Journal of Computational and Applied Mathematics 234 (2010) 2562–2577

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Study of a finite element method for the time-dependent generalized Stokes system associated with viscoelastic flow

J.H. Carneiro de Araujo [a,b], P.D. Gomes [b], V. Ruas [b,c,*]

[a] *Departamento de Ciência da Computação, Universidade Federal Fluminense, Niterói, Rio de Janeiro state, Brazil*

[b] *Programa de Pós-Graduação em Ciência da Computação, Universidade Federal Fluminense, Niterói, Rio de Janeiro state, Brazil*

[c] *UPMC Univ. Paris 6, UMR 7190, Institut Jean Rond d'Alembert/ CNRS, Paris, France*

ARTICLE INFO

Article history:
Received 28 August 2009
Received in revised form 25 February 2010

Keywords:
Explicit solution
Finite elements
Piecewise linear
Stokes system
Three-field methods
Time-dependent
Viscoelastic flows

ABSTRACT

A three-field finite element scheme designed for solving systems of partial differential equations governing time-dependent viscoelastic flows is studied. Once a classical backward Euler time discretization is performed, the resulting three-field system of equations allows for a stable approximation of velocity, pressure and extra stress tensor, by means of continuous piecewise linear finite elements, in both two- and three- dimensional space. This is proved to hold for the linearized form of the system. An advantage of the new formulation is the fact that it provides an algorithm for the explicit iterative resolution of system nonlinearities. Convergence in an appropriate sense applying to these three flow fields is demonstrated.

1. Introduction

The numerical simulation of the flow of viscoelastic liquids is known to be a delicate problem in many respects. First of all, the models most frequently in use involve three strongly coupled fields, namely, the velocity $\mathbf{u}$, the pressure p and the extra stress tensor σ. Furthermore the highly nonlinear system of partial differential equations that govern this kind of flow may change type, according to different parameters or flow conditions. Nevertheless in the past two decades a lot of progress has been accomplished in deriving numerical methodology in order to overcome such difficulties.

As far as multi-field finite element methods suitable for treating this class of problems in a reliable way are concerned, the work of Fortin and collaborators (see e.g. [1]), incorporating a fourth field, namely, the strain rate tensor, is among the most outstanding contributions in this direction. In particular it significantly advanced the numerical simulation of viscolelastic fluid flow in three-dimensional space, which became more widespread in the past decade (cf. [2]). As for the numerical analysis of finite element methods for the complete set of equations governing viscoelastic flows, the contribution of Baranger and Sandri (see e.g. [3]) is the main reference. More recently Codina [4] adopted an interesting stabilizing technique for the equal order finite element approximation of the three-field Stokes system, using sub-scales. Confining their analysis to the linearized case, the authors Carneiro de Araujo and Ruas themselves attempted to bring about valid alternatives to study this class of problems, mostly in the two-dimensional case. This was achieved through drastic reductions of the number of degrees of freedom necessary to obtain reliable approximations (cf. [5]) as compared to other methods in use of the same order (cf. [6]). However in the framework of three-dimensional flows, such an approach is not satisfactory, since the final

* Corresponding address: Universite Pierre et Marie Curie, Institut Jean le Rond d'Alembert, 4 place Jussieu, couloir 55-65, 5e étage., 75252, Paris cedex 05, France. Tel.: +33 1 49570436; fax: +33 1 44275259.

E-mail addresses: jhca@ic.uff.br (J.H. Carneiro de Araujo), dgpatricia@ic.uff.br (P.D. Gomes), vitoriano.ruas@upmc.fr (V. Ruas).

0377-0427/$ – see front matter
doi:10.1016/j.cam.2010.03.025

number of degrees of freedom remains excessively high anyway. That is why the third author also worked on three-field methods using $\mathbf{u}-\sigma$ stabilizing techniques, in order to reduce the number of extra-stress degrees of freedom [7]. The key to this approach is a Galerkin least square formulation proposed and exploited by Franca, Stenberg and collaborators in the mid-eighties (cf. [8,9]).

As far as time-dependent viscolelastic flow is concerned, the numerical analysis of classical time-marching schemes combined with velocity–pressure mixed finite elements suitable for treating incompressible flow were first reported in the mid-nineties (cf. [10,11]). Some other works on this topic have appeared in the past decade, such as [12–15]. Very recent papers [16,17] by Chrispell, Ervin and Jenkins based on the θ-method for the time-integration, combined with Taylor–Hood elements and a SUPG treatment of advective terms supply analyses of the full nonlinear time-dependent system of equations governing viscoelastic flow. However, as we should point out, most of those works avoided the crucial issue of extra stress vs. velocity numerical stability. This was achieved by considering only models incorporating a purely viscous term in the constitutive law, such as Oldroyd's or Johnson-Segalman's model (cf. [18]). It should also be stressed that the same works overlooked error estimates for the pressure, even though this is an essential point to ensure reliability of the flow simulation. Nevertheless, since handling system nonlinearities is a challenge that any numerical methodology must be able to take up efficiently, it can be asserted that the series of works quoted above provided decisive contributions to the simulation of time dependent viscolelastic flow.

Keeping in mind some limitations of the above work pointed out in the previous paragraph, in [19,20] the authors studied a stabilization technique applied to the three-field scheme, based on time integration for solving stationary or time-dependent viscoelastic flow equations in both two- and three-dimensional space. The study was carried out for the time-dependent three-field Stokes system obtained through linearization of the system governing viscolelastic flow, even for stationary problems. Indeed in this case time integration is purely fictitious and plays the role of an iterative solution method. It was shown in [21,20] that such iterations combined with standard piecewise linear representations of the three fields, give rise to convergent approximations when applied to the stationary Stokes problem as a linearized form of the stationary Navier–Stokes equations and the viscoelastic flow equations, respectively. Actually a fundamental ingredient of our numerical strategy described below is aimed at overcoming simultaneously both the difficulties mentioned at the beginning of this Section. More specifically the splitting algorithm in use allows the solution of the three-field system at every time step in an explicit manner, as far as the velocity and the extra stress tensor are concerned, for a time step of the same order as the mesh step size. The pressure in turn is determined as the solution of a consistent pressure Poisson equation, following ideas already exploited by Goldberg & Ruas [22].

2. Maxwell flow equations

Although the techniques to be developed hereafter extend in a straightforward manner to the case of a wide spectrum of viscolelastic constitutive laws, we consider as a model the case of Maxwell fluids. This choice is due to the fact that, in principle, Maxwell models require extra stress–velocity compatible representations. Let Ω be a bounded domain of $\mathbb{R}^N$, $N=2$ or 3, with boundary $\partial\Omega$. Under the action of volumetric forces $\mathbf{f}$, we consider the evolution in time t of the flow in Ω of a viscoelastic liquid obeying a constitutive law of the differential type. Throughout this work we assume that the velocity of the liquid is prescribed on $\partial\Omega$, say $\mathbf{u}=\mathbf{g}$, where $\mathbf{g}$ satisfies the conservation property $\int_{\partial\Omega}\mathbf{g}(.,t)\cdot\vec{\nu}ds=0\ \forall t$, $\vec{\nu}$ being the unit outer normal vector on $\partial\Omega$. Moreover without any loss of essential aspects, just to simplify the presentation, we consider a constitutive law of the upper convected type, which relates the extra stress tensor to the velocity in the following manner:

$$\sigma+\lambda\left[\frac{\partial\sigma}{\partial t}+(\mathbf{u}\cdot\nabla)\,\sigma-(\nabla\mathbf{u})\sigma-\sigma(\nabla\mathbf{u})^{\mathrm{T}}\right]=2\eta\mathrm{D}(\mathbf{u}). \tag{1}$$

In (1) λ is the stress relaxation time of the liquid and η is its reference viscosity, both assumed to be constant; ∇ represents the gradient of a scalar or a vector valued function and $\mathrm{D}(\mathbf{u})$ denotes the strain rate tensor, i.e., $\mathrm{D}(\mathbf{u}):=\frac{1}{2}\left[\nabla\mathbf{u}+(\nabla\mathbf{u})^{\mathrm{T}}\right]$.

Then given a solenoidal velocity $\mathbf{u}^0$ and an extra stress σ^0 at time $t=0$, for $t>0$, in addition to the law (1), the flow is governed by the following system:

$$\left.\begin{array}{l}\dfrac{\partial\mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\,\mathbf{u}-\nabla\cdot\sigma+\nabla p=\mathbf{f}\\ \nabla\cdot\mathbf{u}=0\end{array}\right\}\quad\text{in }\Omega\times(0,T) \tag{2}$$

where T is a given time and the density of the liquid is assumed to be equal to one.

In [21] we treated the steady state case. In this work we will be concerned with the search for time-dependent solutions. Therefore contrary to [21] we treat the case where both $\mathbf{f}$ and $\mathbf{g}$ depend on t.

Now we consider the following semi-implicit discretization in time of system (1)–(2). Let M be a strictly positive integer and $\Delta t=T/M$ be a time step. We denote by $\mathbf{u}^n$, p^n and σ^n the approximations of $\mathbf{u}(n\Delta t)$, $p(n\Delta t)$ and $\sigma(n\Delta t)$, respectively, for a strictly positive integer n. Setting $\mathbf{f}^n(.)=\mathbf{f}(.,n\Delta t)$ and $\mathbf{g}^n(.)=\mathbf{g}(.,n\Delta t)$ for $n=0,1,2,\ldots,M$, starting from $\mathbf{u}^0$ and σ^0, and prescribing $\mathbf{u}^n=\mathbf{g}^n$ on $\partial\Omega$ for every n, $\mathbf{u}^n$, p^n and σ^n for $n=1,2,\ldots,M$, are determined as the solution of the following system in Ω:

$$\begin{cases} \dfrac{\mathbf{u}^n - \mathbf{u}^{n-1}}{\Delta t} + (\mathbf{u}^{n-1} \cdot \nabla)\mathbf{u}^{n-1} - \nabla \cdot \sigma^n + \nabla p^n = \mathbf{f}^n \\ \nabla \cdot \mathbf{u}^n = 0 \\ \sigma^n + \lambda \left[\dfrac{\sigma^n - \sigma^{n-1}}{\Delta t} + (\mathbf{u}^{n-1} \cdot \nabla)\sigma^{n-1} - (\nabla \mathbf{u}^{n-1})\sigma^{n-1} - \sigma^{n-1}(\nabla \mathbf{u}^{n-1})^{\mathrm{T}} \right] = 2\eta \mathrm{D}(\mathbf{u}^n). \end{cases} \tag{3}$$

As one can easily infer, (3) is a linear problem for every n. Actually assuming moderate velocities and velocity gradients, the nonlinear terms may be neglected. In this case we can legitimately linearize (1)–(2) into the system governing the very slow flow of a viscoelastic fluid of the Maxwell type. For the sake of conciseness we introduce our methodology in the context of the following generalized Stokes system, derived from the linearization of the equations that govern the flow of a Maxwell viscolelastic liquid (cf. [6]), namely:

From a given state at time $t = 0$ defined by a given solenoidal velocity $\mathbf{u}^0$ and an extra stress tensor σ^0, for $t > 0$ find $\mathbf{u}$, p, σ that solve the following system, with $\mathbf{u} = \mathbf{g}$ on $\partial\Omega \times (0, T)$:

$$\left. \begin{aligned} &\frac{\partial \mathbf{u}}{\partial t} - \nabla \cdot \sigma + \nabla p = \mathbf{f} \\ &\nabla \cdot \mathbf{u} = 0 \\ &\sigma + \lambda \frac{\partial \sigma}{\partial t} = 2\eta \mathrm{D}(\mathbf{u}) \end{aligned} \right\} \quad \text{in } \Omega \times (0, T). \tag{4}$$

Remark. Strictly speaking, in order to ensure objectivity (cf. [23]), system (4) should hold only for a Lagrangian description of the motion. This means that in practice it can also be viewed as a system governing small deformations of a Maxwell viscoelastic solid.

3. Time discretization and splitting algorithm

Splitting algorithms based on projections onto spaces of solenoidal fields were first proposed by Chorin [24] and Temam [25]. They have since proved to be an efficient tool to solve the incompressible Navier–Stokes equations. One of its main features is handling the two primitive variables, velocity and pressure, in an uncoupled manner. Another important advantage of this kind of approach is that, at least in some versions, it allows for the use of the simplest possible space discretizations for both variables without affecting numerical stability. This property was exploited by many authors (cf. [26] for example). However the main drawback of projection algorithms as reported by different authors (see e.g. [27]) remained a persistent numerical inconsistency in the versions most widely in use. This is especially true of those employing a pressure Poisson equation with unphysical Neumann boundary conditions. In [22] an alternative to this pressure solver aimed at overcoming such a difficulty was proposed. The basic idea was the computation of a post-processed pressure at each time step from the available velocity, by a least-square approach, using the momentum equation. The numerical results certified a considerable improvement of the thus corrected pressure, as compared to the one obtained in a classical way, at least for Reynolds numbers that were not very low. Indeed, the fact that the viscous term was systematically purged from the true boundary conditions for the corrected pressure equation, caused a more significant loss of accuracy the lower the Reynolds number was (cf. [22]). This is because second order derivatives are not computable with classical Lagrangian finite elements. Although remedies to this problem were proposed and tested in [28], a persistent lack of accuracy in pressure computations was systematically reported. Notice that in [29] a modification of the above mentioned pressure correction technique was proposed, in order to circumvent such an inconsistency of the projection algorithms. However the authors do not show that their approach allows for the use of finite element representations violating the classical inf–sup condition (see e.g. [30]), such as continuous piecewise linear interpolations for both variables.

In this Section we describe our algorithm for solving both Newtonian and non-Newtonian flow equations, in the $\mathbf{u}$, p, σ formulation. Although this technique is described here only in the context of problem (4), its adaption to more general cases is straightforward, including for instance the Navier–Stokes equations, or even turbulent flow with turbulent stress models. Indeed in the latter cases it suffices to take $\lambda = 0$, before incorporating nonlinear expressions or terms. It seems however that in the context of viscoelastic flow the new approach appears to be the most promising, since in this case the use of a three-field formulation is mandatory.

