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The existence of multiple positive solutions for multi-point boundary value problems on the half-line

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ABSTRACT

In this paper, we consider the existence of multiple positive solutions for some nonlinear m-point boundary value problems on the half-line

$$(\varphi(u'))' + a(t)f(t, u(t)) = 0, \quad 0 < t < +\infty,$$

$$u(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \qquad u'(\infty) = 0,$$

where $\varphi : R \to R$ is an increasing homeomorphism and positive homomorphism and $\varphi(0) = 0$. Using a fixed-point theorem for operator on a cone, we provide sufficient conditions for the existence of multiple positive solutions to the above boundary value problem.

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Contents

1. Introduction

In this paper we study the existence of multiple positive solutions of the following boundary value problem on a half-line

$$(\varphi(u'))' + a(t)f(t, u(t)) = 0, \quad 0 < t < +\infty, \tag{1.1}$$

$$u(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \quad u'(\infty) = 0, \tag{1.2}$$

where $\varphi : R \to R$ is an increasing homeomorphism and positive homomorphism and $\varphi(0) = 0$, $\xi_i \in (0, +\infty)$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < +\infty$ and α_i satisfy $\alpha_i \in [0, +\infty)$, $0 < \sum_{i=1}^{m-2} \alpha_i < 1$.

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A projection $\varphi : R \to R$ is called an increasing homeomorphism and positive homomorphism, if the following conditions are satisfied:

(1) if $x \le y$, then $\varphi(x) \le \varphi(y)$, for all $x, y \in R$;

(2) φ is a continuous bijection and its inverse mapping is also continuous;

(3) $\varphi(xy) = \varphi(x)\varphi(y)$, for all $x, y \in [0, +\infty)$.

In above definition, we can replace the condition (3) by the following stronger condition:

(4) $\varphi(xy) = \varphi(x)\varphi(y)$, for all $x, y \in R$, where $R = (-\infty, +\infty)$.

Remark 1.1. If conditions (1), (2) and (4) hold, then it implies that φ is homogeneous generating a p-Laplace operator, i.e., $\varphi(x) = |x|^{p-2}x$, for some $p > 1$.

In this paper, we assume that the following conditions are satisfied:

(C_1) $f \in C([0, +\infty) \times [0, +\infty), [0, +\infty))$, $f(t, 0) \not\equiv 0$ on any subinterval of $[0, +\infty)$ and, when u is bounded, $f(t, (1+t)u)$ is bounded on $[0, +\infty) \times [0, +\infty)$;

(C_2) $a(t)$ is a nonnegative measurable function defined in $(0, +\infty)$ and $a(t)$ does not identically vanish on any subinterval of $(0, +\infty)$ and

$$0 < \int_0^{+\infty} a(t)\mathrm{d}t < +\infty.$$

The multi-point boundary value problems for ordinary differential equations arise in a variety of different areas of applied mathematics and physics. The study of multi-point boundary value problems for linear second-order ordinary differential equations was initiated in [3]. Since then, nonlinear multi-point boundary value problems have been studied by several authors. We refer the reader to [1–14] and the references therein. Recently, Liu and Zhang [2] studied the existence of positive solutions of quasi-linear differential equation

$$\begin{cases} (\varphi(x'))' + a(t)f(x(t)) = 0, \quad t \in (0, 1), \\ x(0) - \beta x'(0) = 0, \quad x(1) + \delta x'(1) = 0, \end{cases}$$

subject to linear mixed boundary value conditions by a simple application of a fixed-point index theorem in cones, where $\varphi : R \to R$ is an increasing homeomorphism and positive homomorphism and $\varphi(0) = 0$.

Wang and Hou [9] studied the following boundary value problem

$$\begin{cases} (\phi_p(u'))'(t) + a(t)f(t, u) = 0, \quad 0 < t < 1, \\ \phi(u'(0)) = \sum_{i=1}^{n-2} a_i \phi_p(u'(\xi_i)), \quad u(1) = \sum_{i=1}^{n-2} b_i u(\xi_i), \end{cases}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, the authors proved that the existence of multiple positive solutions to the above boundary value problem by using a fixed-point theorem for operator on a cone.

Lian [17] studied the following boundary value problem of second-order differential equation with a p-Laplacian operator on a half-line

$$\begin{cases} (\varphi_p(u'(t)))' + \phi(t)f(t, u(t), u') = 0, \quad 0 < t < +\infty, \\ \alpha u(0) - \beta u'(0) = 0, \quad u'(\infty) = 0. \end{cases}$$

They showed the existence of at least three positive solutions by using a fixed-point theorem in a cone due to Avery–Peterson.

In the past few years there have been many papers investigated the positive solutions of boundary value problem on the half-line, see [15–18]. They discuss the existence and multiplicity positive solutions to nonlinear differential equations. However, there is few papers concerned with the existence of multiple positive solutions to boundary value problems of differential equation on infinite intervals so far by using fixed-point theorem for operator on a cone. The goal of present paper is to fill the gap in this area.

