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Periodic boundary value problem for first-order impulsive functional differential equations

Zhimin He*, Jianshe Yu

Department of Applied Mathematics, Hunan University, Changsha 410082, Hunan, China

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Abstract

In this paper, by means of the method of upper and lower solutions and the monotone iterative technique, the existence of minimal and maximal solutions of the periodic boundary value problem for first-order impulsive functional differential equations is considered. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

The theory of impulsive differential equations is emerging as an important area of investigation since it is much richer than the corresponding theory of differential equations (see [1,3,4,6,7,9,10,12]). In this paper, we consider the following periodic boundary value problem for first-order impulsive functional differential equations (PBVP)

$$\begin{aligned}x'(t) &= f(t, x(t), x_t), \quad t \neq t_k, \quad t \in [0, T], \\ \Delta x(t_k) &= I_k(x(t_k)), \quad k = 1, 2, \dots, p, \\ x(t) &= x(0), \quad t \in [-\tau, 0], \\ x(0) &= x(T),\end{aligned}\tag{1}$$

where $f \in C([0, T] \times R \times D, R)$, $D = \{\psi: [-\tau, 0] \rightarrow R; \psi \text{ is continuous everywhere except for a finite number of points } \bar{t} \text{ at which } \psi(\bar{t}^-) \text{ and } \psi(\bar{t}^+) \text{ exist and } \psi(\bar{t}^-) = \psi(\bar{t}^+)\}$, $I_k \in C(R, R)$,

* Corresponding author. Corresponding address: Department of Applied Mathematics, Central South University, Changsha 410083, Hunan, China.

$\Delta x(t_k) = x(t_k^+) - x(t_k)$, for all $k = 1, 2, \dots, p$; $0 < t_1 < t_2 < \dots < t_k < \dots < t_p < T$, $\tau = \text{constant} > 0$, for every $t \in [0, T]$, $x_t \in D$ is defined by $x_t(s) = x(t + s)$, $s \in [-\tau, 0]$.

The method of upper and lower solutions coupled with the monotone iterative technique provides an effective mechanism to prove constructive existence results for initial and boundary value problems for nonlinear differential equations in recent years (see [2–6, 8–12]). The basic idea of this method is that using the upper and lower solutions as an initial iteration one can construct monotone sequences from a corresponding linear equation, and these sequences converge monotonically to the maximal and minimal solutions of the nonlinear equation. When the method is applied to impulsive functional differential equations, it usually depends on the theory of comparison principles. In Section 2, we establish a comparison principle, i.e., Lemma 2.2. Then we discuss the existence and uniqueness of the solutions for a linear PBVP for impulsive functional differential equations, i.e., Lemmas 2.3 and 2.4. In Section 3, by use of the monotone iterative technique and the method of upper and lower solutions we obtain the existence theorem of extremal solutions for the PBVP (1).

2. Preliminaries

Let $J \subset R$ be an interval, we define $PC(J, R) = \{x: J \rightarrow R; x(t) \text{ is continuous everywhere except for some } t_k \text{ at which } x(t_k^-) \text{ and } x(t_k^+) \text{ exist and } x(t_k^-) = x(t_k^+)\}$; $PC'(J, R) = \{x \in PC(J, R): x(t) \text{ is continuously differentiable everywhere except for some } t_k \text{ at which } x'(t_k^-) \text{ and } x'(t_k^+) \text{ exist and } x'(t_k^-) = x'(t_k^+)\}$. Let $E = PC([-\tau, T], R) \cap PC'([0, T], R)$. A function $x \in E$ is called a solution of PBVP (1) if it satisfies (1).

Definition 2.1. A function $\alpha \in E$ is called a lower solution of the PBVP (1) if

$$\begin{aligned} \alpha'(t) &\leq f(t, \alpha(t), \alpha_t), \quad t \neq t_k, t \in [0, T], \\ \Delta \alpha(t_k) &\leq I_k(\alpha(t_k)), \quad k = 1, 2, \dots, p, \\ \alpha(t) &= \alpha(0), \quad t \in [-\tau, 0], \\ \alpha(0) &\leq \alpha(T). \end{aligned} \tag{2}$$

Definition 2.2. A function $\beta \in E$ is called an upper solution of the PBVP (1) if

$$\begin{aligned} \beta'(t) &\geq f(t, \beta(t), \beta_t), \quad t \neq t_k, t \in [0, T], \\ \Delta \beta(t_k) &\geq I_k(\beta(t_k)), \quad k = 1, 2, \dots, p, \\ \beta(t) &= \beta(0), \quad t \in [-\tau, 0], \\ \beta(0) &\geq \beta(T). \end{aligned} \tag{3}$$

Lemma 2.1 (Lakshmikantham et al. [7]). Assume that

- (a1) the sequence $\{t_k\}$ satisfies $0 \leq t_0 < t_1 < t_2 < \dots < t_k < \dots$ with $\lim_{k \rightarrow +\infty} t_k = +\infty$;
- (a2) $m \in PC'(R_+, R)$ is left continuous at t_k for $k = 1, 2, \dots$;

(a3) for $k = 1, 2, \dots, t \geq t_0$,

$$m'(t) \geq p(t)m(t) + q(t), \quad t \neq t_k, \quad (4)$$

$$m(t_k^+) \geq d_k m(t_k) + b_k, \quad (5)$$

where $p, q \in C(R_+, R)$, $d_k \geq 0$ and b_k are real constants.

