ELSEVIER

Journal of Computational and Applied Mathematics 123 (2000) 307–321

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

www.elsevier.nl/locate/cam

Scalable preconditioned conjugate gradient inversion of vector finite element mass matrices ☆

Joe Koning[a], Garry Rodrigue[a,*], Dan White[b]

[a] *Department of Applied Science, University of California, Davis, and Institute for Scientific Computing Research, Lawrence Livermore National Laboratory, L-561, P.O. Box 808 L-794, Livermore, CA 94550, USA*

[b] *Center for Advanced Scientific Computing, Lawrence Livermore National Laboratory, L-561, Livermore, CA, USA*

Received 4 June 1999

Abstract

Mass matrices arise in the numerical solution of time-dependent partial differential equations by the Galerkin method. Since these systems must be inverted at each time step, rapid inversion algorithms for these systems are important. When nodal finite elements are used as basis functions, it is known that the mass matrices can be consistently approximated by a diagonal matrix or solved by a scalable conjugate gradient method. This may not be the case for other basis functions. In this paper, we show that the preconditioned conjugate gradient method is scalable when used to invert mass matrices that arise from vector finite element basis functions. These basis functions are particularly important for solving Maxwell's equations on unstructured grids by the Galerkin method.

Keywords: Galerkin method; Mass matrices; Vector finite elements; Conjugate gradient method; Vector wave equation; Maxwell's equations

1. Mass matrices

The Gram matrix of the linear-independent elements $\phi_1, \phi_2, \ldots, \phi_n$ in an inner product space V is the symmetric positive-definite $n \times n$ matrix

☆ This research was supported under the auspices of the United States Department of Energy by Lawrence Livermore National Laboratory contract W-7405-Eng-48.

* Correspondence address: Department of Applied Science, University of California, Davis, and Institute for Scientific Computing Research, Lawrence Livermore National Laboratory, L-561, P.O.Box 808 L-794, Livermore, CA 94550, USA.

E-mail addresses: koning@llnl.gov (J. Koning), rodrigue@llnl.gov (G. Rodrigue), white37@llnl.gov (D. White).

0377-0427/00/$ - see front matter
PII: S0377-0427(00)00407-6

$$G = \begin{bmatrix} (\phi_1, \phi_1) & (\phi_1, \phi_2) & \cdots & (\phi_1, \phi_n) \\ (\phi_2, \phi_1) & (\phi_2, \phi_2) & \cdots & (\phi_2, \phi_n) \\ \vdots & \vdots & \vdots & \vdots \\ (\phi_n, \phi_1) & (\phi_n, \phi_2) & \cdots & (\phi_n, \phi_n) \end{bmatrix}, \tag{1}$$

[5]. A classic example of a Gram matrix is the Hilbert matrix given by $\phi(x) = x^{i-1}$ and the inner product $(u, v) = \int_0^1 u(x)v(x)\,\mathrm{d}x$ in which case $G = [g_{ij}] = [(i+j-1)^{-1}]$. Gram matrices naturally arise in the numerical solution of time-dependent partial differential equations by the Galerkin method [14]. Specifically, given the weak differential equation

$$\left(\frac{\partial u}{\partial t}, v\right) = (L[u], v), \tag{2}$$

a function $\tilde{u}(\boldsymbol{x}, t) = \sum_{i=1}^{n} \alpha_i(t)\phi_1(\boldsymbol{x})$ is sought out in a finite-dimensional subspace spanned by a linearly independent set of basis functions $\phi_1(\boldsymbol{x}), \phi_2(\boldsymbol{x}), \ldots, \phi_n(\boldsymbol{x})$ that approximates the weak solution of Eq. (2). The Galerkin method calculates this approximation by defining $\tilde{u}$ to satisfy

$$\left(\frac{\partial}{\partial t}\tilde{u}, \phi_j\right) = (L[\tilde{u}], \phi_j), \quad j = 1, 2, \ldots, n. \tag{3}$$

Then, if we let $\alpha(t) = [\alpha_1, \alpha_2, \ldots, \alpha_n]^t$, this results in a system of ordinary differential equations

$$G\frac{\mathrm{d}\alpha}{\mathrm{d}t} = F(\alpha), \tag{4}$$

where G is the Gram matrix of Eq. (1) and is called the mass matrix of the Galerkin procedure.

If one approximates Eq. (4) by any numerical time differencing scheme, we see that it is necessary to invert the mass matrix at each time step. Hence, the ease and rapidity of the mass matrix inversion process is an important part of any Galerkin method.

2. Numerical inversion of the mass matrix

Since the mass matrix G is symmetric and positive definite, the natural choice for its inversion is the preconditioned conjugate gradient method. The efficiency of the preconditioned conjugate gradient method relies on the choice of the preconditioner Q [11]. Examples of preconditioners include the incomplete Cholesky factorization [9], the SSOR preconditioner [22], multigrid preconditioners [3] and domain decomposition preconditioners [2].