We have mainly dealt with an explicit splitting algorithm for the time integration or the iterative solution of (4). However, before presenting it we consider the underlying implicit discretization in time of this system, described as follows:

Starting from $\mathbf{u}^0$ and σ^0, for $n = 1, 2, \ldots$, we determine approximations of $p(n\Delta t)$, $\mathbf{u}(n\Delta t)$ and $\sigma(n\Delta t)$, denoted by p^n, $\mathbf{u}^n$ and σ^n respectively, as the solution of the following problem:

$$\left. \begin{cases} \dfrac{\mathbf{u}^n - \mathbf{u}^{n-1}}{\Delta t} - \nabla \cdot \sigma^n + \nabla p^n = \mathbf{f}^n \\ \nabla \cdot \mathbf{u}^n = 0 \\ \sigma^n + \lambda \left(\dfrac{\sigma^n - \sigma^{n-1}}{\Delta t} \right) = 2\eta \mathrm{D}(\mathbf{u}^n) \end{cases} \right\} \quad \text{in } \Omega \tag{5}$$

$$\mathbf{u}^n = \mathbf{g}^n \quad \text{on } \partial\Omega. \tag{6}$$

For the sake of simplicity we assume that Ω has suitable non restrictive regularity properties. Moreover in order to give a precise meaning to system (5)–(6) in the strong sense, we further assume that $\mathbf{f}^n \in L^2(\Omega)^N$, $\mathbf{g}^n \in H^{3/2}(\partial\Omega)^N$, $\forall n$, $\mathbf{u}^0 \in H^2(\Omega)^N$ and $\sigma^0 \in H^1(\Omega)^{N\times N}$ (cf. [31]) (for instance if Ω is a convex polygon or polyhedron this implies that $\mathbf{u}^n \in H^2(\Omega)^N$, $p^n \in H^1(\Omega)$ and $\sigma^n \in H^1(\Omega)^{N\times N}$ $\forall n$). Let also $\langle \cdot, \cdot \rangle_{1/2,\partial\Omega}$ denote the duality product between $H^{1/2}(\partial\Omega)^N$ and $H^{-1/2}(\partial\Omega)^N$, $(\cdot,\cdot)$ and $\|\cdot\|$ denote the standard L^2-inner product and the associated norm, respectively. We further denote by $\|\cdot\|_m$ the standard norm of $H^m(\Omega)$ for $m \in \mathbb{N}$ and by $\|\cdot\|_{s,\partial\Omega}$ the standard norm of $H^s(\partial\Omega)$ for $s \in \mathbb{R}$ (cf [31]), both in scalar and non-scalar version.

Notice that system (5)–(6) can be written in equivalent variational form as follows:

$$\begin{cases} \text{Find } p^n \in Q, \mathbf{u}^n \in \mathbf{V} \text{ and } \sigma^n \in \Sigma \text{ such that} \\ \Delta t^2(\nabla p^n - \nabla\cdot\sigma^n, \nabla q) = \Delta t^2(\mathbf{f}^n, \nabla q) + \Delta t(\mathbf{u}^{n-1}, \nabla q) - \Delta t\langle \mathbf{g}^n, q\vec{\nu}\rangle_{1/2,\partial\Omega} \quad \forall q \in Q, \\ (\mathbf{u}^n - \Delta t(\nabla\cdot\sigma^n - \nabla p^n), \mathbf{v}) = (\mathbf{u}^{n-1} + \Delta t\mathbf{f}^n, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V} \\ \dfrac{\Delta t + \lambda}{2\eta}(\sigma^n, \tau) + \Delta t^2(\nabla\cdot\sigma^n - \nabla p^n, \nabla\cdot\tau) \\ \quad = \dfrac{\lambda}{2\eta}(\sigma^{n-1}, \tau) - \Delta t^2(\mathbf{f}^n, \nabla\cdot\tau) - \Delta t(\mathbf{u}^{n-1}, \nabla\cdot\tau) + \Delta t\langle \mathbf{g}^n, \tau\vec{\nu}\rangle_{1/2,\partial\Omega} \quad \forall \tau \in \Sigma. \end{cases} \tag{7}$$

where $Q := H^1(\Omega) \cap L^2_0(\Omega)$, with $L^2_0(\Omega) := \{q \mid q \in L^2(\Omega), \int_\Omega q\,dx = 0\}$, $\mathbf{V} = L^2(\Omega)^N$ and $\Sigma := \{\sigma,\ \sigma \in \mathbf{H}(\mathrm{div}, \Omega)^N \text{ and } \sigma = \sigma^T\}$ (cf. [32]).

Proposition 1. *System (5)–(6) and (7) are equivalent.*

Proof. Let us first prove that (5)–(6) implies (7): To begin with, we observe that the second equation of (7) is nothing but the first equation of (5) tested with $\mathbf{v} \in \mathbf{V}$. On the other hand the third equation of (7) results from the third equation of (5) tested with $\tau \in \Sigma$, with the addition of terms that stem from the first equation of (5) tested with $\Delta t^2\nabla\cdot\tau \in \mathbf{V}$ (this adds positiveness and in this sense it plays a stabilizing role, similarly to previous works such as [8,9]), using also a well-known identity in order to replace the term $\Delta t(\mathbf{u}^n, \nabla\cdot\tau)$ with $\Delta t[-(\mathrm{D}(\mathbf{u}^n), \tau) + \langle \mathbf{g}^n, \tau\vec{\nu}\rangle_{1/2,\partial\Omega}]$. Finally the first equation of (7) is obtained by testing the first equation of (5) with $\Delta t^2\nabla q \in \mathbf{V}$, and by noticing that, from the second equation of (5) together with (6), we have $\Delta t(\mathbf{u}^n, \nabla q) = \Delta t\langle \mathbf{g}^n, q\vec{\nu}\rangle_{1/2,\partial\Omega}$ $\forall q \in Q$, according to the Green formula recalled at the end of the second part of the proof.

Next we prove that (7) implies (5)–(6): From the second equation of (7) we immediately establish that the first equation of (5) holds. Now in order to derive the third equation of (5) we first take in the third equation of (7) $\tau \in \Sigma \cap \mathcal{D}(\Omega)^N$, $\mathcal{D}(\Omega)$ being the test-function space of Schwartz distributions (see e.g. [33]). In so doing we may add to the right hand side of this equation the term $\Delta t[(\mathrm{D}(\mathbf{u}^n), \tau) - (\mathbf{u}^n, \nabla\cdot\tau)]$, which equals zero from well-known properties of Schwartz distributions. Since the duality term in the resulting relation necessarily vanishes, dropping the terms corresponding to the first equation of (5) tested with $\Delta t^2\nabla\cdot\tau \in \mathcal{D}(\Omega)^N$, we readily establish that the third equation of (5) holds. On the other hand this equation together with the first Korn's inequality (see e.g. [34]), implies that $\mathbf{u}^n$ is necessarily a field of $H^1(\Omega)^N$, and hence by well-known properties of this space (cf. [31]), the trace of $\mathbf{u}^n$ on $\partial\Omega$ belongs to $H^{1/2}(\partial\Omega)^N$. Next coming back to the third equation of (7), taking this time arbitrary tensors $\tau \in \Sigma$, simple manipulations combined with the above conclusions imply in a standard manner that $\Delta t[\langle \mathbf{g}^n - \mathbf{u}^n, \tau\vec{\nu}\rangle_{1/2,\partial\Omega}] = \mathbf{0}$ $\forall\tau \in \Sigma$. Now using the trace theorem for $\mathbf{H}(\mathrm{div}, \Omega)$ (cf. [32]), we note that by convenient choices of the components of τ one may generate any vector $\mathbf{w} \in H^{-1/2}(\partial\Omega)^N$ represented in the form $\tau\vec{\nu}$ for $\tau \in \Sigma$. It readily follows that (6) holds. Finally the first equation of (7) is nothing but the first equation of (5) tested with $\Delta t^2\nabla q \in \mathbf{V}$, except for the term $\Delta t(\mathbf{u}^n, \nabla q)$, which is replaced with $\Delta t\langle \mathbf{g}^n, q\vec{\nu}\rangle_{1/2,\partial\Omega}$. In this manner, the second equation of (5) results from the combination of the first two equations of (7), thanks to (6) and the well-known Green's Formula: $(\mathbf{u}^n, \nabla q) = \langle \mathbf{u}^n, q\vec{\nu}\rangle_{1/2,\partial\Omega} - (\nabla\cdot\mathbf{u}^n, q)$, $\forall q \in Q$. Indeed comparing these two relations it is readily seen that $(\nabla\cdot\mathbf{u}^n, q) = 0$ for every $q \in Q$ and hence by density for every $q \in L^2_0(\Omega)$. Since $\nabla\cdot\mathbf{u}^n \in L^2_0(\Omega)$, owing to the fact that $\int_\Omega \nabla\cdot\mathbf{u}^n dx = \int_{\partial\Omega} \mathbf{g}^n\cdot\vec{\nu}ds = 0$, we may take $q = \nabla\cdot\mathbf{u}^n \in L^2_0(\Omega)$. This implies that $\nabla\cdot\mathbf{u}^n = 0$ in Ω, which completes the proof. □

4. Space discretization

Now we consider the following discrete analogue of (7). Henceforth we assume that Ω is a polygon for $N = 2$ or a polyhedron for $N = 3$, and that $\mathbf{f}$ and $\mathbf{g}$ are smooth enough for the regularity of the unknown fields required in the sequel to hold.

Let then $\mathcal{T}_h$ be a partition of Ω into N-simplices with maximum edge length equal to h. We assume that $\mathcal{T}_h$ satisfies the usual compatibility conditions for finite element meshes, and that it belongs to a quasi-uniform family of partitions. For every subset ω of $\mathbb{R}^N$ we further denote by $P_k(\omega)$ the space of polynomials of degree less than or equal to k defined in ω. In so doing we introduce the following spaces or manifolds associated with $\mathcal{T}_h$:

$$S_h := \left\{ v \mid v \in C^0(\bar{\Omega}) \text{ and } v_{/K} \in P_1(K), \forall K \in \mathcal{T}_h \right\},$$
$$\mathbf{V}_h := \{ \mathbf{v} \mid \forall i \; v_i \in S_h \},$$
$$Q_h := S_h \cap L^2_0(\Omega),$$
$$\Sigma_h := \left\{ \tau \mid \tau \in [S_h]^{N\times N}, \tau = \tau^{\mathrm{T}} \right\}.$$

Then letting $\mathbf{u}_h^0$ be the field of $\mathbf{V}_h$ satisfying $\mathbf{u}_h^0(P) = \mathbf{u}^0(P)$, and σ_h^0 be the tensor of Σ_h satisfying $\sigma_h^0(P) = \sigma^0(P)$, for every vertex P of $\mathcal{T}_h$, we provisionally set the following problem to approximate (7), or yet (5)–(6), for every n, $n = 1, 2, \ldots, M$:

$$\begin{cases} \text{Find } p_h^n \in Q_h, \mathbf{u}_h^n \in \mathbf{V}_h \text{ and } \sigma_h^n \in \Sigma_h \text{ such that} \\ \Delta t^2(\nabla p_h^n - \nabla\cdot\sigma_h^n, \nabla q) = \Delta t^2(\mathbf{f}^n, \nabla q) + \Delta t(\mathbf{u}_h^{n-1}, \nabla q) - \Delta t\langle \mathbf{g}^n, \; q\vec{\nu}\,\rangle_{1/2,\partial\Omega} \quad \forall q \in Q_h, \\ (\mathbf{u}_h^n - \Delta t(\nabla\cdot\sigma_h^n - \nabla p_h^n), \mathbf{v}) = (\mathbf{u}_h^{n-1} + \Delta t\mathbf{f}^n, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}_h, \\ \dfrac{\Delta t + \lambda}{2\eta}(\sigma_h^n, \tau) + \Delta t^2(\nabla\cdot\sigma_h^n - \nabla p_h^n, \nabla\cdot\tau) \\ \quad = \dfrac{\lambda}{2\eta}(\sigma_h^{n-1}, \tau) - \Delta t^2(\mathbf{f}^n, \nabla\cdot\tau) - \Delta t(\mathbf{u}_h^{n-1}, \nabla\cdot\tau) + \Delta t\langle \mathbf{g}^n, \; \tau\vec{\nu}\,\rangle_{1/2,\partial\Omega} \quad \forall \tau \in \Sigma_h. \end{cases} \tag{8}$$

For problem (8) the following result holds, whose proof is given in [19].

Proposition 2. *Problem (8) has a unique solution for every Δt and every n.* $\square$

As a matter of fact we will work with the mass lumping technique for terms of the type (φ, ψ) (see e.g. [35]), where φ and ψ are both either velocities or stress tensors assumed to belong to $L^2(\Omega)^L$, $L \in \mathbb{N}$. This gives rise to an approximate inner product $(\varphi, \psi)_h$ derived from the application of the trapezoidal rule in every N-simplex of $\mathcal{T}_h$ if both arguments happen to belong to $(S_h)^L$. More specifically denoting by π_1^K the standard L^2-projection operator from $L^2(K)^L$ onto $P_1(K)^L$, we define:

$$(\varphi, \psi)_h := \sum_{K\in\mathcal{T}_h} (\varphi, \psi)_K, \quad \text{with } (\varphi, \psi)_K := \frac{meas(K)}{N+1}\sum_{i=1}^{N+1}[\pi_1^K(\varphi_{/K})\cdot\pi_1^K(\psi_{/K})](S_i^K), \tag{9}$$

S_i^K being the vertices of N-simplex K, $i = 1, \ldots, N+1$. We further set for $\varphi, \psi \in L^2(\Omega)^L$, $\|\varphi\|_h := (\varphi, \varphi)_h^{1/2}$ and $\epsilon_h(\varphi, \psi) := (\varphi, \psi)_h - (\varphi, \psi)$.

Next we prove a Lemma for scalar functions, which obviously extends to the case of fields of any kind:

Lemma 3. *$\forall u \in S_h$, $\epsilon_h(u, u) \geq 0$ and if $\epsilon_h(u, u) = 0$ then u is constant all over Ω. Moreover the following relation holds:*

$$\frac{1}{\sqrt{N+2}}\|u\|_h \leq \|u\| \quad \forall u \in S_h. \tag{10}$$

Proof. Since $\pi_1^K(u) = u_{/K}$ if $u \in S_h$, setting $\epsilon_K(u, v) := \sum_{i=1}^{N+1}\frac{u_i^K v_i^K}{N+1}meas(K) - \int_K u_{/K}v_{/K}\mathrm{d}x$, we have $\epsilon_h(u, u) = \sum_{K\in\mathcal{T}_h}\epsilon_K(u, u)$ where $u_i^K = u(S_i^K)$ and $v_i^K = v(S_i^K)$.

We know that $u_{/K} = \sum_{i=1}^{N+1} u_i^K\lambda_i^K$ and $v_{/K} = \sum_{j=1}^{N+1} v_j^K\lambda_j^K$ where $\lambda_1^K, \lambda_2^K, \ldots, \lambda_{N+1}^K$ are the barycentric coordinates of K. Thus we have (cf. [36]):

$$\int_K u_{/K}^2\mathrm{d}x = \sum_{i=1}^{N+1}\sum_{j=1}^{N+1} u_i^K u_j^K \int_K \lambda_i^K\lambda_j^K\mathrm{d}x = \sum_{i=1}^{N+1}\sum_{j=i}^{N+1} u_i^K\; u_j^K\;\frac{2meas(K)}{(N+1)(N+2)}.$$

Hence after straightforward calculations we derive,

$$\epsilon_K(u_{/K}, u_{/K}) = (N+1)\int_K u_{/K}^2\mathrm{d}x - \frac{meas(K)}{N+1}\left(\sum_{j=1}^{N+1} u_j^K\right)^2 \tag{11}$$

which readily yields (10).