Motivated by all the works above, the purpose of this paper is to study the existence of multiple positive solutions for some boundary value problems on the half-line by using a fixed-point theorem for operator on a cone. We emphasize that the results in the paper are new even for the case of $\varphi(u) = u$ and $\varphi(u) = |u|^{p-2}u$, $p > 1$.

By the positive solution of (1.1) and (1.2) one means a function $u(t)$ which is positive on $0 < t < +\infty$ and satisfies the differential equation (1.1) and the boundary value conditions (1.2).

2. The preliminary lemmas

To obtain positive solutions of (1.1) and (1.2) the following fixed-point theorem in cones is fundamental.

Lemma 2.1 ([6]). *Let K be a cone in a Banach space X. Let D be an open bounded set with $D_k = D \cap K \neq \emptyset$ and $\overline{D}_k \neq K$. Let $T : \overline{D}_k \to K$ be a compact map such that $x \neq Tx$ for $x \in \partial D_k$. Then the following results hold.*

(1) *If $\|Tx\| \leq \|x\|$ for $x \in \partial D_k$, then $i_k(T, D_k) = 1$.*
(2) *Suppose there is $e \in K, e \neq 0$ such that $x \neq Tx + \lambda e$ for all $x \in \partial D_k$ and all $\lambda > 0$, then $i_k(T, D_k) = 0$.*
(3) *Let D^1 be open in X such that $\overline{D^1} \subset D_k$. If $i_k(T, D_k) = 1$ and $i_k(T, D^1_k) = 0$, then T has a fixed point in $D_k \setminus \overline{D^1_k}$. Then same result holds if $i_k(T, D_k) = 0$ and $i_k(T, D^1_k) = 1$.*

Lemma 2.2. *For any $x \in C[0, 1], x(t) \geq 0$, the problem*

$$(\varphi(x'))' + a(t)f(t, x(t)) = 0, \quad 0 < t < +\infty, \tag{2.1}$$

$$x(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), \qquad x'(\infty) = 0 \tag{2.2}$$

has a unique solution

$$x(t) = \int_0^t \varphi^{-1}\left(\int_s^{+\infty} a(\tau)f(\tau, x(\tau))\mathrm{d}\tau\right)\mathrm{d}s + \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \varphi^{-1}\left(\int_s^{+\infty} a(\tau)f(\tau, x(\tau))\mathrm{d}\tau\right)\mathrm{d}s}{1 - \sum_{i=1}^{m-2} \alpha_i}.$$

Proof. It is easy to prove, so we omit it here. □

In this paper we will use the following space E which is denoted by

$$E = \left\{u \in C[0, +\infty) : \sup_{0 \leq t < +\infty} \frac{|u(t)|}{1+t} < +\infty\right\}$$

to study (1.1) and (1.2). Then E is a Banach space, equipped with the norm $\|u\| = \sup_{0 \leq t < +\infty} \frac{|u(t)|}{1+t} < +\infty$.

Define cone $K \subset E$ by

$$K = \left\{u \in E : u(t) \text{is a nonnegative concave function on } [0, +\infty) \text{ and } \lim_{t \to +\infty} u'(t) = 0\right\}.$$

Now we define an operator $T : K \to C[0, +\infty)$ by

$$(Tu)(t) = \int_0^t \varphi^{-1}\left(\int_s^{+\infty} a(\tau)f(\tau, u(\tau))\mathrm{d}\tau\right)\mathrm{d}s + \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \varphi^{-1}\left(\int_s^{+\infty} a(\tau)f(\tau, u(\tau))\mathrm{d}\tau\right)\mathrm{d}s}{1 - \sum_{i=1}^{m-2} \alpha_i}. \tag{2.3}$$

Obviously $(Tu)(t) \geq 0$ for $t \in (0, +\infty)$ and $(Tu)'(t) = \varphi^{-1}\left(\int_t^{+\infty} a(\tau)f(\tau, u(\tau))\mathrm{d}\tau\right) \geq 0$, furthermore $(\varphi(Tu)'(t))' = -a(t)f(t, u(t)) \leq 0$. This shows $(TK) \subset K$.

To obtain the complete continuity of T the following lemma is still needed.

Lemma 2.3 ([18]). *Let W be a bounded subset of K. Then W is relatively compact in E if $\left\{\frac{W(t)}{1+t}\right\}$ are equicontinuous on any finite subinterval of $[0, +\infty)$ and for any $\varepsilon > 0$ there exists $N > 0$ such that*

$$\left|\frac{x(t_1)}{1+t_1} - \frac{x(t_2)}{1+t_2}\right| < \varepsilon$$

uniformly with respect to $x \in W$ as $t_1, t_2 \geq N$, where $W(t) = \{x(t) : x \in W\}, t \in [0, +\infty)$.

