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Triple positive solutions to third order three-point BVP with increasing homeomorphism and positive homomorphism[☆]

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ABSTRACT

In this paper, we investigate the existence of triple positive solutions for nonlinear differential equations boundary value problems with increasing homeomorphism and positive homomorphism operator. By using fixed point theorems in cones, we establish results on the existence of three positive solutions with suitable growth conditions imposed on the nonlinear term. As applications, two examples are given to demonstrate our result. The conclusions in this paper essentially extend and improve the known results.

1. Introduction

In this paper, we consider the following third order three-point boundary value problem

$$\begin{cases}(\phi(u''(t)))' + a(t)f(u(t)) = 0, \ 0 < t < 1,\\ u(0) = \beta u(\xi), \qquad u'(1) = 0, \qquad \phi(u''(0)) = \delta\phi(u''(\xi)),\end{cases} \tag{1.1}$$

here $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism and positive homomorphism and $\phi(0) = 0, f, a, \beta, \delta, \xi$ satisfy:

(H_1) $0 < \xi < 1, \ 0 < \beta < 1, \ 0 < \delta < 1$;
(H_2) $f : [0, \infty) \to \mathbb{R}^+$ is continuous, $a \in C([0, 1], \mathbb{R}^+)$ and there exists $t_0 \in [0, 1]$ such that $a(t_0) > 0$, where $\mathbb{R}^+ = [0, \infty)$.

A projection $\phi : \mathbb{R} \to \mathbb{R}$ is called an increasing homeomorphism and a positive homomorphism, if the following conditions are satisfied:

(1) if $x \le y$, then $\phi(x) \le \phi(y)$ for all $x, y \in \mathbb{R}$;
(2) ϕ is a continuous bijection and its inverse mapping ϕ^{-1} is also continuous;
(3) $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in [0, +\infty)$.
In the above definition, we can replace condition (3) by the following stronger condition:
(4) $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in \mathbb{R}$.

Remark 1.1. If conditions (1), (2) and (4) hold, then it implies that ϕ is homogeneous, generating a p-Laplacian operator, i.e., $\phi(x) = |x|^{p-2}x$, for some $p > 1$.

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Third order differential equations arise in a variety of different areas of applied mathematics and physics. In recent years, the existence and multiplicity of positive solutions for nonlinear third order ordinary differential equations with a three-point boundary value problem (BVP for short) have been studied by several authors. An interest in triple solutions evolved from the Leggett–Williams multiple fixed points theorem [1]. And lately, two triple-fixed-point theorems due to Avery [2] and Avery and Peterson [3], have been applied to obtain triple solutions for certain three point boundary value problems of third order ordinary differential equations. For example, one may see [4–10] and references therein. In [5], D. R. Anderson considered the the following third order nonlinear boundary value problem

$$\begin{cases} x'''(t) = f(t, x(t)), \quad t_1 < t < t_3, \\ x(t_1) = x'(t_2) = 0, \qquad \gamma x(t_3) + \delta x''(t_3) = 0. \end{cases}$$

He used the Krasnoselskii and Leggett–Williams fixed point theorems to prove the existence of solutions to the nonlinear boundary value problem. In [9], Sun considered the the following third order nonlinear boundary value problem

$$\begin{cases} u'''(t) = a(t)f(t, u(t), u'(t), u''(t)), \quad 0 < t < 1, \\ u(0) = \delta u(\eta), \qquad u'(\eta) = 0, \qquad u''(1) = 0. \end{cases}$$

He used the fixed point theorems due to Avery and Peterson to establish results on the existence of positive solutions to the nonlinear boundary value problem.

On the other hand, the boundary value problems with a p-Laplacian operator have also been discussed extensively in the literature, for example, see [11–17,10]. In [10], Zhou and Ma studied the existence of positive solutions for the following third order generalized right-focal boundary value problem with a p-Laplacian operator

$$\begin{cases} (\phi_p(u''))'(t) = q(t)f(t, u(t)) = 0, \quad 0 \le t \le 1, \\ u(0) = \sum_{i=1}^{m} \alpha_i u(\xi_i), \qquad u'(\eta) = 0, \qquad u''(1) = \sum_{i=1}^{n} \beta_i u''(\theta_i), \end{cases}$$

where $\phi_p(s) = |s|^{p-2}s$, $1 < p \le 2$. They established a corresponding iterative scheme for the boundary value problem by using the monotone iterative technique.