Then,

$$\begin{aligned} m(t) &\geq m(t_0) \prod_{t_0 < t_k < t} d_k \exp\left(\int_{t_0}^t p(s) \, ds\right) + \int_{t_0}^t \prod_{s < t_k < t} d_k \exp\left(\int_s^t p(\sigma) \, d\sigma\right) q(s) \, ds \\ &\quad + \sum_{t_0 < t_k < t} \prod_{t_k < t_j < t} d_j \exp\left(\int_{t_k}^t p(s) \, ds\right) b_k. \end{aligned} \quad (6)$$

Lemma 2.2. Let $t_0 = 0, t_{p+1} = T$. Assume that $m \in E$ satisfies

$$\begin{aligned} m'(t) &\geq Mm(t) + N \int_{t-\tau}^t m(s) \, ds, \quad t \neq t_k, \quad t \in [0, T], \\ \Delta m(t_k) &\geq L_k m(t_k), \quad k = 1, 2, \dots, p, \\ m(t) &= m(0), \quad t \in [-\tau, 0], \\ m(0) &\geq m(T), \end{aligned} \quad (7)$$

where constants $M > 0, N > 0, L_k \geq 0$ ($k = 1, 2, \dots, p$), $\tau > 0$, and

$$N \left(\tau + \frac{1}{M} (1 - e^{-M\tau}) \right) (e^{MT} + 1) \leq \frac{\{\prod_{0 < t_k < T} (1 + L_k)^{-1}\}^2}{\int_0^T \prod_{0 < t_k < s} (1 + L_k)^{-1} \, ds}. \quad (8)$$

Then $m(t) \leq 0$ for $t \in [-\tau, T]$.

Proof. Set $u(t) = m(t)e^{-Mt}$ for $t \in [-\tau, T]$, then $u \in E$ satisfies

$$\begin{aligned} u'(t) &\geq N \int_{t-\tau}^t u(s) e^{-M(t-s)} \, ds, \quad t \neq t_k, \quad t \in [0, T], \\ \Delta u(t_k) &\geq L_k u(t_k), \quad k = 1, 2, \dots, p, \\ u(t) &= u(0) e^{-Mt}, \quad t \in [-\tau, 0] \\ u(0) &\geq u(T) e^{MT}. \end{aligned} \quad (9)$$

We now prove

$$u(t) \leq 0 \quad \text{for } t \in [0, T]. \quad (10)$$

Suppose that (10) is false. Then, there are two cases:

- (a) there exists $t_1^* \in [0, T]$ such that $u(t_1^*) > 0$ and $u(t) \geq 0$ for $t \in [0, T]$;
- (b) there exists $t_1^*, t_2^* \in [0, T]$ such that $u(t_1^*) > 0$ and $u(t_2^*) < 0$.

In case (a): (9) implies that

$$u'(t) \geq 0, \quad t \neq t_k, \quad t \in [0, T],$$

$$\Delta u(t_k) \geq 0, \quad k = 1, 2, \dots, p.$$

This means that $u(t)$ is nondecreasing in $[0, T]$, and therefore,

$$u(T) \geq u(t_1^*) > 0, \quad (11)$$

and

$$u(T) \geq u(0) \geq u(T)e^{MT}, \quad (12)$$

which contradicts (11).

In case (b): let $\inf_{t \in [0, T]} u(t) = -\lambda$. Then $\lambda > 0$, and there exists $t_i < t_0^* \leq t_{i+1}$ for some i such that $u(t_0^*) = -\lambda$ or $u(t_i^+) = -\lambda$. We may assume that $u(t_0^*) = -\lambda$ since, in case of $u(t_i^+) = -\lambda$, the proof is similar. If $t - \tau \geq 0$, then

$$\begin{aligned} u'(t) &\geq N \int_{t-\tau}^t u(s) e^{-M(t-s)} ds \\ &\geq -\lambda N \int_{t-\tau}^t e^{-M(t-s)} ds \\ &= -\frac{\lambda N}{M} (1 - e^{-M\tau}), \quad t \neq t_k, \quad t \in [\tau, T]. \end{aligned}$$

If $t - \tau < 0$, then

$$\begin{aligned} u'(t) &\geq N \int_{t-\tau}^0 u(s) e^{-M(t-s)} ds + N \int_0^t u(s) e^{-M(t-s)} ds \\ &\geq N \int_{t-\tau}^0 u(0) e^{-Ms} e^{-M(t-s)} ds - \lambda N \int_0^t e^{-M(t-s)} ds \\ &\geq -\lambda N e^{-Mt} (\tau - t) - \frac{\lambda N}{M} (1 - e^{-Mt}) \\ &\geq -\lambda N \left\{ \tau + \frac{1}{M} (1 - e^{-M\tau}) \right\}, \quad t \neq t_k, \quad t \in [0, \tau]. \end{aligned}$$

Hence, for $t \neq t_k$, $t \in [0, T]$

$$\begin{aligned} u'(t) &\geq \min \left\{ -\frac{\lambda N}{M} (1 - e^{-M\tau}), -\lambda N \left(\tau + \frac{1}{M} (1 - e^{-M\tau}) \right) \right\} \\ &= -\lambda N \left(\tau + \frac{1}{M} (1 - e^{-M\tau}) \right) \\ &= -\lambda M_0, \end{aligned}$$

where $M_0 = N(\tau + (1/M)(1 - e^{-M\tau}))$.