Lemma 2.4. *Let (C_1) and (C_2) hold. Then $T : K \to K$ is completely continuous.*

Proof. Firstly it is easy to check that $T : K \to K$ is well defined. From the definition of E, we can choose r_0 such that $\sup_{n\in N\setminus\{0\}} \|u_n\| < r_0$. Let $B_{r_0} = \sup\{f(t,(1+t)u), (t,u) \in [0,+\infty) \times [0,r_0]\}$ and Ω be any bounded subset of K. Then there exists $r > 0$ such that $\|u\| \le r$ for all $u \in \Omega$. Therefore we have

$$\begin{aligned}
\|Tu\| &= \sup_{t\in[0,+\infty)} \frac{1}{1+t}\left|\int_0^t \varphi^{-1}\left(\int_s^{+\infty} a(\tau)f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s + \frac{\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\varphi^{-1}\left(\int_s^{+\infty} a(\tau)f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s}{1-\sum_{i=1}^{m-2}\alpha_i}\right| \\
&\le \sup_{t\in[0,+\infty)} \frac{1}{1+t}\int_0^t \varphi^{-1}\left(\int_s^{+\infty} a(\tau)f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\
&\quad + \sup_{t\in[0,+\infty)} \frac{1}{1+t}\,\frac{\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_{m-2}}\varphi^{-1}\left(\int_s^{+\infty} a(\tau)f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s}{1-\sum_{i=1}^{m-2}\alpha_i} \\
&\le \varphi^{-1}\left(\int_0^{+\infty} a(\tau)f(\tau,u(\tau))\mathrm{d}\tau\right)\left(1+\frac{\sum_{i=1}^{m-2}\alpha_i\xi_{m-2}}{1-\sum_{i=1}^{m-2}\alpha_i}\right) \\
&\le c_1\varphi^{-1}(B_r), \quad \forall\, u \in \Omega,
\end{aligned}$$

where

$$c_1 = \varphi^{-1}\left(\int_0^{+\infty} a(\tau)\mathrm{d}\tau\right)\left(1+\frac{\sum_{i=1}^{m-2}\alpha_i\xi_{m-2}}{1-\sum_{i=1}^{m-2}\alpha_i}\right).$$

So $T\Omega$ is bounded. Moreover for any $T \in (0,+\infty)$ and $t_1, t_2 \in [0,T]$, we have

$$\begin{aligned}
\left|\frac{(Tu)(t_1)}{1+t_1} - \frac{(Tu)(t_2)}{1+t_2}\right| &\le \frac{\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\varphi^{-1}\left(\int_s^{+\infty} a(\tau)f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s}{\left(1-\sum_{i=1}^{m-2}\alpha_i\right)}\left|\frac{1}{1+t_1}-\frac{1}{1+t_2}\right| \\
&\quad + \left|\frac{1}{1+t_1}\int_0^{t_1}\varphi^{-1}\left(\int_\tau^{+\infty} a(s)f(s,u(s))\mathrm{d}s\right)\mathrm{d}\tau\right. \\
&\quad \left. - \frac{1}{1+t_2}\int_0^{t_2}\varphi^{-1}\left(\int_\tau^{+\infty} a(s)f(s,u(s))\mathrm{d}s\right)\mathrm{d}\tau\right| \\
&\le c_2\varphi^{-1}(B_r)\left|\frac{1}{1+t_1}-\frac{1}{1+t_2}\right| + c_3\varphi^{-1}(B_r)\,|t_1-t_2| \\
&\to 0, \quad \text{uniformly as } t_1 \to t_2,
\end{aligned}$$

where

$$c_2 = \frac{\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\varphi^{-1}\left(\int_s^{+\infty} a(\tau)\mathrm{d}\tau\right)\mathrm{d}s}{\left(1-\sum_{i=1}^{m-2}\alpha_i\right)} + T\varphi^{-1}\left(\int_0^{+\infty} a(s)\mathrm{d}s\right)$$

and

$$c_3 = \varphi^{-1}\left(\int_0^{+\infty} a(s)\mathrm{d}s\right).$$

Therefore, we can get $T\Omega$ is equicontinuous on any finite subinterval of $[0,+\infty)$.

Next we prove for any $\varepsilon > 0$, there exists sufficiently large $N > 0$ such that

$$\left|\frac{(Tu)(t_1)}{1+t_1} - \frac{(Tu)(t_2)}{1+t_2}\right| < \varepsilon \quad \text{for all } t_1, t_2 \geq N,\ \forall\, u \in \Omega. \tag{2.4}$$

Since $\int_0^{+\infty} a(\tau)f(u(\tau)\mathrm{d}\tau) < +\infty$, we can choose $N_1 > 0$ such that

$$\frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \varphi^{-1}\left(\int_s^{+\infty} a(\tau)f(u(\tau))\mathrm{d}\tau\right)\mathrm{d}s}{N_1\left(1 - \sum_{i=1}^{m-2}\alpha_i\right)} < \frac{\varepsilon}{5}.$$

