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Convergence of two-dimensional branching recursions

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Abstract

The asymptotic distribution of branching type recursions (L_n) of the form $L_n \stackrel{\mathrm{d}}{=} A\ L_{n-1} + B\ \bar{L}_{n-1}$ is investigated in the two-dimensional case. Here $\bar{L}_{n-1}$ is an independent copy of L_{n-1} and A, B are random matrices jointly independent of $L_{n-1}, \bar{L}_{n-1}$. The asymptotics of L_n after normalization are derived by a contraction method. The limiting distribution is characterized by a fixed point equation. The assumptions of the convergence theorem are checked in some examples using eigenvalue decompositions and computer algebra. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

The aim of this paper is to analyse the asymptotics of the two-dimensional branching type recursive sequence

$$L_n \stackrel{\mathrm{d}}{=} A\ L_{n-1} + B\ \bar{L}_{n-1}, \tag{1.1}$$

where $A = (A_{i,j})$, $B = (B_{i,j})$ are random 2×2 matrices, $\bar{L}_{n-1}$ is an independent copy of L_{n-1} and $\{A, B\}$ are independent of $\{L_{n-1}, \bar{L}_{n-1}\}$. The one-dimensional case of branching type recursions has been studied by Kahane and Peyrière [12], Holley and Liggett [11], Durrett and Liggett [9], Guivarch [10], Rösler [16], and Rachev and Rüschendorf [15]. The case with immigration has been investigated in Cramer and Rüschendorf [8] and Cramer [7]. Several applications of these recursions to iterated function systems, fractal sets, a turbulence model and others can be found there. The multivariate

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case of random affine mappings is also well investigated (see [1–5,13]). In comparison the branching part in (1.1) introduces an additional difficulty by the noncommutativity of the products of A, B. It turns out that the contraction technique developed for the analysis of algorithms in Rachev and Rüschendorf [15] and in Rösler [16] can be applied to branching type recursions. The paper is based on parts of the dissertation of Cramer [6] where details of the calculations can be found.

To calculate the mean and covariances of L_n define

$$\begin{aligned} \ell_n &:= EL_n = (\ell_n^{(1)}, \ell_n^{(2)}), \qquad a = EA, \quad b = EB, \\ C &:= a + b = (C_{ij}), \end{aligned} \tag{1.2}$$

and assume that L_0 has second moments. Then we obtain

$$\ell_n = C^n \ell_0. \tag{1.3}$$

For the calculation of the covariance matrix $\vartheta_n := \mathrm{Cov}(L_n)$ it is useful to vectorize this matrix. We introduce the relevant vector $\boldsymbol{\vartheta}_n$ by

$$\begin{aligned} \boldsymbol{\vartheta}_n &= (\mathrm{Var}(L_n^{(1)}), \mathrm{Cov}(L_n^{(1)}, L_n^{(2)}), \mathrm{Var}(L_n^{(2)}))^{\mathrm{T}} \\ &= (\vartheta_{n,1}, \vartheta_{n,2}, \vartheta_{n,3})^{\mathrm{T}} \end{aligned} \tag{1.4}$$

and the corresponding squared expectation vector

$$\boldsymbol{\ell}_n = ((\ell_n^{(1)})^2, \ell_n^{(1)}\ell_n^{(2)}, (\ell_n^{(2)})^2)^{\mathrm{T}} \tag{1.5}$$

Proposition 1.1.
(a) $\boldsymbol{\ell}_n = N^n \boldsymbol{\ell}_0$ *with*

$$N := \begin{pmatrix} C_{11}^2 & 2C_{11}C_{12} & C_{12}^2 \\ C_{11}C_{21} & C_{11}C_{22} + C_{12}C_{21} & C_{12}C_{22} \\ C_{21}^2 & 2C_{21}C_{22} & C_{22}^2 \end{pmatrix} \tag{1.6}$$

(b) $\boldsymbol{\vartheta}_n = M_2 \boldsymbol{\vartheta}_{n-1} + M_\vartheta \boldsymbol{\ell}_{n-1}$ *with*

$$M_2 = \begin{pmatrix} EA_{11}^2 + EB_{11}^2 & 2E(A_{11}A_{12}) + 2E(B_{11}B_{12}) & EA_{12}^2 + EB_{12}^2 \\ E(A_{11}A_{21}) & E(A_{11}A_{12}) + E(A_{12}A_{22}) & E(A_{12}A_{22}) \\ \quad +E(B_{11}B_{21}) & \quad +E(B_{11}B_{22}) + E(B_{12}B_{21}) & \quad +E(B_{12}B_{22}) \\ EA_{21}^2 + EB_{21}^2 & 2E(A_{21}A_{22}) + 2E(B_{21}B_{22}) & EA_{22}^2 + EB_{22}^2 \end{pmatrix}$$

and

$$M_\vartheta = \begin{pmatrix} \mathrm{Var}(A_{11} + B_{11}) & 2\mathrm{Cov}(A_{11} + B_{11}, A_{12} + B_{12} & \mathrm{Var}(A_{12} + B_{12}) \\ \mathrm{Cov}(A_{11} + B_{11}, & \mathrm{Cov}(A_{11} + B_{11}, A_{22} + B_{22}) & \mathrm{Cov}(A_{12} + B_{12}, \\ \quad A_{21} + B_{21}) & \quad +\mathrm{Cov}(A_{12} + B_{12}, A_{21} + B_{21}) & \quad A_{22} + B_{22}) \\ \mathrm{Var}(A_{21} + B_{21}) & 2\mathrm{Cov}(A_{21} + B_{21}, A_{22} + B_{22}) & \mathrm{Var}(A_{22} + B_{22}) \end{pmatrix}. \tag{1.7}$$

Furthermore, with $M := M_\vartheta + N - M_2$

$$\begin{aligned}\boldsymbol{\vartheta}_n &= M_2^n \boldsymbol{\vartheta}_0 + \sum_{i=1}^n M_2^{i-1} M_\vartheta N^{n-i} \boldsymbol{\ell}_0 \\ &= M_2^n(\boldsymbol{\vartheta}_0 + \boldsymbol{\ell}_0) - N^n \boldsymbol{\ell}_0 + \sum_{i=1}^n M_2^{i-1} M N^{n-i} \boldsymbol{\ell}_0. \end{aligned} \tag{1.8}$$

Proof. (a) and (b) follow by straightforward calculation using $\mathrm{Cov}(UX, VY) = EUV\,\mathrm{Cov}(X,Y) + \mathrm{Cov}(U,V)\,EX\,EY$ for real r.v. such that $\{U,V\}$, $\{X,Y\}$ are independent. □

An equivalent form of M is

$$M = \begin{pmatrix} 2E(A_{11}B_{11}) & 2E(A_{11}B_{12}) + 2E(A_{12}B_{11}) & 2E(A_{12}B_{12}) \\ E(A_{11}B_{21}) & E(A_{11}B_{22}) + E(A_{22}B_{11}) & E(A_{12}B_{22}) \\ +E(A_{21}B_{11}) & +E(A_{12}B_{21}) + E(A_{21}B_{12}) & +E(A_{22}B_{12}) \\ 2E(A_{21}B_{21}) & 2E(A_{21}B_{22}) + 2E(A_{22}B_{21}) & 2E(A_{22}B_{22}) \end{pmatrix}. \tag{1.9}$$

2. Limit theorem for L_n

The aim of this section is to prove convergence of a standardized version $\tilde{L}_n$ of L_n where

$$\tilde{L}_n = \left(\frac{L_n^{(i)} - \ell_n^{(i)}}{\sqrt{\mathrm{Var}\, L_n^{(i)}}} \right) = (\tilde{L}_n^{(i)}) \tag{2.1}$$