However, to the best of our knowledge, for the increasing homeomorphism and positive homomorphism operator the research has proceeded slowly. In [18], Liu and Zhang studied the existence of positive solutions of quasilinear differential equation

$$\begin{cases} (\phi(x'))' + a(t)f(x(t)) = 0, \quad 0 < t < 1, \\ x(0) - \beta x'(0) = 0, \qquad x(1) + \delta x'(1) = 0, \end{cases}$$

here $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism and positive homomorphism and $\phi(0) = 0$. They obtained the existence of one or two positive solutions by using a fixed-point theorem in cones. For other results which involved an increasing homeomorphism and positive homomorphism operator, the readers are referred to [19–24].

However, there are not many results concerning the existence of triple positive solutions to the third order three-point boundary value problems of nonlinear differential equations with increasing homeomorphism and positive homomorphism operator so far. Whether or not we can obtain triple positive solutions to these kinds of boundary value problems still remains unknown. Motivated greatly by the results mentioned above, especially reference [9], in this paper, we will consider the existence of positive solutions(at least three) to BVP (1.1) by using fixed-point theorems in cones. We improve and generate a p-Laplacian operator and establish some criteria for the existence of triple positive solutions to BVP (1.1).

The methods used in our work will depend on applications of a fixed point theorem due to Avery–Peterson [3] which deals with fixed points of a cone-preserving operator defined on an ordered Banach space, and another fixed point theorem which can be found in [25]. The emphasis here is the differential equation with increasing homeomorphism and the positive homomorphism operator.

The paper is planned as follows. In Section 2, for convenience of the readers we give some definitions and lemmas in order to prove our main results. Section 3 is developed to present and prove our main results. As applications, two examples are given to demonstrate our results in Section 4.

2. Preliminaries and lemmas

In this section, we provide some background materials cited from the cone theory in Banach spaces, and we then state two triple fixed points theorem for a cone preserving operator. The following definitions and lemmas can be found in the monograph by Deimling [26] as well as the monograph by Guo and Lakshmikanthan [25].

Definition 2.1. Let $(E, \|\cdot\|)$ be a real Banach space. A nonempty, closed, convex set $P \subset E$ is said to be a cone provided the following are satisfied:

(a) if $y \in P$ and $\lambda \ge 0$, then $\lambda y \in P$;
(b) if $y \in P$ and $-y \in P$, then $y = 0$.

If $P \subset E$ is a cone, we denote the order induced by P on E by $\leq$, that is, $x \leq y$ if and only if $y - x \in P$.

Definition 2.2. A map α is said to be a nonnegative, continuous, concave(resp. convex) functional in a cone P of a real Banach space E, if $\alpha : P \to [0, \infty)$ is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y) \quad \text{for all } x, y \in P \text{ and } t \in [0, 1],$$
$$(\alpha(tx + (1-t)y) \leq t\alpha(x) + (1-t)\alpha(y) \quad \text{for all } x, y \in P \text{ and } t \in [0, 1], \text{ respectively}).$$

Let γ and θ be nonnegative continuous convex functionals on P, Let α be a nonnegative continuous concave functional on P, and let ψ be a nonnegative continuous functional on P. Then for positive numbers a, b, k and c, we define the following convex sets of P :

$$P(\gamma, c) = \{x \in P | \gamma(x) < c\},$$
$$P(\alpha, b; \gamma, c) = \{x \in P | b \leq \alpha(x), \gamma(x) \leq c\},$$
$$P(\alpha, b; \theta, k; \gamma, c) = \{x \in P | b \leq \alpha(x), \theta(x) \leq k, \gamma(x) \leq c\},$$

and a closed set

$$R(\psi, a; \gamma, c) = \{x \in P | a \leq \psi(x), \gamma \leq c\}.$$

Theorem 2.1 ([3]). *Let P be a cone in a real Banach space E. Let γ and θ be nonnegative continuous convex functionals on P, let α be a nonnegative continuous concave functional on P, and let ψ be a nonnegative continuous functional on P satisfying $\psi(\lambda x) \leq \lambda\psi(x)$forall $0 \leq \lambda \leq 1$, such that for some positive numbers M and c,*

$$\alpha(x) \leq \psi(x), \qquad \|x\| \leq M\gamma(x) \quad \textit{forall } x \in \overline{P(\gamma, c)}.$$

Suppose that $T : \overline{P(\gamma, c)} \to \overline{P(\gamma, c)}$ is completely continuous and there exist positive numbers a, b and k with $0 < a < b$ such that