We can also select $N_2, N_3 > 0$ large enough so that

$$N_2 > \frac{5\int_0^{+\infty} \varphi^{-1}\left(\int_s^{+\infty} a(\tau)f(u(\tau))\mathrm{d}\tau\right)\mathrm{d}s}{\varepsilon}, \quad \varphi^{-1}\left(\int_{N_3}^{+\infty} a(\tau)f(u(\tau))\mathrm{d}\tau\right) < \frac{\varepsilon}{5}$$

are satisfied respectively. Then let $N = \max\{N_1, N_2, N_3\}$. Without loss of generality, we assume $t_2 > t_1 \geq N$. So it follows that

$$\begin{aligned}
\left|\frac{(Tu)(t_1)}{1+t_1} - \frac{(Tu)(t_2)}{1+t_2}\right| &\leq \int_0^{+\infty} \varphi^{-1}\left(\int_s^{+\infty} a(\tau)f(u(\tau))\mathrm{d}\tau\right)\mathrm{d}s \left|\frac{1}{1+t_1} - \frac{1}{1+t_2}\right| \\
&\quad + \frac{\int_{t_1}^{t_2} \varphi^{-1}\left(\int_{t_1}^{+\infty} a(s)f(u(s))\mathrm{d}s\right)\mathrm{d}s}{1+t_2} + \frac{\sum_{i=1}^{m-2}\alpha_i \int_0^{\xi_i}\varphi^{-1}\left(\int_s^{+\infty} a(\tau)f(u(\tau))\mathrm{d}\tau\right)\mathrm{d}s}{(1+t_1)\left(1 - \sum_{i=1}^{m-2}\alpha_i\right)} \\
&\quad + \frac{\sum_{i=1}^{m-2}\alpha_i \int_0^{\xi_i}\varphi^{-1}\left(\int_s^{+\infty} a(\tau)f(u(\tau))\mathrm{d}\tau\right)\mathrm{d}s}{(1+t_2)\left(1 - \sum_{i=1}^{m-2}\alpha_i\right)} \\
&\leq \frac{2\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} = \varepsilon.
\end{aligned}$$

That is, (3.4) holds. By Lemma 2.3 $T\Omega$ is relatively compact. Therefore we know that T is a compact operator.

Thirdly we prove that T is continuous. Let $u_n \to u$ as $n \to +\infty$ in K. Then by the Lebesgue dominated convergence theorem and continuity of f, we can get

$$\left|\int_t^{+\infty} a(s)f(u_n(s))\mathrm{d}s - \int_t^{+\infty} a(s)f(u(s))\mathrm{d}s\right| \leq \int_t^{+\infty} a(s)\left|f(u_n(s)) - f(u(s))\right|\mathrm{d}s \to 0 \quad \text{as } n \to +\infty,$$

i.e.,

$$\int_t^{+\infty} a(s)f(u_n(s))\mathrm{d}s \to \int_t^{+\infty} a(s)f(u(s))\mathrm{d}s \quad \text{as } n \to +\infty.$$

Moreover

$$\varphi^{-1}\left(\int_t^{+\infty} a(s)f(u_n(s))\mathrm{d}s\right) \to \varphi^{-1}\left(\int_t^{+\infty} a(s)f(u(s))\mathrm{d}s\right) \quad \text{as } n \to +\infty.$$

So

$$\begin{aligned}
\|Tu_n - Tu\| \leq &\sup_{t\in[0,+\infty)} \frac{1}{1+t}\int_0^t \left|\varphi^{-1}\left(\int_\tau^{+\infty} a(s)f(u_n(s))\mathrm{d}s\right) - \varphi^{-1}\left(\int_\tau^{+\infty} a(s)f(u(s))\mathrm{d}s\right)\right|\mathrm{d}\tau \\
&+ \frac{\sum_{i=1}^{m-2}\alpha_i\xi_{m-2}}{1 - \sum_{i=1}^{m-2}\alpha_i}\left|\varphi^{-1}\left(\int_s^{+\infty} a(\tau)f(u_n(\tau))\mathrm{d}\tau\right) - \varphi^{-1}\left(\int_s^{+\infty} a(\tau)f(u(\tau))\mathrm{d}\tau\right)\right| \\
\to &\ 0 \quad \text{as } n \to +\infty.
\end{aligned}$$

Therefore T is continuous. In sum, $T : K \to K$ is completely continuous. $\square$

Lemma 2.5. *Let $u \in K$ and $[a, b]$ be any finite closed interval of $(0, +\infty)$. Then $u(t) \geq \lambda(t)\|u\|$, where*

$$\lambda(t) = \begin{cases} \sigma, & t \geq \sigma, \\ t, & t \leq \sigma, \end{cases}$$

and $\sigma = \inf\left\{\xi \in [0, +\infty) : \sup_{t\in[0,+\infty)} \frac{|u(t)|}{1+t} = \frac{u(\xi)}{1+\xi}\right\}$.