(we assume that for $n \geqslant n_0, \mathrm{Var}\, L_n^{(i)} > 0$). Then

$$\mathrm{Cov}(\tilde{L}_n) = \begin{pmatrix} 1 & \varrho_n \\ \varrho_n & 1 \end{pmatrix} \tag{2.2}$$

where

$$\varrho_n := \frac{\mathrm{Cov}\,(L_n^{(1)}, L_n^{(2)})}{\sqrt{\mathrm{Var}\,(L_n^{(1)})\ \mathrm{Var}\,(L_n^{(2)})}}$$

is the correlation coefficient of $L_n^{(1)}$, $L_n^{(2)}$. $(\tilde{L}_n)$ satisfies the modified recursion

$$\tilde{L}_n \stackrel{\mathrm{d}}{=} A_n \tilde{L}_{n-1} + B_n L_{n-1}^* + V_n A\ \ell_{n-1} + V_n B \ell_{n-1} - V_n \ell_n \tag{2.3}$$

where L_{n-1}^* is an independent copy of $\tilde{L}_{n-1}$ and with

$$V_n = \begin{pmatrix} (\mathrm{Var}\, L_n^{(1)})^{-1/2} & 0 \\ 0 & (\mathrm{Var}\, L_n^{(2)})^{-1/2} \end{pmatrix}.$$

$$
\begin{aligned}
A_n &= V_n A V_{n-1}^{-1}\\
&= \begin{pmatrix} \sqrt{\dfrac{\operatorname{Var} L_{n-1}^{(1)}}{\operatorname{Var} L_n^{(1)}}} A_{11} & \sqrt{\dfrac{\operatorname{Var} L_{n-1}^{(2)}}{\operatorname{Var} L_n^{(1)}}} A_{12} \\[3ex] \sqrt{\dfrac{\operatorname{Var} L_{n-1}^{(1)}}{\operatorname{Var} L_n^{(2)}}} A_{21} & \sqrt{\dfrac{\operatorname{Var} L_{n-1}^{(2)}}{\operatorname{Var} L_n^{(2)}}} A_{22} \end{pmatrix}
\end{aligned}
$$

and similarly

$$B_n = V_n B V_{n-1}^{-1}$$

and (A,B), $\tilde{L}_{n-1}$, L_{n-1}^* are stochastically independent. The modified recursion implies for the coefficients of $\tilde{L}_n$

$$
\begin{aligned}
\tilde{L}_n^{(1)} \stackrel{\mathrm{d}}{=}& \sqrt{\frac{\operatorname{Var} L_{n-1}^{(1)}}{\operatorname{Var} L_n^{(1)}}} A_{11} \tilde{L}_{n-1}^{(1)} + \sqrt{\frac{\operatorname{Var} L_{n-1}^{(2)}}{\operatorname{Var} L_n^{(1)}}} A_{12} \tilde{L}_{n-1}^{(2)}\\
&+ \sqrt{\frac{\operatorname{Var} L_{n-1}^{(1)}}{\operatorname{Var} L_n^{(1)}}} B_{11} (L_{n-1}^*)^{(1)} + \sqrt{\frac{\operatorname{Var} L_{n-1}^{(2)}}{\operatorname{Var} L_n^{(1)}}} B_{12} (L_{n-1}^*)^{(2)}\\
&+ \frac{1}{\sqrt{\operatorname{Var} L_n^{(1)}}} (A_{11} \ell_{n-1}^{(1)} + A_{12} \ell_{n-1}^{(2)} - \ell_n^{(1)} + B_{11} \ell_{n-1}^{(1)} + B_{12} \ell_{n-1}^{(2)})
\end{aligned}
\tag{2.4}
$$

and

$$
\begin{aligned}
\tilde{L}_n^{(2)} \stackrel{\mathrm{d}}{=}& \sqrt{\frac{\operatorname{Var} L_{n-1}^{(1)}}{\operatorname{Var} L_n^{(2)}}} (A_{21} \tilde{L}_{n-1}^{(1)} + B_{21} (L_{n-1}^*)^{(1)})\\
&+ \sqrt{\frac{\operatorname{Var} L_{n-1}^{(2)}}{\operatorname{Var} L_n^{(2)}}} (A_{22} \tilde{L}_{n-1}^{(2)} + B_{22} (L_{n-1}^*)^{(2)})\\
&+ \frac{1}{\sqrt{\operatorname{Var} L_n^{(2)}}} (A_{21} \ell_{n-1}^{(1)} + A_{22} \ell_{n-1}^{(2)} + B_{21} \ell_{n-1}^{(1)} + B_{22} \ell_{n-1}^{(2)} - \ell_n^{(2)}).
\end{aligned}
\tag{2.5}
$$

For convergence of $\tilde{L}_{n-1}^{(i)}$ to hold we would expect that the coefficients in (2.4) and (2.5) should converge. We therefore assume existence of the following limits:

Assumption A.

$$\begin{aligned}
&(1)\ c_{11}:=\lim_{n\to\infty}\sqrt{(\operatorname{Var}L_{n-1}^{(1)}/\operatorname{Var}L_n^{(1)})}>0,\\
&(2)\ \beta:=\lim_{n\to\infty}\sqrt{(\operatorname{Var}L_{n}^{(1)}/\operatorname{Var}L_n^{(2)})}>0,\\
&(3)\ c_{\ell 1}:=\lim(\ell_n^{(1)}/\sqrt{\operatorname{Var}L_n^{(1)}}),\\
&(4)\ c_{\ell 2}=\lim(\ell_n^{(2)}/\sqrt{\operatorname{Var}L_n^{(2)}}),\\
&(5)\ \varrho=\lim\varrho_n.
\end{aligned}\tag{2.6}$$

Remark. Assumptions (A1)–(A5) imply existence of

$$\begin{aligned}
c_{22}&:=\lim\sqrt{(\operatorname{Var}L_{n-1}^{(2)}/\operatorname{Var}L_n^{(2)})}=c_{11},\\
c_{12}&:=\lim\sqrt{(\operatorname{Var}L_{n-1}^{(2)}/\operatorname{Var}L_n^{(1)})}=\frac{c_{11}}{\beta},\\
c_{21}&:=\lim\sqrt{(\operatorname{Var}L_{n-1}^{(1)}/\operatorname{Var}L_n^{(2)})}=\beta c_{22}.
\end{aligned}\tag{2.7}$$

The limiting values of A_n, B_n then are given by

$$A_\infty:=\begin{pmatrix} c_{11}A_{11} & c_{12}A_{12}\\ c_{21}A_{21} & c_{22}A_{22}\end{pmatrix}=(\bar{A}_{ij}),$$

$$B_\infty:=\begin{pmatrix} c_{11}B_{11} & c_{12}B_{12}\\ c_{21}B_{21} & c_{22}B_{22}\end{pmatrix}=(\bar{B}_{ij}).$$

Therefore, by passing in (2.3) to the limit as $n\to\infty$ one obtains a limiting equation of the form

$$G\stackrel{\mathrm{d}}{=}T(G),\tag{2.8}$$

where G is the distribution of $\tilde{L}_\infty$ and the limiting operator T is defined by

$$T(F)\stackrel{\mathrm{d}}{=}A_\infty V+B_\infty\bar{V}+(A_\infty+B_\infty-I_2)\begin{pmatrix}c_{\ell 1}\\ c_{\ell 2}\end{pmatrix}\tag{2.9}$$

where $V\stackrel{\mathrm{d}}{=}F$, $\bar{V}$ an independent copy of V. The limiting distribution is a fixed point of the operator T. We use in the following also the notation $T(X)$ for $T(F)$ if $X\stackrel{\mathrm{d}}{=}F$.

The proper domain of the limiting operator T is

$$M_{0,2}:=\{F;F \text{ is a distribution on } (\mathbb{R}^2,\mathscr{B}^2) \text{ with expectation 0 and } F \text{ is square integrable}\}.\tag{2.10}$$

Proposition 2.1. *For $G\in M_{0,2}$ holds $T(G)\in M_{0,2}$*, i.e. $T: M_{0,2}\to M_{0,2}$.

Proof. Using independence properties of $V, \bar{V}$, (A_∞, B_∞) we obtain

$$E(T(G)) = EA_\infty EV + EB_\infty E\bar{V} + E(A_\infty + B_\infty - I_2)\begin{pmatrix} c_{\ell 1} \\ c_{\ell 2} \end{pmatrix}$$
$$= \begin{pmatrix} (c_{11}EA_{11} + c_{11}EB_{11} - 1)c_{\ell 1} + (c_{12}EA_{12} + c_{12}EB_{12})c_{\ell 2} \\ (c_{21}EA_{21} + c_{21}EB_{21})c_{\ell 1} + (c_{22}EA_{22} + c_{22}EB_{22} - 1)c_{\ell 2} \end{pmatrix}, \tag{2.11}$$

as $EV = E\bar{V} = 0$. Further

$$\ell_n^{(1)} = (a_{11} + b_{11})\ell_{n-1}^{(1)} + (a_{12} + b_{12})\ell_{n-1}^{(2)}$$

implies

$$\frac{\ell_{n-1}^{(1)}}{\sqrt{\operatorname{Var} L_{n-1}^{(1)}}}\sqrt{\frac{\operatorname{Var} L_{n-1}^{(1)}}{\operatorname{Var} L_n^{(1)}}}(a_{11} + b_{11}) + \frac{\ell_{n-1}^{(2)}}{\sqrt{\operatorname{Var} L_{n-1}^{(2)}}}\sqrt{\frac{\operatorname{Var} L_{n-1}^{(2)}}{\operatorname{Var} L_n^{(1)}}}(a_{12} + b_{12}) - \frac{\ell_n^{(1)}}{\sqrt{\operatorname{Var} L_n^{(1)}}} = 0.$$

In the limit for $n \to \infty$ this yields $c_{\ell 1}c_{11}(a_{11}+b_{11}) + c_{\ell 2}c_{12}(a_{12}+b_{12}) - c_{\ell 1} = 0$, i.e., the first component in (2.11) is zero. Similarly, the second component is zero. The square integrability of $T(G)$ follows by direct calculation. □