On the other hand, for $\vec{e} = (e_1, e_2, \ldots, e_{N+1})$ with $e_j = 1$, $\forall j$ we have:

$$\left(\sum_{j=1}^{N+1} u_j^K\right)^2 \leq \sum_{j=1}^{N+1}\left(u_j^K\right)^2\sum_{j=1}^{N+1} e_j^2. \tag{12}$$

Thus plugging (12) into (11), after performing some elementary manipulations, we conclude that $\epsilon_K(u_{/K}, u_{/K}) \geq 0$ $\forall K \in \mathcal{T}_h$, that is, $\epsilon_h(u, u) \geq 0$.

Finally, if $\epsilon_h(u,u) = 0$, necessarily from (12), $\left(\sum_{j=1}^{N+1} u_j^K e_j\right)^2 = \sum_{j=1}^{N+1} \left(u_j^K\right)^2 \sum_{j=1}^{N+1} e_j^2$. But in this case $\vec{u}^K := (u_1^K, u_2^K, \ldots, u_{N+1}^K)$ is parallel to $\vec{e}$ and thus u is constant in K. Since u is continuous, it must be constant in the whole Ω. $\square$

Next we consider a lumped mass version of system (8), that we will actually employ in this work, namely:

$$\begin{cases} \text{Find } p_h^n \in Q_h, \mathbf{u}_h^n \in \mathbf{V}_h, \text{ and } \sigma_h^n \in \Sigma_h \text{ such that} \\ \Delta t^2 \left[(\nabla p_h^n, \nabla q) - (\nabla \cdot \sigma_h^n, \nabla q) \right] = \Delta t^2 (\mathbf{f}_h^n, \nabla q) + \Delta t (\mathbf{u}_h^{n-1}, \nabla q) - \Delta t \langle \mathbf{g}_h^n, q\vec{\nu} \rangle_{1/2,\partial\Omega} \quad \forall q \in Q_h, \\ (\mathbf{u}_h^n, \mathbf{v})_h + \Delta t \left[(\nabla p_h^n, \mathbf{v}) - (\nabla \cdot \sigma_h^n, \mathbf{v}) \right] = (\mathbf{u}_h^{n-1}, \mathbf{v})_h + \Delta t (\mathbf{f}_h^n, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}_h, \\ \dfrac{\Delta t + \lambda}{2\eta} (\sigma_h^n, \tau)_h + \Delta t^2 \left[(\nabla \cdot \sigma_h^n, \nabla \cdot \tau) - (\nabla p_h^n, \nabla \cdot \tau) \right] \\ \quad = \dfrac{\lambda}{2\eta} (\sigma_h^{n-1}, \tau)_h - \Delta t^2 (\mathbf{f}_h^n, \nabla \cdot \tau) - \Delta t (\mathbf{u}_h^{n-1}, \nabla \cdot \tau) + \Delta t \langle \mathbf{g}_h^n, \tau\vec{\nu} \rangle_{1/2,\partial\Omega} \quad \forall \tau \in \Sigma_h. \end{cases} \tag{13}$$

where $\mathbf{g}_h^n$ and $\mathbf{f}_h^n$ are suitable approximations of $\mathbf{g}^n$ and $\mathbf{f}^n$.

Proposition 4. *Given $\mathbf{u}_h^{n-1}, \sigma_h^{n-1}, \mathbf{f}_h^n, \mathbf{g}_h^n$, problem (13) has a unique solution $(p_h^n, \mathbf{u}_h^n, \sigma_h^n)$.*

Proof. Since Q_h, $\mathbf{V}_h$ and Σ_h, are finite dimensional spaces and (13) is equivalent to a linear system of J_h equations with J_h unknowns, where $J_h = dim\ Q_h + dim\ \mathbf{V}_h + dim\ \Sigma_h$, it suffices to prove that $\mathbf{f}_h^n = \mathbf{0}$, $\mathbf{g}^n = \mathbf{g}_h^n = \mathbf{0}$, $\mathbf{u}_h^{n-1} = \mathbf{0}$ and $\sigma_h^{n-1} = \mathcal{O}$ implies that $p_h^n = 0$, $\mathbf{u}_h^n = \mathbf{0}$, $\sigma_h^n = \mathcal{O}$.

Under this hypothesis we take $q = p_h^n \in Q_h$, $\mathbf{v} = \mathbf{u}_h^n \in \mathbf{V}_h$ and $\tau = \sigma_h^n$. This gives:

$$\begin{cases} \Delta t^2 (\nabla p_h^n - \nabla \cdot \sigma_h^n, \nabla p_h^n) = 0 \\ \left\| \mathbf{u}_h^n \right\|_h^2 + \Delta t (\nabla p_h^n - \nabla \cdot \sigma_h^n, \mathbf{u}_h^n) = 0 \\ \dfrac{\lambda + \Delta t}{2\eta} \left\| \sigma_h^n \right\|_h^2 + \Delta t^2 (\nabla \cdot \sigma_h^n - \nabla p_h^n, \nabla \cdot \sigma_h^n) = 0. \end{cases} \tag{14}$$

Adding up the three relations above it is readily seen that

$$\Delta t^2 \left\| \nabla p_h^n - \nabla \cdot \sigma_h^n \right\|^2 + \left\| \mathbf{u}_h^n \right\|_h^2 + \frac{\lambda + \Delta t}{\eta} \left\| \sigma_h^n \right\|_h^2 + \left\| \Delta t (\nabla p_h^n - \nabla \cdot \sigma_h^n) + \mathbf{u}_h^n \right\|^2 + \epsilon_h (\mathbf{u}_h^n, \mathbf{u}_h^n) = 0.$$

This trivially yields $\mathbf{u}_h^n = \mathbf{0}$, $\sigma_h^n = \mathcal{O}$ and $\nabla p_h^n = \mathbf{0}$. Since $p_h^n \in L_0^2(\Omega)$ this implies that $p_h^n = 0$ too. $\square$

Let us now consider the following splitting algorithm for solving explicitly system (13) at every time step.

Set for every $n \geq 0$, $\sigma_h^{n,0} = \sigma_h^{n-1}$. Then for $s = 1, 2, \ldots$ determine approximations $p_h^{n,s} \in Q_h$, $\mathbf{u}_h^{n,s} \in \mathbf{V}_h$ and $\sigma_h^{n,s} \in \Sigma_h$ of p_h^n, $\mathbf{u}_h^n$ and σ_h^n by solving successively the following problems:

$$\begin{cases} \Delta t^2 (\nabla p_h^{n,s}, \nabla q) = \Delta t^2 [(\mathbf{f}_h^n, \nabla q) + (\nabla \cdot \sigma_h^{n,s-1}, \nabla q)] + \Delta t (\mathbf{u}_h^{n-1}, \nabla q) - \Delta t \langle \mathbf{g}_h^n, q\vec{\nu} \rangle_{1/2,\partial\Omega} \quad \forall q \in Q_h \\ (\mathbf{u}_h^{n,s}, \mathbf{v})_h = \Delta t (\mathbf{f}_h^n + \nabla \cdot \sigma_h^{n,s-1} - \nabla p_h^{n,s}, \mathbf{v}) + (\mathbf{u}_h^{n-1}, \mathbf{v})_h \quad \forall \mathbf{v} \in \mathbf{V}_h \\ \dfrac{\lambda + \Delta t}{2\eta} (\sigma_h^{n,s}, \tau)_h = \dfrac{\lambda}{2\eta} (\sigma_h^{n,s-1}, \tau)_h - \Delta t^2 (\mathbf{f}_h^n + \nabla \cdot \sigma_h^{n,s-1} - \nabla p_h^{n,s}, \nabla \cdot \tau) \\ \quad - \Delta t \left[(\mathbf{u}_h^{n-1}, \nabla \cdot \tau) - \langle \mathbf{g}_h^n, \tau\vec{\nu} \rangle_{1/2,\partial\Omega} \right] \quad \forall \boldsymbol{\tau} \in \Sigma_h. \end{cases} \tag{15}$$

This algorithm is unlikely to generate converging sequence of approximations of $(p^n, \mathbf{u}^n, \sigma^n)$ as s goes to infinity in the analogous continuous case (7). However, here it is applied in the framework of the discrete counterpart of (7) defined by replacing Q, $\mathbf{V}$ and Σ with finite dimensional spaces Q_h, $\mathbf{V}_h$ and Σ_h, for which the classical inverse inequalities hold. In our case we shall employ the following one (cf. [37]):

There exists a constant C independent of h such that

$$\| \nabla \cdot \tau \| \leq \frac{C}{h} \| \tau \| \quad \forall \tau \in \Sigma_h. \tag{16}$$

In this way we are able to prove:

Proposition 5. *Let ε be a parameter satisfying $0 < \varepsilon \leq 1$, and C be the constant of the inverse inequality (16). Provided Δt is chosen such that*

$$\Delta t \leq \frac{h}{C} \sqrt{\frac{\lambda\varepsilon}{2\eta(1+\varepsilon)}}, \tag{17}$$

the sequence $\{(p_h^{n,s}, \mathbf{u}_h^{n,s}, \sigma_h^{n,s})\}_s$ defined by (15) converges to the solution $(p_h^n, \mathbf{u}_h^n, \sigma_h^n)$ of (13) in $Q_h \times \mathbf{V}_h \times \Sigma_h$ as s goes to infinity.

Proof. Let us set $\bar{\mathbf{u}}_h^{n,s} := \mathbf{u}_h^{n,s} - \mathbf{u}_h^n$; $\bar{p}_h^{n,s} := p_h^{n,s} - p_h^n$; $\bar{\sigma}_h^{n,s} := \sigma_h^{n,s} - \sigma_h^n$

Comparing (13) and (15), $\bar{\mathbf{u}}_h^{n,s}, \bar{p}_h^{n,s}$ and $\bar{\sigma}_h^{n,s}$ are easily found to satisfy:

$$\begin{cases} \Delta t^2(\nabla \bar{p}_h^{n,s}, \nabla q) = \Delta t^2(\nabla \cdot \bar{\sigma}_h^{n,s-1}, \nabla q) \quad \forall q \in Q_h \\ (\bar{\mathbf{u}}_h^{n,s}, \mathbf{v})_h = -\Delta t(\nabla \bar{p}_h^{n,s}, \mathbf{v}) + \Delta t(\nabla \cdot \bar{\sigma}_h^{n,s-1}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}_h \\ \dfrac{\lambda + \Delta t}{2\eta}(\bar{\sigma}_h^{n,s}, \tau)_h = \Delta t^2(\nabla \bar{p}_h^{n,s}, \nabla \cdot \tau) - \Delta t^2(\nabla \cdot \bar{\sigma}_h^{n,s-1}, \nabla \cdot \tau) \quad \forall \boldsymbol{\tau} \in \Sigma_h. \end{cases} \tag{18}$$

Taking $\mathbf{v} = \bar{\mathbf{u}}_h^{n,s}$, $q = \bar{p}_h^{n,s}$ and $\boldsymbol{\tau} = \bar{\boldsymbol{\sigma}}_h^{n,s}$, we obtain:

$$\begin{cases} \Delta t^2 \left\|\nabla \bar{p}_h^{n,s}\right\|^2 = \Delta t^2(\nabla \cdot \bar{\sigma}_h^{n,s-1}, \nabla \bar{p}_h^{n,s}) \\ \left\|\bar{\mathbf{u}}_h^{n,s}\right\|_h^2 = -\Delta t(\nabla \bar{p}_h^{n,s}, \bar{\mathbf{u}}_h^{n,s}) + \Delta t(\nabla \cdot \bar{\sigma}_h^{n,s-1}, \bar{\mathbf{u}}_h^{n,s}) \\ \dfrac{\lambda + \Delta t}{2\eta} \left\|\bar{\sigma}_h^{n,s}\right\|_h^2 = \Delta t^2(\nabla \bar{p}_h^{n,s}, \nabla \cdot \bar{\sigma}_h^{n,s}) - \Delta t^2(\nabla \cdot \bar{\sigma}_h^{n,s-1}, \nabla \cdot \bar{\sigma}_h^{n,s}). \end{cases} \tag{19}$$

Adding up the three relations in (19), we derive:

$$\begin{aligned} &\Delta t^2 \left\|\nabla \bar{p}_h^{n,s}\right\|^2 - \Delta t^2(\nabla \cdot \bar{\sigma}_h^{n,s-1}, \nabla \bar{p}_h^{n,s}) + \left\|\bar{\mathbf{u}}_h^{n,s}\right\|_h^2 + \Delta t(\nabla \bar{p}_h^{n,s} - \nabla \cdot \bar{\sigma}_h^{n,s-1}, \bar{\mathbf{u}}_h^{n,s}) \\ &\times \frac{\lambda + \Delta t}{2\eta} \left\|\bar{\sigma}_h^{n,s}\right\|_h^2 + \Delta t^2(\nabla \cdot \bar{\sigma}_h^{n,s-1}, \nabla \cdot \bar{\sigma}_h^{n,s}) - \Delta t^2(\nabla \bar{p}_h^{n,s}, \nabla \cdot \bar{\sigma}_h^{n,s}) = 0 \end{aligned} \tag{20}$$

which yields:

$$\begin{aligned} &\Delta t^2 \left\|\nabla \bar{p}_h^{n,s}\right\|^2 - 2\Delta t^2(\nabla \bar{p}_h^{n,s}, \nabla \cdot \bar{\sigma}_h^{n,s}) + \Delta t^2(\nabla \cdot \bar{\sigma}_h^{n,s} - \nabla \cdot \bar{\sigma}_h^{n,s-1}, \nabla \bar{p}_h^{n,s}) \\ &+ \Delta t^2 \left\|\nabla \cdot \bar{\sigma}_h^{n,s}\right\|^2 - \Delta t^2(\nabla \cdot \bar{\sigma}_h^{n,s} - \nabla \cdot \bar{\sigma}_h^{n,s-1}, \nabla \cdot \bar{\sigma}_h^{n,s}) + \Delta t(\nabla \bar{p}_h^{n,s} - \nabla \cdot \bar{\sigma}^{n,s}, \bar{\mathbf{u}}_h^{n,s}) \\ &+ \left\|\bar{\mathbf{u}}_h^{n,s}\right\|_h^2 + \Delta t(\nabla \cdot \bar{\sigma}_h^{n,s} - \nabla \cdot \bar{\sigma}_h^{n,s-1}, \bar{\mathbf{u}}_h^{n,s}) + \frac{\lambda + \Delta t}{2\eta} \left\|\bar{\sigma}_h^{n,s}\right\|_h^2 = 0. \end{aligned} \tag{21}$$