Proof. From the definition of K we know that $u(t)$ is increasing on $[0, +\infty)$. Moreover $u'(\infty) = 0$ implies that the function $\frac{u(t)}{1+t}$ achieves its maximum at $\xi \in [0, +\infty)$. We divide the proof into three steps:

Step (1). If $\sigma \in [0, a]$, then we have $t \geq \sigma$, for $t \in [a, b]$. Since $u(t)$ is increasing on $[0, +\infty)$. So, we have

$$u(t) \geq u(\sigma) = (1 + \sigma)\|u\| > \sigma\|u\|, \quad \text{for } t \in [a, b].$$

Step (2). If $\sigma \in [a, b]$, then we have $t \leq \sigma$, for $t \in [a, \sigma]$. By the concavity of $u(t)$ we can obtain

$$\frac{u(t) - u(0)}{t} \geq \frac{u(\sigma) - u(0)}{\sigma},$$

i.e.

$$\frac{u(t)}{t} \geq \frac{u(\sigma)}{\sigma} - \frac{u(0)}{\sigma} + \frac{u(0)}{t} \geq \frac{u(\sigma)}{1+\sigma} = \|u\|.$$

Therefore $u(t) \geq t\|u\|$, for $a \leq t \leq \sigma$. If $t \in [\sigma, b]$, similarly to Step (1), we have

$$u(t) \geq \sigma\|u\| \quad \text{for } \sigma \leq t \leq b.$$

Step (3). If $\sigma \in [b, +\infty)$, similarly by the concavity of $u(t)$ we also have

$$\frac{u(t) - u(0)}{t} \geq \frac{u(\sigma) - u(0)}{\sigma},$$

which yields $u(t) \geq t\|u\|$, for $a \leq t \leq b \leq \sigma$. The proof is complete. □

Remark 2.1. It is easy to see that

(i) $\lambda(t)$ is nondecreasing on $[a, b]$;

(ii) $0 < \lambda(t) < 1$, for $t \in [a, b] \subset (0, 1)$.

For any $k > 1$ be a fixed constant and we choose $a = \frac{1}{k}$, $b = k$. We define

$$\gamma = \lambda\left(\frac{1}{k}\right) \frac{\lambda\left(\frac{1}{k}\right) \frac{1}{1+\frac{1}{k}} \int_0^{\frac{1}{k}} \varphi^{-1}\left(\int_{\frac{1}{k}}^{k} a(\tau)\mathrm{d}\tau\right) \mathrm{d}s}{\varphi^{-1}\left(\int_0^{+\infty} a(\tau)\mathrm{d}\tau\right)\left(1 + \frac{\sum_{i=1}^{m-2} \alpha_i \xi_{m-2}}{1 - \sum_{i=1}^{m-2} \alpha_i}\right)},$$

$$\gamma_1 = \frac{\lambda\left(\frac{1}{k}\right) \frac{1}{1+\frac{1}{k}} \int_0^{\frac{1}{k}} \varphi^{-1}\left(\int_{\frac{1}{k}}^{k} a(\tau)\mathrm{d}\tau\right) \mathrm{d}s}{\varphi^{-1}\left(\int_0^{+\infty} a(\tau)\mathrm{d}\tau\right)\left(1 + \frac{\sum_{i=1}^{m-2} \alpha_i \xi_{m-2}}{1 - \sum_{i=1}^{m-2} \alpha_i}\right)},$$

$$K_\rho = \{u \in K : \|u\| \leq \rho\},$$

$$\Omega_\rho = \{u \in K : \min_{t\in\left[\frac{1}{k},k\right]} \frac{u(t)}{1+t} < \gamma\rho\} = \{u \in K : \gamma\|u\| \leq \min_{t\in\left[\frac{1}{k},k\right]} \frac{u(t)}{1+t} < \gamma\rho\}.$$

Lemma 2.6 ([6]). *Ω_ρ has the following properties:*

(a) *Ω_ρ is open relative to K.*

(b) *$K_{\gamma\rho} \subset \Omega_\rho \subset K_\rho$.*

(c) *$u \in \partial\Omega_\rho$ if and only if $\min_{t\in\left[\frac{1}{k},k\right]} \frac{u(t)}{1+t} = \gamma\rho$.*

(d) *$u \in \partial\Omega_\rho$, then $\gamma\rho \leq \frac{u(t)}{1+t} \leq \rho$ for $t \in \left[\frac{1}{k}, k\right]$.*

Now, we introduce the following notations. Let

$$f_{\gamma\rho}^{\rho}=\min\left\{\frac{f(t,(1+t)u)}{\varphi(\rho)}:t\in\left[\frac{1}{k},k\right],\ u\in[\gamma\rho,\rho]\right\},$$

$$f_{0}^{\rho}=\sup\left\{\frac{f(t,(1+t)u)}{\varphi(\rho)}:\ t\in[0,+\infty),\ \ u\in[0,\rho]\right\},$$

$$f^{\alpha}=\lim_{u\to\alpha}\sup\left\{\frac{f(t,(1+t)u)}{\varphi(\rho)}:t\in[0,+\infty)\right\},$$

$$f_{\alpha}=\lim_{u\to\alpha}\min\left\{\frac{f(t,(1+t)u)}{\varphi(\rho)}:t\in\left[\frac{1}{k},k\right]\right\}\quad(\alpha:=\infty\text{ or }0^{+}),$$

$$\frac{1}{m}=\varphi^{-1}\left(\int_0^{+\infty}a(\tau)\mathrm{d}\tau\right)\left(1+\frac{\sum_{i=1}^{m-2}\alpha_i\xi_{m-2}}{1-\sum_{i=1}^{m-2}\alpha_i}\right),$$

$$\frac{1}{M}=\lambda\left(\frac{1}{k}\right)\frac{1}{1+\frac{1}{k}}\int_0^{\frac{1}{k}}\varphi^{-1}\left(\int_{\frac{1}{k}}^{k}a(\tau)\mathrm{d}\tau\right)\mathrm{d}s.$$

Remark 2.2. It is easy to see that $0<m,M<\infty$ and $M\gamma=M\gamma_1\lambda\left(\frac{1}{k}\right)=\lambda\left(\frac{1}{k}\right)m<m$.

