \frac{x_j^{n+1} - x_0^{n+1}}{(\xi - x_0^{n+1})^2} f_i\left(\ln\left(\frac{x_j^{n+1} - x_0^{n+1}}{\xi - x_0^{n+1}}\right)\right) d\xi\right].\\
&\leq \frac{1}{4}\left[\exp\left(\mu_i + \frac{\varrho_i^2}{2}\right) + \exp\left(2\mu_i + 2\varrho_i^2\right) + \frac{\exp\left(\mu_i + \frac{\varrho_i^2}{2}\right)}{\varrho_i\sqrt{2\pi}}\right] \sum_j \left(e_j^{n+1}(i)\right)^2 \frac{x_{j+1}^{n+1} - x_{j-1}^{n+1}}{2}.
\end{aligned}
$$

Similarly, we can obtain that

$$
\begin{aligned}
&(\text{Term 2})\\
&= \left|\sum_j \sum_k \left[e_k^{n+1}(i) e_j^{n+1}(i) \frac{x_{j+1}^{n+1} - x_{j-1}^{n+1}}{4} \int_{x_k^{n+1}}^{x_{k+1}^{n+1}} \frac{x_j^{n+1} - x_0^{n+1}}{(\xi - x_0^{n+1})^2} f_i\left(\ln\left(\frac{x_j^{n+1} - x_0^{n+1}}{\xi - x_0^{n+1}}\right)\right) d\xi\right]\right|\\
&\leq \frac{1}{4}\left[\exp\left(\mu_i + \frac{\varrho_i^2}{2}\right) + \exp\left(2\mu_i + 2\varrho_i^2\right) + \frac{\exp\left(\mu_i + \frac{\varrho_i^2}{2}\right)}{\varrho_i\sqrt{2\pi}}\right] \sum_j \left(e_j^{n+1}(i)\right)^2 \frac{x_{j+1}^{n+1} - x_{j-1}^{n+1}}{2}.
\end{aligned}
$$

So we estimate (48) as

$$
\begin{aligned}
|(\mathrm{ii})| \quad \leq \quad & \frac{1}{2}\Delta\tau_n\lambda_i\left[\exp(\mu_i+\frac{\varrho_i^2}{2})+\exp(2\mu_i+2\varrho_i^2)+\frac{\exp(\mu_i+\frac{\varrho_i^2}{2})}{\varrho_i\sqrt{2\pi}}\right] \\
& \cdot\sum_j\left(e_j^{n+1}(i)\right)^2\frac{x_{j+1}^{n+1}-x_{j-1}^{n+1}}{2}. \qquad (54)
\end{aligned}
$$

Incorporating the estimations (47) and (54) into (46) and applying Lemma 2.1 and Lemma 2.2 give that

$$
\begin{aligned}
& (1-C\Delta\tau_n)\left\{\sum_{l=1}^{d}\left[\sum_j\left(e_j^{n+1}(l)\right)^2\frac{x_{j+1}^{n+1}-x_{j-1}^{n+1}}{2}\right]^{1/2}\right\} \\
& \leq\ (1+C\Delta\tau_n)\left\{\sum_{l=1}^{d}\left[\sum_j\left(e_j^{n}(l)\right)^2\frac{x_{j+1}^{n}-x_{j-1}^{n}}{2}\right]^{1/2}\right\} \\
& +C\Delta\tau_n(M^{-1}+N^{-2}). \qquad (55)
\end{aligned}
$$

Finally using Lemma 2.3 for (55), we complete the proof of this theorem. $\square$

3 Numerical examples

In this section, we carry out several numerical examples to verify the convergence rates of the moving FDM (18). The values of parameters used in the computation are listed in the corresponding examples. Since the exact solution of the problem is not known, we shall use the following formulas (57) and (58), which are given by Ma and Zhou [19], to calculate the convergence rates for time and space.

Let $G^s(x,i),\ i\in\mathcal{D}$ (state of regime), be the continuous form of the computational solutions at $\tau=T$ of scheme (18) for Example 3.1 with respect to the number of time and space mesh points $M_s,\ N_s$ $(s=1,2,\ldots)$. Denote the error between two adjacent levels of computational values by

$$
\mathrm{Error}(s,s+1)\equiv\|G^s-G^{s+1}\|_{M_s},
$$

where the norm is defined by (42).