(S_1) *$\{x \in P(\alpha, b; \theta, k; \gamma, c) | \alpha(x) > b\} \neq \emptyset$ and $\alpha(Tx) > b$ for $x \in P(\alpha, b; \theta, k; \gamma, c)$;*
(S_2) *$\alpha(Tx) > b$ for $x \in P(\alpha, b; \gamma, c)$ with $\theta(Tx) > k$;*
(S_3) *$0 \notin R(\psi, a; \gamma, c)$ and $\psi(Tx) < a$ for $x \in R(\psi, a; \gamma, c)$ with $\psi(x) = a$.*

Then T has at least three fixed points x_1, x_2, and $x_3 \in \overline{P(\gamma, c)}$ such that

$$\gamma(x_i) \leq c, \quad i = 1, 2, 3; \qquad b < \alpha(x_1); \qquad a < \psi(x_2) \quad with\, \alpha(x_2) < b; \;\; \psi(x_3) < a.$$

Theorem 2.2 ([25]). *Let A be a bounded closed convex subset of a Banach space E. Assume that A_1, A_2 are disjoint closed convex subsets of A and U_1, U_2 are nonempty open subsets of A with $U_1 \subset A_1$ and $U_2 \subset A_2$. Suppose that $T : A \to A$ is completely continuous and the following conditions hold:*

(i) $T(A_1) \subset A_1,\ T(A_2) \subset A_2$;
(ii) *T has no fixed points in $(A_1 \setminus U_1) \cup (A_2 \setminus U_2)$.*

Then T has at least three fixed points x_1, x_2 and x_3 such that $x_1 \in U_1, x_2 \in U_2$ and $x_3 \in A \setminus (A_1 \cup A_2)$.

Lemma 2.1. *If condition (H_1) holds, then for $h \in C([0, 1], \mathbb{R})$, the boundary value problem*

$$\begin{cases} u''(t) + h(t) = 0, & 0 < t < 1, \\ u(0) = \beta u(\xi), & u'(1) = 0 \end{cases} \tag{2.1}$$

has a unique solution

$$u(t) = \int_0^t (1-s)h(s) + \frac{\beta \int_0^\xi (1-s)h(s)\mathrm{d}s}{1-\beta}. \tag{2.2}$$

Proof. By calculating, we can easily get (2.2). So we omit it. □

Lemma 2.2. *If condition (H_1) holds, then for $h \in C([0, 1], \mathbb{R})$, the boundary value problem*

$$\begin{cases} (\phi(u''(t)))' + h(t) = 0, & 0 < t < 1, \\ u(0) = \beta u(\xi), \quad u'(1) = 0, & \phi(u''(0)) = \delta\phi(u''(\xi)) \end{cases} \tag{2.3}$$

has a unique solution

$$u(t)=\int_0^t (1-s)\phi^{-1}\left(\int_0^s h(r)\mathrm{d}r+C\right)\mathrm{d}s+\frac{\beta\int_0^{\xi}(1-s)\phi^{-1}\left(\int_0^s h(r)\mathrm{d}r+C\right)\mathrm{d}s}{1-\beta},$$

where $C=\frac{\delta\int_0^{\xi}h(r)\mathrm{d}r}{1-\delta}$, $\phi^{-1}(s)$ *is the inverse function to* $\phi(s)$.

Proof. Integrating both sides of equation in (2.3) on $[0,t]$, we have

$$\phi(u''(t))=\phi(u''(0))-\int_0^t h(r)\mathrm{d}r, \tag{2.4}$$

so

$$\phi(u''(\xi))=\phi(u''(0))-\int_0^{\xi} h(r)\mathrm{d}r.$$

By the boundary value condition $\phi(u''(0))=\delta\phi(u''(\xi))$, we have

$$\phi(u''(0))=-\frac{\delta\int_0^{\xi}h(r)\mathrm{d}r}{1-\delta}. \tag{2.5}$$

By (2.4) and (2.5) we know

$$u''(t)=-\phi^{-1}\left(\frac{\delta\int_0^{\xi}h(r)\mathrm{d}r}{1-\delta}+\int_0^t h(r)\mathrm{d}r\right).$$

This together with Lemma 2.1 imply that

$$u(t)=\int_0^t (1-s)\phi^{-1}\left(\int_0^s h(r)\mathrm{d}r+C\right)\mathrm{d}s+\frac{\beta\int_0^{\xi}(1-s)\phi^{-1}(\int_0^s h(r)\mathrm{d}r+C)\mathrm{d}s}{1-\beta},$$

where $C=\frac{\delta\int_0^{\xi}h(r)\mathrm{d}r}{1-\delta}$. The proof is complete. □