Lemma 2.7. *If f satisfies the condition*

$$f_0^{\rho}\le\varphi(m)\quad\text{and}\quad u\neq Tu\quad\text{for } u\in\partial K_{\rho},\tag{2.5}$$

then $i_k(T,K_{\rho})=1$.

Proof. By (2.3) and (2.5), we have for $u(t)\in\partial K_{\rho}$, then $\|u\|=\sup_{0\le t<+\infty}\frac{|u(t)|}{1+t}=\rho$, from the definition of f_0^{ρ} we have

$$f(t,u)\le\varphi(\rho)\varphi(m)=\varphi(\rho m).$$

Therefore,

$$\begin{aligned}
\|Tu\| &= \sup_{t\in[0,+\infty)}\frac{1}{1+t}\left|\int_0^t\varphi^{-1}\left(\int_s^{+\infty}a(\tau)f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s+\frac{\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\varphi^{-1}\left(\int_s^{+\infty}a(\tau)f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s}{1-\sum_{i=1}^{m-2}\alpha_i}\right|\\
&\le \sup_{t\in[0,+\infty)}\frac{1}{1+t}\int_0^t\varphi^{-1}\left(\int_s^{+\infty}a(\tau)f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s\\
&\quad+\sup_{t\in[0,+\infty)}\frac{1}{1+t}\frac{\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_{m-2}}\varphi^{-1}\left(\int_s^{+\infty}a(\tau)f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s}{1-\sum_{i=1}^{m-2}\alpha_i}\\
&\le \varphi^{-1}\left(\int_0^{+\infty}a(\tau)f(\tau,u(\tau))\mathrm{d}\tau\right)\left(1+\frac{\sum_{i=1}^{m-2}\alpha_i\xi_{m-2}}{1-\sum_{i=1}^{m-2}\alpha_i}\right)\\
&< \varphi^{-1}(\varphi(m)\varphi(\rho))\varphi^{-1}\left(\int_0^{+\infty}a(\tau)\mathrm{d}\tau\right)\left(1+\frac{\sum_{i=1}^{m-2}\alpha_i\xi_{m-2}}{1-\sum_{i=1}^{m-2}\alpha_i}\right)\\
&= m\rho\varphi^{-1}\left(\int_0^{+\infty}a(\tau)\mathrm{d}\tau\right)\left(1+\frac{\sum_{i=1}^{m-2}\alpha_i\xi_{m-2}}{1-\sum_{i=1}^{m-2}\alpha_i}\right)=\rho=\|u\|.
\end{aligned}$$

This implies that $\|Tu\| < \|u\|$ for $u(t) \in \partial K_\rho$. By Lemma 2.1(1) we have $i_k(T, K_\rho) = 1$. □

Lemma 2.8. *If f satisfies the condition*

$$f^{\rho}_{\gamma\rho} \geq \varphi(M\gamma) \quad \text{and} \quad u \neq Tu \quad \text{for } u \in \partial\Omega_\rho, \tag{2.6}$$

then $i_k(T, \Omega_\rho) = 0$.

Proof. Let $e(t) \equiv 1$ for $t \in [0, +\infty)$. Then $e \in \partial K_1$, we claim that

$$u \neq Tu + \lambda e, \qquad u \in \partial\Omega_\rho, \quad \lambda > 0.$$

In fact, if not, there exist $u_0 \in \partial\Omega_\rho$ and $\lambda_0 > 0$ such that $u_0 = Tu_0 + \lambda_0 e$. By (2.3) and (2.6) we have

$$\begin{aligned}
u_0 &= Tu_0(t) + \lambda_0 e \geq \lambda\left(\frac{1}{k}\right)\|Tu_0\| + \lambda_0 \\
&= \lambda\left(\frac{1}{k}\right)\sup_{t\in[0,+\infty)}\frac{1}{1+t}\left|\int_0^t \varphi^{-1}\left(\int_s^{+\infty} a(\tau)f(\tau,u_0(\tau))\mathrm{d}\tau\right)\mathrm{d}s\right. \\
&\quad \left. + \frac{\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\varphi^{-1}\left(\int_s^{+\infty} a(\tau)f(\tau,u_0(\tau))\mathrm{d}\tau\right)\mathrm{d}s}{1-\sum_{i=1}^{m-2}\alpha_i}\right| + \lambda_0 \\
&\geq \lambda\left(\frac{1}{k}\right)\frac{1}{1+\frac{1}{k}}\int_0^{\frac{1}{k}}\varphi^{-1}\left(\int_{\frac{1}{k}}^{k} a(\tau)f(\tau,u_0(\tau))\mathrm{d}\tau\right)\mathrm{d}s + \lambda_0 \\
&> \lambda\left(\frac{1}{k}\right)\frac{1}{1+\frac{1}{k}}\varphi^{-1}(\varphi(M\gamma)\varphi(\rho))\int_0^{\frac{1}{k}}\varphi^{-1}\left(\int_{\frac{1}{k}}^{k} a(\tau)\mathrm{d}\tau\right)\mathrm{d}s + \lambda_0 \\
&= \lambda\left(\frac{1}{k}\right)\frac{1}{1+\frac{1}{k}}M\gamma\rho\int_0^{\frac{1}{k}}\varphi^{-1}\left(\int_{\frac{1}{k}}^{k} a(\tau)\mathrm{d}\tau\right)\mathrm{d}s + \lambda_0 \\
&= \gamma\rho + \lambda_0.
\end{aligned}$$