Let the sequence of M_s, N_s $(s=1,2,\ldots)$ satisfy

$$
M_{s+1}-M_s=M_{s+2}-M_{s+1},\quad N_{s+1}-N_s=N_{s+2}-N_{s+1},\quad s=1,2,\ldots. \qquad (56)
$$

Then the convergence rates can be roughly estimated by, fixing the number of space mesh points $N_s=N,\ s=1,2,\ldots,$

$$
\text{Rate for time}=\frac{\log\left[M_s\mathrm{Error}(s,s+1)/(M_{s+1}\mathrm{Error}(s+1,s+2))\right]}{\log(M_{s+2}/M_{s+1})}, \qquad (57)
$$

and, fixing the number of time mesh points $M_s=M,\ s=1,2,\ldots,$

$$
\text{Rate for space}=\frac{\log\left[N_s\mathrm{Error}(s,s+1)/(N_{s+1}\mathrm{Error}(s+1,s+2))\right]}{\log(N_{s+2}/N_{s+1})}. \qquad (58)
$$

The codes for the examples in this section are run in MATLAB R2014a on a PC with the configuration: Intel(R)Core(TM), CPU i7-8550U@1.80 GHz 2.0GHz and 8.0 GB RAM.

Example 3.1 *In this example, we compare the moving mesh methods in this paper with that in Dang et al. [8] for 2-state regime-switching jump-diffusion model. In this example, the jump size* $Y_i\,(i=1,2)$ *follow double-exponential distributions, whose probability density functions (pdfs) are*

$$f_i(y) = p_i\epsilon_{1i}e^{-\epsilon_{1i}y}\mathbb{I}_{y\geq 0} + (1-p_i)\epsilon_{2i}e^{\epsilon_{2i}y}\mathbb{I}_{y<0}, \;\; i = 1, 2, \tag{59}$$

where $\kappa_i = p_i\frac{\epsilon_{1i}}{\epsilon_{1i}-1} + (1-p_i)\frac{\epsilon_{2i}}{\epsilon_{2i}+1} - 1,\ \epsilon_{11} = \epsilon_{12} = 3.0465,\ \epsilon_{21} = \epsilon_{22} = 3.0775,\ p_1 = p_2 = 0.3445$. *The parameters of the hidden process are taken as* $K = 100,\ X = 2.3,\ r_1 = r_2 = 0.05,\ \sigma_1 = 0.15,\ \sigma_2 = 0.25,\ \delta_1 = \delta_2 = 0,\ T = 1,\ \lambda_1 = 5,\ \lambda_2 = 2,\ -q_{11} = q_{12} = q_{21} = -q_{22} = 0.5.$

Tables 1 presents numerical results for the prices of Asian options at time $t = 0$. Columns 3 and 4 in Table 1 are from Dang et al [8] who use 1000 time steps and a 50 by 50 grids of cubic finite elements in the S- and I-directions to obtain the results. In column 5 of Table 1, the moving mesh methods of this paper only use 600 time steps and 200 spatial mesh points to obtain the results that have two digit accuracy after decimal point, the average CPU time spent by the moving mesh methods is 89.56 seconds for regime 1, and 90.33 seconds for regime 2. The moving mesh methods of this paper use less mesh nodes while obtain almost the same accuracy as the approach of Dang et al. [8].

Table 1: Prices for Asian options at $t = 0$ for Example 3.1.

S_0	Regime	Dang et al. Alg.1a	Dang et al. Alg.2	Moving mesh Alg.
92		17.45	17.42	17.42
100	Regime 1	21.55	21.52	21.55
108		26.25	26.21	26.27
92		11.78	11.75	11.79
100	Regime 2	15.67	15.63	15.68
108		20.42	20.39	20.44

Example 3.2 *In this example, we compare the moving mesh methods in this paper with that in Dang et al. [8] for the 3-state regime-switching jump-diffusion model.*