Lemma 2.3. *Let condition* (H_1) *hold. If* $h\in C([0,1],\mathbb{R}^+)$, *then the unique solution* $u(t)$ *of* (2.3) *satisfies*

$$u(t)\geq 0,\quad t\in[0,1].$$

Proof. By $u''(t)=-\phi^{-1}\left(\frac{\delta\int_0^{\xi}h(r)\mathrm{d}r}{1-\delta}+\int_0^t h(r)\mathrm{d}r\right)\leq 0$, we know that the graph of $u(t)$ is concave down on (0,1) and $u'(t)$ is nonincreasing on [0,1]. This together with the assumption that the boundary condition $u'(1)=0$ implies that $u'(t)\geq 0$ for $t\in[0,1]$. This implies that

$$\|u\|=u(1),\qquad \min_{t\in[0,1]}u(t)=u(0).$$

So we only prove $u(0)\geq 0$. By condition (H_1) we have

$$u(0)=\frac{\beta\int_0^{\xi}(1-s)\phi^{-1}\left(\int_0^s h(r)\mathrm{d}r+C\right)\mathrm{d}s}{1-\beta}\geq 0.\quad □$$

3. Main results

In this section, two existence results of triple positive solutions to BVP (1.1) are established by imposing some conditions on f and defining a suitable Banach space and a cone.

Let $E=C([0,1])$ be endowed with the ordering $x\leq y$ if $x(t)\leq y(t)$ for all $t\in[0,1]$, and $\|u\|=\max_{t\in[0,1]}|u(t)|$ is defined as usual by maximum norm. Clearly, it follows that $(E,\|u\|)$ is a Banach space.

We define a cone $P\subset E$ by

$$P=\{u: u\in E, u(t) \text{ is concave, nondecreasing and nonnegative on } [0,1],\ u'(1)=0\}.$$

Let $\eta\geq\frac{1}{2}$, and fix $l\in[0,1]$ such that $0<\eta<l<1$, and define the nonnegative continuous convex functionals γ and θ, the nonnegative continuous concave functional α, and the nonnegative continuous functional ψ on the cone P by

$$\gamma(u)=\theta(u)=\max_{t\in[0,l]}u(t)=u(l),$$

$$\alpha(u)=\min_{t\in[\eta,1]}u(t)=u(\eta),\qquad \psi(u)=\max_{t\in[0,\eta]}=u(\eta).$$

For notational convenience, denote by

$$\hat{C} = \frac{\delta \int_0^\xi a(r)\mathrm{d}r}{1-\delta},$$

$$m_\eta = \int_0^\eta (1-s)\phi^{-1}\left(\int_0^\xi a(r)\mathrm{d}r + \hat{C}\right)\mathrm{d}s, \qquad m = \frac{\beta \int_0^\xi (1-s)\phi^{-1}\left(\int_0^s a(r)\mathrm{d}r + \hat{C}\right)\mathrm{d}s}{1-\beta}$$

$$M_l = \int_0^l (1-s)\phi^{-1}\left(\int_0^s a(r)\mathrm{d}r + \hat{C}\right)\mathrm{d}s + \frac{\beta \int_0^\xi (1-s)\phi^{-1}\left(\int_0^s a(r)\mathrm{d}r + \hat{C}\right)\mathrm{d}s}{1-\beta}$$

$$M_\eta = \int_0^\eta (1-s)\phi^{-1}\left(\int_0^s a(r)\mathrm{d}r + \hat{C}\right)\mathrm{d}s + \frac{\beta \int_0^\xi (1-s)\phi^{-1}\left(\int_0^s a(r)\mathrm{d}r + \hat{C}\right)\mathrm{d}s}{1-\beta}$$

$$M = \int_0^1 (1-s)\phi^{-1}\left(\int_0^s a(r)\mathrm{d}r + \hat{C}\right)\mathrm{d}s + \frac{\beta \int_0^\xi (1-s)\phi^{-1}\left(\int_0^s a(r)\mathrm{d}r + \hat{C}\right)\mathrm{d}s}{1-\beta}.$$

In our main results we will make use of the following lemmas.

Lemma 3.1 ([27]). *If $u \in P$, then*
(1) $u(t) \geq t\|u\|$ *for all* $t \in [0, 1]$;
(2) $\frac{u(s)}{s} \geq \frac{u(t)}{t}$, *for* $t, s \in [0, 1]$ *with* $s \leq t$.