This implies that $\gamma\rho \geq \gamma\rho + \lambda_0$ which is a contradiction. Hence by Lemma 2.2(2), we have $i_k(T, \Omega_\rho) = 0$. □

3. Main results

The main results in this paper are the following.

Theorem 3.1. *Assume that one of the following conditions holds:*

(C_3) *There exist $\rho_1, \rho_2, \rho_3 \in (0, \infty)$ with $\rho_1 < \gamma\rho_2$ and $\rho_2 < \rho_3$ such that*

$$f_0^{\rho_1} \leq \varphi(m), \quad f^{\rho_2}_{\gamma\rho_2} \geq \varphi(M\gamma), \quad u \neq Tu \quad \text{for } u \in \partial\Omega_{\rho_2} \quad \text{and} \quad f_0^{\rho_3} \leq \varphi(m).$$

(C_4) *There exist $\rho_1, \rho_2, \rho_3 \in (0, \infty)$ with $\rho_1 < \rho_2 < \gamma\rho_3$ such that*

$$f^{\rho_1}_{\gamma\rho_1} \geq \varphi(M\gamma), \quad f_0^{\rho_2} \leq \varphi(m), \quad u \neq Tu \quad \text{for } u \in \partial K_{\rho_2} \quad \text{and} \quad f^{\rho_3}_{\gamma\rho_3} \geq \varphi(M\gamma).$$

Then (1.1) and (1.2) have two positive solutions in K. Moreover if in (C_3) $f_0^{\rho_1} \leq \varphi(m)$ is replaced by $f_0^{\rho_1} < \varphi(m)$, then (1.1) and (1.2) have a third positive solution $u_3 \in K_{\rho_1}$.

Proof. The proof is similar to that given for Theorem 2.10 in [6]. We omit it here. □

As a special case of Theorem 3.1 we obtain the following result.

Corollary 3.1. *If there exists $\rho > 0$ such that one of the following conditions holds:*

(C_5) $0 \leq f^0 < \varphi(m)$, $f^{\rho}_{\gamma\rho} \geq \varphi(M\gamma)$, $u \neq Tu$ *for* $u \in \partial\Omega_\rho$ *and* $0 \leq f^\infty < \varphi(m)$,

(C_6) $\varphi(M) < f_0 \leq \infty$, $f_0^{\rho} \geq \varphi(m)$, $u \neq Tu$ *for* $u \in \partial K_\rho$ *and* $\varphi(M) < f_\infty \leq \infty$,

then (1.1) and (1.2) has two positive solutions in K.

Proof. We show that (C_5) implies (C_3). It is easy to verify that $0 \le f^0 < \varphi(m)$ implies that there exists $\rho_1 \in (0, \gamma\rho)$ such that $f_0^{\rho_1} < \varphi(m)$. Let $a \in (f^\infty, \varphi(m))$. Then there exists $r > \rho$ such that $\sup_{t\in[0,+\infty)} f(t, (1+t)u) \le a\varphi(u)$ for $u \in [r, \infty)$ since $0 \le f^0 < \varphi(m)$. Let

$$\beta = \max\left\{\sup_{t\in[0,+\infty)} f(t, (1+t)u) : 0 \le u \le r\right\} \quad \text{and} \quad \rho_3 > \varphi^{-1}\left(\frac{\beta}{\varphi(m) - a}\right).$$

Then we have

$$\sup_{t\in[0,+\infty)} f(t, (1+t)u) \le k\varphi(u) + \beta \le k\varphi(\rho_3) + \beta < \varphi(m)\varphi(\rho_3) \quad \text{for } u \in [0, \rho_3].$$

This implies that $f_0^{\rho_3} < \varphi(m)$ and (C_3) holds. Similarly, (C_6) implies (C_4). □

By an argument similar to that of Theorem 3.1 we obtain the following results.

Theorem 3.2. *Assume that one of the following conditions holds:*

(C_7) *There exist* $\rho_1, \rho_2 \in (0, \infty)$ *with* $\rho_1 < \gamma\rho_2$ *such that* $f_0^{\rho_1} \le \varphi(m)$ *and* $f_{\gamma\rho_2}^{\rho_2} \ge \varphi(M\gamma)$.
(C_8) *There exist* $\rho_1, \rho_2 \in (0, \infty)$ *with* $\rho_1 < \rho_2$ *such that* $f_{\gamma\rho_1}^{\rho_1} \ge \varphi(M\gamma)$ *and* $f_0^{\rho_2} \le \varphi(m)$.