An efficient preconditioner must possess three properties:

1. The preconditioner must be relatively easy to solve.
2. The matrix $Q^{-1}G$ must "approximate the identity".
3. The preconditioner must yield a "scalable" method in the sense that the number of iterations to convergence must approach a constant as the size of the matrix n approaches infinity.

For the preconditioned conjugate gradient method, the spectral condition number ratio

$$\kappa(Q^{-1}G) = \frac{\lambda_{\max}(Q^{-1}G)}{\lambda_{\min}(Q^{-1}G)}$$

of the largest to the smallest eigenvalue of $Q^{-1}G$ enters into the upper bound for the error

$$\frac{\|e^k\|_G}{\|e^0\|_G} \leqslant 2\left[\frac{\kappa^{1/2}-1}{\kappa^{1/2}+1}\right]^k, \tag{5}$$

where the G-norm of the error $\|e^k\|_G$, is defined as $(e^k)^t G e^k$. The bound in Eq. (5) is not sharp for the conjugate gradient method. A sharp error bound for the conjugate gradient method is more complicated [10], involving the distribution of the eigenvalues of $Q^{-1}G$. However, a spectral condition number close to 1 and bounded from above as the size n approaches infinity is sufficient to ensure fast and scalable convergence of the conjugate gradient algorithm.

In this paper we concentrate on determining preconditioners that yield scalable conjugate gradient algorithms. That is we seek preconditioners such that

$$\lim_{n\to\infty} \kappa(Q^{-1}G) < C$$

for some constant C independent of n.

Condition number bounds can sometimes be achieved by obtaining a bound on the condition number of an associated matrix and then "comparing" it to the original system. Unfortunately, there are few theoretical comparison results for the condition number of preconditioned systems. An exception is the case of diagonal and block diagonal preconditioners. Van der Sluis [19] proved the following theorem about diagonal scaling of a symmetric positive matrix G.

Theorem (Van der Sluis [19]). *Let D be the diagonal of the symmetric positive-definite matrix G, and let $\hat{D}$ be any other positive-definite diagonal matrix. Then*

$$\kappa(D^{-1}G) \leqslant m\kappa(\hat{D}^{-1}G),$$

where m is the maximum number of nonzeros in any row of G.

When the matrix G has property-A, that is when G can be permuted in the form

$$G = \begin{bmatrix} D_1 & B \\ B^t & D_2 \end{bmatrix},$$

where D_1 and D_2 are diagonal matrices, a stronger result holds [8].

Theorem (Forsythe and Strauss [8]). *Using the above notation, if the symmetric positive-definite matrix G has property-A, then*

$$\kappa(D^{-1}G) \leqslant \kappa(\hat{D}^{-1}G).$$

A generalization of the Van der Sluis theorem has also been proved for block diagonal preconditioners [6].

Theorem (Demmel [6]). *Let D be the block diagonal of the symmetric positive-definite matrix G, and let $\hat{D}$ be any other symmetric positive-definite block diagonal matrix with same size blocks. Then*

$$\kappa(D^{-1}G) \leqslant b\kappa(\hat{D}^{-1}G),$$

where b is the number of blocks in D.

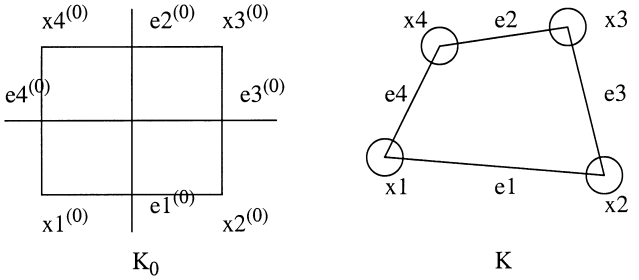

Fig. 1. Numbering configuration for reference element K_0 and quadrilateral element K.

A result similar to that of Forsythe and Strauss has also been proved for block diagonal preconditioners [7], when the matrix G is block 2-cyclic and is permuted in the form

$$G = \begin{bmatrix} D_1 & C \\ C^t & D_2 \end{bmatrix}, \tag{6}$$

where $D_i,\ i = 1,2$, is a block diagonal matrix with diagonal blocks $D_{i,j},\ j = 1,2,\ldots,r_i$.

Theorem (Eisenstat et al. [7]). *Let G be the form in Eq.* (6) *and let D be the block diagonal matrix whose diagonal blocks are $\{D_{1,1},\ldots,D_{1,r_1},D_{2,1},\ldots,D_{2,r_2}\}$. Let $\hat{D}$ be any other block diagonal matrix with same size blocks. Then*

$$\kappa(D^{-1}G) \leqslant \kappa(\hat{D}^{-1}G).$$

3. The finite element Galerkin method

The finite element Galerkin method is a systematic technique for constructing the basis functions ϕ_i for the Galerkin method based around a numerical grid. Irregular domains and mixed boundary conditions are easily accommodated and the resulting equations describing the discrete model are generally well-conditioned [1].