It follows that:

$$\begin{aligned} &\Delta t^2 \left\|\nabla \bar{p}_h^{n,s} - \nabla \cdot \bar{\sigma}_h^{n,s}\right\|^2 + \left\|\bar{\mathbf{u}}_h^{n,s}\right\|_h^2 + \Delta t(\nabla \bar{p}_h^{n,s} - \nabla \cdot \bar{\sigma}_h^{n,s}, \bar{\mathbf{u}}_h^{n,s}) + \frac{\lambda + \Delta t}{2\eta} \left\|\bar{\sigma}_h^{n,s}\right\|_h^2 \\ &= -\Delta t(\nabla \cdot \bar{\sigma}_h^{n,s} - \nabla \cdot \bar{\sigma}_h^{n,s-1}, \bar{\mathbf{u}}_h^{n,s} + \Delta t(\nabla \bar{p}_h^{n,s} - \nabla \cdot \bar{\sigma}_h^{n,s})). \end{aligned} \tag{22}$$

Taking into account that $(f, g) \leq \frac{1}{2}[\|f\|^2 + \|g\|^2], \forall f, g$, we may write:

$$\begin{aligned} &\frac{\Delta t^2}{2} \left\|\nabla \bar{p}_h^{n,s} - \nabla \cdot \bar{\sigma}_h^{n,s}\right\|^2 + \frac{1}{2} \left\|\bar{\mathbf{u}}_h^{n,s}\right\|_h^2 + \frac{1}{2} \left\|\bar{\mathbf{u}}_h^{n,s} + \Delta t \left(\nabla \bar{p}_h^{n,s} - \nabla \cdot \bar{\sigma}_h^{n,s}\right)\right\|^2 + \frac{1}{2}\epsilon_h(\bar{\mathbf{u}}_h^{n,s}, \bar{\mathbf{u}}_h^{n,s}) \\ &+ \frac{\lambda + \Delta t}{2\eta} \left\|\bar{\sigma}_h^{n,s}\right\|_h^2 \leq \frac{1}{2}\Delta t^2\|\nabla \cdot \bar{\sigma}_h^{n,s} - \nabla \cdot \bar{\sigma}_h^{n,s-1}\|^2 + \frac{1}{2}\|\bar{\mathbf{u}}_h^{n,s} + \Delta t(\nabla \bar{p}_h^{n,s} - \nabla \cdot \bar{\sigma}_h^{n,s})\|^2. \end{aligned} \tag{23}$$

Furthermore, owing to Lemma 3 we have:

$$\Delta t^2 \left\|\nabla \bar{p}_h^{n,s} - \nabla \cdot \bar{\sigma}_h^{n,s}\right\|^2 + \left\|\bar{\mathbf{u}}_h^{n,s}\right\|_h^2 + \frac{\lambda + \Delta t}{\eta} \left\|\bar{\sigma}_h^{n,s}\right\|_h^2 \leq \Delta t^2 \left\|\nabla \cdot \bar{\sigma}_h^{n,s} - \nabla \cdot \bar{\sigma}_h^{n,s-1}\right\|^2. \tag{24}$$

This further gives:

$$\Delta t^2 \left\|\nabla \bar{p}_h^{n,s} - \nabla \cdot \bar{\sigma}_h^{n,s}\right\|^2 + \left\|\bar{\mathbf{u}}_h^{n,s}\right\|_h^2 + \frac{\lambda + \Delta t}{\eta} \left\|\bar{\sigma}_h^{n,s}\right\|_h^2 \leq 2\Delta t^2 \left(\|\nabla \cdot \bar{\sigma}_h^{n,s}\|^2 + \|\nabla \cdot \bar{\sigma}_h^{n,s-1}\|^2\right). \tag{25}$$

Recalling (16), we come up with:

$$\Delta t^2\|\nabla \bar{p}_h^{n,s} - \nabla \cdot \bar{\sigma}_h^{n,s}\|^2 + \|\bar{\mathbf{u}}^{n,s}\|_h^2 + \frac{\lambda + \Delta t}{\eta}\|\bar{\sigma}_h^{n,s}\|_h^2 \leq 2\Delta t^2\frac{C^2}{h^2}(\|\bar{\sigma}_h^{n,s}\|^2 + \|\bar{\sigma}_h^{n,s-1}\|^2) \tag{26}$$

or yet:

$$\Delta t^2\|\nabla \bar{p}_h^{n,s} - \nabla \cdot \bar{\sigma}_h^{n,s}\|^2 + \|\bar{\mathbf{u}}_h^{n,s}\|^2 + \left(\frac{\lambda + \Delta t}{\eta} - 2\frac{C^2\Delta t^2}{h^2}\right)\|\bar{\sigma}_h^{n,s}\|_h^2 \leq 2\frac{C^2\Delta t^2}{h^2}\|\bar{\sigma}_h^{n,s-1}\|_h^2. \tag{27}$$

Now setting $a = \frac{2C^2\Delta t^2}{h^2}$ and $b = \frac{\lambda}{\eta}$ we momentarily assume that $b \geq a$. In this way:

$$\|\bar{\sigma}_h^{n,s}\|_h^2 \leq \rho^2 \|\bar{\sigma}_h^{n,s-1}\|_h^2 \quad \text{where } \rho^2 = \frac{a}{b-a+\Delta t/\eta}. \tag{28}$$

Applying (28) iteratively for $s = 1, 2, \ldots$ we derive $\|\bar{\sigma}_h^{n,s}\|_h \leq \rho^s \|\bar{\sigma}_h^{n,0}\|_h = \rho^s \|\sigma_h^n - \sigma_h^{n-1}\|_h$, $\forall s$. Then letting s go to infinity, $\|\bar{\sigma}_h^{n,s}\|_h$ will tend to zero if $\rho < 1$ and by (16), $\nabla \cdot \bar{\sigma}_h^{n,s}$ tends to zero as well. Then since $\bar{\sigma}_h^{n,s} \to \mathcal{O}$ it immediately follows that $\mathbf{u}_h^{n,s} \to \mathbf{0}$ and $\left(\nabla \bar{p}_h^{n,s} - \nabla \cdot \bar{\sigma}_h^{n,s}\right) \to \mathbf{0}$. Therefore, $\nabla \bar{p}_h^{n,s} \to \mathbf{0}$, and thus $\bar{p}_h^{n,s}$ tends to a constant. However, $\bar{p}_h^{n,s} \in L_0^2(\Omega)$ $\forall s$, which implies that $\bar{p}_h^{n,s}$ tends to zero too.

Finally in order to ensure convergence, we must have $\rho < 1$. Now if $a \leq \frac{\varepsilon b}{1+\varepsilon}$ necessarily $b > a$ and $\frac{a}{b-a} \leq \varepsilon$. As a result if Δt and h satisfy (17), then $\rho^2 < \varepsilon \leq 1$. This completes the proof. $\square$

5. Stability

Remark. Henceforth the letter C combined or not with other symbols will represent different strictly positive constants independent of Δt and h.

In this Section we proceed to the stability analysis of scheme (13). For this purpose it is convenient to assume that we are solving a more general problem, namely:

$$\begin{cases} \Delta t^2 \left(\nabla p_h^n, \nabla q\right) - \Delta t^2 (\nabla \cdot \sigma_h^n, \nabla q) = \Delta t (\mathbf{u}_h^{n-1}, \nabla q) - \Delta t G_h^n (q\vec{\nu}) + \Delta t^2 L_h^{p,n}(\nabla q) \quad \forall q \in Q_h \\ (\mathbf{u}_h^n, \mathbf{v})_h + \Delta t (\nabla p_h^n - \nabla \cdot \sigma_h^n, \mathbf{v}) = (\mathbf{u}_h^{n-1}, \mathbf{v}) + \Delta t L_h^{\mathbf{u},n}(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}_h \\ \dfrac{\Delta t}{2\eta}(\sigma_h^n, \tau)_h + \dfrac{\lambda}{2\eta}(\sigma_h^n, \tau)_h + \Delta t^2 (\nabla \cdot \sigma_h^n, \nabla \cdot \tau) - \Delta t^2 (\nabla p_h^n, \nabla \cdot \tau) \\ \quad = \dfrac{\lambda}{2\eta}(\sigma_h^{n-1}, \tau)_h - \Delta t (u_h^{n-1}, \tau)_h + \Delta t G_h^n(\tau\vec{\nu}) - \Delta t^2 L_h^{p,n}(\nabla \cdot \tau) + \Delta t L_h^{\sigma,n}(\tau). \end{cases} \tag{29}$$

We assume that $L_h^{\mathbf{u},n}$, $L_h^{p,n}$, $L_h^{\sigma,n}$ and G_h^n are linear functionals satisfying:

$$\begin{cases} L_h^{\mathbf{u},n}(\mathbf{d}) \leq |L_h^{\mathbf{u},n}| \, \|\mathbf{d}\|, \quad \forall \mathbf{d} \in \mathbf{D}_h, \\ L_h^{p,n}(\mathbf{v}) \leq |L_h^{p,n}| \, \|\mathbf{d}\|, \quad \forall \mathbf{d} \in \mathbf{D}_h, \\ L_h^{\sigma,n}(\tau) \leq |L_h^{\sigma,n}| \, \|\tau\|, \quad \forall \tau \in \Sigma_h, \\ G_h^n(\mathbf{w}) \leq [G_h^n] \, \|\mathbf{w}\|_{-1/2,\partial\Omega}, \quad \forall \mathbf{w} \in \Gamma_h \end{cases} \tag{30}$$

where $|\cdot|$ and $[\cdot]$ denote standard functional norms, $\mathbf{D}_h := \{\mathbf{v} | \mathbf{v}_{/K} \in P_1(K)^N, \forall K \in \mathcal{T}_h\}$, $\Gamma_h := \{\mathbf{w} : \partial\Omega \longrightarrow \mathbb{R}^N \mid \mathbf{w}_{/F} \in P_1(F)^N \ \forall$ face or edge F of $K \in \mathcal{T}_h$ such that $F \subset \partial\Omega\}$.

Notice that in practice $L_h^{\mathbf{u},n}(\mathbf{v}) = (\mathbf{f}_h^n, \mathbf{v})$ $\forall \mathbf{v} \in L^2(\Omega)^N$, $L_h^{p,n} = L_h^{\mathbf{u},n}$, $L_h^{\sigma,n} = 0$ and $G_h^n(\mathbf{w}) = \langle \mathbf{g}_h^n, \mathbf{w} \rangle_{1/2,\partial\Omega}$ $\forall \mathbf{w} \in H^{-1/2}(\partial\Omega)^N$.

Theorem 6. *Assuming that $\Delta t < \frac{1}{2}$ the following stability result holds for scheme (29):*

$$\begin{aligned} \forall n \leq M : \|\mathbf{u}_h^n\|^2 + \epsilon_h(\mathbf{u}_h^n, \mathbf{u}_h^n) + \frac{\Delta t^2}{2} \|\nabla p_h^n - \nabla \cdot \sigma_h^n\|^2 + \frac{\lambda}{2\eta} \|\sigma_h^n\|_h^2 \\ \leq e^{4T} \left[\|\mathbf{u}_h^0\|^2 + \epsilon_h(\mathbf{u}_h^0, \mathbf{u}_h^0) + \frac{\lambda}{2\eta} \|\sigma_h^0\|_h^2 + \tilde{C} \Delta t \sum_{i=1}^n \left(|L_h^{p,i}|^2 + |L_h^{\mathbf{u},i}|^2 + |L_h^{\sigma,i}|^2 + \frac{[G_h^i]^2}{\Delta t^2} \right) \right]. \end{aligned} \tag{31}$$

Proof. Setting $\mathbf{v} = \mathbf{u}_h^n$, $q = p_h^n$ and $\tau = \sigma_h^n$ in (29) we derive:

$$\begin{cases} \Delta t^2 \| \nabla p_h^n \|^2 - \Delta t^2 (\nabla \cdot \sigma_h^n, \nabla p_h^n) = \Delta t^2 L_h^{p,n}(\nabla p_h^n) - \Delta t \, G_h^n(p_h^n \vec{\nu}) + \Delta t (\mathbf{u}_h^{n-1}, \nabla p_h^n) \\ \|\mathbf{u}_h^n\|_h^2 + \Delta t (\nabla p_h^n - \nabla \cdot \sigma_h^n, \mathbf{u}_h^n) = \Delta t L_h^{\mathbf{u},n}(\mathbf{u}_h^n) + (\mathbf{u}_h^{n-1}, \mathbf{u}_h^n)_h \\ \dfrac{\lambda + \Delta t}{2\eta} \|\sigma_h^n\|_h^2 + \Delta t^2 \|\nabla \cdot \sigma_h^n\|^2 - \Delta t^2 (\nabla p_h^n, \nabla \cdot \sigma_h^n) = \dfrac{\lambda}{2\eta}(\sigma_h^n, \sigma_h^{n-1})_h \\ \quad - \Delta t (\mathbf{u}_h^{n-1}, \nabla \cdot \sigma_h^n) + \Delta \, t G_h^n(\sigma_h^n \vec{\nu}) - \Delta t^2 L_h^{p,n}(\nabla \cdot \sigma_h^n) + \Delta t L_h^{\sigma,n}(\sigma_h^n). \end{cases} \tag{32}$$

Adding up the three relations above we come up with:

$$\|\mathbf{u}_h^n\|_h^2 + \Delta t (\nabla p_h^n - \nabla \cdot \sigma_h^n, \mathbf{u}_h^n) + \Delta t^2 \|\nabla p_h^n - \nabla \cdot \sigma_h^n\|^2 + \frac{\lambda}{2\eta} \|\sigma_h^n\|_h^2 + \frac{\Delta t}{2\eta} \|\sigma_h^n\|_h^2$$

$$= (\mathbf{u}_h^{n-1}, \mathbf{u}_h^n)_h + \Delta t(\nabla p_h^n - \nabla\cdot\sigma_h^n, \mathbf{u}_h^{n-1}) + \frac{\lambda}{2\eta}(\sigma_h^n, \sigma_h^{n-1})_h$$
$$- G_h^n(p_h^n I - \sigma_h^n) + \Delta t^2 L_h^{p,n}(\nabla p_h^n - \nabla\cdot\sigma_h^n) + \Delta t\, L_h^{\mathbf{u},n}(\mathbf{u}_h^n) + \Delta\, t L_h^{\sigma,n}(\sigma_h^n). \tag{33}$$