Then (1.1) *and* (1.2) *have a positive solution in* K.

As a special case of Theorem 3.2 we obtain the following result.

Corollary 3.2. *Assume that one of the following conditions holds:*

(C_9) $0 \le f^0 < \varphi(m)$ *and* $\varphi(M) < f_\infty \le \infty$.
(C_{10}) $0 \le f^\infty < \varphi(m)$ *and* $\varphi(M) < f_0 \le \infty$.

Then (1.1) *and* (1.2) *have a positive solution in* K.

Remark 3.1. If $\varphi(u) = u$, the problem is second boundary value problem. If $\varphi(u) = |u|^{p-2}u$, $p > 1$, the problem is boundary value problem with p-Laplacian. Then our results of Theorems 3.1 and 3.2 are also new.

4. Example

Example 4.1. As an example we mention the boundary value problem

$$\begin{cases} (\varphi(u'))' + a(t)f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = \dfrac{1}{6}u(1) + \dfrac{1}{6}u(3), \quad u'(\infty) = 0, \end{cases} \tag{4.1}$$

where

$$\varphi(u) = \begin{cases} \dfrac{u^5}{1+u^2}, & u \le 0, \\ u^2, & u > 0, \end{cases}$$

and

$$f(t, u) = \begin{cases} 10^{-5}|\sin t| + \left(\dfrac{u}{1+t}\right)^8, & u \le 2, \\ 10^{-5}|\sin t| + \left(\dfrac{2}{1+t}\right)^8, & u \ge 2, \end{cases}$$

we take $k = 2$, $\lambda(t) = t$ and $\int_0^{+\infty} a(t)\mathrm{d}t = 4$, $\int_{\frac{1}{2}}^{2} a(t)\mathrm{d}t = 1$. It is easy to see by calculating that

$$\frac{1}{m} = \varphi^{-1}\left(\int_0^{+\infty} a(\tau)\mathrm{d}\tau\right)\left(1 + \frac{\sum_{i=1}^{m-2}\alpha_i\xi_{m-2}}{1 - \sum_{i=1}^{m-2}\alpha_i}\right) = \frac{7}{2},$$

$$\frac{1}{M} = \lambda\left(\frac{1}{k}\right)\frac{1}{1+\frac{1}{k}}\int_0^{\frac{1}{k}} \varphi^{-1}\left(\int_{\frac{1}{k}}^{k} a(\tau)\mathrm{d}\tau\right)\mathrm{d}s = \frac{1}{6},$$

$$\gamma=\lambda\left(\frac{1}{k}\right)\frac{\lambda\left(\frac{1}{k}\right)\frac{1}{1+\frac{1}{k}}\int_0^{\frac{1}{k}}\varphi^{-1}\left(\int_{\frac{1}{k}}^{k}a(\tau)\mathrm{d}\tau\right)\mathrm{d}s}{\varphi^{-1}\left(\int_0^{+\infty}a(\tau)\mathrm{d}\tau\right)\left(1+\frac{\sum_{i=1}^{m-2}\alpha_i\xi_{m-2}}{1-\sum_{i=1}^{m-2}\alpha_i}\right)}=\frac{2}{21},$$

thus $m=\frac{2}{7}$, $M=6$ and let $\rho_1=\frac{1}{2}$, $\rho_2=21$, $\rho_3=70$. After some simple calculation we have

$$f(t,(1+t)u)\le 10^{-5}+\frac{1}{256}<\frac{1}{49}=\varphi(m\rho_1)=\varphi(m)\varphi(\rho_1),\quad (t,u)\in[0,+\infty)\times\left[0,\frac{1}{2}\right];$$

this shows $f_0^{\rho_1}<\varphi(m)$. On the other hand,

$$f(t,(1+t)u)\ge 2^8=256>144=\varphi(M\gamma\rho_2)=\varphi(M)\varphi(\gamma\rho_2),\quad (t,u)\in\left[\frac{1}{2},2\right]\times[2,21];$$

we have $f_{\gamma\rho_2}^{\rho_2}>\varphi(M\gamma)$. At last

$$f(t,(1+t)u)\le 10^{-5}+2^8<257<400=\varphi(m\rho_3)=\varphi(m)\varphi(\rho_3),\quad (t,u)\in[0,+\infty)\times[0,70];$$

so we have $f_0^{\rho_3}<\varphi(m)$. Then the condition (C_3) in Theorem 3.1 is satisfied. So boundary value problem (4.1) has at least three positive solutions in K.

Remark 4.1. From the Example 4.1, we can see that φ is not odd, then the boundary value problem with p-Laplacian operator [8,9,12,18] do not apply to Example 4.1. So, we generalize a p-Laplace operator for some $p>1$ and the function φ which we defined above is more comprehensive and general than p-Laplace operator.

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