Formally, a finite element (K, P_K, A_K) is defined as follows [4]:

1. K, a quadrilateral domain.
2. $P_K = (P_1)^N = P_1 \otimes \cdots \otimes P_N$, a vector space consisting of the tensor product of a polynomial vector spaces P_i defined on K. P_K has a basis $\{\Psi_1, \Psi_2, \Psi_3, \Psi_4\}$.
3. A_K, a set of linear functionals defined on P_K having a basis $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ (called degrees of freedom).

Each finite element (K, P_K, A_K) will be isoparametrically equivalent to a single reference finite element (K_0, P_0, A_0) where $K_0 = \{-1 \leqslant x,\ y \leqslant 1\}$. If we assume the numbering configuration for the nodes and edges of a given quadrilateral in Fig. 1, then the isoparametric mapping is given by

$$F_K(\xi,\eta) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} N_1 + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} N_2 + \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} N_3 + \begin{bmatrix} x_4 \\ y_4 \end{bmatrix} N_4,$$

where K is the quadrilateral with vertices $\{(x_i, y_i),\ i = 1,2,3,4\}$ and

$$\begin{aligned} N_1(\xi,\eta) &= \tfrac{1}{4}(1-\xi)(1-\eta),\\ N_2(\xi,\eta) &= \tfrac{1}{4}(1+\xi)(1-\eta),\\ N_3(\xi,\eta) &= \tfrac{1}{4}(1+\xi)(1+\eta),\\ N_4(\xi,\eta) &= \tfrac{1}{4}(1-\xi)(1+\eta). \end{aligned} \tag{7}$$

Then, P_K is defined by

$$P_K = \{p = p_0 \cdot F_K^{-1} : p_0 \in P_0\} \tag{8}$$

and the basis of P_K is given by $\Psi_i = \Psi_i^{(0)} \cdot F_K^{-1}$ where

$$P_0 = \operatorname{span}[\Psi_1^{(0)}, \Psi_2^{(0)}, \Psi_3^{(0)}, \Psi_4^{(0)}].$$

A finite element is said to be unisolvent if the set of degrees of freedom A_K determines a unique polynomial in P_K. If this is the case, then for any function f defined on K, there exists a unique interpolant $\pi(f) \in P$ such that $\alpha(f) = \alpha[\pi(f)]$ for all $\alpha \in A_K$.

The element mass matrix is defined to be the 4×4 matrix

$$M_K = \left[\int_K \Psi_i \cdot \Psi_j \,\mathrm{d}K\right] = \left[\int_{-1}^{1}\int_{-1}^{1} \Psi_i^{(0)} \cdot \Psi_j^{(0)} \det(J_K)\,\mathrm{d}\xi\,\mathrm{d}\eta\right],$$

where

$$J_K = \begin{bmatrix} x_\xi & y_\xi \\ x_\eta & y_\eta \end{bmatrix}.$$

The mass matrix is then given by

$$M = \sum_K M_K,$$

where the matrix behind the summation signs are expanded or augmented by zero filling.

4. Nodal finite elements

Here, the polynomial space $P_K = P = \operatorname{span}[1, x, y, xy]$. The four degrees of freedom are

$$A_K = \operatorname{span}[\alpha_i(p) = p(x_i, y_i), \quad p \in P_K,\ \ i = 1,2,3,4], \tag{9}$$

where (x_i, y_i) are the coordinates of the nodes of K. Clearly, the finite element is unisolvent under the degrees of freedom in Eq. (9). For the reference element, $P_0 = \operatorname{span}[N_1, N_2, N_3, N_4]$ (the N_i are defined in Eq. (7)). An important result regarding the scalability of the conjugate gradient method for solving mass matrix systems arising from nodal finite elements is the following.

Theorem (Ciarlet [4]). *Assume*

- *Ω a polynomial domain in R^2.*
- *Γ the boundary of Ω.*

- *G_h a quadrilateral decomposition of Ω, i.e., a decomposition of Ω into a set $G_h = K_1, K_2, \ldots, K_m$ of nonoverlapping quadrilaterals K_i such that $\Omega = \bigcup_{K \in G_h} K$ and no vertex of one quadrilateral lies on the edge of another quadrilateral.*
- *$h = \max_{K \in G_h} \operatorname{diam}(K)$ where $\operatorname{diam}(K)$ is the longest side of quadrilateral K.*
- *There exists positive constants β_1, β_2 independent of h such that for all $K \in G_h$,*

$$\beta_1 h \leqslant h_K = \operatorname{diam}(K) \leqslant \beta_2 h.$$

- *$\phi_1, \phi_2, \ldots, \phi_n$ are a nodal basis functions of V_h.*

Then if $M = \operatorname{matrix}(\int_\Omega \phi_i \phi_j d\Omega)$ is the mass matrix, there exist constants C_1, C_2 depending only on β_1, β_2 such that

$$\kappa(M) \leqslant \frac{C_2}{C_1}.$$

Hence, we see that if a sequence of grids satisfies the previous theorem, then the preconditioned conjugate gradient will attain a constant number of iterations as the number of grid points increases whenever the preconditioner satisfies any of the theorems in Section 2.