This leads to

$$\frac{1}{2}[\|\mathbf{u}_h^n\|_h^2 + \Delta t^2\|\nabla p_h^n - \nabla\cdot\sigma_h^n\|^2 + \|\mathbf{u}_h^n + \Delta t(\nabla p_h^n - \nabla\cdot\sigma_h^n)\|^2 + \frac{\lambda}{\eta}\|\sigma_h^n\|_h^2 + \frac{\Delta t}{\eta}\|\sigma_h^n\|_h^2] + \epsilon_h(\mathbf{u}_h^n, \mathbf{u}_h^n)$$
$$= \epsilon_h(\mathbf{u}_h^{n-1}, \mathbf{u}_h^n) + \frac{\lambda}{2\eta}(\sigma_h^n, \sigma_h^{n-1}) + (\mathbf{u}_h^{n-1}, \mathbf{u}_h^n + \Delta t(\nabla p_h^n - \nabla\cdot\sigma_h^n)) - \Delta t\; G_h^n(p_h^n I - \sigma_h^n) + \Delta t L_h^{\sigma,n}(\sigma_h^n)$$
$$+ \Delta t L_h^{p,n}(\mathbf{u}_h^n + \Delta t(\nabla p_h^n - \nabla\cdot\sigma_h^n)) + \Delta t(L_h^{\mathbf{u},n} - L_h^{p,n})(\mathbf{u}_h^n). \tag{34}$$

Using the Cauchy–Schwarz inequality, the inequality $|yz| \le \frac{y^2}{2\kappa} + \frac{\kappa}{2}z^2, \forall\kappa > 0, \forall y, z \in \mathbb{R}$ together with the properties of functionals $L_h^{p,n}$, $L_h^{\mathbf{u},n}$, $L_h^{\sigma,n}$ and G_h^n, for $\alpha, \beta, \gamma, \delta > 0$, provided $\alpha + \Delta t\beta = 1$, we obtain:

$$\left\{\begin{array}{l} \dfrac{1}{2}\left[\|\mathbf{u}_h^n\|^2 + \epsilon_h(\mathbf{u}_h^n, \mathbf{u}_h^n) + \Delta t^2\|\nabla p_h^n - \nabla\cdot\sigma_h^n\|^2 + \dfrac{\lambda}{2\eta}\|\sigma_h^n\|_h^2 + \dfrac{\Delta t}{\eta}\|\sigma_h^n\|_h^2\right] \\ \quad \le \dfrac{1}{2}\left\{\dfrac{1}{\alpha}\|\mathbf{u}_h^{n-1}\|^2 + \dfrac{\Delta t}{\beta}|L_h^{p,n}|^2 + \epsilon_h(\mathbf{u}_h^{n-1}, \mathbf{u}_h^{n-1}) + \dfrac{\lambda}{2\eta}\|\sigma_h^{n-1}\|_h^2 + \Delta t\left\{\dfrac{[G_h^n]^2}{\gamma} + \gamma\|(p_h^n I - \sigma_h^n)\vec{\nu}\|_{-1/2,\partial\Omega}^2\right\}\right. \\ \qquad \left. + \Delta t\left(\dfrac{|L_h^{\sigma,n}|^2}{\delta} + \delta\|\sigma_h^n\|_h^2\right) + \Delta t\left[\dfrac{|L_h^{p,n}|^2}{\beta} + (|L_h^{\mathbf{u},n}| + |L_h^{p,n}|)^2 + \|\mathbf{u}_h^n\|^2\right]\right\}. \end{array}\right. \tag{35}$$

Now assuming $\Delta t < \frac{1}{2}$ we take $\alpha = 1 - \Delta t$ and $\beta = 1$.

On the other hand from a standard result (cf. Girault–Raviart [32]) we have:

$$\|(p_h^n I - \sigma_h^n)\vec{\nu}\|_{-1/2,\partial\Omega} \le C_1\left\{\| p_h^n I - \sigma_h^n \| + \| \nabla\cdot\left(p_h^n I - \sigma_h^n\right) \|\right\} \le C_1\left\{\| p_h^n \| + \| \sigma_h^n \| + \| \nabla p_h^n - \nabla\cdot\sigma_h^n \|\right\}. \tag{36}$$

By a well-known result (see e.g. [38] p.33), $\exists C_2$ such that $\| q \| \le C_2\| \nabla q \|, \forall q \in Q$.

Applying this relation to (36) we obtain:

$$\|(p_h^n I - \sigma_h^n)\vec{\nu}\|_{-1/2,\partial\Omega} \le C_1\left\{C_2\| \nabla p_h^n \| + \| \sigma_h^n \| + \| \nabla p_h^n - \nabla\cdot\sigma_h^n \|\right\}$$
$$\le C_1\left[C_2\| \nabla\cdot\sigma_h^n \| + \| \sigma_h^n \| + (C_2 + 1)\| \nabla p_h^n - \nabla\cdot\sigma_h^n \|\right].$$

Recalling (16) we are led to

$$\|(p_h^n I - \sigma_h^n)\vec{\nu}\|_{-1/2,\partial\Omega} \le \frac{C_3}{h}\| \sigma_h^n \| + C_4\| \nabla p_h^n - \nabla\cdot\sigma_h^n \|$$

where $C_3 = CC_2 + C_1 H$ with $H = \max_{\tau_h\subset\mathcal{P}} h$, $\mathcal{P}$ being the family of quasi-uniform partitions in use and $C_4 = C_1(1 + C_2)$.

Next we choose γ and δ such that $\gamma \times \frac{C_3^2}{h^2} + \frac{\delta}{2} \le \frac{1}{2\eta}$ and $\gamma C_4^2 \le \frac{\Delta t}{4}$. Taking $\delta = \frac{1}{2\eta}$ and setting $\Delta t = \mu h$ we choose $\gamma = \min\left\{\frac{1}{C_4^2}, \frac{\Delta t}{\mu^2 C_3^2\eta}\right\}\frac{\Delta t}{4}$.

In so doing we obtain:

$$(1 - \Delta t)\|\mathbf{u}_h^n\|^2 + \epsilon_h(\mathbf{u}_h^n, \mathbf{u}_h^n) + \frac{\Delta t^2}{2}\|\nabla p_h^n - \nabla\cdot\sigma_h^n\|^2 + \frac{\lambda}{2\eta}\|\sigma_h^n\|_h^2$$
$$\le \frac{\|\mathbf{u}_h^{n-1}\|^2}{(1 - \Delta t)} + \epsilon_h(\mathbf{u}_h^{n-1}, \mathbf{u}_h^{n-1}) + \frac{\lambda}{2\eta}\|\sigma_h^{n-1}\|_h^2 + \bar{C}\left[\Delta t\left(|L_h^{p,n}|^2 + |L_h^{\mathbf{u},n}|^2 + |L_h^{\sigma,n}|^2\right) + \frac{[G_h^n]^2}{\Delta t}\right]. \tag{37}$$

Using the relation $\frac{1}{1-\Delta t} \le 1 + 2\Delta t$ for $\Delta t < \frac{1}{2}$, after straightforward calculations we obtain:

$$\|\mathbf{u}_h^n\|^2 + \epsilon_h(\mathbf{u}_h^n, \mathbf{u}_h^n) + \frac{\Delta t^2}{2}\|\nabla p_h^n - \nabla\cdot\sigma_h^n\|^2 + \frac{\lambda}{2\eta}\|\sigma_h^n\|_h^2 \le (1 + 2\Delta t)^2\left[\|\mathbf{u}_h^{n-1}\|^2 + \epsilon_h(\mathbf{u}_h^{n-1}, \mathbf{u}_h^{n-1}) + \frac{\lambda}{2\eta}\|\sigma_h^{n-1}\|_h^2\right.$$
$$\left. + \tilde{C}\Delta t\left(|L_h^{p,n}|^2 + |L_h^{\mathbf{u},n}|^2 + |L_h^{\sigma,n}|^2 + \frac{[G_h^n]^2}{\Delta t^2}\right)\right]. \tag{38}$$

Now we may follow the main lines of Gronwall's Lemma [39]. More specifically, applying (38) iteratively from $n = 1$, we derive:

$$\| \mathbf{u}_h^n \|^2 + \epsilon_h(\mathbf{u}_h^n, \mathbf{u}_h^n) + \frac{\Delta t^2}{2}\| \nabla p_h^n - \nabla\cdot\sigma_h^n \|^2 + \frac{\lambda}{2\eta}\| \sigma_h^n \|_h^2 \le (1 + 2\Delta t)^{2n}\left[\| \mathbf{u}_h^0 \|^2 + \epsilon_h(\mathbf{u}_h^0, \mathbf{u}_h^0) + \frac{\lambda}{2\eta}\| \sigma_h^0 \|_h^2\right]$$

$$+\tilde{C}\Delta t\sum_{i=1}^{n}(1+2\Delta t)^{2(n-i+1)}\left(|L_h^{p,i}|^2+|L_h^{\mathbf{u},i}|^2+|L_h^{\sigma,i}|^2+\frac{[G_h^i]^2}{\Delta t^2}\right). \tag{39}$$

Since $(1+2\Delta t)^{2n}\leq\left(1+\frac{2T}{M}\right)^{2M}\leq e^{4T}\ \forall n$, we finally obtain the stability result (31). □

Corollary 7. *Provided $\Delta t<1/2$, stability holds in the following sense for scheme (13):*

$$\begin{aligned}&\forall n\leq M:\|\mathbf{u}_h^n\|^2+\epsilon_h(\mathbf{u}_h^n,\mathbf{u}_h^n)+\frac{\Delta t^2}{2}\sum_{i=1}^{n}\|\nabla p_h^i-\nabla\cdot\sigma_h^i\|^2+\frac{\lambda}{2\eta}\|\sigma_h^n\|_h^2\\&\quad\leq e^{4T}\left[\|\mathbf{u}_h^0\|^2+\epsilon_h(\mathbf{u}_h^0,\mathbf{u}_h^0)+\frac{\lambda}{2\eta}\|\sigma_h^0\|_h^2+\tilde{C}\Delta t\sum_{i=1}^{n}\left(2\|\mathbf{f}_h^i\|_2^2+\frac{1}{\Delta t^2}\|\mathbf{g}_h^i\|_{1/2,\partial\Omega}^2\right)\right].\end{aligned} \tag{40}$$

Proof. Recalling that in the case under study $L_h^{u,n}(\mathbf{v})=L_h^{p,n}(\mathbf{v})=(\mathbf{f}_h^n,\mathbf{v})$, $L_h^{\sigma,n}=0$ and $G_h^n(\mathbf{w})=\langle\mathbf{g}_h^n,\mathbf{w}\rangle_{1/2,\partial\Omega}$ this result is a mere consequence of (31). □

6. Consistency

As a preparatory step to prove the convergence of our scheme, we establish in this Section that it is consistent in an appropriate sense. In order to do so we define $\forall n$,

$$\tilde{\mathbf{u}}_h^n=\tilde{\mathbf{u}}_h(n\Delta t) \tag{41}$$

where $\tilde{\mathbf{u}}_h(t)\in\mathbf{V}_h$ is given by

$$(\tilde{\mathbf{u}}_h(t),\mathbf{v})_h=(\mathbf{u}(t),\mathbf{v})_h\quad\forall\mathbf{v}\in\mathbf{V}_h, \tag{42}$$

together with the pair $[\tilde{p}_h^n,\tilde{\sigma}_h^n]\in Q_h\times\Sigma_h$ defined by

$$\begin{aligned}&\frac{\lambda+\Delta t}{2\eta}(\tilde{\sigma}_h^n,\tau)_h+\Delta t^2(\nabla\tilde{p}_h^n-\nabla\cdot\tilde{\sigma}_h^n,\nabla q-\nabla\cdot\tau)\\&\quad=\frac{\lambda+\Delta t}{2\eta}(\sigma^n,\tau)_h+\Delta t^2(\nabla p^n-\nabla\cdot\sigma^n,\nabla q-\nabla\cdot\tau)\quad\forall[q,\tau]\in Q_h\times\Sigma_h.\end{aligned} \tag{43}$$

Proposition 8. *If $\mathbf{u}(t)\in H^2(\Omega)^N$, $t\in[0,T]$, the following estimate holds:*

$$\|\mathbf{u}(t)-\tilde{\mathbf{u}}_h(t)\|\leq Ch^2\|\mathbf{u}(t)\|_2. \tag{44}$$

Proof. Let π_h denote the standard $\mathbf{V}_h$-interpolation operator of any field with components in $H^2(\Omega)$. By the definition of $\tilde{\mathbf{u}}_h(t)$ we may write $(\mathbf{u}(t)-\tilde{\mathbf{u}}_h(t),\pi_h[\mathbf{u}(t)]-\tilde{\mathbf{u}}_h(t))_h=0$, which yields,

$$\|\mathbf{u}(t)-\tilde{\mathbf{u}}_h(t)\|_h\leq\|\mathbf{u}(t)-\pi_h[\mathbf{u}(t)]\|_h. \tag{45}$$

Next we note that, owing to (11) together with the classical estimate $\|[\pi_h(\mathbf{v})-\mathbf{v}]_{/K}\|_{0,K}\leq Ch^2\|\mathbf{v}\|_{2,K}$, for every $K\in\mathcal{T}_h$ and $\forall\mathbf{v}\in H^2(\Omega)^N$, we may write,

$$(\mathbf{w}-\pi_h(\mathbf{w}),\mathbf{v}-\pi_h(\mathbf{v}))_K\leq C\|\mathbf{w}\|_{2,K}\|\mathbf{v}\|_{2,K}\quad\forall\mathbf{w},\mathbf{v}\in[H^2(K)^N]^2,$$

where $(\cdot,\cdot)_K$ is defined in (9). Then Ciarlet's Lemma for bilinear forms [37] yields:

$$\|\mathbf{u}(t)-\pi_h[\mathbf{u}(t)]\|_h\leq Ch^2\|\mathbf{u}(t)\|_2. \tag{46}$$

On the other hand, since $\|\mathbf{v}\|\leq\|\mathbf{v}\|_h\ \forall\mathbf{v}\in\mathbf{V}_h$ we have

$$\|\mathbf{u}(t)-\tilde{\mathbf{u}}_h(t)\|\leq\|\pi_h[\mathbf{u}(t)]-\mathbf{u}(t)\|+\|\pi_h[\mathbf{u}(t)]-\tilde{\mathbf{u}}_h(t)\|_h. \tag{47}$$

Then from (45) and (47) we easily establish that,

$$\|\mathbf{u}(t)-\tilde{\mathbf{u}}_h(t)\|\leq\|\pi_h[\mathbf{u}(t)]-\mathbf{u}(t)\|+2\|\pi_h[\mathbf{u}(t)]-\mathbf{u}(t)\|_h.$$

Finally combining this relation with (46) and using the standard estimate $\|\mathbf{u}(t)-\pi_h[\mathbf{u}(t)]\|\leq Ch^2\|\mathbf{u}(t)\|_2$ (cf. [37]) the result follows. □