For the analysis of the operator T which describes the limiting equation we use the minimal ℓ_2-metric defined by

$$\ell_2(F, G) = \inf\{(E\,\|X - Y\|^2)^{1/2};\ X \stackrel{d}{=} F,\ Y = G\} \tag{2.12}$$

where $\|\,\|$ denotes the euclidean metric. ℓ_2 defines a complete metric on $M_{0,2}$ and convergence w.r.t. ℓ_2 is equivalent to weak convergence plus convergence of second moments (see [14]). We introduce the following notation:

$$e := E[\bar{A}_{11}^2 + \bar{B}_{11}^2 + \bar{A}_{21}^2 + \bar{B}_{21}^2],$$
$$f := E[\bar{A}_{11}\bar{A}_{12} + \bar{B}_{11}\bar{B}_{12} + \bar{A}_{21}\bar{A}_{22} + \bar{B}_{21}\bar{B}_{22}],$$
$$g := E[\bar{A}_{12}^2 + \bar{B}_{12}^2 + \bar{A}_{22}^2 + \bar{B}_{22}^2].$$

Proposition 2.2 (Contraction property of T). *For $F, G \in M_{0,2}$ holds*

$$\ell_2(T(F), T(G)) \leqslant \sqrt{\gamma}\,\ell_2(F, G) \tag{2.13}$$

where

$$\gamma := \sup_{\substack{\mathscr{L}(U),\mathscr{L}(V)\in M_{0,2} \\ \mathscr{L}(U)\neq\mathscr{L}(V)}} \frac{eE(U_1 - V_1)^2 + 2fE(U_1 - V_1)(U_2 - V_2) + gE(U_2 - V_2)^2}{E(U_1 - V_1)^2 + E(U_2 - V_2)^2}. \tag{2.14}$$

Proof. For $F, G \in M_{0,2}$ let $U \stackrel{d}{=} F,\ V \stackrel{d}{=} G$ be chosen such that

$$\ell_2^2(F, G) = E\,\|U - V\|^2 = E(U_1 - V_1)^2 + E(U_2 - V_2)^2.$$

Also let $(\bar U,\bar V)\overset{\mathrm{d}}{=}(U,V)$ and let $(\bar U,\bar V),(U,V),(A_\infty,B_\infty)$ be independent. Then using the definition of ℓ_2,γ and the independence properties

$$\begin{aligned}\ell_2^2(T(F),T(G)) &\leqslant L_2\left(A_\infty U+B_\infty\bar U+(A_\infty+B_\infty-I_2)\begin{pmatrix}c_{\ell 1}\\ c_{\ell 2},\end{pmatrix},\right.\\ &\qquad\left. A_\infty V+B_\infty\bar V+(A_\infty+B_\infty-I_2)\begin{pmatrix}c_{\ell 1}\\ c_{\ell 2}\end{pmatrix}\right)\\ &= E\,\|A_\infty(U-V)+B_\infty(\bar U-\bar V)\|^2\\ &= eE(U_1-V_1)^2+2fE(U_1-V_1)((U_2-V_2)+gE(U_2-V_2)^2\\ &\leqslant \gamma(E(U_1-V_1)^2+E(U_2-V_2)^2)\\ &= \gamma\ell_2^2(F,G). \qquad\square\end{aligned}$$

In consequence of Proposition 2.2 there exists exactly one fixed point $F^*\in M_{0,2}$ of T in $M_{0,2}$ if $\gamma<1$ and the iteration (T^nF) converges exponentially fast to F^* for any $F\in M_{0,2}$. In the next step we determine the covariance matrix of the fixed point distribution F^*.

Proposition 2.3. *If $\gamma<1$ and $Z\overset{\mathrm{d}}{=}F^*\in M_{0,2}$ then*

$$\mathrm{Cov}(Z)=\begin{pmatrix}1 & \varrho\\ \varrho & 1\end{pmatrix}. \tag{2.15}$$

Proof. The proof is given in two steps:

Step 1: If $G\in M_{0,2}$ and $V\overset{\mathrm{d}}{=}G$,

$$\mathrm{Cov}\,V=\begin{pmatrix}1 & \varrho\\ \varrho & 1\end{pmatrix}$$

then

$$\mathrm{Cov}\,\tilde V=\begin{pmatrix}1 & \varrho\\ \varrho & 1\end{pmatrix}$$

for $\tilde V\overset{\mathrm{d}}{=}T(G)$. For the proof of Step 1 we verify the following formulas using (2.6) (for details see [6]):

(a) $c_{11}^2[EA_{11}^2+EB_{11}^2]+c_{12}^2[EA_{12}+EB_{12}^2]+2\varrho c_{11}c_{12}[E(A_{11}A_{12})+E(B_{11}B_{12})]$

$$\begin{aligned}&=\lim_{n\to\infty}\frac{1}{\mathrm{Var}\,L_n^{(1)}}[\mathrm{Var}\,L_{n-1}^{(1)}(EA_{11}^2+EB_{11}^2)+\mathrm{Var}\,L_{n-1}^{(2)}(EA_{12}^2+EB_{12}^2)\\ &\qquad +2\,\mathrm{Cov}(L_{n-1}^{(1)},L_{n-1}^{(2)})[E(A_{11}A_{12})+E(B_{11}B_{12})]],\end{aligned} \tag{2.16}$$

(b) $c_{\ell 1}^2[\bar A_{11}+\bar B_{11}]^2+2c_{\ell 1}c_{\ell 2}E(\bar A_{11}+\bar B_{11})(\bar A_{12}\bar B_{12})+c_{\ell 2}^2E(\bar A_{12}+B_{12})^2$

$$\begin{aligned}&=\lim_{n\to\infty}\frac{1}{\mathrm{Var}\,L_n^{(1)}}[\ell_{n-1}^{(1)2}E(A_{11}+B_{11})^2+2\ell_{n-1}^{(1)}\ell_{n-1}^{(2)}E(A_{11}+B_{11})(A_{12}+B_{12})\\ &\qquad +\ell_{n-1}^{(2)2}E(A_{12}+B_{12})^2],\end{aligned} \tag{2.17}$$

(c) $c_{\ell 1}^2(2E\bar{A}_{11} + 2E\bar{B}_{11} - 1) + 2c_{\ell 1}c_{\ell 2}E(\bar{A}_{12} + \bar{B}_{12})$

$$= \lim_{n\to\infty} \frac{1}{\operatorname{Var} L_n^{(1)}} [\ell_{n-1}^{(1)2}[E(A_{11} + B_{11})]^2 + 2\ell_{n-1}^{(1)}\ell_{n-1}^{(2)}E(A_{11} + B_{11})E(A_{12} + B_{12}) + \ell_{n-1}^{(2)2}[E(A_{12} + B_{12})]^2]. \tag{2.18}$$

(a)–(c) imply

$$\begin{aligned}
1 &= \lim_{n\to\infty} \frac{\operatorname{Var} L_n^{(1)}}{\operatorname{Var} L_n^{(1)}} \\
&= \lim_{n\to\infty} \frac{1}{\operatorname{Var} L_n^{(1)}} [\operatorname{Var} L_{n-1}^{(1)}(EA_{11}^2 + EB_{11}^2) + \operatorname{Var} L_{n-1}^{(2)}(EA_{12}^2 + EB_{12}^2) \\
&\qquad + 2\operatorname{Cov}(L_{n-1}^{(1)}, L_{n-1}^{(2)})[E(A_{11}A_{12}) + E(B_{11}B_{12})] \\
&\qquad + \ell_{n-1}^{(1)2}\operatorname{Var}(A_{11} + B_{11}) + \ell_{n-1}^{(1)2}\operatorname{Var}(A_{12} + B_{12}) \\
&\qquad + 2\ell_{n-1}^{(1)}\ell_{n-1}^{(2)}\operatorname{Cov}(A_{11} + B_{11}, A_{12} + B_{12})] \\
&= c_{11}^2(EA_{11}^2 + EB_{11}^2) + 2\varrho c_{11}c_{12}E(A_{11}A_{12}) + E(B_{11}B_{12}) + c_{12}^2(EA_{12}^2 + EB_{12}^2) \\
&\quad + c_{\ell 1}^2 E(\bar{A}_{11} + \bar{B}_{11})^2 + 2c_{\ell 1}c_{\ell 2}E(\bar{A}_{11} + \bar{B}_{11})(\bar{A}_{12} + \bar{B}_{12}) \\
&\quad + c_{\ell 2}^2 E(\bar{A}_{12} + \bar{B}_{12})^2 - [c_{\ell 1}^2(2E\bar{A}_{11} + 2E\bar{B}_{11} - 1) + 2c_{\ell 1}c_{\ell 2}E(\bar{A}_{12} + \bar{B}_{12})] \\
&= E\tilde{V}_1^2 \qquad \text{by direct calculation}
\end{aligned}$$