Another important property of nodal mass matrices is that they can be consistently "lumped" [15]. That is, they can be consistently approximated by a diagonal matrix.

5. Motivation – the vector wave equation

The two-dimensional Maxwell's equations consist of two equations that relate the vector electric field $\boldsymbol{E} = [E_1, E_2]$, a scalar magnetic field H and a divergence condition [12].

$$\nabla \times \boldsymbol{E} = -\frac{\partial H}{\partial t}, \tag{10}$$

$$\vec{\nabla} \times H = \frac{\partial \boldsymbol{D}}{\partial t}, \tag{11}$$

$$\nabla \cdot \boldsymbol{D} = 0, \tag{12}$$

where

$$\vec{\nabla} \times H = \left[\frac{\partial H}{\partial y}, -\frac{\partial H}{\partial x}\right]^t, \quad \nabla \times \boldsymbol{E} = \frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y}.$$

Two constitutive relations are required to close Maxwell's equations,

$$\boldsymbol{D} = \varepsilon \boldsymbol{E}, \quad B = \mu H, \tag{13}$$

where the dielectric permittivity ε and the magnetic permeability μ are scalar functions of position.

The magnetic field is eliminated by applying the operation $\vec{\nabla} \times$ to Eq. (10) and applying the identities Eqs. (11) and (13) to obtain the vector wave equation for the electric field

$$\varepsilon \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = -\vec{\nabla} \times \frac{1}{\mu} \nabla \times \boldsymbol{E} \tag{14}$$

The Galerkin method for solving Eq. (14) computes an approximation

$$\tilde{\boldsymbol{E}} = [\tilde{E}_1, \tilde{E}_2]^t = \sum_{i=1}^{n} \alpha_i(t)\, \vec{\Phi}(x, y),$$

such that

$$\int_\Omega \varepsilon \left[\frac{\partial^2}{\partial t^2}\tilde{\boldsymbol{E}}\right]^t \vec{\Phi}_j \,\mathrm{d}\Omega = -\int_\Omega \left[\vec{\nabla} \times \frac{1}{\mu}\nabla \times \tilde{\boldsymbol{E}}\right]^t \vec{\Phi}_j \,\mathrm{d}\Omega$$
$$= -\int_\Omega \frac{1}{\mu}[\nabla \times \boldsymbol{E}]^t [\nabla \times \vec{\Phi}_j]\,\mathrm{d}\Omega,$$

where the second equality follows from Green's second vector theorem [20]. Substituting the expansion for $\boldsymbol{E}$ we get a square system of equations

$$\sum_i \left(\int_\Omega \vec{\Phi}_i^{\,t} \vec{\Phi}_j \,\mathrm{d}\Omega\right) \frac{\partial^2 \alpha_i}{\partial t^2} = -\sum_i \left(\int_\Omega [\nabla \times \vec{\Phi}_i]^t [\nabla \times \vec{\Phi}_j]\,\mathrm{d}\Omega\right) \alpha_i$$

yielding the system of ordinary differential equations in Eq. (4) where the mass matrix G is given by

$$G = \left[\int_\Omega \vec{\Phi}_i^{\,t} \vec{\Phi}_j \,\mathrm{d}\Omega\right].$$

One could, of course, use the nodal finite elements to provide Galerkin vector approximations to the vector wave equation in Eq. (14). This has the advantage that the mass matrices can be consistently lumped or be solved by a scalable preconditioned conjugate gradient algorithm. Unfortunately, the continuity of the nodal finite element approximations turns out to be a liability when applied to the vector wave equation when the dielectric ε has a jump discontinuity. In this case, it is known that the tangential component of the electric field is continuous across the discontinuity while its normal component may be discontinuous. Consequently, an important property of electric fields that should be preserved in any numerical approximation is the following: *The tangential component of E across an interface is continuous but the normal component f E across the same interface may be discontinuous*. To ensure this, the tangential component of the numerical approximation $\tilde{E}$ should be continuous along the edges of each quadrilateral of the grid but its normal component need not be. More specifically, if K_1 and K_2 are two elements with a common edge e then the tangential components of $\pi_1(u)$ and $\pi_2(u)$ are the same on e for all $u \in C^\infty(K_1 \cup K_2)$ [16]. Finite elements with this property are said to be conformal. Since the tangential and normal components of the Galerkin approximation provided by the nodal finite elements are continuous, nonphysical spurious oscillations have been observed when they are used to solve Eq. (14) [13].