Define an operator $T : P \to E$ by

$$Tu(t) = \int_0^t (1-s)\phi^{-1}\left(\int_0^s a(r)f(u(r))\mathrm{d}r + \tilde{C}\right)\mathrm{d}s + \frac{\beta \int_0^\xi (1-s)\phi^{-1}\left(\int_0^s a(r)f(u(r))\mathrm{d}r + \tilde{C}\right)\mathrm{d}s}{1-\beta},$$

where $\tilde{C} = \frac{\delta \int_0^\xi a(r)f(u(r))\mathrm{d}r}{1-\delta}$. Then u is a solution of boundary value problem (1.1) if and only if u is a fixed point of operator T. Obviously, for $u \in P$ one has $(Tu)(t) \geq 0$ for $t \in [0, 1]$. In addition, $(Tu)''(t) \leq 0$ for $t \in [0, 1]$ and $(Tu)'(1) = 0$, this implies $TP \subset P$. With standard argument one may show that $T : P \to P$ is completely continuous.

Theorem 3.1. *Suppose that conditions* (H_1) *and* (H_2) *hold, and there exist positive numbers* a, b, c *with* $a < \eta b < b < lc$, $M_l b < mc$ *such that*
(B_1) $f(u) \leq \phi\left(\frac{c}{M_l}\right)$, $u \in \left[0, \frac{c}{l}\right]$;
(B_2) $f(u) > \phi\left(\frac{b}{m\eta}\right)$, $u \in \left[b, \frac{b}{l^2}\right]$;
(B_3) $f(u) < \phi\left(\frac{a}{M_\eta}\right)$, $u \in \left[0, \frac{a}{\eta}\right]$.

Then the BVP (1.1) has at least three positive solutions u_1, u_2 *and* $u_3 \in \overline{P(\gamma, c)}$ *satisfying*

$$\gamma(u_i) \leq c, \quad i = 1, 2, 3,$$

and

$$b < \alpha(u_1), \qquad \alpha(u_2) < b, \qquad a < \psi(u_2), \qquad \psi(u_3) < a.$$

Proof. BVP (1.1) has a solution $u = u(t)$ if and only if u solves the operator equation $u = Tu$. Thus we set out to verify that the operator T satisfies the fixed point theorem, which then implies the existence of three fixed points of T.

Based on Lemma 3.1, it is clear that for $u \in P$ and $\lambda \in [0, 1]$, there are $\alpha(u) = \psi(u)$, $\psi(\lambda u) = \lambda\psi(u)$ and $\|u\| \leq \frac{1}{l}u(l) = \frac{1}{l}\gamma(u)$. Furthermore, $\psi(0) = 0 < a$ and therefore $0 \notin R(\psi, a; \gamma, c)$.

For $u \in \overline{P(\gamma, c)}$, we have $0 \leq u \leq \|u\| \leq \frac{1}{l}\gamma(u) \leq \frac{1}{l}c$. By condition (B_1) one derives

$$\begin{aligned}\gamma(Tu) &= Tu(l) = \int_0^l (1-s)\phi^{-1}\left(\int_0^s a(r)f(u(r))\mathrm{d}r + \tilde{C}\right)\mathrm{d}s + \frac{\beta \int_0^\xi (1-s)\phi^{-1}(\int_0^s a(r)f(u(r))\mathrm{d}r + \tilde{C})\mathrm{d}s}{1-\beta}\\ &\leq \frac{c}{M_l}\left(\int_0^l (1-s)\phi^{-1}\left(\int_0^s a(r)\mathrm{d}r + \tilde{C}\right)\mathrm{d}s + \frac{\beta \int_0^\xi (1-s)\phi^{-1}\left(\int_0^s a(r)\mathrm{d}r + \tilde{C}\right)\mathrm{d}s}{1-\beta}\right)\\ &= c.\end{aligned}$$

Therefore, $T : \overline{P(\gamma, c)} \to \overline{P(\gamma, c)}$.