Similarly, $E\tilde{V}_2^2 = 1$. To prove $E\tilde{V}_1\tilde{V}_2 = \varrho$, observe that by simple calculus

$$\begin{aligned}
E(\tilde{V}_1, \tilde{V}_2) &= c_{11}c_{21}E(A_{11}A_{21} + B_{11}B_{21}) + c_{12}c_{22}E(A_{12}A_{22} + B_{12}B_{22}) \\
&\quad + \varrho c_{11}c_{22}E(A_{11}A_{22} + B_{11}B_{22}) + \varrho c_{12}c_{21}E(A_{12}A_{21} + B_{12}B_{21}) \\
&\quad + c_{\ell 1}^2 c_{11}c_{21}E(A_{11} + B_{11})(A_{21} + B_{21}) \\
&\quad + c_{\ell 2}^2 c_{12}c_{22}E(A_{12} + B_{12})(A_{22} + B_{22}) \\
&\quad + c_{\ell 1}c_{\ell 2}c_{12}c_{21}E(A_{12} + B_{12})(A_{21} + B_{21}) \\
&\quad + c_{\ell 1}c_{\ell 2}c_{11}c_{22}E(A_{11} + B_{11})(A_{22} + B_{22}) \\
&\quad + c_{\ell 1}c_{\ell 2}[1 - E(\bar{A}_{11} + \bar{B}_{11} + \bar{A}_{22} + \bar{B}_{22})] \\
&\quad - c_{\ell 1}^2 E(\bar{A}_{21} + \bar{B}_{21}) - c_{\ell 2}^2 E(\bar{A}_{12} + \bar{B}_{12}).
\end{aligned} \tag{2.19}$$

The statement $E\tilde{V}_1\tilde{V}_2 = \varrho$ is a consequence of the following three formulas which are consequences of (2.6):

(a) $c_{11}c_{21}E(A_{11}A_{21} + B_{11}B_{21}) + c_{12}c_{22}E(A_{12}A_{22} + B_{12}B_{22})$
$+ \varrho c_{11}c_{22}E(A_{11}A_{22} + B_{11}B_{22}) + \varrho c_{12}c_{21}E(A_{12}A_{21} + B_{12}B_{21})$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{\operatorname{Var} L_n^{(1)} \operatorname{Var} L_n^{(2)}}} [\operatorname{Var} L_{n-1}^{(1)}E(A_{11}A_{21} + B_{11}B_{21}) + \operatorname{Var} L_{n-1}^{(2)}E(A_{12}A_{22} + B_{12}B_{22}) + \operatorname{Cov}(L_{n-1}^{(1)}, L_{n-1}^{(2)})E(A_{11}A_{22} + B_{11}B_{22} + A_{12}A_{21} + B_{12}B_{21})], \tag{2.20}$$

(b) $c_{\ell 1}^2 c_{11}c_{21}E(A_{11}+B_{11})(A_{21}+B_{21})+c_{\ell 2}^2 c_{12}c_{22}E(A_{12}+B_{12})(A_{22}+B_{22})$

$$\begin{aligned}
&+c_{\ell 1}c_{\ell 2}c_{11}c_{22}E(A_{11}+B_{11})(A_{22}+B_{22})+c_{\ell 1}c_{\ell 2}c_{12}c_{21}E(A_{12}+B_{12})(A_{21}+B_{21})\\
&\quad=\lim_{n\to\infty}\frac{1}{\sqrt{\operatorname{Var}L_n^{(1)}\operatorname{Var}L_n^{(2)}}}[\ell_{n-1}^{(1)2}E(A_{11}+B_{11})(A_{21}+B_{21})+\ell_{n-1}^{(2)2}E(A_{12}+B_{12})(A_{22}+B_{22})\\
&\qquad+\ell_{n-1}^{(1)}\ell_{n-1}^{(2)}[E(A_{11}+B_{11})(A_{22}+B_{22})\\
&\qquad+E(A_{12}+B_{12})(A_{21}+B_{21})]] \qquad (2.21)
\end{aligned}$$

and

(c) $c_{\ell 1}c_{\ell 2}[1-E(\bar A_{11}+\bar B_{11}+\bar A_{22}+\bar B_{22})]-c_{\ell 1}^2E(\bar A_{21}+\bar B_{21})-c_{\ell 2}^2E(\bar A_{12}+\bar B_{12})$

$$\begin{aligned}
&=\lim_{n\to\infty}\frac{-1}{\sqrt{\operatorname{Var}L_n^{(1)}\operatorname{Var}L_n^{(2)}}}[\ell_{n-1}^{(1)2}(A_{11}+B_{11})(A_{21}+B_{21})\\
&\qquad+\ell_{n-1}^{(1)}\ell_{n-1}^{(2)}[E(A_{11}+B_{11})E(A_{22}+B_{22})\\
&\qquad+E(A_{12}+B_{12})E(A_{21}+B_{21})]\\
&\qquad+\ell_{n-1}^{(2)2}E(A_{12}+B_{12})E(A_{22}+B_{22})]. \qquad (2.22)
\end{aligned}$$

(a)–(c) imply using $\boldsymbol{\vartheta}_n=M_2\boldsymbol{\vartheta}_{n-1}+M_\vartheta N^{n-1}\ell_0$.

$$\begin{aligned}
\varrho&=\lim_{n\to\infty}\varrho_n\\
&=\lim_{n\to\infty}\frac{\operatorname{Cov}(L_n^{(1)},L_n^{(2)})}{\sqrt{\operatorname{Var}L_n^{(1)}\operatorname{Var}L_n^{(2)}}}\\
&=\lim_{n\to\infty}\frac{1}{\sqrt{\operatorname{Var}L_n^{(1)}\operatorname{Var}L_n^{(2)}}}\\
&\quad\cdot[\operatorname{Var}L_{n-1}^{(1)}E(A_{11}A_{21}+B_{11}B_{21})\\
&\qquad+\operatorname{Cov}(L_n^{(1)},L_n^{(2)})E(A_{11}A_{22}+A_{12}A_{21}+B_{11}B_{22}+B_{12}B_{21})\\
&\qquad+\operatorname{Var}L_{n-1}^{(2)}E(A_{12}A_{22}+B_{12}B_{22})\\
&\qquad+\ell_{n-1}^{(1)2}\operatorname{Cov}(A_{11}+B_{11},A_{21}+B_{21})\\
&\qquad+\ell_{n-1}^{(1)}\,\ell_{n-1}^{(2)}\\
&\qquad\quad\cdot[\operatorname{Cov}(A_{11}+B_{11},A_{22}+B_{22})+\operatorname{Cov}(A_{12}+B_{12},A_{21}+B_{21})]\\
&\qquad+\ell_{n-1}^{(2)2}\operatorname{Cov}(A_{12}+B_{12},A_{22}+B_{22})]\\
&=c_{11}c_{21}E(A_{11}A_{21}+B_{11}B_{21})+c_{12}c_{22}E(A_{12}A_{22}+B_{12}B_{22})\\
&\quad+\varrho c_{11}c_{22}E(A_{11}A_{22}+B_{11}B_{22})+\varrho c_{12}c_{21}E(A_{12}A_{21}+B_{12}B_{21})\\
&\quad+c_{\ell 1}^2c_{11}c_{21}E(A_{11}+B_{11})(A_{21}+B_{21})+c_{\ell 2}^2c_{12}c_{22}E(A_{12}+B_{12})(A_{22}+B_{22})\\
&\quad+c_{\ell 1}c_{\ell 2}c_{11}c_{22}E(A_{11}+B_{11})(A_{22}+B_{22})+c_{\ell 1}c_{\ell 2}c_{11}c_{22}E(A_{12}+B_{12})(A_{21}+B_{21})\\
&\quad+c_{\ell 1}c_{\ell 2}[1-E(\bar A_{11}+\bar B_{11}+\bar A_{22}+\bar B_{22})]-c_{\ell 1}^2\,E(\bar A_{21}+\bar B_{21})-c_{\ell 2}^2\,E(\bar A_{12}+\bar B_{12}). \qquad (2.23)
\end{aligned}$$

Comparing (2.23) with (2.19) yields $E\tilde{V}_1\tilde{V}_2 = \varrho$.

Step 2

$$\operatorname{Cov}(Z) = \begin{pmatrix} 1 & \varrho \\ \varrho & 1 \end{pmatrix}$$

for $Z \stackrel{\mathrm{d}}{=} F^*$.