The pair $[\tilde{\sigma}_h^n,\tilde{p}_h^n]$ in turn is a sort of orthogonal projection of $[\sigma^n,p^n]$. Thus by similar arguments to those employed in the proof of Proposition 8, we can prove the following:

Proposition 9. *If $\sigma^n \in H^2(\Omega)^{N\times N}$ and $p^n \in H^2(\Omega)$ we have:*

$$\| \tilde{\sigma}_h^n - \sigma^n \|_h \le Ch\Delta t \left[\|p^n\|_2 + (1+h/\Delta t)\|\sigma^n\|_2\right] \tag{48}$$

$$\| \nabla\tilde{p}_h^n - \nabla p^n - \nabla\cdot\tilde{\sigma}_h^n + \nabla\cdot\sigma^n \| \le Ch\left[\|p^n\|_2 + (1+h/\Delta t)\|\sigma^n\|_2\right]. \quad \square \tag{49}$$

Next we apply scheme (13) to the triple $(\tilde{\mathbf{u}}_h^n, \tilde{p}_h^n, \tilde{\sigma}_h^n) \in \mathbf{V}_h \times Q_h \times \Sigma_h$ assuming that $\tilde{\mathbf{u}}_h^{n-1}$ and $\tilde{\sigma}_h^{n-1}$, $n = 1, 2, \ldots, M$ are known. More specifically, taking $\tilde{\mathbf{u}}_h^0 = \pi_h(\mathbf{u}^0)$ and $\tilde{\sigma}_h^0 = \pi_h(\sigma^0)$, we determine the residuals in (13), when $(\mathbf{u}_h^n, p_h^n, \sigma_h^n)$ is replaced with $(\bar{\mathbf{u}}_h^n, \bar{p}_h^n, \bar{\sigma}_h^n) := (\tilde{\mathbf{u}}_h^n - \mathbf{u}_h^n, \tilde{p}_h^n - p_h^n, \tilde{\sigma}_h^n - \sigma_h^n)$ and $[\mathbf{u}_h^{n-1}, \sigma_h^{n-1}]$ is replaced with $[\bar{\mathbf{u}}_h^{n-1}, \bar{\sigma}_h^{n-1}] := [\tilde{\mathbf{u}}_h^{n-1} - \mathbf{u}_h^{n-1}, \tilde{\sigma}_h^{n-1} - \sigma_h^{n-1}]$.

By definition we have:

$$\begin{cases} \Delta t^2\left[(\nabla\bar{p}_h^n, \nabla q) - (\nabla\cdot\bar{\sigma}_h^n, \nabla q)\right] - \Delta t(\mathbf{u}_h^{n-1}, \nabla q) = -\Delta t\mathcal{S}_h^n(q\vec{\nu}) + \Delta t^2\mathcal{R}_h^{p,n}(\nabla q) \\ (\bar{\mathbf{u}}_h^n, \mathbf{v})_h + \Delta t(\nabla\bar{p}_h^n - \nabla\cdot\bar{\sigma}_h^n, \mathbf{v})_h - (\bar{\mathbf{u}}_h^{n-1}, \mathbf{v})_h = \Delta t\mathcal{R}_h^{\mathbf{u},n}(\mathbf{v}) \\ \dfrac{\Delta t+\lambda}{2\eta}(\bar{\sigma}_h^n, \tau)_h - \dfrac{\lambda}{2\eta}(\bar{\sigma}_h^{n-1}, \tau)_h + \Delta t^2\left[(\nabla\cdot\bar{\sigma}_h^n, \nabla\cdot\tau) - (\nabla\bar{p}_h^n, \nabla\cdot\tau)\right] \\ \quad + \Delta t(\mathbf{u}_h^{n-1}, \nabla\cdot\tau) = \Delta t\mathcal{S}_h^n(\tau\vec{\nu}) - \Delta t^2\mathcal{R}_h^{p,n}(\nabla\cdot\tau) + \Delta t\mathcal{R}_h^{\sigma,n}(\tau) \end{cases} \tag{50}$$

where $\mathcal{R}_h^{\mathbf{u},n}$, $\mathcal{R}_h^{p,n}$ and $\mathcal{R}_h^{\sigma,n}$ are functionals standing for the residuals with respect to domain integrals and $\mathcal{S}_h^n$ is the one representing the residuals related to boundary integrals.

As for the second equation of (50) we have:

$$\begin{aligned} &(\tilde{\mathbf{u}}_h^n, \mathbf{v})_h + \Delta t(\nabla\tilde{p}_h^n - \nabla\cdot\tilde{\sigma}_h^n, \mathbf{v}) - (\tilde{\mathbf{u}}_h^{n-1}, \mathbf{v})_h = (\tilde{\mathbf{u}}_h^n - \mathbf{u}^n, \mathbf{v})_h \\ &\quad + \Delta t(\nabla\tilde{p}_h^n - \nabla\cdot\tilde{\sigma}_h^n - \nabla p^n + \nabla\cdot\sigma^n, \mathbf{v}) - (\tilde{\mathbf{u}}_h^{n-1} - \mathbf{u}^{n-1}, \mathbf{v})_h \\ &\quad + (\mathbf{u}^n, \mathbf{v}) + \Delta t(\nabla p^n - \nabla\cdot\sigma^n, \mathbf{v}) - (\mathbf{u}^{n-1}, \mathbf{v}) + \epsilon_h(\mathbf{u}^n - \mathbf{u}^{n-1}, \mathbf{v}). \end{aligned} \tag{51}$$

On the other hand for every field $\mathcal{F}$ sufficiently smooth we have $\forall n$ (cf. [39]):

$$\mathcal{F}^n - \mathcal{F}^{n-1} = \Delta t\mathcal{F}_t^n - \int_{(n-1)\Delta t}^{n\Delta t}[s-(n-1)\Delta t]\,\mathcal{F}_{tt}(s)\mathrm{d}s, \quad \text{with } \mathcal{F}^n(\cdot) = \mathcal{F}(\cdot, n\Delta t). \tag{52}$$

Hence using (4) it follows that:

$$(\mathbf{u}^n - \mathbf{u}^{n-1} + \Delta t(\nabla p^n - \nabla\cdot\sigma^n), \mathbf{v}) = \Delta t(\mathbf{f}^n, \mathbf{v}) - \left(\int_{(n-1)\Delta t}^{n\Delta t}[s-(n-1)\Delta t]\,\mathbf{u}_{tt}(s)\mathrm{d}s, \mathbf{v}\right).$$

Then since $(\tilde{\mathbf{u}}_h^n - \mathbf{u}^n, \mathbf{v})_h = 0$ and $(\tilde{\mathbf{u}}_h^{n-1} - \mathbf{u}^{n-1}, \mathbf{v})_h = 0$, recalling (51) and (13), we easily derive:

$$\begin{cases} \mathcal{R}_h^{\mathbf{u},n}(\mathbf{v}) = (\nabla\tilde{p}_h^n - \nabla\cdot\tilde{\sigma}_h^n - \nabla p^n + \nabla\cdot\sigma^n, \mathbf{v}) + (\mathbf{f}^n - \mathbf{f}_h^n, \mathbf{v}) \\ \quad -\dfrac{1}{\Delta t}\left(\displaystyle\int_{(n-1)\Delta t}^{n\Delta t}[s-(n-1)\Delta t]\,\mathbf{u}_{tt}(s)\mathrm{d}s, \mathbf{v}\right) + \dfrac{1}{\Delta t}\epsilon_h(\mathbf{u}^n - \mathbf{u}^{n-1}, \mathbf{v}). \end{cases} \tag{53}$$

As for the first and third equations of (50), taking into account (43) we easily obtain:

$$\begin{aligned} &\frac{\lambda+\Delta t}{2\eta}(\tilde{\sigma}_h^n, \tau)_h - \frac{\lambda}{2\eta}(\tilde{\sigma}_h^{n-1}, \tau)_h + \Delta t^2[(\nabla\tilde{p}_h^n - \nabla\cdot\tilde{\sigma}_h^n, \nabla q) - (\nabla\tilde{p}_h^n - \nabla\cdot\tilde{\sigma}_h^n, \nabla\cdot\tau)] - \Delta t(\tilde{\mathbf{u}}_h^{n-1}, \nabla q - \nabla\cdot\tau) \\ &\quad = -\Delta t(\tilde{\mathbf{u}}_h^{n-1} - \mathbf{u}^{n-1}, \nabla q - \nabla\cdot\tau) - \frac{\lambda}{2\eta}(\tilde{\sigma}_h^{n-1} - \sigma^{n-1}, \tau)_h + \Delta t^2\left[(\nabla p^n - \nabla\cdot\sigma^n, \nabla q - \nabla\cdot\tau)\right] \\ &\quad\quad + \frac{\lambda+\Delta t}{2\eta}\left[(\sigma^n, \tau) + \epsilon_h(\sigma^n, \tau)\right] - \Delta t(\mathbf{u}^n, \nabla q - \nabla\cdot\tau) \\ &\quad\quad + \Delta t(\mathbf{u}^n - \mathbf{u}^{n-1}, \nabla q - \nabla\cdot\tau) - \frac{\lambda}{2\eta}\left[(\sigma^{n-1}, \tau) + \epsilon_h(\sigma^{n-1}, \tau)\right]. \end{aligned}$$

Then applying integration by parts together with (52), we derive:

$$\begin{aligned} &\frac{\lambda+\Delta t}{2\eta}(\tilde{\sigma}_h^n, \tau)_h - \frac{\lambda}{2\eta}(\tilde{\sigma}_h^{n-1}, \tau)_h + \Delta t^2(\nabla\tilde{p}_h^n - \nabla\cdot\tilde{\sigma}_h^n, \nabla q - \nabla\cdot\tau) - \Delta t(\tilde{\mathbf{u}}_h^{n-1}, \nabla q - \nabla\cdot\tau) \\ &\quad = -\Delta t(\tilde{\mathbf{u}}_h^{n-1} - \mathbf{u}^{n-1}, \nabla q - \nabla\cdot\tau) - \frac{\lambda}{2\eta}(\tilde{\sigma}_h^{n-1} - \sigma^{n-1}, \tau)_h + \Delta t^2(\mathbf{u}_t^n + \nabla p^n - \nabla\cdot\sigma^n, \nabla q - \nabla\cdot\tau) \\ &\quad\quad - \Delta t\left(\int_{(n-1)\Delta t}^{n\Delta t}[s-(n-1)\Delta t]\,\mathbf{u}_{tt}(s)\mathrm{d}s, \nabla q - \nabla\cdot\tau\right) + \frac{\lambda+\Delta t}{2\eta}\epsilon_h(\sigma^n, \tau) \end{aligned}$$

$$- \frac{\lambda}{2\eta}\epsilon_h(\sigma^{n-1},\tau) + \frac{\Delta t}{2\eta}\left(\sigma^n + \lambda\sigma_t^n - \frac{\lambda}{\Delta t}\int_{(n-1)\Delta t}^{n\Delta t}[s-(n-1)\Delta t]\,\sigma_{tt}(s)\mathrm{d}s, \tau\right)$$
$$- \Delta t[\langle \mathbf{g}^n,\ (qI-\tau)\vec{\nu}\,\rangle_{1/2,\partial\Omega} - (\nabla\cdot\mathbf{u}^n, q) + (\mathrm{D}(\mathbf{u}^n),\tau)].$$

Since $\mathbf{u}_t^n + \nabla p^n - \nabla\cdot\sigma^n = \mathbf{f}^n$, $\nabla\cdot\mathbf{u}^n = 0$ and $\sigma^n + \lambda\sigma_t^n = 2\eta\mathrm{D}(\mathbf{u}^n)$, recalling (13) we obtain,

$$\frac{\lambda+\Delta t}{2\eta}(\bar{\sigma}_h^n,\tau)_h - \frac{\lambda}{2\eta}(\bar{\sigma}_h^{n-1},\tau)_h + \Delta t^2(\nabla\bar{p}_h^n - \nabla\cdot\bar{\sigma}_h^n, \nabla q - \nabla\cdot\tau) - \Delta t(\bar{\mathbf{u}}_h^{n-1}, \nabla q - \nabla\cdot\tau)$$
$$= \Delta t^2(\mathbf{f}^n - \mathbf{f}_h^n, \nabla q - \nabla\cdot\tau) - \Delta t(\tilde{\mathbf{u}}_h^{n-1} - \mathbf{u}^{n-1}, \nabla q - \nabla\cdot\tau) - \frac{\lambda}{2\eta}(\tilde{\sigma}^{n-1} - \sigma^{n-1},\tau)_h + \frac{\lambda+\Delta t}{2\eta}\epsilon_h(\sigma^n,\tau)$$
$$- \frac{\lambda}{2\eta}\epsilon_h(\sigma^{n-1},\tau) - \Delta t\left(\int_{(n-1)\Delta t}^{n\Delta t}[s-(n-1)\Delta t]\,\mathbf{u}_{tt}(s)\mathrm{d}s, \nabla q - \nabla\cdot\tau\right)$$
$$- \frac{\lambda}{2\eta}\left(\int_{(n-1)\Delta t}^{n\Delta t}[s-(n-1)\Delta t]\,\sigma_{tt}(s)\mathrm{d}s, \tau\right) - \Delta t\langle \mathbf{g}^n - \mathbf{g}_h^n,\ (qI-\tau)\vec{\nu}\,\rangle_{1/2,\partial\Omega}.$$

This means that:

$$\begin{cases}\mathcal{R}_h^{p,n}(\nabla q - \nabla\cdot\tau) = \left(\mathbf{f}^n - \mathbf{f}_h^n + \frac{1}{\Delta t}\left(\mathbf{u}^{n-1} - \tilde{\mathbf{u}}_h^{n-1} - \int_{(n-1)\Delta t}^{n\Delta t}[s-(n-1)\Delta t]\,\mathbf{u}_{tt}(s)\mathrm{d}s\right), \nabla q - \nabla\cdot\tau\right).\end{cases} \tag{54}$$

$$\begin{cases}\mathcal{R}_h^{\sigma,n}(\tau) = & \frac{\lambda}{2\eta\Delta t}\left[(\sigma^{n-1} - \tilde{\sigma}_h^{n-1},\tau)_h + \epsilon_h(\sigma^n, \tau - \sigma^{n-1},\tau) - \left(\left(\int_{(n-1)\Delta t}^{n\Delta t}[s-(n-1)\Delta t]\,\sigma_{tt}(s)\mathrm{d}s, \tau\right)\right)\right] \\ & + \frac{1}{2\eta}\epsilon_h(\sigma^n,\tau)\end{cases} \tag{55}$$

$$\mathcal{S}_h^n((qI-\tau)\vec{\nu}) = \langle \mathbf{g}^n - \mathbf{g}_h^n,\ (qI-\tau)\vec{\nu}\,\rangle_{1/2,\partial\Omega}. \tag{56}$$

Now we endeavour to estimate the norms of $\mathcal{R}_h^{\mathbf{u},n}$, $\mathcal{R}_h^{p,n}$, $\mathcal{R}_h^{\sigma,n}$ and $\mathcal{S}_h^n$, $\mathbf{f}_h^n$ being defined as the piecewise constant interpolate of $\mathbf{f}^n$ and $\mathbf{g}_h^n$ being the quadratic interpolate of $\mathbf{g}^n$ at the vertices and edge mid-points of the boundary edges or faces of the mesh.