To check condition (S_1) of Theorem 2.1, We choose $u \equiv \frac{b}{l}$ and $k = \frac{b}{l}$, it is easy to see that

$$\alpha(u) = u(\eta) = \frac{b}{l} > b, \qquad \theta(u) = u(l) = \frac{b}{l}, \qquad \gamma(u) = \frac{b}{l} < c,$$

which means $\{u \in P(\alpha, b; \theta, \frac{b}{l}; \gamma, c) | \alpha(u) > b\} \neq \emptyset$. Hence for $u \in P(\alpha, b; \theta, \frac{b}{l}; \gamma, c)$, we have $b \leq u(t) \leq \frac{b}{l^2}$ for $t \in [\eta, 1]$. It follows from condition (B_2) that

$$\begin{aligned}\alpha(Tu) &= Tu(\eta) = \int_0^\eta (1-s)\phi^{-1}\left(\int_0^s a(r)f(u(r))\mathrm{d}r + \tilde{C}\right)\mathrm{d}s + \frac{\beta\int_0^\xi (1-s)\phi^{-1}(\int_0^s a(r)f(u(r))\mathrm{d}r + \tilde{C})\mathrm{d}s}{1-\beta}\\ &\geq \frac{b}{m_\eta}\int_0^\eta (1-s)\phi^{-1}\left(\int_0^s a(r)\mathrm{d}r + \hat{C}\right)\mathrm{d}s\\ &= b.\end{aligned}$$

i.e.,$\alpha(Tu) > b$ for all $u \in P(\alpha, b; \theta, \frac{b}{l}; \gamma, c)$, This shows that condition (S_1) of Theorem 2.1 is satisfied.

Moreover, if $u \in P(\alpha, b; \gamma, c)$ and $\theta(Tu) > c$, then due to (2) of Lemma 3.1 we have

$$\alpha(Tu) = Tu(\eta) \geq \frac{\eta}{l}(Tu)(l) = \frac{\eta}{l}\theta(Tu) > \frac{\eta c}{l} > \frac{\eta b}{l^2} > b.$$

Thus condition (S_2) of Theorem 2.1 is satisfied.

Finally, we show that condition (S_3) of Theorem 2.1 holds as well. Clearly, $0 \notin R(\psi, a; \gamma, c)$ since $\psi(0) = 0 < a$. Suppose that $u \in R(\psi, a; \gamma, c)$ with $\psi(u) = a$, then $0 \leq u \leq \|u\| \leq \frac{1}{\eta}u(\eta) = \frac{1}{\eta}\psi(u) = \frac{a}{\eta}$. By condition (B_3), we obtain that

$$\begin{aligned}\psi(Tu) &= Tu(\eta) = \int_0^\eta (1-s)\phi^{-1}\left(\int_0^s a(r)f(u(r))\mathrm{d}r + \tilde{C}\right)\mathrm{d}s + \frac{\beta\int_0^\xi (1-s)\phi^{-1}\left(\int_0^s a(r)f(u(r))\mathrm{d}r + \tilde{C}\right)\mathrm{d}s}{1-\beta}\\ &\leq \frac{a}{M_\eta}\left(\int_0^l (1-s)\phi^{-1}\left(\int_0^s a(r)\mathrm{d}r + \tilde{C}\right)\mathrm{d}s + \frac{\beta\int_0^\xi (1-s)\phi^{-1}\left(\int_0^s a(r)\mathrm{d}r + \tilde{C}\right)\mathrm{d}s}{1-\beta}\right)\\ &= a.\end{aligned}$$

Hence, we have $\psi(Tu) \leq a$. So condition (S_3) of Theorem 2.1 is satisfied.

Since all conditions of Theorem 2.1 are satisfied, BVP (1.1) has at least three positive solutions $u_1, u_2,$ and $u_3 \in \overline{P(\gamma, d)}$ such that

$$\gamma(u_i) \leq d, \quad i = 1, 2, 3; \qquad b < \alpha(u_1); \qquad a < \psi(u_2) \quad \text{with}\ \alpha(u_2) < b; \qquad \psi(u_3) < a.$$

The proof is complete. □

Theorem 3.2. *Suppose that condition* (H_1) *and* (H_2) *hold. Let* $0 < a < b < c$, $Mb < mc$ *and*

(C_1) $f(u) < \phi(\frac{a}{M}), u \in [0, a]$;
(C_2) *there exist an number* $d > c$ *such that* $f(u) < \phi(\frac{d}{M}), u \in [0, d]$;
(C_3) $\phi(\frac{b}{m}) < f(u) < \phi(\frac{c}{M}), u \in [b, c]$.