Consider the inequalities

$$\begin{aligned} u'(t) &\geq -\lambda M_0, \quad t \neq t_k, \quad t \in [0, T], \\ u(t_k^+) &\geq (1 + L_k)u(t_k), \quad k = 1, 2, \dots, p. \end{aligned}$$

By Lemma 2.1, we have

$$u(t) \geq u(0) \prod_{0 < t_k < t} (1 + L_k) + \int_0^t \prod_{s < t_k < t} (1 + L_k)(-\lambda M_0) \, ds. \quad (13)$$

Let $t = t_0^*$ in (13), then

$$u(0) \leq -\lambda \prod_{0 < t_k < t_0^*} (1 + L_k)^{-1} + \lambda M_0 \int_0^{t_0^*} \prod_{0 < t_k < s} (1 + L_k)^{-1} \, ds. \quad (14)$$

If $u(0) > 0$, then (14) gives

$$M_0 > \frac{\prod_{0 < t_k < t_0^*} (1 + L_k)^{-1}}{\int_0^{t_0^*} \prod_{0 < t_k < s} (1 + L_k)^{-1} \, ds} \geq \frac{\prod_{0 < t_k < T} (1 + L_k)^{-1}}{\int_0^T \prod_{0 < t_k < s} (1 + L_k)^{-1} \, ds},$$

which contradicts (8). So, we have $u(0) \leq 0$, and by (9), $u(T) \leq u(0)e^{-MT} \leq 0$. Hence $0 < t_1^* < T$. Let $t_j < t_1^* \leq t_{j+1}$ for some j .

We first assume that $t_1^* < t_0^*$. So $j \leq i$. Consider the inequalities

$$\begin{aligned} u'(t) &\geq -\lambda M_0, \quad t \neq t_k, \quad t \in [t_1^*, T], \\ u(t_k^+) &\geq (1 + L_k)u(t_k), \quad k = j + 1, j + 2, \dots, p. \end{aligned}$$

By Lemma 2.1, we have

$$u(t) \geq u(t_1^*) \prod_{t_1^* < t_k < t} (1 + L_k) + \int_{t_1^*}^t \prod_{s < t_k < t} (1 + L_k)(-\lambda M_0) \, ds. \quad (15)$$

Let $t = t_0^*$ in (15), then

$$u(t_0^*) \geq u(t_1^*) \prod_{t_1^* < t_k < t_0^*} (1 + L_k) + \int_{t_1^*}^{t_0^*} \prod_{s < t_k < t_0^*} (1 + L_k)(-\lambda M_0) \, ds. \quad (16)$$

From (16) we have

$$0 < u(t_1^*) \leq -\lambda \prod_{t_1^* < t_k < t_0^*} (1 + L_k)^{-1} + \lambda M_0 \int_{t_1^*}^{t_0^*} \prod_{t_1^* < t_k < s} (1 + L_k)^{-1} \, ds, \quad (17)$$

which gives

$$M_0 > \frac{\prod_{t_1^* < t_k < t_0^*} (1 + L_k)^{-1}}{\int_{t_1^*}^{t_0^*} \prod_{t_1^* < t_k < s} (1 + L_k)^{-1} \, ds} \geq \frac{\prod_{0 < t_k < T} (1 + L_k)^{-1}}{\int_0^T \prod_{0 < t_k < s} (1 + L_k)^{-1} \, ds},$$

and this contradicts (8).

Next assume that $t_0^* < t_1^*$. So $i \leq j$. Let $t = T$ in (15); we then have

$$u(T) \geq u(t_1^*) \prod_{t_1^* < t_k < T} (1 + L_k) + \int_{t_1^*}^T \prod_{s < t_k < T} (1 + L_k)(-\lambda M_0) \, ds,$$

which gives

$$0 < u(t_1^*) \leq u(T) \prod_{t_1^* < t_k < T} (1 + L_k)^{-1} + \lambda M_0 \int_{t_1^*}^T \prod_{t_1^* < t_k < s} (1 + L_k)^{-1} \, ds. \quad (18)$$

From (18), we get

$$u(T) \prod_{t_1^* < t_k < T} (1 + L_k)^{-1} > -\lambda M_0 \int_{t_1^*}^T \prod_{t_1^* < t_k < s} (1 + L_k)^{-1} \, ds.$$