First we need the following technical result

Lemma 10. *$\exists C_E > 0$ such that $\forall\mathbf{u}\in H^1(\Omega)^N$ and $\forall\mathbf{v}\in\mathbf{V}_h$ we have*

$$|\epsilon_h(\mathbf{u},\mathbf{v})| \le C_E h\|\mathbf{u}\|_1\|\mathbf{v}\|. \tag{57}$$

Proof. Recalling (9) let us first extend the definition of ϵ_K in Lemma 3 in order to accommodate $\mathbf{u}\in H^1(K)^N$ and $\mathbf{v}\in P_1(K)^N$, by setting: $\epsilon_K(\mathbf{u},\mathbf{v}) := (\pi_1^K(\mathbf{u}),\mathbf{v})_K - \int_K \pi_1^K(\mathbf{u})\cdot\mathbf{v}$. From the semi-positive-definiteness of ϵ_K established in the proof of Lemma 3, we know that $\epsilon_K(\pi_1^K(\mathbf{u}),\mathbf{v})^2 \le \epsilon_K(\pi_1^K(\mathbf{u}),\pi_1^K(\mathbf{u}))\epsilon_K(\mathbf{v},\mathbf{v})$. Thus using (11) we readily derive $\epsilon_K(\mathbf{u},\mathbf{v}) \le (N+1)\|\pi_1^K(\mathbf{u})\|_{0,K}\|\mathbf{v}\|_{0,K} \le (N+1)\|\mathbf{u}\|_{1,K}\|\mathbf{v}\|_{0,K}$. Moreover, as one can easily check, $\epsilon_K(\mathbf{u},\mathbf{v}) = 0, \forall\mathbf{u}\in P_0(K)^N$ and $\forall\mathbf{v}\in P_1(K)^N$. Then (57) is a direct consequence of Ciarlet's Lemma for bilinear forms (cf. [37]). □

Proposition 11. *Assuming that $\mathbf{f}(t)\in H^1(\Omega)^N$ $\forall t$, $p^n\in H^2(\Omega)$, $\mathbf{u}_t^n\in H^1(\Omega)^N$, $\sigma^n\in\left[H^2(\Omega)\right]^{N\times N}$ $\forall n$ and $\mathbf{u}_{tt}\in L^\infty\left[(0,T),L^2(\Omega)\right]^N$ (cf. [40]), it holds that*

$$|\mathcal{R}^{\mathbf{u},n}| \le C\alpha_{\mathbf{u}}^n(h + \Delta t + h^2/\Delta t) \tag{58}$$

where $\alpha_{\mathbf{u}}^n = \|\mathbf{f}^n\|_1 + \|p^n\|_2 + \|\mathbf{u}_t^n\|_1 + \sup\mathrm{ess}_{0\le s\le T}\ \|\mathbf{u}_{tt}(s)\|$. □

Proof. According to (53) we have:

$$|\mathcal{R}^{\mathbf{u},n}| \le \|\nabla\tilde{p}_h^n - \nabla p^n - \nabla\cdot\tilde{\sigma}_h^n + \nabla\cdot\sigma^n\| + \|\mathbf{f}^n - \mathbf{f}_h^n\| + \Delta t\sup_{0\le s\le T}\mathrm{ess}\ \|\mathbf{u}_{tt}(s)\| + \sup_{\mathbf{v}\in\mathbf{V}_h-\{\mathbf{0}\}}\frac{|\epsilon_h(\mathbf{u}^n - \mathbf{u}^{n-1},\mathbf{v})|}{\Delta t\|\mathbf{v}\|}.$$

Using Lemma 10 together with (52), it is possible to derive in a straightforward manner,

$$\sup_{\mathbf{v}\in\mathbf{V}_h-\{\mathbf{0}\}}\frac{|\epsilon_h(\mathbf{u}^n - \mathbf{u}^{n-1},\mathbf{v})|}{\Delta t\|\mathbf{v}\|} \le C\left(h\|\mathbf{u}_t^n\|_1 + \Delta t\sup_{0\le s\le T}\mathrm{ess}\ \|\mathbf{u}_{tt}(s)\|\right).$$

Then the remainder of the proof is a consequence of (49), together with the classical estimate $\|\mathbf{f}^n - \mathbf{f}_h^n\| \le Ch\|f^n\|_1$. □

In an entirely analogous manner one can prove:

Proposition 12. *Under the assumptions of Proposition 11 together with the regularity hypotheses that $\sigma_t^n \in H^1(\Omega)^{N\times N}$ and $\sigma_{tt} \in L^\infty[(0,T), L^2(\Omega)]^{N\times N}$ (cf. [40]), it holds that:*

$$|\mathcal{R}^{p,n}| \le C\alpha_p^n(h + \Delta t + h^2/\Delta t) \tag{59}$$

where $\alpha_p^n = \|\mathbf{f}^n\|_1 + \|\mathbf{u}^{n-1}\|_2 + \sup\text{ess}_{0\le s\le T}\ \|\mathbf{u}_{tt}(s)\|$, *and*

$$|\mathcal{R}^{\sigma,n}| \le C\alpha_\sigma^n(h + \Delta t + h^2/\Delta t) \tag{60}$$

where $\alpha_\sigma^n = \|p^n\|_2 + \|\sigma_t^n\|_1 + \sup\text{ess}_{0\le s\le T}\ \|\sigma_{tt}(s)\|$. $\square$

Finally by the classical interpolation theory (cf. [37]) the following estimate holds:

Proposition 13. *If* $\mathbf{g}^n \in H^{5/2}(\partial\Omega)^N$ *then:*

$$[\mathcal{S}_h^n] \le Ch^2\|\mathbf{g}^n\|_{5/2,\partial\Omega}. \quad \square \tag{61}$$

7. Convergence

In this Section we prove the convergence of the method in natural norms. The essential step for this is the application of the stability result (40) to problem (50). Indeed thanks to (58)–(61) and Lemma 3, we have

Proposition 14. *Let $\Delta t = T/M$ with $M > 2T$. Then under the assumptions of Propositions 11–13, μ being the ratio $\Delta t/h$ considered to be fixed, we have $\forall n \le M$:*

$$\|\tilde{\mathbf{u}}_h^n - \mathbf{u}_h^n\|^2 + \frac{\Delta t^2}{2}\sum_{i=1}^n \|\nabla(\tilde{p}_h^i - p_h^i) - \nabla\cdot(\tilde{\sigma}_h^i - \sigma_h^i)\|^2 + \frac{\lambda}{2\eta}\|\tilde{\sigma}_h^n - \sigma_h^n\|^2$$
$$\le \frac{C(T)h^2}{\mu^2}\max_{1\le i\le n}\left\{\left[(\alpha_{\mathbf{u}}^i)^2 + (\alpha_p^i)^2 + (\alpha_\sigma^i)^2\right](\mu^2+\mu+1)^2 + \|\mathbf{g}^i\|_{5/2,\partial\Omega}\right\}. \quad \square \tag{62}$$

Before pursuing this we need the following technical lemma:

Lemma 15. *If $\sigma^n \in H^2(\Omega)^{N\times N}$ and $p^n \in H^2(\Omega)$ then,*

$$\|\sigma^n - \tilde{\sigma}_h^n\| \le C(h^2 + h\Delta t)(\|\sigma^n\|_2 + \|p^n\|_2). \tag{63}$$

Proof. First we write $\|\sigma^n - \tilde{\sigma}_h^n\| \le \|\sigma^n - \pi_h(\sigma^n)\| + \|\pi_h(\sigma^n) - \tilde{\sigma}_h^n\|$. Then

$$\|\sigma^n - \tilde{\sigma}_h^n\| \le Ch^2\|\sigma^n\|_2 + \|\pi_h(\sigma^n) - \tilde{\sigma}_h^n\|. \tag{64}$$

On the other hand, by the definition of $\tilde{\sigma}_h^n$ we have $\forall q \in Q_h$ and $\forall \tau \in \Sigma_h$:

$$\frac{\lambda+\Delta t}{2\eta}(\tilde{\sigma}_h^n - \pi_h(\sigma^n), \tau)_h + \Delta t^2(\nabla(\tilde{p}_h^n - \pi_h(p^n)) - \nabla\cdot(\tilde{\sigma}_h^n - \pi_h(\sigma^n)), \nabla q - \nabla\cdot\tau)$$
$$= \frac{\lambda+\Delta t}{2\eta}(\sigma^n - \pi_h(\sigma^n), \tau)_h + \Delta t^2(\nabla(p^n - \pi_h(p^n)) - \nabla\cdot(\sigma^n - \pi_h(\sigma^n)), \nabla q - \nabla\cdot\tau).$$

Thus taking $\tau = \tilde{\sigma}_h^n - \pi_h(\sigma^n)$ and $q = \tilde{p}_h^n - \pi_h(p^n)$ and using the Cauchy-Schwartz inequality we obtain:

$$\frac{\lambda+\Delta t}{2\eta}\|\pi_h(\sigma^n) - \tilde{\sigma}_h^n\|_h^2 + \Delta t^2\|\nabla(\tilde{p}_h^n - \pi_h(p^n)) - \nabla\cdot(\tilde{\sigma}_h^n - \pi_h(\sigma^n))\|^2$$
$$\le \frac{\lambda+\Delta t}{2\eta}\|\pi_h(\sigma^n) - \sigma^n\|_h^2 + \Delta t^2\|\nabla(p^n - \pi_h(p^n)) - \nabla\cdot(\sigma^n - \pi_h(\sigma^n))\|^2$$
$$\le C(h^4 + \Delta t^2h^2)(\|p^n\|_2^2 + \|\sigma^n\|_2^2).$$

Recalling (64) and noticing that from Lemma 3 we have $\|\tau\| \le \|\tau\|_h$ $\forall \tau \in \Sigma_h$, the proof is complete. $\square$

We are now ready to give the main convergence results:

Theorem 16. *Let $\Delta t = T/M < 1/2$, and $\Delta t = \mu h$. Under the regularity assumptions on the data and the solution of problem (4) made throughout this paper, there exists a constant $C_T^\mu > 0$ such that $\forall n \le M$ the following estimate holds:*

$$\|\mathbf{u}^n - \mathbf{u}_h^n\|^2 + \Delta t^2\sum_{i=1}^n\|\nabla(p^i - p_h^i) - \nabla\cdot(\sigma^i - \sigma_h^i)\|^2 + \frac{\lambda}{2\eta}\|\sigma^n - \sigma_h^n\|^2 \le C_T^\mu h^2. \tag{65}$$

More specifically we have:

$$C_T^{\mu} = C(\mu, T)\left\{\max_{1\leq i\leq M}[\|\mathbf{u}^i\|_2^2 + \|p^i\|_2^2 + \|\sigma^i\|_2^2 + \|\mathbf{f}^i\|_1^2 + \|\mathbf{g}^i\|_{5/2,\partial\Omega}^2 \right.$$
$$\left. + \|\mathbf{u}_t^i\|_1^2 + \|\sigma_t^i\|_1^2] + \operatorname*{sup\,ess}_{0\leq s\leq T} \|\mathbf{u}_{tt}(s)\|^2 + \operatorname*{sup\,ess}_{0\leq s\leq T} \|\sigma_{tt}(s)\|^2\right\}.$$

Proof. Using $\|\mathbf{u}^n - \mathbf{u}_h^n\|^2 \leq 2[\|\mathbf{u}^n - \tilde{\mathbf{u}}_h^n\|^2 + \|\tilde{\mathbf{u}}^n - \mathbf{u}_h^n\|^2]$ and analogous inequalities for both $\|\sigma^n - \sigma_h^n\|^2$ and $\sum_{i=1}^n \|\nabla(\tilde{p}_h^i - p_h^i) - \nabla\cdot(\tilde{\sigma}_h^i - \sigma_h^i)\|^2$, this result directly follows from Lemma 15, together with Proposition 14, (44), (48) and (49). □

As a consequence of Theorem 16, first order convergence in the sense of $L^2(\Omega)$ of $\mathbf{u}_h^n$ to $\mathbf{u}(t)$, and of σ_h^n to $\sigma(t)$ as n goes to ∞ and h goes to zero was established, provided $t = n\Delta t$ remains fixed. Next we give another result stating the convergence of p_h^n to p in a weaker sense. More precisely we mean the sense of the discrete $L^2[(0, T), L^2(\Omega)]$-norm denoted by $\|\cdot\|_M$ given by,

$$\|q\|_M = \left[\sum_{i=1}^{M} \Delta t\|q^i\|^2\right]^{1/2}, \quad \text{with } q^i = q(i\Delta t), i = 1, 2, \ldots, M, \forall q \in C^0\{[0, T], L^2(\Omega)\}.$$

Here we assume that q varies linearly with t between $i\Delta t$ and $(i + 1)\Delta t$ for every i, if the function q is defined only at times $i\Delta t$ for $i = 1, 2, \ldots, M$.

Theorem 17. *Under the same regularity assumptions on the solution of (4) made in Theorem 16 and for $\Delta t = T/M < 1/2$, $\mu = \Delta t/h$ being fixed $\exists\, C_M$ such that:*

$$\|p - p_h\|_M^2 \leq C_M h \tag{66}$$

where p_h is defined by $p_h(i\Delta t) = p_h^i$, $i = 1, 2, \ldots, M$.