Then BVP (1.1) *has at least three positive solutions* $u_1, u_2,$ and u_3 *such that*

$$b < u_1(t) < c, \qquad \|u_2\| < a, \quad \text{and} \quad \|u_3\| > a.$$

Proof. We first show that $T(\overline{P_a}) \subseteq P_a \subseteq \overline{P_a}$ if condition (C_1) holds. If $u \in \overline{P_a}$, then $0 \leq u \leq \|u\| \leq a$, which implies $f(u) < \phi(\frac{a}{M})$. We have

$$\begin{aligned}\|Tu\| &= Tu(1)\\ &= \int_0^1 (1-s)\phi^{-1}\left(\int_0^s a(r)f(u(r))\mathrm{d}r + \tilde{C}\right)\mathrm{d}s + \frac{\beta\int_0^\xi (1-s)\phi^{-1}\left(\int_0^s a(r)f(u(r))\mathrm{d}r + \tilde{C}\right)\mathrm{d}s}{1-\beta}\\ &\leq \frac{a}{M}\left(\int_0^1 (1-s)\phi^{-1}\left(\int_0^s a(r)\mathrm{d}r + \tilde{C}\right)\mathrm{d}s + \frac{\beta\int_0^\xi (1-s)\phi^{-1}\left(\int_0^s a(r)\mathrm{d}r + \tilde{C}\right)\mathrm{d}s}{1-\beta}\right)\\ &= a.\end{aligned}$$

This implies that $T(\overline{P_a}) \subseteq P_a \subseteq \overline{P_a}$.

Next, condition (C_2) indicates that there exists $d > c$ such that $T(\overline{P_d}) \subseteq P_d$. Now we let

$$A = \overline{P_d}, \qquad A_1 = [\phi_b, \phi_c], \qquad U_1 = \mathrm{int}(A_1), \qquad A_2 = \overline{P_a}, \qquad U_2 = P_a,$$

where for a real number b, $\phi_b : [0, 1] \to [0, \infty)$ is continuous, $\phi_b(t) = b$, for $t \in [0, 1]$; $\mathrm{int}(A_1)$ is the interior of A_1. Then we have $T(A) \subset A$, $T(A_2) \subset A_2$. Moreover, $T(\overline{P_a}) \subseteq P_a \subseteq \overline{P_a}$ means $T(A_2) \subseteq U_2 \subseteq A_2$. Thus T has no fixed point in $(A_2 \setminus U_2)$.

To show $T(A_1) \subset A_1$ and T has no fixed point in $(A_1 \setminus U_1)$. Set $u \in A_1$, following the definition of ϕ_b, we can know $b \le u(t) \le c$, for $t \in [0, 1]$. Condition (C_3) then gives rise to $\phi(\frac{b}{m}) < f(u) < \phi(\frac{c}{M})$, which in turn produces

$$\begin{aligned}
(Tu)(t) &\ge (Tu)(0) \\
&= \frac{\beta\int_0^{\xi}(1-s)\phi^{-1}\left(\int_0^s a(r)f(u(r))\mathrm{d}r + \tilde{C}\right)\mathrm{d}s}{1-\beta} \\
&> \frac{b}{m}\,\frac{\beta\int_0^{\xi}(1-s)\phi^{-1}\left(\int_0^s a(r)\mathrm{d}r + \tilde{C}\right)\mathrm{d}s}{1-\beta} \\
&= b
\end{aligned}$$

and

$$\begin{aligned}
(Tu)(t) &\le (Tu)(1) \\
&= \int_0^1 (1-s)\phi^{-1}\left(\int_0^s a(r)f(u(r))\mathrm{d}r + \tilde{C}\right)\mathrm{d}s + \frac{\beta\int_0^{\xi}(1-s)\phi^{-1}\left(\int_0^s a(r)f(u(r))\mathrm{d}r + \tilde{C}\right)\mathrm{d}s}{1-\beta} \\
&\le \frac{c}{M}\left(\int_0^1 (1-s)\phi^{-1}\left(\int_0^s a(r)\mathrm{d}r + \tilde{C}\right)\mathrm{d}s + \frac{\beta\int_0^{\xi}(1-s)\phi^{-1}\left(\int_0^s a(r)\mathrm{d}r + \tilde{C}\right)\mathrm{d}s}{1-\beta}\right) \\
&= c.
\end{aligned}$$

Combining the above two inequalities one achieves $\phi_b(t) = b < (Tu)(t) < c = \phi_c(t)$, for $t \in [0, 1]$. That is, $Tu \in U_1$. So $T(A_1) \subseteq U_1 \subset A_1$ and T has no fixed point in $(A_1 \setminus U_1)$. Therefore, all conditions of Theorem 2.2 are fulfilled and BVP (1.1) has at least three positive solutions u_1, u_2, u_3 such that

$$b < u_1(t) < c, \qquad \|u_2\| < a, \quad \text{and} \quad \|u_3\| > a. \quad \square$$

4. Examples

In this section, we present two examples to demonstrate our main results.