By (9), we obtain

$$-\lambda M_0 \int_{t_1^*}^T \prod_{t_1^* < t_k < s} (1 + L_k)^{-1} \, ds < u(0)e^{-MT} \prod_{t_1^* < t_k < T} (1 + L_k)^{-1}. \quad (19)$$

From (14) and (19), we have

$$\begin{aligned} -\lambda M_0 \int_{t_1^*}^T \prod_{t_1^* < t_k < s} (1 + L_k)^{-1} \, ds &< e^{-MT} \prod_{t_1^* < t_k < T} (1 + L_k)^{-1} \left\{ -\lambda \prod_{0 < t_k < t_0^*} (1 + L_k)^{-1} \right. \\ &\quad \left. + \lambda M_0 \int_0^{t_0^*} \prod_{0 < t_k < s} (1 + L_k)^{-1} \, ds \right\}, \end{aligned}$$

or

$$\begin{aligned} &\prod_{t_1^* < t_k < T} (1 + L_k)^{-1} \prod_{0 < t_k < t_0^*} (1 + L_k)^{-1} \\ &< M_0 \prod_{t_1^* < t_k < T} (1 + L_k)^{-1} \int_0^{t_0^*} \prod_{0 < t_k < s} (1 + L_k)^{-1} \, ds + M_0 e^{MT} \int_{t_1^*}^T \prod_{t_1^* < t_k < s} (1 + L_k)^{-1} \, ds. \end{aligned}$$

Hence

$$\begin{aligned} \left\{ \prod_{0 < t_k < T} (1 + L_k)^{-1} \right\}^2 &\leq \prod_{t_1^* < t_k < T} (1 + L_k)^{-1} \prod_{0 < t_k < t_0^*} (1 + L_k)^{-1} \prod_{0 < t_k < T} (1 + L_k)^{-1} \\ &< M_0 \prod_{t_1^* < t_k < T} (1 + L_k)^{-1} \prod_{0 < t_k < T} (1 + L_k)^{-1} \int_0^{t_0^*} \prod_{0 < t_k < s} (1 + L_k)^{-1} \, ds \\ &\quad + M_0 e^{-MT} \prod_{0 < t_k < T} (1 + L_k)^{-1} \int_{t_1^*}^T \prod_{t_1^* < t_k < s} (1 + L_k)^{-1} \, ds \\ &< M_0 (e^{MT} + 1) \int_0^T \prod_{0 < t_k < s} (1 + L_k)^{-1} \, ds, \end{aligned}$$

which contradicts (8). This proof is complete.

Corollary 2.1. Let $\delta = \max\{t_k - t_{k-1} : k = 1, 2, \dots, p+1\}$ (where $t_0 = 0, t_{p+1} = T$). Assume that $m \in E$ satisfies (7), and constants $M > 0$, $N > 0$, $L_k \geq 0$ ($k = 1, 2, \dots, p$), $\tau > 0$. If

$$N \left(\tau + \frac{1}{M}(1 - e^{-M\tau}) \right) (e^{MT} + 1)\delta \leq \frac{\{\prod_{k=1}^p (1 + L_k)^{-1}\}^2}{1 + \sum_{n=1}^p \prod_{k=1}^n (1 + L_k)^{-1}}, \quad (20)$$

then $m(t) \leq 0$ for $t \in [-\tau, T]$.

Proof. Assuming that inequality (20) holds, we have

$$\begin{aligned} \int_0^T \prod_{0 < t_k < s} (1 + L_k)^{-1} ds &= \int_{t_0^+}^{t_1} \prod_{0 < t_k < s} (1 + L_k)^{-1} ds + \int_{t_1^+}^{t_2} \prod_{0 < t_k < s} (1 + L_k)^{-1} ds + \dots \\ &+ \int_{t_k^+}^{t_{k+1}} \prod_{0 < t_k < s} (1 + L_k)^{-1} ds + \dots + \int_{t_p^+}^{t_{p+1}} \prod_{0 < t_k < s} (1 + L_k)^{-1} ds \\ &= (t_1 - t_0) + (1 + L_1)^{-1}(t_2 - t_1) + \dots \\ &+ (1 + L_1)^{-1}(1 + L_2)^{-1} \dots (1 + L_k)^{-1}(t_{k+1} - t_k) + \dots \\ &+ (1 + L_1)^{-1}(1 + L_2)^{-1} \dots (1 + L_p)^{-1}(t_{p+1} - t_p) \\ &= (t_1 - t_0) + \sum_{n=1}^p \prod_{k=1}^n (1 + L_k)^{-1}(t_{n+1} - t_n) \\ &\leq \delta \left\{ 1 + \sum_{n=1}^p \prod_{k=1}^n (1 + L_k)^{-1} \right\}. \end{aligned} \quad (21)$$

From the above inequality and (20), we found that inequality (8) holds. So Lemma 2.2 implies that $m(t) \leq 0$ for $t \in [-\tau, T]$.