6. Edge elements

Finite elements that enforce continuity of the electric field across edges have been recently discovered and analyzed [16,17]. Basically, these "vector finite element" assign degrees of freedom to the edges rather than to the nodes of the elements. For this reason, they are called edge elements. Although these types of elements were described in [21], as early as 35 years ago, their use and

importance in electromagnetics was not realized until recently. Extensive investigations as well as some very successful applications have been carried over the past few years [18,20]. In this section, we introduce the edge elements (K, P_K, A_K) in two dimensions and analyze the mass matrices that arise from their use in the Galerkin procedure.

The degrees of freedom A_K for the edge elements are the line integrals

$$\alpha_i(\boldsymbol{p}) = \int_{e_{i_i}} \boldsymbol{p} \cdot \boldsymbol{t}_i \, \mathrm{d}e_i, \quad \boldsymbol{p} \in P$$

where $\boldsymbol{t}_i$ is the unit tangent along edge e_i, $i = 1, 2, 3, 4$ [13]. The fact that these elements are conforming is found in [17]. On the reference element,

$$P_0 = \{a + b\eta\} \otimes \{c + \mathrm{d}\xi\}$$

and the conditions

$$\alpha_i(\boldsymbol{p}) = \int_{e_{i_j}} \boldsymbol{p} \cdot \boldsymbol{t}_j \, \mathrm{d}\sigma = \delta_{ij}, \quad \boldsymbol{p} \in P_0$$

yields the basis functions

$$\Psi_1^{(0)}(\xi, \eta) = \frac{1}{4}(1 - \eta) \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\Psi_2^{(0)}(\xi, \eta) = \frac{1}{4}(1 + \eta) \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\Psi_3^{(0)}(\xi, \eta) = \frac{1}{4}(1 - \xi) \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$\Psi_4^{(0)}(\xi, \eta) = \frac{1}{4}(1 + \xi) \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then, $P_K = \mathrm{span}[\Psi_1, \Psi_2, \Psi_3, \Psi_4]$ where $\Psi_i(x, y) = \Psi_i^{(0)}[F_K^{-1}(x, y)]$. Note that if $\boldsymbol{t}_j$ is the unit tangent vector along edge e_j, then

$$\int_{e_j} (\Psi_i(x, y) \cdot \boldsymbol{t}_j) \, \mathrm{d}e_j = \delta_{ij}.$$

The element mass matrix is given by

$$\int_K \Psi_i^t \Psi_j \, \mathrm{d}K = \int_{-1}^{1} \int_{-1}^{1} (\Psi_i^0)^t \Psi_j^{(0)} \det(J_K) \, \mathrm{d}\xi \, \mathrm{d}\eta.$$

Table 1
Mass matrix condition number for n edge elements

n	$\kappa(M)$
12	2.37
40	3.87
144	3.92
544	3.97
2112	4.0
5320	4.0
33024	4.07

7. Edge element mass matrices

7.1. Uniform grid

We first consider the edge element mass matrices generated on a uniform grid of grid size h. Here, $K = \{(x_i \leqslant x \leqslant x_i + h,\ y_i \leqslant y \leqslant y_i + h)\}$ and the element matrix is given by

$$M_K = \frac{h^2}{6}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

This yields a block diagonal mass matrix

$$M = \frac{h^2}{6}\begin{bmatrix} A_1 & & & & \\ & A_2 & & & \\ & & \ddots & & \\ & & & A_{n-1} & \\ & & & & A_n \end{bmatrix}, \quad A_i = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

If we estimate the eigenvalues of the mass matrix M using Gerschgorin discs, we get the following result.

Theorem 1. *If M is the mass matrix generated from vector edge elements on a uniform rectangular grid, then*

$$\kappa(M) \leqslant 6. \tag{15}$$

Table 1 tabulates the actual condition number for a variety of matrix sizes. We see that the bound on the condition number in Eq. (15) appears to be an over-estimate.

Of course, a natural question to ask would be whether the mass matrix can be consistently approximated by a diagonal matrix much in the same manner as is commonly done using mass lumping techniques for nodal elements. In this regard, if the trapezoid rule is used to evaluate the inner product integrals, we get the following result.