To prove Step 2 let $H \in M_{0,2}$ and $W \stackrel{\mathrm{d}}{=} H$ such that

$$\operatorname{Cov}(W) = \begin{pmatrix} 1 & \varrho \\ \varrho & 1 \end{pmatrix}.$$

Then by Step 1

$$\operatorname{Cov}(T^n H) = \begin{pmatrix} 1 & \varrho \\ \varrho & 1 \end{pmatrix}$$

and $\ell_2(T^n H, F^*) \to 0$ which implies that

$$\operatorname{Cov} Z = \begin{pmatrix} 1 & \varrho \\ \varrho & 1 \end{pmatrix}. \qquad \square$$

In our main result we prove convergence of $\tilde{L}_n$ to Z the unique fixed point in distribution of the operator T defined in (2.10)

Theorem 2.4 (Convergence theorem). *Assume conditions* (A1)–(A5) *and let* $\gamma < 1$, *then*

$\ell_2(\tilde{L}_n, Z) \to 0$ *where Z is the unique fixed point of T in $M_{0,2}$.*

Proof. Let $\tilde{L}^*_{n-1}$ be an independent copy of $\tilde{L}_{n-1}$, $(\tilde{L}^*_{n-1}, Z) \stackrel{\mathrm{d}}{=} (L^*_{n-1}, \bar{Z})$ such that $E \parallel \tilde{L}^*_{n-1} - Z \parallel^2 = \ell_2^2(\tilde{L}_{n-1}, Z)$ and $(A_\infty, B_\infty), (\tilde{L}^*_{n-1}, Z)$, $(L^*_{n-1}, \bar{Z})$ are independent. Then by the recursion formula

$$\begin{aligned}
\ell_2^2(\tilde{L}_n, Z) &\leqslant E \left\| A_n \tilde{L}^*_{n-1} + B_n L^*_{n-1} + V_n A \ell_{n-1} + V_n B \ell_{n-1} - V_n \ell_n \right. \\
&\qquad \left. - \left[A_\infty Z + B_\infty \bar{Z} + (A_\infty + B_\infty - I_2) \begin{pmatrix} c_{\ell 1} \\ c_{\ell 2} \end{pmatrix} \right] \right\|^2 \\
&= E \left\| (A_n \tilde{L}^*_{n-1} - A_\infty Z) + (B_n L^*_{n-1} - B_\infty \bar{Z}) \right. \\
&\qquad \left. + \left[V_n [(A+B)\ell_{n-1} - \ell_n] - (A_\infty + B_\infty - I_2) \begin{pmatrix} c_{\ell 1} \\ c_{\ell 2} \end{pmatrix} \right] \right\|^2 \\
&\leqslant E \parallel A_n \tilde{L}^*_{n-1} - A_\infty Z \parallel^2 + E \parallel B_n L^*_{n-1} - B_\infty \bar{Z} \parallel^2 \\
&\qquad + E \left\| V_n [(A+B)\ell_{n-1} - \ell_n] - (A_\infty + B_\infty - I_2) \begin{pmatrix} c_{\ell 1} \\ c_{\ell 2} \end{pmatrix} \right\|^2 .
\end{aligned}$$

As in the proof of Proposition 2.2

$$\begin{aligned}
E \parallel A_\infty (\tilde{L}^*_{n-1} - Z) \parallel^2 + E \parallel B_\infty (L^*_{n-1} - \bar{Z}) \parallel^2 &= E \parallel A_\infty (\tilde{L}^*_{n-1} - Z) + B_\infty (\tilde{L}^*_{n-1} - \bar{Z}) \parallel^2 \\
&\leqslant \gamma \, \ell_2^2(\tilde{L}_{n-1}, Z).
\end{aligned}$$

This implies by the triangle inequality the reduction inequality:

$$\ell_2^2(\tilde{L}_n, Z) \leqslant \gamma \ell_2^2(\tilde{L}_{n-1}, Z) + b_n \tag{2.24}$$

where

$$b_n := E\,\|(A_n - A_\infty)\tilde{L}_{n-1}\|^2 + 2(R_{A_1} + R_{A_2}) + E\,\|(B_n - B_\infty)L^*_{n-1}\|^2$$
$$+ 2(R_{B_1} + R_{B_2}) + E\left\| V_n[(A+B)\ell_{n-1} - \ell_n] - (A_\infty + B_\infty - I_2)\begin{pmatrix} c_{\ell 1} \\ c_{\ell 2} \end{pmatrix}\right\|^2$$

and

$$R_{A_i} = E[(A_\infty(\tilde{L}_{n-1} - Z))_i((A_n - A_\infty)\tilde{L}_{n-1})_i], \quad i = 1, 2,$$

R_{B_i} defined similarly. We next prove that as $n \to \infty$

$$b_n \to 0. \tag{2.25}$$

Consider the first part of b_n. Using

$$\mathrm{Cov}(\tilde{L}_{n-1}) = \begin{pmatrix} 1 & \varrho_{n-1} \\ \varrho_{n-1} & 1 \end{pmatrix}$$

and independence we obtain

$$\begin{aligned} E\,\|(A_n - A_\infty)\tilde{L}_{n-1}\|^2 = {} & E(A_{n11} - \bar{A}_{11})^2 + E(A_{n12} - \bar{A}_{12})^2 \\ & + 2\varrho_{n-1} E(A_{n11} - \bar{A}_{11})(A_{n12} - \bar{A}_{12}) \\ & + E(A_{n21} - \bar{A}_{21})^2 + E(A_{n22} - \bar{A}_{22})^2 \\ & + 2\,\varrho_{n-1} E(A_{n21} - \bar{A}_{21})(A_{n22} - \bar{A}_{22}). \end{aligned}$$

Since

$$E(A_{nij} - A_{ij})^2 = EA_{ij}^2 \left(\sqrt{\frac{\mathrm{Var}\, L_{n-1}^{(j)}}{\mathrm{Var}\, L_n^{(i)}}} - c_{ij} \right)^2 \to 0$$

we obtain

$$\begin{aligned} E\,\|(A_n - A_\infty)\tilde{L}_{n-1}\|^2 \leqslant {} & 2[E(A_{n11} - \bar{A}_{11})^2 + E(A_{n12} - \bar{A}_{12})^2 \\ & + E(A_{n21} - \bar{A}_{21})^2 + E(A_{n22} - \bar{A}_{22})^2] \to 0. \end{aligned}$$

Similarly, $E\,\|(B_n - B_\infty)L^*_{n-1}\|^2 \to 0$. Also $R_{A_i} \to 0$, $R_{B_i} \to 0$. Consider, e.g.

$$\begin{aligned} R_{A_1} &= E([\bar{A}_{11}(\tilde{L}_{n-1}^{(1)} - Z_1) + \bar{A}_{12}(\tilde{L}_{n-1}^{(2)} - Z_2)] \\ &\qquad \times [(A_{n11} - \bar{A}_{11})\tilde{L}_{n-1}^{(1)} + (A_{n12} - \bar{A}_{12})\tilde{L}_{n-1}^{(2)}]) \\ &= E\left[c_{11}\left(\sqrt{\frac{\mathrm{Var}\, L_{n-1}^{(1)}}{\mathrm{Var}\, L_n^{(1)}}} - c_{11} \right) A_{11}^2(\tilde{L}_{n-1}^{(1)2} - \tilde{L}_{n-1}^{(1)} Z_1) \right. \\ &\qquad \left. + c_{11}\left(\sqrt{\frac{\mathrm{Var}\, L_{n-1}^{(2)}}{\mathrm{Var}\, L_n^{(1)}}} - c_{12} \right) A_{11} A_{12}(\tilde{L}_{n-1}^{(1)}\tilde{L}_{n-1}^{(2)} - \tilde{L}_{n-1}^{(2)} Z_1) \right. \end{aligned}$$

$$
\begin{aligned}
&+c_{12}\left(\sqrt{\frac{\operatorname{Var} L_{n-1}^{(1)}}{\operatorname{Var} L_n^{(1)}}}-c_{11}\right) A_{11} A_{12}(\tilde{L}_{n-1}^{(1)}\tilde{L}_{n-1}^{(2)}-\tilde{L}_{n-1}^{(1)}Z_2)\\
&\left.+c_{12}\left(\sqrt{\frac{\operatorname{Var} L_{n-1}^{(2)}}{\operatorname{Var} L_n^{(1)}}}-c_{12}\right) A_{12}^2(\tilde{L}_{n-1}^{(2)2}-\tilde{L}_{n-1}^{(2)}Z_2)\right]\\
&\to 0 \quad \text{by (2.6) and (2.7),}
\end{aligned}
$$

since $|E(\tilde{L}_{n-1}^{(1)}\tilde{L}_{n-1}^{(2)}-\tilde{L}_{n-1}^{(1)}Z_2)| \leqslant \varrho_n+\sqrt{\operatorname{Var}\tilde{L}_{n-1}^{(1)}\operatorname{Var} Z_2} \leqslant 2$. Finally

$$
E\left\| V_n[(A+B)\ell_{n-1}-\ell_n]-(A_\infty+B_\infty-E_2)\begin{pmatrix} c_{\ell 1}\\ c_{\ell 2}\end{pmatrix}\right\|^2 = R_1+R_2
$$

where

$$
\begin{aligned}
R_1 = E&\left[\frac{1}{\sqrt{\operatorname{Var} L_n^{(1)}}}((A_{11}+B_{11})\ell_{n-1}^{(1)}+(A_{12}+B_{12})\ell_{n-1}^{(2)}-\ell_n^{(1)})\right.\\
&\left.-((\bar{A}_{11}+\bar{B}_{11}-1)c_{\ell 1}+(\bar{A}_{12}+\bar{B}_{12})c_{\ell 2})\right]^2
\end{aligned}
$$

and

$$
\begin{aligned}
R_2 = E&\left[\frac{1}{\sqrt{\operatorname{Var} L_n^{(2)}}}((A_{21}+B_{21})\ell_{n-1}^{(1)}+(A_{22}+B_{22})\ell_{n-1}^{(2)}-\ell_n^{(2)})\right.\\
&\left.-((\bar{A}_{21}+\bar{B}_{21})c_{\ell 1}+(\bar{A}_{22}+\bar{B}_{22}-1)c_{\ell 2})\right]^2.
\end{aligned}
$$