Let us consider the following periodic boundary value problem of a linear impulsive functional differential equation (PBVP)

$$\begin{aligned} u'(t) - Mu(t) &= N \int_{t-\tau}^t u(s) ds + \sigma(t), \quad t \neq t_k, \quad t \in [0, T], \\ \Delta u(t_k) &= L_k u(t_k) + I_k(\eta(t_k)) - L_k \eta(t_k), \quad k = 1, 2, \dots, p, \\ u(t) &= u(0), \quad t \in [-\tau, 0], \\ u(0) &= u(T), \end{aligned} \quad (22)$$

where $M > 0$, $N > 0$ and $L_k \geq 0$ ($k = 1, 2, \dots, p$), $I_k \in C([0, T], R)$, $\sigma \in PC([0, T], R)$ and $\eta \in E$.

Lemma 2.3. Let $M > 0$, $N > 0$ and $L_k \geq 0$ ($k = 1, 2, \dots, p$), $I_k \in C([0, T], R)$, $\sigma \in PC([0, T], R)$ and $\eta \in E$. Then $u \in E$ is a solution of the PBVP (22) if and only if $u \in PC([-\tau, T], R)$ is a

solution of the following impulsive integral equation:

$$u(t) = \begin{cases} - \int_0^T G(t,s) \left(N \int_{s-\tau}^s u(r) dr + \sigma(s) \right) ds \\ - \sum_{0 < t_k < T} G(t, t_k) (L_k u(t_k) + I_k(\eta(t_k)) - L_k \eta(t_k)), & t \in [0, T], \\ u(0), & t \in [-\tau, 0), \end{cases} \quad (23)$$

where

$$G(t,s) = \frac{1}{e^{MT} - 1} \begin{cases} e^{M(t-s)}, & 0 \leq s \leq t \leq T, \\ e^{M(T+t-s)}, & 0 \leq t \leq s \leq T. \end{cases}$$

Proof. Assume that $u \in E$ is a solution of the PBVP (22). By the variation of parameters formula, we get

$$\begin{aligned} u(t) = & u(0)e^{Mt} + \int_0^t e^{M(t-s)} \left(N \int_{s-\tau}^s u(r) dr + \sigma(s) \right) ds \\ & + \sum_{0 < t_k < t} e^{M(t-t_k)} (L_k u(t_k) + I_k(\eta(t_k)) - L_k \eta(t_k)). \end{aligned} \quad (24)$$

Let $t = T$ in (24); we then have

$$\begin{aligned} u(T) = & u(0)e^{MT} + \int_0^T e^{M(T-s)} \left(N \int_{s-\tau}^s u(r) dr + \sigma(s) \right) ds \\ & + \sum_{0 < t_k < T} e^{M(T-t_k)} (L_k u(t_k) + I_k(\eta(t_k)) - L_k \eta(t_k)). \end{aligned} \quad (25)$$

From the boundary condition $u(0) = u(T)$, we obtain

$$\begin{aligned} u(0) = & -\frac{1}{e^{MT} - 1} \left\{ \int_0^T e^{M(T-s)} \left(N \int_{s-\tau}^s u(r) dr + \sigma(s) \right) ds \right. \\ & \left. + \sum_{0 < t_k < T} e^{M(T-t_k)} (L_k u(t_k) + I_k(\eta(t_k)) - L_k \eta(t_k)) \right\}. \end{aligned} \quad (26)$$

Substituting (26) into (24) and using the initial condition $u(t) = u(0)$ for $t \in [-\tau, 0]$, we see that $u \in PC([-\tau, T], R)$ satisfies (23).

If $u \in PC([-\tau, T], R)$ is a solution of (23), then $u \in PC'([0, T], R)$ and

$$u'(t) - Mu(t) = N \int_{t-\tau}^t u(s) ds + \sigma(t), \quad t \neq t_k, \quad t \in [0, T],$$

$$\Delta u(t_k) = L_k u(t_k) + I_k(\eta(t_k)) - L_k \eta(t_k), \quad k = 1, 2, \dots, p,$$

$$u(t) = u(0), \quad t \in [-\tau, 0].$$

Let $t = 0, T$ in (23), respectively, we then have

$$\begin{aligned} u(T) &= -\frac{1}{e^{MT} - 1} \left\{ \int_0^T e^{M(T-s)} \left(N \int_{s-\tau}^s u(r) dr + \sigma(s) \right) ds \right. \\ &\quad \left. + \sum_{0 < t_k < T} e^{M(T-t_k)} [L_k u(t_k) + I_k(\eta(t_k)) - L_k \eta(t_k)] \right\} \\ &= u(0). \end{aligned}$$

Therefore, $u \in E$ is a solution of the PBVP (22). Thus Lemma 2.3 is proved. \square

Lemma 2.4. Assume that $M > 0$, $N > 0$ and $L_k \geq 0$ for $k = 1, 2, \dots, p$, $I_k \in C([0, T], R)$, $\sigma \in PC([0, T], R)$, $\eta \in E$, and the following inequality holds:

$$\frac{N(T + \tau)}{M} + \frac{e^{MT}}{e^{MT} - 1} \sum_{k=1}^p L_k < 1. \quad (27)$$

Then PBVP (22) possesses a unique solution in $PC([- \tau, T], R)$.