Table 2
Condition number of element mass matrix for $f = 0.44$

n	$f = 0.49$	$f = 0.47$	$f = 0.44$	$f = 0.35$
144	3.86	4.41	5.27	10.31
544	3.92	4.95	6.76	20.28
2112	3.97	5.49	8.35	45.35
5320	3.99	6.15	10.69	86.18
33024	4.0	7.2	15.72	230

Table 3
Condition number calculations for diagonally preconditioned systems

n	$f = 0.49$	$f = 0.47$	$f = 0.44$	$f = 0.35$
40	3.0	3.15	3.33	4.08
144	3.0	3.18	3.43	4.43
544	3.0	3.28	3.64	5.58
2112	3.0	3.43	3.95	10.87
5320	3.0	3.63	4.3	16.28

Theorem 2.

$$\frac{h}{64} I = M_K + \mathrm{O}(h^2). \tag{16}$$

A corollary to Theorem 2 is that the diagonal approximation in Eq. (16) yields the well-known Yee's method which is totally consistent with the vector wave equation [13].

7.2. Non-uniform grid

We now examine the edge element mass matrices based upon a nonuniform grid. In this case, no consistent mass lumping procedure is known to exist and matrix inversion of the mass matrix is necessary to use the Galerkin procedure. In this section, we examine the condition numbers of the preconditioned mass matrices to determine if a scalable preconditioned conjugate gradient method exists.

The nonuniform grids were constructed by recursively forming four new quadrilaterals out of one initial quadrilateral. Along each edge of the quad, a random position is chosen using: $x_{\text{newnode}} = sx_{\text{node } i} + (1 - s)x_{\text{node } j}$, $y_{\text{newnode}} = sy_{\text{node } i} + (1 - s)y_{\text{node } j}$, where s is defined by a user chosen variable f as $s = f + (1 - 2f)\,\text{rand}(\cdot)$ and rand($\cdot$) is a random number between 0 and 1. These four new nodes are used to define the center by finding random positions between the new left and right nodes, as well as the new top and bottom nodes, thus giving four new quadrilaterals. This operation is performed on each new quadrilateral until the desired number of elements is reached (see Fig. 2).

Tables 2 and 3 list the condition number of the unconditioned and diagonally preconditioned mass matrices. Inner product integrations were performed using a four-point Gaussian quadrature rule.

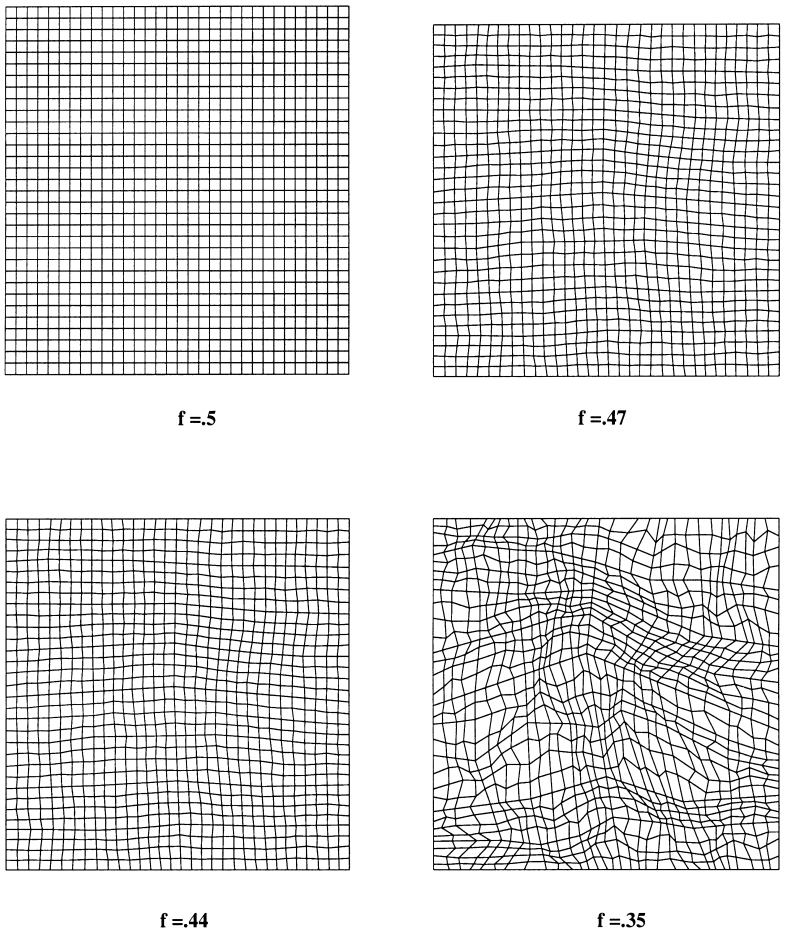

Fig. 2. 32×32 numerical grids for different f values.

Table 4
Number of preconditioned conj. grad. iterations for $f = 0.44$

n	Jacobi	ILU
144	14	5
544	15	6
2112	16	6
33024	16	7

Table 4 list the number of iterations for convergence of the preconditioned conjugate gradient algorithm for the mass matrices generated on the unstructured grids generated when $f = 0.44$. The preconditioners used were Jacobi diagonal scaling and the Incomplete-LU.