Similar to the argument above we obtain

$$
\begin{aligned}
R_2 = E&\left[\left(\frac{\ell_{n-1}^{(1)}}{\sqrt{\operatorname{Var} L_n^{(2)}}}-c_{21}c_{\ell 1}\right)(A_{21}+B_{21})+\left(\frac{\ell_{n-1}^{(2)}}{\sqrt{\operatorname{Var} L_n^{(2)}}}-c_{22}c_{\ell 2}\right)(A_{22}+B_{22})\right.\\
&\left.+\left(c_{\ell 2}-\frac{\ell_n^{(2)}}{\sqrt{\operatorname{Var} L_n^{(2)}}}\right)\right]^2 \to 0 \quad \text{and also } R_1 \to 0.
\end{aligned}
$$

From the reduction inequality (2.24) we obtain by iteration and using $\gamma<1$, $b_n \to 0$, that

$$
\ell_2^2(\tilde{L}_n, Z) \leqslant \gamma^{n-n_0}\,\ell_2^2(\tilde{L}_{n_0}, Z)+\sum_{k=0}^{n-n_0-1}\gamma^k\, b_{n-k} \to 0. \tag{2.26}
$$

Here n_0 is chosen such that $\operatorname{Var}(L_n)>0$ for $n \geqslant n_0$. □

The contraction factor γ from Theorem 2.4 can be calculated explicitly in terms of first moments of A, B.

Proposition 2.5. *The contraction factor γ has the explicit form*

$$\gamma = \frac{e+g+\sqrt{4f^2+(e-g)^2}}{2}. \tag{2.27}$$

Proof. In the first step we prove

$$\gamma = \sup_{t\in[0,\pi/2]} (e\sin^2 t + 2\,|f|\sin t\cos t + g\cos^2 t). \tag{2.28}$$

For the proof let (for $U,V \in M_{0,2},\ t\in(0,\pi/2]$)

$$\sin^2 t = \frac{E(U_1-V_1)^2}{E(U_1-V_1)^2+E(U_2-V_2)^2}, \quad \cos^2 t = \frac{E(U_2-V_2)^2}{\|U-V\|^2}.$$

By Cauchy–Schwarz

$$\frac{E(U_1-V_1)(U_2-V_2)}{E\|U-V\|^2} \leqslant \sin t\cos t$$

which implies that the r.h.s. is an upper bound for γ. Conversely, for the two point distribution

$$F_{p,q} = \tfrac{1}{2}\,\delta_{(\sqrt{p},\sqrt{q})} + \tfrac{1}{2}\,\delta_{(-\sqrt{p},-\sqrt{q})}, \quad p,q\geqslant 0,$$

applied in definition (2.14) we obtain

$$\gamma \geqslant \frac{ep+2f\sqrt{pq}+gq}{p+q}$$

and for

$$\tilde{F}_{p,q} := \tfrac{1}{2}\delta_{(\sqrt{p},-\sqrt{q})} + \tfrac{1}{2}\delta_{(-\sqrt{p},\sqrt{q})},$$

we obtain

$$\gamma \geqslant \frac{ep-2f\sqrt{pq}+gq}{p+q}.$$

This implies with $p=\sin^2 t,\ q=\cos^2 t$ the other inequality.

To prove (2.27) consider at first the case $f=0$. Then by (2.28)

$$\gamma = \max(e,g) = \frac{e+g+\sqrt{(e-g)^2}}{2}$$

i.e. (2.27) holds. For $|f|>0$ consider

$$F(t) = e\sin^2 t + 2|f|\sin t\cos t + g\cos^2 t \quad \text{on } [0,\pi/2].$$

It is easy to see that $F(t)$ has a maximum for $\tilde{t} = \arctan(\sqrt{1+d^2}+d)$ with $d=(e-g)/(2|f|)$. Using $g-e=-2|f|d$ one finds by some calculus that

$$\gamma = F(\tilde{t}\,) = \frac{e+g+\sqrt{4f^2+(e-g)^2}}{2}. \qquad \square$$

Note that by Cauchy–Schwarz $f^2\leqslant eg$ and so

$$\gamma \leqslant e+g. \tag{2.29}$$

The simple condition $e+g<1$ is therefore sufficient for contraction.

3. Examples

In some simple examples the mean and covariances can be evaluated explicitly and the conditions (A1)–(A5) can be checked directly.

Example 3.1. Consider a random variable X with $EX=1$, $EX^2=2$ and consider independent matrices

$$A=\begin{pmatrix}1+X & 1\\ 1 & 1+X\end{pmatrix},$$

$B\stackrel{\mathrm{d}}{=}A$ and let

$$L_0=\begin{pmatrix}1\\0\end{pmatrix}.$$

Then the matrices C,N,M_2,M_ϑ,M are given by

$$C=\begin{pmatrix}4&2\\2&4\end{pmatrix},\quad N=4\cdot\begin{pmatrix}4&4&1\\2&5&2\\1&4&4\end{pmatrix},\quad M_2=2\cdot\begin{pmatrix}5&4&1\\2&6&2\\1&4&5\end{pmatrix},$$

$$M_\vartheta=2\cdot\begin{pmatrix}2&0&0\\0&2&0\\0&0&2\end{pmatrix}=2\cdot I_3,\quad\text{and}\quad M=\begin{pmatrix}8&8&2\\4&10&4\\2&8&8\end{pmatrix}.$$

Also by induction in n one obtains the powers

(a) $C^n=2^{n-1}\begin{pmatrix}3^n+1 & 3^n-1\\ 3^n-1 & 3^n+1\end{pmatrix}$,

(b) $$N^n=36^n\begin{pmatrix}\frac14+\frac{1}{2\cdot3^n}+\frac{1}{4\cdot9^n} & \frac12-\frac{1}{2\cdot9^n} & \frac14-\frac{1}{2\cdot3^n}+\frac{1}{4\cdot9^n}\\ \frac14-\frac{1}{4\cdot9^n} & \frac12+\frac{1}{2\cdot9^n} & \frac14-\frac{1}{4\cdot9^n}\\ \frac14-\frac{1}{2\cdot3^n}+\frac{1}{4\cdot9^n} & \frac12-\frac{1}{2\cdot9^n} & \frac14+\frac{1}{2\cdot3^n}+\frac{1}{4\cdot9^n}\end{pmatrix}, \tag{3.1}$$

(c) $$M_2^n=20^n\begin{pmatrix}\frac14+\frac{2^{n+1}+1}{4\cdot5^n} & \frac12-\frac{1}{2\cdot5^n} & \frac14-\frac{2^{n+1}-1}{4\cdot5^n}\\ \frac14-\frac{1}{4\cdot5^n} & \frac12+\frac{1}{2\cdot5^n} & \frac14-\frac{1}{4\cdot5^n}\\ \frac14-\frac{2^{n+1}-1}{4\cdot5^n} & \frac12-\frac{1}{2\cdot5^n} & \frac14+\frac{2^{n+1}+1}{4\cdot5^n}\end{pmatrix},$$

and one obtains

$$\ell_n = C^n \ell_0 = 2^{n-1} \begin{pmatrix} 3^n+1 & 3^n-1 \\ 3^n-1 & 3^n+1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 6^n \begin{pmatrix} \dfrac{1}{2} + \dfrac{1}{2\cdot 3^n} \\ \dfrac{1}{2} - \dfrac{1}{2\cdot 3^n} \end{pmatrix}$$

$$\sim \frac{1}{2} 6^n \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{3.2}$$

As $\ell_0 = (1,0,0)^{\mathrm{T}}$ we obtain

$$\vartheta_n = \sum_{i=1}^n M_2^{i-1} M_\vartheta N^{n-i} \ell_0 = 2 \sum_{i=1}^n M_2^{i-1} 36^{n-i} \frac{1}{4} \begin{pmatrix} 1 + \dfrac{2}{3^{n-i}} + \dfrac{1}{9^{n-i}} \\ 1 - \dfrac{1}{9^{n-i}} \\ 1 - \dfrac{2}{3^{n-i}} + \dfrac{1}{9^{n-i}} \end{pmatrix}$$

which turns out after some calculations to be

$$\vartheta_n = 36^n \frac{1}{32} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - 20^n \frac{1}{32} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 12^n \frac{1}{4} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - 8^n \frac{1}{4} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + n \cdot 4^n \frac{1}{8} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\sim 36^n \frac{1}{32} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{3.3}$$