Proof. Let $E_0 = \{u \in PC([- \tau, T], R): u(t) \equiv u(0) \text{ for } t \in [- \tau, 0]\}$ with norm $\|u\| = \sup\{|u(t)|: t \in [- \tau, T]\}$, then E_0 is a Banach space.

For any $u \in E_0$, consider the operator F defined by the formula

$$(Fu)(t) = \begin{cases} -\int_0^T G(t, s) \left(N \int_{s-\tau}^s u(r) dr + \sigma(s) \right) ds, \\ -\sum_{0 < t_k < T} G(t, t_k) (L_k u(t_k) + I_k(\eta(t_k)) - L_k \eta(t_k)), & t \in [0, T], \\ (Fu)(0), & t \in [- \tau, 0]. \end{cases}$$

Then $(Fu) \in E_0$, i.e., $FE_0 \subset E_0$.

For every $u, v \in E_0$, $t \in [0, T]$, we have

$$\begin{aligned} |(Fu)(t) - (Fv)(t)| &\leq N \int_0^T G(t, s) \int_{s-\tau}^s |u(r) - v(r)| dr ds + \sum_{0 < t_k < T} G(t, t_k) L_k |u(t_k) - v(t_k)| \\ &\leq \left\{ \frac{N(T + \tau)}{M} + \frac{e^{MT}}{e^{MT} - 1} \sum_{k=1}^p L_k \right\} \|u - v\|. \end{aligned}$$

Hence,

$$\begin{aligned} \|Fu - Fv\| &= \sup_{t \in [- \tau, T]} |(Fu)(t) - (Fv)(t)| \\ &= \sup_{t \in [0, T]} |(Fu)(t) - (Fv)(t)| \\ &\leq \alpha \|u - v\|, \end{aligned}$$

where

$$\alpha = \frac{N(T + \tau)}{M} + \frac{e^{MT}}{e^{MT} - 1} \sum_{k=1}^p L_k < 1.$$

Thus, the operator F is a contraction on E_0 . That is, there is a unique element $u \in E_0$ such that $u = Fu$. This u is the unique solution of PBVP (22).

The proof of Lemma 2.4 is complete.

3. The main result

Theorem 3.1. *Let the following conditions hold.*

(i) *The functions $\alpha, \beta \in E \cap E_0$ are lower and upper solutions of PBVP (1) such that*

$$\beta(t) \leq \alpha(t) \quad \text{on } [-\tau, T].$$

(ii) *The function $f \in C([0, T] \times R \times D, R)$ satisfies*

$$f(t, x, \phi) - f(t, \bar{x}, \bar{\phi}) \leq M(x - \bar{x}) + N \int_{t-\tau}^t [\phi(s) - \bar{\phi}(s)] ds$$

whenever $\beta(t) \leq \bar{x} \leq x \leq \alpha(t)$, $\beta_t(\theta) \leq \bar{\phi}(\theta) \leq \phi(\theta) \leq \alpha_t(\theta)$, $t \in [0, T]$, $\theta \in [-\tau, 0]$, where $M > 0$, $N > 0$.

(iii) *The functions $I_k \in C(R, R)$ satisfy*

$$I_k(x) - I_k(y) \leq L_k(x - y)$$

whenever $\beta(t_k) \leq y \leq x \leq \alpha(t_k)$ ($k = 1, 2, \dots, p$), where $0 \leq L_k$ ($k = 1, 2, \dots, p$).

(iv) *Inequalities (8) and (27) hold.*

Then, there exist monotone sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ with $\alpha_0 = \alpha$, $\beta_0 = \beta$ such that $\lim_{n \rightarrow \infty} \alpha_n(t) = r(t)$, $\lim_{n \rightarrow \infty} \beta_n(t) = \rho(t)$ uniformly on $[-\tau, T]$ and ρ, r are the minimal and the maximal solutions of PBVP (1), respectively, such that $\beta_0 \leq \beta_1 \leq \beta_2 \leq \dots \leq \beta_n \leq \rho \leq x \leq r \leq \alpha_n \leq \dots \leq \alpha_2 \leq \alpha_1 \leq \alpha_0$ on $[-\tau, T]$, where x is any solution of PBVP (1) such that $\beta(t) \leq x(t) \leq \alpha(t)$ on $[\tau, T]$.

Proof. Let $[\beta, \alpha] = \{x \in E \cap E_0: \beta(t) \leq x(t) \leq \alpha(t), t \in [-\tau, T]\}$. For any $\eta \in [\beta, \alpha]$, consider PBVP (22) with

$$\sigma(t) = f(t, \eta(t), \eta_t) - M\eta(t) - N \int_{t-\tau}^t \eta(s) ds.$$

By Lemmas 2.2 and 2.3, PBVP (22) possesses a unique solution $u \in E \cap E_0$. We define an operator A by $u = A\eta$, then the operator A has the following properties.