As one can be seen from the condition number computations for $f = 0.35$ in Table 3, the condition number of the preconditioned does not seem to be approaching a constant as would be hoped. The

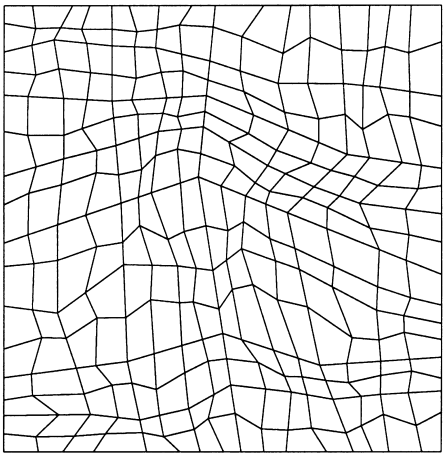

Fig. 3. 16×16 numerical grid for $f = 0.35$.

Table 5
Ratio of maximum zone area to minimum zone area for different grid sizes

n	$f = 0.47$	$f = 0.44$	$f = 0.35$
40	0.86	0.72	0.44
144	0.77	0.59	0.24
544	0.69	0.47	0.12
2112	0.62	0.37	0.07
5320	0.54	0.29	0.03
33024	0.48	0.23	0.01

Table 6
Ratio of maximum edge length to minimum edge length for different grid sizes

n	$f = 0.47$	$f = 0.44$	$f = 0.35$
40	0.856	0.733	0.441
144	0.796	0.626	0.293
544	0.719	0.509	0.172
2112	0.657	0.424	0.107
5320	0.597	0.351	0.069
33024	0.547	0.294	0.42

reason for this is that, unlike uniform grids, the fundamental structure of the grid is not the same as the number of grid points is increased, compare the grids in Figs. 2 and 3. This becomes evident when one compares the zone sizes and edge lengths of the different grids. Tables 5 and 6 list ratios of maximum to minimum zone sizes and edge lengths of the different grid sizes. In this case, n refers to the number of edges in the grid.

In order to determine if a result holds that is similar to the Ciarlet Theorem for nodal finite elements, the previous computations were carried out on a sequence of grids whose diameters are

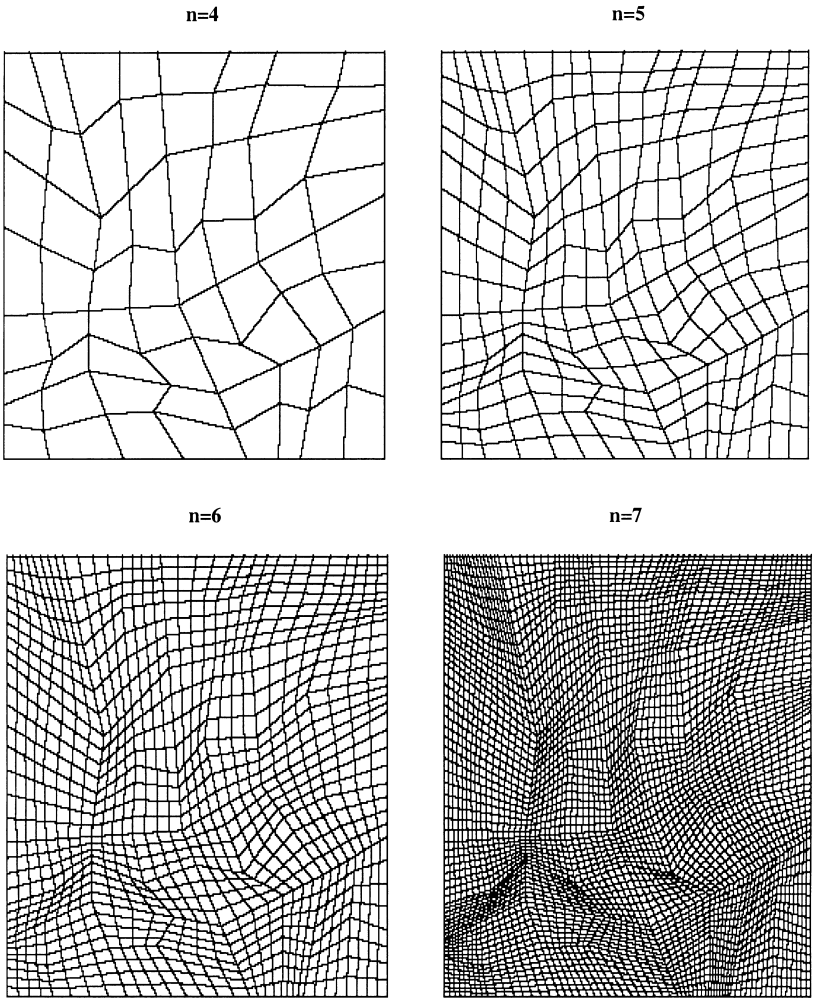

Fig. 4. $2^{n-1}+1 \times 2^{n-1}+1$ grids.

related (see Fig. 4). The initial coarse grid was constructed using a seed of $f = 0.35$. Mesh metrics, condition numbers and preconditioned conjugate gradient iterations are given in Tables 7–9, respectively.