This implies

$$\sqrt{\frac{\operatorname{Var} L_{n-1}^{(1)}}{\operatorname{Var} L_n^{(1)}}} \to \sqrt{\frac{32}{36^n} \cdot \frac{36^{n-1}}{32}} = \frac{1}{6} = c_{11}, \tag{3.4}$$

and similarly $c_{12} = c_{21} = c_{22} = \frac{1}{6}$, $\beta = 1$,

$$\varrho_n = \frac{\operatorname{Cov}(L_n^{(1)}, L_n^{(2)})}{\sqrt{\operatorname{Var} L_n^{(1)} \cdot \operatorname{Var} L_n^{(2)}}} \to 1 = \varrho. \tag{3.5}$$

and $c_{\ell 1} = c_{\ell 2} = 2\sqrt{2}$. Altogether (A1)–(A5) are fulfilled. Direct calculation yields $e = \frac{1}{3}$, $f = \frac{2}{9}$, $g = \frac{1}{3}$ and, therefore, $\gamma = \frac{5}{9} < 1$. Our main result implies that $\ell_2(\tilde{L}_n, Z) \to 0$ where Z is the unique solution of

$$Z \stackrel{\mathrm{d}}{=} \frac{1}{6} \begin{pmatrix} 1+X & 1 \\ 1 & 1+X \end{pmatrix} Z + \frac{1}{6} \begin{pmatrix} 1+\bar{X} & 1 \\ 1 & 1+\bar{X} \end{pmatrix} \bar{Z} + 2\sqrt{2} \left(\frac{X+\bar{X}}{6} - \frac{1}{3} \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{3.6}$$

$$EZ = 0 \quad \text{and} \quad \operatorname{Cov} Z = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Exponential convergence in Theorem 2.4 implies that the solution of the fixed point equation can be obtained by simulation with a few iteration steps.

In general it is not easy to check the conditions of Theorem 2.4. The following (negative) result gives some conditions leading to $\gamma \geqslant 1$.

Proposition 3.2. *Let* $\ell_0 = 0$ *or* $M_\vartheta = 0$ *or* $N = 0$ *and assume* (A1)–(A5) *are satisfied, then any of the following three conditions* (a)–(c), *implies that* $\gamma \geqslant 1$.

(a) $E(A_{11}^2 + B_{11}^2) = E(A_{22}^2 + B_{22}^2)$

$$\text{(b) } E(A_{11}^2 + B_{11}^2) > E(A_{22}^2 + B_{22}^2) \quad \textit{and} \quad \beta^2 E(A_{21}^2 + B_{21}^2) \geqslant \beta^{-2} E(A_{12}^2 + B_{12}^2) \tag{3.7}$$

(c) $E(A_{11}^2 + B_{11}^2) < E(A_{22}^2 + B_{22}^2)$ *and* $\beta^2 E(A_{21}^2 + B_{21}^2) \leqslant \beta^{-2} E(A_{12}^2 + B_{12}^2)$.

For the proof we refer to Cramer [6].

Remark 3.3. (a) The case of two independent one-dimensional recursions

$$L_n^{(i)} \stackrel{\mathrm{d}}{=} X_1^{(i)} L_{n-1}^{(i)} + X_2^{(i)} \bar{L}_{n-1}^{(i)}, \quad L_0^{(i)} = 1 \tag{3.8}$$

can be imbedded in the two-dimensional case with $L_n = (L_n^{(1)}, L_n^{(2)})$. The conditions $a_{(i)} = E((X_1^{(i)})^2 + (X_2^{(i)})^2) < c_{(i)}^2 = (E(X_1^{(i)} + X_2^{(i)}))^2$ imply convergence of the normalized versions $\tilde{L}_n$. The conditions of Theorem 2.4 are checked easily in this case and correspond to conditions in the one-dimensional case.

(b) Consider the one-dimensional branching recursion with stationary immigration

$$L_n^{(1)} \stackrel{\mathrm{d}}{=} X_1 L_{n-1}^{(1)} + X_2 \bar{L}_{n-1}^{(1)} + Y, \quad L_0^{(1)} \stackrel{\mathrm{d}}{=} Y \tag{3.9}$$

where (X_1, X_2, Y) are independent of $L_{n-1}^{(1)}, \bar{L}_{n-1}^{(1)}$. Then $L_n = (L_n^{(1)}, 1)^{\mathrm{T}}$ satisfies the two-dimensional recursion of branching type

$$L_0 \stackrel{\mathrm{d}}{=} \begin{pmatrix} Y \\ 1 \end{pmatrix}, \quad L_n \stackrel{\mathrm{d}}{=} A L_{n-1} + B \bar{L}_{n-1} \tag{3.10}$$

with

$$A = \begin{pmatrix} X_1 & Y \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} X_2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Assume that $\operatorname{Var} L_n^{(1)} > 0$, $\forall n$ and consider the normalized version

$$\tilde{L}_n = \begin{pmatrix} \dfrac{L_n^{(1)} - \ell_n^{(1)}}{\sqrt{\operatorname{Var}(L_n^{(1)})}} \\ 0 \end{pmatrix}.$$

If $a_{(1)} = E(X_1^2 + X_2^2)$, $c_{(1)} = EX_1 + X_2 > 1$ and $a_{(1)} < c_{(1)}^2$, then the convergence theorem in a slightly modified form remains applicable and $\ell_2(\tilde{L}_n, Z) \to 0$ where Z is fixed point of the operator T where

$$A_\infty = \begin{pmatrix} X_1/c_{(1)} & 0 \\ 0 & 0 \end{pmatrix}, \quad B_\infty = \begin{pmatrix} X_2/c_{(1)} & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.11}$$

This covers a result of Cramer and Rüschendorf [8] and Cramer [7].

In the following example we use eigenvalue theory and computer algebra to check the conditions of Theorem 2.4.

Example 3.4. Consider A, B stochastically independent

$$A = \begin{pmatrix} 1+X & 1+(X/2) \\ 1-(X/2) & 1+X \end{pmatrix},$$

$B \stackrel{d}{=} A$, $EX = 1$, $EX^2 = 2$. Then

$$C = \begin{pmatrix} 4 & 3 \\ 1 & 4 \end{pmatrix}, \quad N = \begin{pmatrix} 16 & 24 & 9 \\ 4 & 19 & 12 \\ 1 & 8 & 16 \end{pmatrix}, \tag{3.12}$$

$$M_\vartheta = \begin{pmatrix} 2 & 2 & \frac{1}{2} \\ -1 & \frac{3}{2} & 1 \\ \frac{1}{2} & -2 & 2 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 10 & 14 & 5 \\ 1 & 11 & 7 \\ 1 & 2 & 10 \end{pmatrix}.$$

The characteristic polynoms and its roots are calculated with help of MAPLE

(a) $$\begin{aligned} P_N(z) &:= \det(z \cdot E_3 - N) = z^3 - 51z^2 + 663z - 2197 \\ &= (z - [19 + 8\sqrt{3}])(z - 13)(z - [19 - 8\sqrt{3}]). \end{aligned} \tag{3.13}$$

(b) $$\begin{aligned} P_{M_2}(z) &:= \det(z \cdot E_3 - M_2) = z^3 - 31z^2 + 287z - 873 \\ &= (z - \mu_1)(z - \mu_2)(z - \bar{\mu}_2), \end{aligned}$$

where $\mu_1 \in \mathbb{R}$, $|\mu_1| > |\mu_2|$ and

$$\mu_1 = p^{1/3} + \tfrac{100}{9} p^{-1/3} + \tfrac{31}{3}, \quad p = \tfrac{1540}{27} + \tfrac{20}{9}\sqrt{381},$$

$$\mu_2 = -\tfrac{1}{2} p^{1/3} - \tfrac{50}{9} p^{-1/3} + \tfrac{31}{3} + \imath \tfrac{3}{2}(p^{1/3} + \tfrac{100}{9} p^{-1/3}).$$

For a matrix $D \in \mathbb{R}^{3\times 3}$ with three different eigenvalues μ_1, μ_2, μ_3 the eigenvalue decomposition yields

$$D^k = \mu_1^k v_1 w_1 + \mu_2^k v_2 w_2 + \mu_3^k v_3 w_3 \tag{3.14}$$

where v_i, w_i are normed right resp. left eigenvectors of μ_i with $w_i v_i = 1$. This yields for M_2, N

$$\begin{aligned} M_2 &= \mu_1 A + \mu_2 B + \mu_3 C, \\ N &= v_1 \tilde{A} + v_2 \tilde{B} + v_3 \tilde{C} \end{aligned} \tag{3.15}$$

with $A, \tilde{A}, \ldots$ products of left and right eigenvectors and $v_1 = 19 + 8\sqrt{3}$, $v_2 = 13$, $v_3 = 19 - 8\sqrt{3}$.