(a) $\beta \leq A\beta$, $A\alpha \leq \alpha$;

(b) A is monotone nondecreasing on $[\beta, \alpha]$, i.e., for any

$$\eta_1, \eta_2 \in [\beta, \alpha], \quad \eta_1 \leq \eta_2 \quad \text{implies} \quad A\eta_1 \leq A\eta_2.$$

To prove (a), set $m = \beta_0 - \beta_1$, where $\beta_1 = A\beta_0$. Then from (i) and (22), we have

$$\begin{aligned} m'(t) &= \beta'_0(t) - \beta'_1(t) \\ &\geq f(t, \beta_0(t), \beta_{0t}) - \left(M\beta_1(t) + N \int_{t-\tau}^t \beta_1(s) \, ds \right. \\ &\quad \left. + f(t, \beta_0(t), \beta_{0t}) - M\beta_0(t) - N \int_{t-\tau}^t \beta_0(s) \, ds \right) \\ &= Mm(t) + N \int_{t-\tau}^t m(s) \, ds, \quad t \neq t_k, \quad t \in [0, T], \\ \Delta m(t_k) &= \Delta \beta_0(t_k) - \Delta \beta_1(t_k) \\ &\geq I_k(\beta_0(t_k)) - [L_k \beta_1(t_k) + I_k(\beta_0(t_k)) - L_k \beta_0(t_k)] \\ &= L_k m(t_k), \quad k = 1, 2, \dots, p, \end{aligned}$$

$$m(t) = m(0) \geq m(T), \quad t \in [-\tau, 0].$$

By Lemma 2.2, we get $m(t) \leq 0$ on $[-\tau, T]$, i.e., $\beta \leq A\beta$. Similar arguments show that $A\alpha \leq \alpha$.

To prove (b), let $u_1 = A\eta_1$, $u_2 = A\eta_2$, where $\eta_1 \leq \eta_2$ on $[-\tau, T]$ and $\eta_1, \eta_2 \in [\beta, \alpha]$. Set $m = u_1 - u_2$. Using (ii), (iii) and (22), we get

$$\begin{aligned} m'(t) &= u'_1(t) - u'_2(t) \\ &= \left(Mu_1(t) + N \int_{t-\tau}^t u_1(s) \, ds + f(t, \eta_1(t), \eta_{1t}) - M\eta_1(t) - N \int_{t-\tau}^t \eta_1(s) \, ds \right) \\ &\quad - \left(Mu_2(t) + N \int_{t-\tau}^t u_2(s) \, ds + f(t, \eta_2(t), \eta_{2t}) - M\eta_2(t) - N \int_{t-\tau}^t \eta_2(s) \, ds \right) \\ &\geq M(u_1(t) - u_2(t)) + N \int_{t-\tau}^t (u_1(s) - u_2(s)) \, ds \\ &= Mm(t) + N \int_{t-\tau}^t m(s) \, ds, \quad t \neq t_k, \quad t \in [0, T], \\ \Delta m(t_k) &= \Delta u_1(t_k) - \Delta u_2(t_k) \\ &= [L_k u_1(t_k) + I_k(\eta_1(t_k)) - L_k \eta_1(t_k)] - [L_k u_2(t_k) + I_k(\eta_2(t_k)) - L_k \eta_2(t_k)] \\ &\geq L_k m(t_k), \quad k = 1, 2, \dots, p, \end{aligned}$$

$$m(t) = m(0) = m(T), \quad t \in [-\tau, 0].$$

In view of Lemma 2.2, we have $m(t) \leq 0$ on $[-\tau, T]$, i.e., $u_1 \leq u_2$.

It is now easy to define the sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ with $\alpha_0 = \alpha$, $\beta_0 = \beta$ such that $\alpha_{n+1} = A\alpha_n$, $\beta_{n+1} = A\beta_n$. From (a) and (b), the functions $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ satisfy the inequalities

$$\beta_0 \leq \beta_1 \leq \beta_2 \leq \dots \leq \beta_n \leq \dots \leq \alpha_n \leq \dots \leq \alpha_2 \leq \alpha_1 \leq \alpha_0 \quad \text{on } [-\tau, T]$$

and each $\alpha_n, \beta_n \in E \cap E_0$ ($n = 1, 2, \dots$) satisfies

$$\alpha_n(t) = \begin{cases} - \int_0^T G(t, s) \left(N \int_{s-\tau}^s \alpha_n(r) dr + \sigma_{n-1}(s) \right) ds \\ - \sum_{0 < t_k < T} G(t, t_k) (L_k \alpha_n(t_k) + I_k(\alpha_{n-1}(t_k)) - L_k \alpha_{n-1}(t_k)), & t \in [0, T], \\ \alpha_n(0), & t \in [-\tau, 0], \end{cases}$$

$$\beta_n(t) = \begin{cases} - \int_0^T G(t, s) \left(N \int_{s-\tau}^s \beta_n(r) dr + \bar{\sigma}_{n-1}(s) \right) ds \\ - \sum_{0 < t_k < T} G(t, t_k) (L_k \beta_n(t_k) + I_k(\beta_{n-1}(t_k)) - L_k \beta_{n-1}(t_k)), & t \in [0, T], \\ \beta_n(0), & t \in [-\tau, 0], \end{cases}$$

where

$$\sigma_{n-1}(t) = f(t, \alpha_{n-1}(t), \alpha_{n-1t}) - M\alpha_{n-1}(t) - N \int_{t-\tau}^t \alpha_{n-1}(s) ds,$$

$$\bar{\sigma}_{n-1}(t) = f(t, \beta_{n-1}(t), \beta_{n-1t}) - M\beta_{n-1}(t) - N \int_{t-\tau}^t \beta_{n-1}(s) ds.$$