8. Conclusions

In this paper we have established computationally that the condition number of the diagonally preconditioned mass edge element matrix essentially remains constant as the size of a grid increases provided the ratio of the mesh lengths remains constant. This is useful when the preconditioned conjugate gradient algorithm is used to invert the edge element mass matrix in Galerkin procedures for solving Maxwell's equations.

Table 7
Mesh metrics

n	$\frac{\max(\text{area})}{\min(\text{area})}$	$\frac{\max(\text{diam})}{\min(\text{diam})}$
4	0.237794	0.293104
5	0.217542	0.293099
6	0.207517	0.293091
7	0.197316	0.293072
8	0.192447	0.293035

Table 8
Condition numbers of mass matrix M and diagonally preconditioned matrix $Q^{-1}M$

n	$\kappa(M)$	$\kappa(Q^{-1}M)$
4	20.29	4.44
5	25.32	4.56
6	30.05	4.71
7	34.1	4.91
8	36.75	5.17

Table 9
Number of iterations for diagonally scaled conj. grad. and ILU conj. grad.

n	Jacobi	ILU
4	17	7
5	17	8
6	18	9
7	18	9
8	18	9

Acknowledgements

The authors would like to thank the Institute for Scientific Computing Research at the Lawrence Livermore National Laboratory for partial support of this research.

References

[1] A. Axelsson, V. Barker, Finite Element Solution of Boundary Value Problems, Academic Press, New York, 1984.
[2] P.E. Bjorstad, O.B. Widlund, Iterative methods for the solution of elliptic problems on regions partitioned into substructures, SIAM J. Numer. Anal. 23(6) (1986) 1097–1120.
[3] A. Brandt, Multilevel adaptive solutions to boundary value problems, Math. Comput. 31 (1977) 333–390.
[4] P. Ciarlet, The Finite Element Methods for Elliptic problems, North-Holland, Amsterdam, 1987.
[5] P. Davis, Interpolation and Approximation, Blaisdell, Waltham, MA, 1963.
[6] J. Demmel, The condition number of equivalence transformations that block diagonalize matrix pencils, SIAM J. Numer. Anal. 20 (1983) 599–610.
[7] S. Eisenstat, J. Lewis, M. Schultz, Optimal block diagonal scaling of blick 2-cyclic matrices, Linear Algebra Appl. 44 (1982) 181–186.

[8] G.E. Forsythe, E.G. Strauss, On best conditioned matrices, Proc. Amer. Math. Soc. 6 (1955) 340–345.
[9] G. Golub, C. Van Loan, Matrix Computations, John Hopkins University Press, Baltimore, 1983.
[10] A. Greenbaum, Comparison of splitting used with the conjugate gradient method, Numer. Math. d 33 (1979) 181–194.
[11] A. Greenbaum, G. Rodrigue, Optimal preconditioners of a given sparsity pattern, BIT 29 (1989) 610–634.
[12] J.D. Jackson, Classical Electrodynamics, Wiley, New York, 1975.
[13] J. Jin, The Finite Element Method in Electromagnetics, Wiley, New York, 1993.
[14] C. Johnson, Numerical Solution of Partial Differential Equations by the Finite Element Method, Cambridge University Press, Cambridge, 1987.
[15] L. Lapidus, G. Pinder, Numerical Solution of Partial Differential Equations in Science and Engineering, Wiley, New York, 1982.
[16] J.C. Nedelec, Mixed finite elements in R3, Numer. Math. 35 (1980) 315–341.
[17] J.C. Nedelec, A new family of mixed finite elements in R3, Numer. Math. 50 (1986) 57–81.
[18] G. Rodrigue, D. White, A vector finite element time-domain method for solving Maxwell's equations on unstructured hexahedral grids, submitted for publication.
[19] A. Van der Sluis, Condition numbers and equilibration matrices, Numer. Math. 14 (1969) 14–23.
[20] D. White, G. Rodrigue, Improved vector FEM solutions of Maxwell's equations using grid pre-conditioning, Internat. J. Numer. Methods Eng. 40 (1997) 3815–3837.
[21] H. Whitney, Geometric Integration Theory, Princeton University Press, Princeton, NJ, 1957.
[22] D.M. Young, Iteration Solution of Large Linear Systems, Academic Press, New York, 1971.