For regime 1, the pdf of the jump size is given by (2) with $i = 1,\ \varrho_1 = 0.3,\ \mu_1 = -0.1$. *For regime 2, the pdf of the jump size is given by (59) with* $i = 2,\ \epsilon_{12} = 3.0465,\ \epsilon_{22} = 3.0775,\ p_2 = 0.3445$. *For regime 3, the pdf of the jump size is given by*

$$f_3(y) = p_3\frac{1}{\sqrt{2\pi}v_{13}}\exp\left\{-\frac{(y-a_{13})^2}{2v_{13}^2}\right\} + (1-p_3)\frac{1}{\sqrt{2\pi}v_{23}}\exp\left\{-\frac{(y-a_{23})^2}{2v_{23}^2}\right\},$$

where $\kappa_3 = p_3e^{a_{13}+\frac{1}{2}v_{13}^2}+(1-p_3)e^{a_{23}+\frac{1}{2}v_{23}^2}-1,\ a_{13} = 0.3753,\ v_{13} = 0.18,\ a_{23} = -0.5503,\ v_{23} = 0.6944,\ p_3 = 0.3445$. *The parameters of the hidden process are taken as* $K = 100,\ X = 2.3,\ r_1 = r_2 = r_3 = 0.05,\ \delta_1 = \delta_2 = \delta_3 = 0,\ \lambda_1 = 1,\ \lambda_2 = 5,\ \lambda_3 = 2,\ \sigma_1 = 0.2,\ \sigma_2 = 0.15,\ \sigma_3 = 0.25,\ T = 1,\ q_{12} = q_{13} = q_{21} = q_{23} = q_{31} = q_{32} = 1/3,\ q_{11} = q_{22} = q_{33} = -2/3.$

Columns 3 and 4 in Table 2 are from Dang et al [8] who use 2000 time steps and a 100 by 100 grids of cubic finite elements to obtain the results. In column 5 of Table 2, the moving mesh methods of this paper only use 600 time steps and 200 spatial mesh points to obtain the results that have two digit accuracy after decimal point, the average CPU time spent by the moving mesh methods is 277.27 seconds for regime 1, 278.64 seconds for regime 2, and 277.85 seconds for regime 3. The moving mesh methods of this paper use less mesh nodes while obtain almost the same accuracy as the approach of Dang et al. [8].

Table 2: Prices for Asian options at $t = 0$ for Example 3.2.

S_0	Regime	Dang et al. Alg.1a	Dang et al. Alg.2	Moving mesh Alg.
92		6.62	6.61	6.65
96		8.46	8.44	8.48
100	Regime 1	10.63	10.62	10.66
104		13.13	13.11	13.15
108		15.90	15.88	15.92
92		17.02	16.99	17.01
96		19.01	18.98	19.01
100	Regime 2	21.16	21.13	21.18
104		23.47	23.43	23.50
108		25.93	25.88	25.97
92		14.09	14.08	14.16
96		16.25	16.23	16.32
100	Regime 3	18.59	18.57	18.68
104		21.12	21.09	21.22
108		23.80	23.78	23.92

Example 3.3 *Consider the system of PIDEs (6) with initial and boundary conditions (7) – (9). Functions* f_i $(i = 1, 2)$ *are given by (2) and the values of parameters are given by* $X = 1.5,\ r_1 = r_2 = 0.05,\ \sigma_1 = 0.15,\ \sigma_2 = 0.25,\ \delta_1 = \delta_2 = 0,\ T = 1,\ \lambda_1 = 1,\ \lambda_2 = 2,\ \mu_1 = \mu_2 = -0.1,\ \varrho_1 = \varrho_2 = 0.3,\ -q_{11} = q_{12} = q_{21} = -q_{22} = 1.$

Tables 3 and 4 show that the convergence rates are 1 in time and 2 in space, which are consistent with the theoretical results of Theorem 2.1.

Example 3.4 *Consider the system of PIDEs (6) with initial and boundary conditions (7) – (9). Functions* f_i $(i = 1, 2, 3)$ *are given by (2) and the values of parameters are given by* $X = 1.5,\ r_1 = r_2 = r_3 = 0.05,\ \sigma_1 = 0.2,\ \sigma_2 = 0.15,\ \sigma_3 = 0.25,\ \delta_1 = \delta_2 = \delta_3 = 0,\ T = 1,\ \lambda_1 = 1,\ \lambda_2 = 5,\ \lambda_3 = 2,\ \mu_1 = -0.1,\ \mu_2 = -0.15,\ \mu_3 = -0.05,\ \varrho_1 = 0.3,\ \varrho_2 = 0.25,\ \varrho_3 = 0.35,\ q_{12} = q_{13} = q_{21} = q_{23} = q_{31} = q_{32} = 1/3,\ q_{11} = q_{22} = q_{33} = -2/3.$

Tables 5 and 6 show that the convergence rates are 1 in time and 2 in space, which are consistent with the theoretical results of Theorem 2.1.