Proof. First we rewrite:

$$\|p - p_h\|_M^2 = \sum_{i=1}^{M} \Delta t\left[\sup_{q\in L_0^2(\Omega)-\{0\}} \frac{|(p^i - p_h^i, q)|}{\|q\|}\right]^2.$$

By a classical result (cf. [32]) $\forall q \in L_0^2(\Omega), q \neq 0, \exists \mathbf{v} \in H_0^1(\Omega)^N$ such that $\nabla\cdot\mathbf{v} = q$ and $\|\nabla\mathbf{v}\| \leq C\|q\|$. Hence using suitable Green's formulae and the Friedrichs–Poincaré inequality, we successively obtain:

$$\begin{aligned}
\sup_{q\in L_0^2(\Omega)-\{0\}} \frac{|(p^i - p_h^i, q)|}{\|q\|} &\leq C \sup_{\mathbf{v}\in H_0^1(\Omega)^N-\{\mathbf{0}\}} \frac{|(p^i - p_h^i, \nabla\cdot\mathbf{v})|}{\|\nabla\mathbf{v}\|} \\
&= C \sup_{\mathbf{v}\in H_0^1(\Omega)^N-\{\mathbf{0}\}} \left[\frac{|(\nabla(p^i - p_h^i) - \nabla\cdot(\sigma^i - \sigma_h^i), \mathbf{v}) - (\sigma^i - \sigma_h^i, \nabla\mathbf{v})|}{\|\nabla\mathbf{v}\|}\right] \\
&\leq C_P\left[\|\nabla(p^i - p_h^i) - \nabla\cdot(\sigma^i - \sigma_h^i)\| + \|\sigma^i - \sigma_h^i\|\right].
\end{aligned}$$

This leads to

$$\|p - p_h\|_M^2 \leq 2C_P\Delta t^{-1}\sum_{i=1}^{M} \Delta t^2[\|\nabla(p^i - p_h^i) - \nabla\cdot(\sigma^i - \sigma_h^i)\|^2 + \|\sigma^i - \sigma_h^i\|^2].$$

Finally using (65) we conclude that $\|p - p_h\|_M^2 \leq C_U[\Delta t^{-1}h^2 + h^2]$ and the result follows. □

8. Numerical aspects

Since questions may arise on the optimality of the error estimates obtained in this work, the authors would like to address a few considerations about this point.

First of all it is worthwhile stressing the fact that the stability result (40) derived for our scheme holds independently of the discretization parameters h and Δt. This is not surprising at all since such a result was derived for a fully implicit scheme applied to a linear problem. However the condition that Δt be bounded by a constant multiplied by h must be satisfied if the algorithm (15) providing an explicit solution procedure is employed at every time step. Again this is no surprise owing to the dominant hyperbolic nature of the three-field system under study for λ not so small.

As for convergence, error bounds proportional to $h + \Delta t$ are the best one can hope for. Indeed, on the one hand this is natural for a first order Euler time integration scheme. On the other hand, contrary to the $\mathcal{O}(h^2)$ estimate for the error

Table 1
Relative errors in the L^2-norm for $\lambda = 10$, $\eta = 1$ and $\mathbf{t} = \mathbf{0.1}$.

M	$\mathbf{u}$	p	σ
2	0.20424657E−04	0.87849969E+00	0.11637832E−04
4	0.31002633E−01	0.44702548E+00	0.83090181E−05
8	0.18373583E−01	0.14006621E+00	0.71100717E−05
16	0.90056909E−02	0.41192174E−01	0.54772777E−05

Table 2
Relative errors in the L^2-norm for $\lambda = 10$, $\eta = 1$ and $\mathbf{t} = \mathbf{1.0}$.

M	$\mathbf{u}$	p	σ
2	0.56647707E−03	0.29960172E+01	0.97030262E−03
4	0.68327049E−02	0.54365712E+00	0.33282136E−03
8	0.31364798E−02	0.18664163E+00	0.13640762E−03
16	0.13763716E−02	0.70842154E−01	0.67042529E−04

introduced by lumping the masses that is known to hold for the heat equation (cf. [39]), in the case under study only an $\mathcal{O}(h)$ bound for such error can be exploited. This is because here we have to deal with the L^2-norm instead of the H^1-norm of the velocity and extra stress test functions. In this respect we refer to Lemma 10. Besides this limitation, the error of the piecewise linear approximations of p and σ in the H^1-norm, necessary to derive the error estimates, is again no better than $\mathcal{O}(h)$, which becomes clear from Proposition 9. In particular estimate (49) indicates that $\Delta t = \mathcal{O}(h)$ is the optimal choice for our method. Nevertheless in order to check this out we performed some three-dimensional computations. More specifically we approximated with our method system (4) in the domain $\Omega \times (0, T)$, Ω being the unit cube $(0, 1)^3$ and $T = 1$. We present below some relevant results for the particular case where the exact solution is given by:

$$\begin{aligned}
&\mathbf{u}(x, y, z, t) = [x(y - z), y(z - x), z(x - y)]^{\mathrm{T}} t \\
&p(x, y, z, t) = [x^2(z - y) + y^2(x - z) + z^2(y - x)]/2 \quad \forall t \\
&\sigma(x, y, z, t) = \eta(t - \lambda) \begin{bmatrix} 2(y - z) & x - y & z - x \\ x - y & 2(z - x) & y - z \\ z - x & y - z & 2(x - y) \end{bmatrix}.
\end{aligned}$$

The corresponding right hand side is given by $\mathbf{f}(x, y, z, t) = [y^2 - z^2, z^2 - x^2, x^2 - y^2]^{\mathrm{T}}/2\ \forall t$, while the prescribed boundary velocity $\mathbf{g}$ and the initial data $\mathbf{u}^0$ and σ^0 are obvious. We solved this problem with uniform tetrahedral meshes obtained by first subdividing Ω into M^3 equal cubes with edge length $h = 1/M$, each one of them being in turn subdivided into six tetrahedra in a classical manner. We display in Tables 1 and 2 the relative errors in the L^2-norm of the approximate velocity, pressure and extra stress for different values of M, corresponding to t equal to 0.1 and $t = 1$ respectively. In all cases we kept $\lambda = 10$ and $\eta = 1$, and Δt was taken equal to $h/50$.

As one can infer from both tables, the predicted convergence rates are roughly confirmed by the numerical results, although only one iterative substep was used in these computations. This means that we actually implemented a fully explicit time integration scheme. Surprisingly enough, this seems to perform rather well, and in this respect we also refer to the general conclusions hereafter.

Remark 18. Similarly to the case considered in this Section, we solved with our method viscoelastic flow problems with known analytical solutions taking into account all the nonlinear terms. The results obtained for such problems up to moderate values of λ, indicate a convergence behavior similar to the one observed for the linear case. However, further stabilization is needed for larger values of this parameter (i.e. higher Deborah numbers) and in this respect we refer to the last paragraph of the next Section. □

9. Discussions and final remarks

We conclude this work by pointing out some of its aspects that in our opinion are worth being emphasized.

First of all we would like to stress the fact that convergence results for the pressure were obtained, even if the roughest possible type of space–time finite element approximations were employed for solving a three-field Stokes system. Moreover, all the results derived here avoided the widespread assumption that the viscoelastic liquid behaves partially as a Newtonian fluid and partially as a non-Newtonian fluid, as most authors of similar work have done so far. As a matter of fact our approach is also designed for purely viscoelastic constitutive laws such as Maxwell's. As is well-known, in this case handling the extra stress tensor properly in the numerical approach is mandatory, if one wishes to perform reliable flow simulations (see e.g. [6]).

Another important remark concerns the iterative substeps to run at each time step: as we observed in our numerical experiments for stationary problems, convergence of this iterative procedure to a reasonably small tolerance occurs after about two iterations, except for the very first values of n (cf. [19]). This is an interesting point, since it means that we are practically dealing with an explicit scheme stable and convergent for values of the time step of the same order as the mesh step size. Actually the authors are currently adapting the present numerical approach, in order to further exploit it in the framework of time-dependent viscoelastic flow problems of practical interest at possibly high Deborah numbers. Corresponding results should be reported in the near future.

Acknowledgements

The authors J.H. Carneiro de Araujo and V. Ruas gratefully acknowledge the financial support provided by the agency CNPq through grants 304518/2002-6 and 307996/2008-5.

References

[1] M. Fortin, R. Guénette, R. Pierre, Numerical analysis of the modified EVSS method, Computer Methods in Applied Mechanics and Engineering 143 (1–2) (1997) 79–95.
[2] A. Ainser, D. Dupuy, G. Panasenko, L. Sirakov, Flow in a wavy tube structure: asymptotic analysis and numerical simulation, Comptes Rendus de l'Académie des Sciences de Paris II-b (331–9) (2003) 609–615.
[3] J. Baranger, D. Sandri, Approximation par éléments finis d'écoulements de fluides viscoélastiques. Existence de solutions approchées et majorations d'erreur, Comptes Rendus de l'Académie des Sciences de Paris I (312) (1991) 541–544.
[4] R. Codina, Finite element approximation of the three-field formulation of the Stokes problem using arbitrary interpolations, SIAM J. Numerical Analysis 47 (1) (2009) 699–718.
[5] V. Ruas, J.H. Carneiro de Araujo, M.A.M. Silva Ramos, Approximation of the three-field Stokes system via optimized quadrilateral finite elements, Mathematical Modelling and Numerical Analysis 27 (1) (1993) 107–127.
[6] J.M. Marchal, M. Crochet, A new mixed finite element for calculating viscoelastic flow, Journal of Non Newtonian Fluid Mechanics 26 (1987) 77–117.
[7] V. Ruas, Galerkin least-squares finite element methods for the Stokes system in $\mathbb{R}^3$, Computer Methods in Applied Mechanics and Engineering 143 (3) (1997) 235–256.
[8] L. Franca, T.J.R. Hughes, A.F.D. Loula, I. Miranda, A new family of stable element for nearly incompressible elasticity based on a mixed Petrov-Galerkin finite element formulation, Numerische Mathematik 53 (1988) 123–141.
[9] L. Franca, R. Stenberg, Error analysis of some Galerkin least-squares methods for the elasticity equations, SIAM Journal of Numerical Analysis 26 (6) (1991) 1680–1697.
[10] J. Baranger, S. Wardi, Numerical analysis of a FEM for a transient viscoelastic flow, Computer Methods in Applied Mechanics and Engineering 125 (1–4) (1995) 171–185.
[11] P. Saramito, A new θ-scheme algorithm and incompressible FEM for viscoelastic fluid flows, RAIRO Modélisation Mathématique et Analyse Numérique 28 (1) (1994) 1–35.
[12] D. Esselaoui, A. Machmoum, Finite element approximation of viscoelastic flow using characteristics method, Comupter Methods in Applied Mechanicas and Engineering 190 (42) (2001) 5603–5618.
[13] V.J. Ervin, W.W. Miles, Approximation of time-dependent viscoelastic fluid flow: SUPG approximation, SIAM J. Numerical Analysis 41 (2) (2003) 457–486.
[14] V.J. Ervin, N. Heuer, Approximation of time-dependent viscoelastic flow: Crank–Nicolson finite element approximation, Numerical Methods for Partial Differential Equations 20 (2) (2004) 248–283.
[15] A. Bonito, P. Clément, M. Picasso, Mathematical and numerical analysis of a simplified time-dependent viscoelastic flow, Numerische Mathematik 107 (2) (2007) 213–255.
[16] J.C. Chrispell, V.J. Ervin, E.W. Jenkins, A fractional step θ-method for viscolelastic fluid flow using a SUPG approximation, Int. J. of Comp. Sci. 2–3 (2008) 336–351.
[17] J.C. Chrispell, V.J. Ervin, E.W. Jenkins, A fractional step θ-method approximation of time-dependent viscolelastic fluid flow, Journal of Computational and Applied Mathematics 232 (2009) 159–175.
[18] R.B. Bird, R.C. Armstrong, O. Hassager, Dynamics of Polymeric Liquids, Wiley-Interscience, 1987.
[19] A.P. Brasil Jr., J.H. Carneiro de Araujo, V. Ruas, A new algorithm for simulating viscoelastic flows accommodating piecewise linear finite elements, Journal of Computational and Applied Mathematics 215 (2008) 311–319.
[20] J.H. Carneiro de Araujo, P. Dias Gomes, V. Ruas, Solving stationary incompressible flow problems in stress–velocity–pressure formulation with linear finite elements, in: E.N. Mamiya (Ed.), Proceedings of COBEM 2007, (CD-ROM), Brasilia, Brazil, 2007.
[21] A.P. Brasil Jr., V. Ruas, Explicit solution of the incompressible Navier–Stokes equations with linear finite elements, Applied Mathematics Letters 20 (2007) 1005–1010.
[22] D. Goldberg, V. Ruas, A numerical study of projection algorithms in the finite element simulation of three-dimensional viscous incompressible flow, International Journal for Numerical Methods in Fluids 30 (1999) 233–256.
[23] G. Astarita, G. Marucci, Principle of Non-Newtonian Fluid Mechanics, Mc Graw Hill, 1974.
[24] A. Chorin, Numerical simulation of the Navier–Stokes equations, Mathematics of Computation 22 (1966) 325–352.
[25] R. Temam, Une méthode d'approximation de la solution des équations de Navier–Stokes, Bulletin de la Société des Mathématiques de France 98 (1968) 115–152.
[26] R. Codina, O.C. Zienkiewicz, CBS versus GLS stabilization of the incompressible Navier–Stokes equations and the role of the time step as a stabilization parameter, Communications in Numerical Methods in Engineering 18 (2002) 99–112.
[27] R. Rannacher, On Chorin's Projection Method for the Incompressible Navier–Stokes Equations, in: Lecture Notes in Mathematics, 1530, Springer Verlag, Berlin, 1991.
[28] H. Rahmoun, V. Ruas, Simulation numérique d'écoulements instationnaires de fluides incompressibles visqueux par la méthode des éléments finis appliquée dans divers domaines, Rapport de stage de DEA de mécanique, Université Pierre et Marie Curie, Paris, 1993.
[29] J.L. Guermond, J. Shen, A new class of truly consistent splitting schemes for incompressible flows, Journal of Computational Physics 192 (2003) 262–276.
[30] F. Brezzi, M. Fortin, Mixed and Hybrid Finite Element Methods, Springer-Verlag, Berlin, 1991.
[31] R.A. Adams, Sobolev Spaces, Academic Press, New York, 1975.
[32] V. Girault, P.A. Raviart, Finite Element Methods for Navier–Stokes Equations, Springer-Verlag, Berlin, 1986.
[33] J.L. Lions, Problèmes aux limites dans les équations aux dérivées partielles, Presses de l'Université de Montréal, Montreal, 1962.
[34] G. Duvaut, J.L. Lions, Les inéquations en mécanique et en physique, Dunod, Paris, 1972.
[35] G. Strang, G. Fix, An Analysis of the Finite Element Method, Prentice Hall, Englewood Cliffs, 1973.
[36] O.C. Zienkiewicz, The Finite Element Method in Structural and Continuum Mechanics, McGraw Hill, London, 1971.
[37] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam, 1977.
[38] D. Braess, Finite Elements, Cambridge University Press, Cambridge UK, 1997.
[39] V. Thomée, Galerkin Finite Element Methods for Parabolic Problems, in: Springer Series in Computational Mathematics, vol. 25, Springer, Berlin, 1997.
[40] H. Fujita, T. Suzuki, Evolution problems, in: P.G. Ciarlet, J.L. Lions (Eds.), Handbook of Numerical Analysis, vol. II. Finite Element Methods (Part 1), North Holland, New York, 1991, pp. 789–928.