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Heisenberg uncertainty principle for a fractional power of the Dunkl transform on the real line

Sami Ghazouani * and Fethi Bouzeffour †

Abstract

The aim of this paper is to prove Heisenberg-Pauli-Weyl inequality for a fractional power of the Dunkl transform on the real line for which there is an index law and a Plancherel theorem.

Keywords: Heisenberg uncertainty principle, Dunkl transform, Fractional Fourier transform, Generalized Hermite polynomials and functions, Generalized Sobolev spaces.

1 Introduction

Dunkl operators are differential-difference operators associated with finite reflection groups in a euclidean space. The first class of such operators were introduced by C. F. Dunkl in a series of papers [5, 6, 7], where he built up the framework for a theory of special functions and integral transforms in several variables related with reflection groups. In addition to the multidimensional case, one-dimensional Dunkl operators are also of great interest. For example, a number of works have recently appeared that develop the harmonic analysis results associated with the one-dimensional Dunkl operator. One of them is the Heisenberg-Weyl type inequality for the one-dimensional Dunkl transform established by Rösler and Voit [15].

The objective of this paper is two-folded: firstly, we develop an harmonic analysis related to a Dunkl type operator on the real line. More precisely, we consider a singular differential-difference operator Λ_μ^α on $\mathbb{R}$ which includes as a particular case the one-dimensional Dunkl operator. The eigenfunction $K_{\mu,\alpha}$ of this operator permits to define a fractional power D_μ^α of the Dunkl transform on $\mathbb{R}$ that reduces to the Dunkl transform, fractional Hankel transform and the fractional Fourier transform for particular cases of the parameters. Next, we develop an L^1 and L^2 theory for this transform. For L^1 theory, we give Riemann-Lebesgue lemma, inversion formula, index additivity property, which is of central importance: without it, we could hardly think of D_μ^α as being the αth power of D_μ, and operational formula. As for as L^2 theory, we prove that the fractional Dunkl transform D_μ^α, initially defined on $L^1(\mathbb{R}, |x|^{2\mu+1}dx)$, have a unique extension to an unitary operator of $L^2(\mathbb{R}, |x|^{2\mu+1}dx)$ and if the extension is also denoted by D_μ^α then the family $\left\{D_\mu^\alpha\right\}_{\alpha\in\mathbb{R}}$ which is parameterized by the parameter $\alpha \in \mathbb{R}$ have a group structure, called the elliptic group. It is like a rotation group since $D_\mu^\alpha \circ D_\mu^\beta = D_\mu^{\alpha+\beta}$ and D_μ^0 is the identity and the inverse is obviously $\left(D_\mu^\alpha\right)^{-1} = D_\mu^{-\alpha}$. We present also the subject of eigenvalues and eigenfunctions. We show that the generalized Hermite functions, which were introduced by Szegö [16] and studied by Chihara [2, 3] and Rosenblum [13], form an orthonormal basis of eigenfunctions of D_μ^α on $L^2(\mathbb{R}, |x|^{2\mu+1}dx)$. As a consequence, we prove that the family $\left\{D_\mu^\alpha\right\}_{\alpha\in\mathbb{R}}$ is a $\mathcal{C}_0$-group of unitary operators and we derive their infinitesimal generators. Secondly, we extend the Heisenberg-Pauli-Weyl uncertainty inequalities established by Rösler and Voit (Theorem 4.1,[15]) to the case of fractional Dunkl transform D_μ^α as follows:

$$\mathrm{var}_\mu(D_\mu^\alpha(f))\ \mathrm{var}_\mu(D_\mu^\beta(f)) \geq \sin^2(\alpha-\beta)\left(\left(\mu+\frac{1}{2}\right)\left(\|f_e\|_{2,\mu}^2 - \|f_o\|_{2,\mu}^2\right) + \frac{1}{2}\right)^2, \tag{1.1}$$

where $\alpha \in \mathbb{R}\backslash\pi\mathbb{Z}$ and $\beta \in \mathbb{R}$. For this purpose, we introduce Sobolev type spaces $H_2^{\mu,\alpha}(\mathbb{R})$ naturally associated to $\Lambda_\mu^{-\alpha}$ and we obtain their basic properties such as the imbedding theorems. We prove that:
• For every $\alpha \in \mathbb{R}\backslash\pi\mathbb{Z}$ and $-\frac{1}{2} \leq \mu < 0$, $H_2^{\mu,\alpha}(\mathbb{R}) \hookrightarrow \mathcal{C}_0(\mathbb{R})$ and the injection map is continuous.