Therefore, there exist ρ, r such that $\lim_{n \rightarrow \infty} \alpha_n(t) = r(t)$, $\lim_{n \rightarrow \infty} \beta_n(t) = \rho(t)$ uniformly on $[-\tau, T]$. Clearly ρ, r satisfy the PBVP (1). We shall prove that ρ, r are minimal and maximal solutions of PBVP (1). Let us assume that $x(t)$ is any solution of PBVP (1) such that $x \in [\beta, \alpha]$ and that there exists a positive integer n such that $\beta_n(t) \leq x(t) \leq \alpha_n(t)$ on $[-\tau, T]$. Then, setting $m = \beta_{n+1} - x$, we have

$$\begin{aligned} m'(t) &= \beta'_{n+1}(t) - x'(t) \\ &= \left(M\beta_{n+1}(t) + N \int_{t-\tau}^t \beta_{n+1}(s) ds + f(t, \beta_n(t), \beta_{nt}) \right. \\ &\quad \left. - M\beta_n(t) - N \int_{t-\tau}^t \beta_n(s) ds \right) - f(t, x(t), x_t) \\ &\geq Mm(t) + N \int_{t-\tau}^t m(s) ds, \quad t \neq t_k, \quad t \in [0, T], \\ \Delta m(t_k) &= \Delta \beta_{n+1}(t_k) - \Delta x(t_k) \\ &= [L_k \beta_{n+1}(t_k) + I_k(\beta_n(t_k)) - L_k \beta_n(t_k)] - I_k(x(t_k)) \\ &\geq L_k m(t_k), \quad k = 1, 2, \dots, p, \end{aligned}$$

$$m(t) = m(0) = m(T), \quad t \in [-\tau, 0].$$

By Lemma 2.2, $m(t) \leq 0$ on $[-\tau, T]$, i.e., $\beta_{n+1}(t) \leq x(t)$ on $[-\tau, T]$. Similarly, we get $x(t) \leq \alpha_{n+1}(t)$ on $[-\tau, T]$. Since $\beta_0(t) \leq x(t) \leq \alpha_0(t)$ on $[-\tau, T]$, by induction we obtain $\beta_n(t) \leq x(t) \leq \alpha_n(t)$ on $[-\tau, T]$ for every n . Therefore, $\rho(t) \leq x(t) \leq r(t)$ on $[-\tau, T]$ by taking limit as $n \rightarrow \infty$. The proof of the theorem is complete. \square

References

- [1] D.D. Bainov, P.S. Simeonov, *Impulsive Differential Equations: Periodic Solutions and Applications*, Longman, Harlow, 1993.
- [2] D.J. Guo, V. Lakshmikantham, X.Z. Liu, *Nonlinear Integral Equations in Abstract Spaces*, Kluwer Academic Publishers, London, 1996.
- [3] G.S. Hristova, D.D. Bainov, Application of the monotone iterative-techniques of V. Lakshmikantham for solving the initial value problem for impulsive differential-difference equations, *J. Rocky. Mountain. Math.* 23 (1993) 609–618.
- [4] G.S. Hristova, D.D. Bainov, Monotone iterative techniques of V. Lakshmikantham for a boundary value problem for systems of impulsive differential-difference equations, *J. Math. Anal. Appl.* 197 (1996) 1–13.
- [5] G.S. Ladde, V. Lakshmikantham, A.S. Vatsala, *Monotone Iterative Techniques for Nonlinear Differential equations*, Pitman, London, 1985.
- [6] G.S. Ladde, S. Sathananthan, Periodic boundary value problem for impulsive integro-differential equations of Volterra type, *J. Math. Ph. Sci.* 25 (1991) 119–129.
- [7] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.
- [8] S. Leela, M.N. Oğuztöreli, Periodic boundary value problem for differential equations with delay and monotone iterative method, *J. Math. Anal. Appl.* 122 (1987) 301–307.
- [9] X.Z. Liu, Iterative methods for solutions of impulsive functional differential systems, *Appl. Anal.* 44 (1992) 171–182.
- [10] X.Z. Liu, D.J. Guo, Periodic boundary value problems for impulsive integro-differential equations of mixed type in Banach spaces, *Chin. Ann. Math.* 19B (1998) 517–528.
- [11] E. Liz, Periodic boundary value problems for a class of functional differential equations, *J. Math. Anal. Appl.* 200 (1996) 680–686.
- [12] Z.L. Wei, Solutions of periodic boundary value problems for first order nonlinear impulsive differential equations in Banach spaces, *J. Systems Sci. Math. Sci.* 19 (1999) 378–384.