For the covariance vector we then obtain

$$\begin{aligned} \boldsymbol{\vartheta}_n &= M_2^n \boldsymbol{\vartheta}_0 + \sum_{i=1}^{n} M_2^{i-1} M_\vartheta N^{n-i} \boldsymbol{\ell}_0 \\ &= (\mu_1^n A + \mu_2^n B + \mu_3^n C)\boldsymbol{\vartheta}_0 \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{n}(\mu_1^{(i-1)}A+\mu_2^{(i-1)}B+\mu_3^{(i-1)}C)M_\vartheta(v_1^{(n-i)}\tilde{A}+v_2^{(n-i)}\tilde{B}+v_3^{(n-i)}\tilde{C})\ell_0\\
&=v_1^n\left[\frac{1-\left(\frac{\mu_1}{v_1}\right)^n}{v_1-\mu_1}AM_\vartheta\tilde{A}\ell_0+\frac{1-\left(\frac{\mu_2}{v_1}\right)^n}{v_1-\mu_2}BM_\vartheta\tilde{A}\ell_0+\frac{1-\left(\frac{\mu_3}{v_1}\right)^n}{v_1-\mu_3}CM_\vartheta\tilde{A}\ell_0\right]\\
&\quad+\mu_1^n\left[\frac{1-\left(\frac{v_2}{\mu_1}\right)^n}{\mu_1-v_2}AM_\vartheta\tilde{B}\ell_0+A\boldsymbol{\vartheta}_0+\frac{1-\left(\frac{v_3}{\mu_1}\right)^n}{\mu_1-v_3}AM_\vartheta\tilde{C}\ell_0\right]\\
&\quad+v_2^n\left[\frac{1-\left(\frac{\mu_2}{v_2}\right)^n}{v_2-\mu_2}BM_\vartheta\tilde{B}\ell_0+\frac{1-\left(\frac{\mu_3}{v_2}\right)^n}{v_2-\mu_3}CM_\vartheta\tilde{B}\ell_0\right]\\
&\quad+\mu_2^n\left[\frac{1-\left(\frac{v_3}{\mu_2}\right)^n}{\mu_2-v_3}BM_\vartheta\tilde{C}\ell_0+B\boldsymbol{\vartheta}_0\right]+\mu_3^n\left[\frac{1-\left(\frac{v_3}{\mu_3}\right)^n}{\mu_3-v_3}CM_\vartheta\tilde{C}\ell_0+C\boldsymbol{\vartheta}_0\right].
\end{aligned}
\tag{3.16}
$$

Observing that $|v|>|\mu_1|>|v_2|>|\mu_2|=|\mu_3|>|v_3|$ the leading term of $\boldsymbol{\vartheta}_n$ is

$$\left[\frac{1}{v_1-\mu_1}A+\frac{1}{v_1-\mu_2}B+\frac{1}{v_1-\mu_3}C\right]M_\vartheta\tilde{A}\ell_0. \tag{3.17}$$

By MAPLE

$$\tilde{A}=\begin{pmatrix}\frac{1}{4} & \frac{\sqrt{3}}{2} & \frac{3}{4}\\ \frac{1}{4\sqrt{3}} & \frac{1}{2} & \frac{\sqrt{3}}{4}\\ \frac{1}{12} & \frac{1}{2\sqrt{3}} & \frac{1}{4}\end{pmatrix}. \tag{3.18}$$

A,B,C can be calculated numerically, $C=\bar{B}$ as $\bar{\mu}_2=\mu_3$. One obtains (exact up to 9 digits).

$$\frac{1}{v_1-\mu_1}A+2\Re\left[\frac{1}{v_1-\mu_2}B\right]=:D$$

$$D=\begin{pmatrix}0.045977212 & 0.031246478 & 0.019627381\\ 0.002827081 & 0.048993672 & 0.015623239\\ 0.002258945 & 0.005654161 & 0.045977212\end{pmatrix}$$

and

$$DM_\vartheta\tilde{A}=:\tilde{Z}=\begin{pmatrix}0.039792832 & 0.137846414 & 0.119378496\\ 0.004836010 & 0.016752430 & 0.014508030\\ 0.002295040 & 0.007950254 & 0.006885122\end{pmatrix}. \tag{3.19}$$

So the leading term of the covariance vector ϑ_n is given by $v_1^n \tilde{Z} \ell_o$. In particular for $\ell_o = (2, -\sqrt{2}, 1)^{\mathrm{T}}$ one obtains

$$\vartheta_n \sim v_1^n (0.004019892, 0.000488536, 0.002138460)^{\mathrm{T}}. \tag{3.20}$$

This allows to establish (A1), (A2) and (A5) with

$$\begin{aligned} & c_{11}^2 = c_{22}^2 = \frac{1}{v_1} = \frac{1}{19 + 8\sqrt{3}}, \quad c_{12}^2 = c_{11}^2/\beta^2, \qquad c_{21}^2 = c_{22}^2 \beta^2, \\ & \beta = \sqrt{\frac{0.004019892}{0.002138460}} = 1.371060525, \\ & \varrho = \frac{0.000488536}{\sqrt{0.004019892 \cdot 0.002138460}} = 0.163033633. \end{aligned} \tag{3.21}$$

With $N^n = v_1^n \tilde{A} + v_2 n \tilde{B} + v_3^n \tilde{C}$ we then obtain

$$\ell_n \sim v_1^n \tilde{A} \ell_0 = v_1^n \left(\frac{5 - 2\sqrt{6}}{4}, \frac{5}{12}\sqrt{3} - \frac{1}{\sqrt{2}}, \frac{5 - 2\sqrt{6}}{12} \right)^{\mathrm{T}}, \tag{3.22}$$

and (A3) and (A4) hold with

$$\begin{aligned} c_{\ell 1} &= \frac{\sqrt{5 - 2\sqrt{6}}}{2} \Big/ \sqrt{0.004019892} = 2.506499371, \\ c_{\ell 2} &= \sqrt{\frac{5 - 2\sqrt{6}}{12}} \Big/ \sqrt{0.00213846} = 1.984100194. \end{aligned} \tag{3.23}$$

Finally

$$\begin{aligned} e &= c_{11}^2 10 + c_{22}^2 \beta^2 1 = 0.361567445, \\ f &= (c_{11}^2 \beta^{-1} 14 + c_{22}^2 \beta 2)/2 = 0.19711825, \\ g &= c_{11}^2 \beta^{-2} 5 + c_{22}^2 10 = 0.385308344. \end{aligned} \tag{3.24}$$

This yields the contraction condition

$$\gamma = 0.570913239 < 1. \tag{3.25}$$

So we obtain by Theorem 2.4 that $\ell_2(\tilde{L}_n, Z) \to 0$ where Z is the unique fixed point of the operator T in $M_{0,2}$ with

$$A_\infty = \begin{pmatrix} \dfrac{1 + X}{\sqrt{19 + 8\sqrt{3}}} & \dfrac{1 + (X/2)}{\beta\sqrt{19 + 8\sqrt{3}}} \\[2ex] \dfrac{\beta(1 - (X/2))}{\sqrt{19 + 8\sqrt{3}}} & \dfrac{1 + X}{\sqrt{19 + 8\sqrt{3}}} \end{pmatrix}, \qquad B_\infty \stackrel{\mathrm{d}}{=} A_\infty, \tag{3.26}$$

A_∞, B_∞ independent and

$$\operatorname{Cov}(Z) = \begin{pmatrix} 1 & \varrho \\ \varrho & 1 \end{pmatrix}, \qquad EZ = 0. \quad \square$$

A remarkable point of this example is the combination of eigenvalue theory with computer algebra. The calculations done numerically are more that necessary to check (exactly) the validity of the assumptions of our convergence theorem. This combination should also be applicable in further examples.

Remark 3.5. Using a different norming, the recursion can be decorrelated.

Define (for $|\varrho_n| < 1$)

$$L_n' := \begin{pmatrix} \dfrac{1}{\sqrt{\operatorname{Var} L_n^{(1)}}} & 0 \\ \dfrac{-\varrho_n}{\sqrt{1-\varrho_n^2}\sqrt{\operatorname{Var} L_n^{(1)}}} & \dfrac{1}{\sqrt{1-\varrho_n^2}\sqrt{\operatorname{Var} L_n^{(2)}}} \end{pmatrix} (L_n - EL_n). \tag{3.27}$$

Then with

$$R_n := \begin{pmatrix} 1 & 0 \\ \dfrac{-\varrho_n}{\sqrt{1-\varrho_n^2}} & \dfrac{1}{\sqrt{1-\varrho_n^2}} \end{pmatrix}$$

$$L_n' = R_n \tilde{L}_n, \qquad \operatorname{Cov}(L_n') = I_2$$

and (for $|\varrho| < 1$) under the conditions of Theorem 2.4

$$\ell_2(L_n', Z') \to 0 \tag{3.28}$$

where Z' is uniquely characterized (in distribution) by the fixed point equation in $M_{0,2}$

$$\begin{aligned} Z' &\stackrel{\mathrm{d}}{=} RA_\infty R^{-1} Z' + RB_\infty R^{-1} \bar{Z}' \\ &\quad + (RA_\infty R^{-1} + RB_\infty R^{-1} - E_2) R \begin{pmatrix} c_{\ell 1} \\ c_{\ell 2} \end{pmatrix} \end{aligned} \tag{3.29}$$

with

$$R := \begin{pmatrix} 1 & 0 \\ \dfrac{-\varrho}{\sqrt{1-\varrho^2}} & \dfrac{1}{\sqrt{1-\varrho^2}} \end{pmatrix}.$$

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