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• For every $\alpha \in \mathbb{R}\backslash\pi\mathbb{Z}$, $\mu \geq 0$ and $f \in H_2^{\mu,\alpha}(\mathbb{R})$, there exists a function $\psi \in \mathcal{C}(\mathbb{R}\backslash\{0\})$ such that $f(x) = \psi(x)$, $a.\ e$ and for all $x \in \mathbb{R}\backslash\{0\}$,

$$|\psi(x)| \leq c\ \|f|H_2^{\mu,\alpha}(\mathbb{R})\|_{2,\mu} \begin{cases} |x|^{-\mu} & \text{if } \mu > 0, \\ |\ln|x||^{\frac{1}{2}} & \text{if } \mu = 0, \end{cases}$$

where $c = c(\mu, \alpha) > 0$. As applications on these spaces, we will show that (1.1) holds with equality if and only if $f(x) = \lambda\ e^{(i\cot(\beta)-b)\frac{x^2}{2}} E_\mu(ax)$, where λ, a and b are suitable parameters.

This paper is organized as follows. Section 2 presents an overview of the Heisenberg's inequality for various Fourier transform on the real line. Section 3 we introduce the fractional Dunkl transform D_μ^α on the real line with parameter $\alpha \in \mathbb{R}$. Riemann-Lebesgue lemma, inversion formula, an index additivity property and operational formulae are derived in section 4. Section 5 is devoted to the extension of the fractional Dunkl transform D_μ^α as an isometry from $L_\mu^2(\mathbb{R})$ to itself and the intimate relationship between the fractional Dunkl transform and generalized Hermite polynomials and functions. In section 6, we study the Sobolev spaces $H_2^{\mu,\alpha}(\mathbb{R})$ associated to $\Lambda_\mu^{-\alpha}$ and we derive a Heisenberg uncertainty principle for the fractional Dunkl transform and the fractional Hankel transform.

2 A brief survey of the Heisenberg's inequality for various Fourier transform

In this section, we give an overview of the Heisenberg's inequality for various Fourier transform on the real line.

2.1 The Heisenberg's inequality for Fourier transform and fractional Fourier transform

• The Fourier transform (FT) can be defined in many ways. For us, three different formulations are in particular important. In its most formulation, the FT is given by the integral transform

$$\mathcal{F}(f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ix\xi}\ dx.$$

Alternatively, one can rewrite the transform as

$$\mathcal{F}(f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} K(x,\xi) f(x)\ dx, \tag{2.1}$$

where $K(x, \xi)$ is the unique solution of the system of PDEs

$$\begin{cases} \dfrac{d}{dx} K(x,\xi) = -i\xi\ K(x,\xi), \\ k(0,\xi) = 1. \end{cases} \tag{2.2}$$

A third formulation is given by

$$\mathcal{F} = e^{\frac{i\pi}{4}} e^{\frac{i\pi}{4}(\Delta - x^2)}, \tag{2.3}$$

with $\Delta = \frac{d^2}{dx^2}$. The classical Heisenberg-Pauli-Weyl inequality [8] states that for $f \in L^2(\mathbb{R})$ and for any $a, b \in \mathbb{R}$,

$$\int_{-\infty}^{+\infty} (x-a)^2 |f(x)|^2\ dx.\ \int_{-\infty}^{+\infty} (\xi - b)^2 |\mathcal{F}(f)(\xi)|^2\ d\xi \geq \frac{1}{4} \left(\int_{-\infty}^{+\infty} |f(x)|^2\ dx \right)^2.$$

It is well known that there is a probabilistic interpretation to the previous inequality in terms of the variance. Let $f \in L^2(\mathbb{R})$ and suppose $\|f\|_{L^2(\mathbb{R})} = 1$. By the Parseval identity, $\|\mathcal{F}(f)\|_{L^2(\mathbb{R})} = 1$. Then $|f|^2$ and $|\mathcal{F}(f)|^2$ are both probability density functions on $\mathbb{R}$. The variance of f and the variance of $\mathcal{F}(f)$ are defined by

$$var(f) = \inf_{a\in\mathbb{R}} \int_{-\infty}^{+\infty} (x-a)^2 |f(x)|^2\ dx, \quad var(\mathcal{F}(f)) = \inf_{b\in\mathbb{R}} \int_{-\infty}^{+\infty} (x-b)^2 |\mathcal{F}(f)(x)|^2\ dx.$$