Example 4.1. Consider the following third order three-point boundary value problem

$$\begin{cases} (\phi(u''(t)))' + a(t)f(u(t)) = 0, \quad 0 \le t \le 1, \\ u(0) = \dfrac{1}{3}u\left(\dfrac{1}{2}\right), \qquad u'(1) = 0, \qquad \phi(u''(0)) = \dfrac{1}{4}\phi\left(u''\left(\dfrac{1}{2}\right)\right), \end{cases} \tag{4.1}$$

where $\phi(x) = x$, $a(t) \equiv 1$, $\beta = \frac{1}{3}$, $\delta = \frac{1}{4}$, $\xi = \frac{1}{2}$.

We choose $\eta = \frac{1}{2}$, by calculating we know $m_\eta = \frac{11}{24}$, $M_l = \frac{351}{256}$, $M_\eta = \frac{33}{48}$. Let $a = 100$, $b = 245$, $c = 770$, $l = \frac{7}{8}$, then $a < \eta b < b < lc$. Obviously, $M_l b < mc$. We define a nonlinearity f as follows:

$$f(u) = \begin{cases} 140, & u \in [0, 200], \\ 140 + \dfrac{410}{45}(u - 200), & u \in [200, 245], \\ 550, & u \in [245, 320], \\ 550 + \dfrac{5}{560}(u - 320), & u \in [320, +\infty). \end{cases}$$

Then by the definition of f, we have

(i) $f(u) \le \phi(\frac{c}{M_l}) \approx 557.2$, $u \in [0, 880]$;
(ii) $f(u) > \phi(\frac{b}{m_\eta}) \approx 534.2$, $u \in [245, 320]$;
(iii) $f(u) < \phi(\frac{a}{M_\eta}) \approx 145.4$, $u \in [0, 200]$.

By Theorem 3.1, BVP (4.1) has at least three positive solutions.

Example 4.2. Consider the following third order three point boundary value problem

$$\begin{cases}(\phi(u''(t)))' + a(t)f(u(t)) = 0, \quad 0 \le t \le 1,\\ u(0) = \dfrac{1}{3}u\left(\dfrac{1}{2}\right), \qquad u'(1) = 0, \qquad \phi(u''(0)) = \dfrac{1}{4}\phi\left(u''\left(\dfrac{1}{2}\right)\right),\end{cases} \tag{4.2}$$

where $\phi(x) = x$, $a(t) \equiv 1$, $\beta = \frac{1}{3}$, $\delta = \frac{1}{4}$, $\xi = \frac{1}{2}$, $\eta = \frac{1}{2}$.

By calculating we can know $m = \frac{11}{48}$, $M = \frac{83}{48}$. Let $a = 7$, $b = 12$, $c = 336$, $l = \frac{7}{8}$, then $a < b < b < lc$. Obviously, $Mb < mc$. We define a nonlinearity f as follows:

$$f(u) = \begin{cases}3, & u \in []0, 7],\\ 3 + \dfrac{97}{25}(u-7)^2, & u \in [7, 12],\\ 100, & u \in [12, 336],\\ 100 + \dfrac{1100}{1764}(u - 336), & u \in [336, +\infty).\end{cases}$$

Then by the definition of f, we have

(i) $f(u) \le \phi(\frac{a}{M}) \approx 4.2$, $u \in [0, 7]$;
(ii) and there exists $d = 2100 > c$ such that $f(u) \le \phi\left(\frac{d}{M}\right) \approx 1214.4$, $u \in [0, 2100]$;
(iii) $\phi(\frac{b}{m}) \approx 52.4 < f(u) < \phi(\frac{c}{M}) \approx 194.3$, $u \in [12, 336]$.

By Theorem 3.2, BVP (4.2) has at least three positive solutions.

Remark 4.1. Consider the following nonlinear three point boundary value problem

$$\begin{cases}(\phi(u''(t)))' + a(t)f(u(t)) = 0, \quad 0 < t < 1,\\ u(0) = \beta u(\xi), \qquad u'(1) = 0, \qquad \phi(u''(0)) = \delta\phi(u''(\xi)),\end{cases}$$

where

$$\phi(u) = \begin{cases}u^3, & u \le 0,\\ u^2, & u > 0,\end{cases}$$

f and a satisfy the condition (H_1) and (H_2). It is clear that $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism and homomorphism and $\phi(0) = 0$. Because p-Laplacian operators are odd, they do not apply to our example. Hence we generalize the boundary value problem with the p-Laplacian operator and the results [16,17] do not apply to the example.

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