4 Conclusions

The value function of the Asian option under the state-dependent regime-switching jump-diffusion models satisfies a system of two-dimensional PIDEs. The simple boundary con-

Table 3: Convergence rates for time for Example 3.3.

$N = 200$				
	Regime 1		Regime 2	
M	Error$(j, j+1)$	Rate	Error$(j, j+1)$	Rate
400	1.426254e-04	0.99	1.444959e-04	1.00
450	1.141044e-04	0.99	1.155678e-04	1.00
500	9.336132e-05	0.99	9.453640e-05	1.00
550	7.780361e-05	0.99	7.876727e-05	1.00
600	6.583578e-05	0.99	6.663996e-05	1.00
650	5.643220e-05	0.99	5.711323e-05	1.00
700	4.890913e-05	0.99	4.949310e-05	1.00
750	4.279646e-05	—	4.330264e-05	—

Table 4: Convergence rates for space for Example 3.3.

$M = 2000$				
	Regime 1		Regime 2	
N	Error$(j, j+1)$	Rate	Error$(j, j+1)$	Rate
100	3.523968e-05	2.26	2.011614e-05	2.25
150	1.228002e-05	2.17	7.026342e-06	2.17
200	5.675010e-06	2.13	3.249841e-06	2.13
250	3.080402e-06	2.10	1.764733e-06	2.10
300	1.856605e-06	2.08	1.063874e-06	2.08
350	1.204698e-06	2.07	6.904139e-07	2.07
400	8.257972e-07	2.06	4.733093e-07	2.06
450	5.906188e-07	—	3.385376e-07	—

Table 5: Convergence rates for time for Example 3.4.

$N = 200$						
	Regime 1		Regime 2		Regime 3	
M	Error$(j, j+1)$	Rate	Error$(j, j+1)$	Rate	Error$(j, j+1)$	Rate
400	1.593156e-04	1.00	1.857532e-04	1.00	1.669601e-04	1.00
450	1.274482e-04	1.00	1.485454e-04	1.00	1.335273e-04	1.00
500	1.042732e-04	1.00	1.214992e-04	1.00	1.092227e-04	1.00
550	8.689262e-05	1.00	1.012232e-04	1.00	9.100034e-05	1.00
600	7.352340e-05	1.00	8.563173e-05	1.00	7.698711e-05	1.00
650	6.301927e-05	1.00	7.338499e-05	1.00	6.597933e-05	1.00
700	5.461614e-05	1.00	6.359013e-05	1.00	5.717490e-05	1.00
750	4.778872e-05	—	5.563358e-05	—	5.002256e-05	—

Table 6: Convergence rates for space for Example 3.4.

	$M = 2000$					
	Regime 1		Regime 2		Regime 3	
N	Error$(j, j+1)$	Rate	Error$(j, j+1)$	Rate	Error$(j, j+1)$	Rate
100	2.728436e-05	2.25	2.628736e-05	2.28	1.459717e-05	2.24
150	9.525623e-06	2.17	9.100082e-06	2.19	5.101504e-06	2.17
200	4.405026e-06	2.13	4.191181e-06	2.14	2.360008e-06	2.12
250	2.391816e-06	2.10	2.270508e-06	2.11	1.281653e-06	2.10
300	1.441841e-06	2.08	1.366750e-06	2.09	7.726880e-07	2.08
350	9.356722e-07	2.07	8.860789e-07	2.08	5.014616e-07	2.07
400	6.414318e-07	2.06	6.070095e-07	2.07	3.437815e-07	2.06
450	4.587819e-07	—	4.339341e-07	—	2.458957e-07	—

ditions for these two-dimensional PIDEs are difficult to be constructed. This fact causes difficulty in the numerical solutions. In this paper, the two-dimensional PIDEs are converted into a moving boundary problem of one-dimensional PIDEs and the exact moving boundary conditions are derived. Moving FDMs are constructed to solve the moving boundary problem. The convergence rates of the moving FDMs are proved. Compared to Dang et al. [8], this paper solves one-dimensional problems instead of two-dimensional problems and analyzes the convergence rates. To the best of our knowledge, for Asian option pricing using PDE approach, only the moving mesh methods' convergence theory has been proved in the literature.

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