With these definitions, Heisenberg's inequality states that for $f \in L^2(\mathbb{R})$ such that $\|f\|_{L^2(\mathbb{R})} = 1$,

$$var(f)\ var(\mathcal{F}(f)) \geq \frac{1}{4}.$$

Note that if the integral defining the variance of f is finite for one value a, then it is finite for every $a \in \mathbb{R}$. In this case $\int_{-\infty}^{+\infty}(x-a)^2|f(x)|^2\ dx$ is a quadratic function of a whose minimum occurs when a is the mean of f, given by $\int_{-\infty}^{+\infty} x|f(x)|^2\ dx = \langle xf, f\rangle$; hence

$$var(f) = \int_{-\infty}^{+\infty} (x - \langle xf, f\rangle)^2\,|f(x)|^2\ dx = \|(x - \langle xf, f\rangle)\,f\|^2_{L^2(\mathbb{R})}$$

and using the fact that $\xi\mathcal{F}(f)(\xi) = \mathcal{F}(\frac{df}{dx})(\xi)$ and the Parseval identity,

$$var(\mathcal{F}(f)) = \int_{-\infty}^{+\infty} (x - \langle x\mathcal{F}(f), \mathcal{F}(f)\rangle)^2\,|\mathcal{F}(f)(x)|^2\ dx = \left\|\left(\frac{d}{dx} - \left\langle \frac{df}{dx}, f\right\rangle\right) f\right\|^2_{L^2(\mathbb{R})}.$$

• The fractional FT is a generalization of the classical FT. It is usually defined by [11]

$$\mathcal{F}^\alpha f(x) = \begin{cases} \frac{e^{i(\hat{\alpha}\pi/4-(\alpha-2n\pi)/2)}}{\sqrt{2\pi|\sin(\alpha)|}} \displaystyle\int_{-\infty}^{+\infty} e^{-\frac{i}{2}(x^2+y^2)\cot(\alpha)+\frac{ixy}{\sin(\alpha)}} f(y)\ dy, & (2n-1)\pi < \alpha < (2n+1)\pi, \\ f(x), & \alpha = 2n\pi, \\ f(-x), & \alpha = (2n+1)\pi, \end{cases} \tag{2.4}$$

with $n \in \mathbb{Z}$ and $\hat{\alpha} = \mathrm{sgn}(\sin(\alpha))$. Following the formulation (2.1) of the ordinary FT, we can rewrite this transformation as

$$\mathcal{F}^\alpha f(x) = \begin{cases} \frac{e^{i(\hat{\alpha}\pi/4-(\alpha-2n\pi)/2)}}{\sqrt{2\pi|\sin(\alpha)|}} \displaystyle\int_{-\infty}^{+\infty} K_\alpha(x,y) f(y)\ dy, & (2n-1)\pi < \alpha < (2n+1)\pi, \\ f(x), & \alpha = 2n\pi, \\ f(-x), & \alpha = (2n+1)\pi, \end{cases}$$

where, for $\alpha \in \mathbb{R}\backslash\pi\mathbb{Z}$, $K_\alpha(x,y) = e^{-\frac{i}{2}(x^2+y^2)\cot(\alpha)+\frac{ixy}{\sin(\alpha)}}$ is the unique solution of the system of PDEs

$$\begin{cases} \left(\frac{d}{dx} + i\cot(\alpha)x\right) K_\alpha(x,y) = \frac{iy}{\sin(\alpha)} K_\alpha(x,y), \\ K_\alpha(0,y) = e^{-\frac{i}{2}y^2\cot(\alpha)}. \end{cases} \tag{2.5}$$

The exponential expression (2.3) takes for the fractional FT the following form:

$$\mathcal{F}^\alpha = e^{\frac{i\alpha}{2}}\, e^{\frac{i\alpha}{2}(\Delta - x^2)}.$$

In [12], the authors gave an uncertainty principle for the fractional FT. They proved the following uncertainty inequalities:

$$\int_{-\infty}^{+\infty}(x-a)^2|\mathcal{F}^\alpha f(x)|^2\ dx.\ \int_{-\infty}^{+\infty}(x-b)^2|\mathcal{F}^\beta(f)(x)|^2\ dx \geq \frac{\sin^2(\alpha-\beta)}{4}\left(\int_{-\infty}^{+\infty}|f(x)|^2\ dx\right)^2.$$

2.2 The Heisenberg's inequality for the Hankel transform and fractional Hankl transform

• The Hankel transform is a generalization of the classical FT. It is defined by

$$\mathcal{H}_\mu(f)(x) = \frac{1}{2^\mu\Gamma(\mu+1)}\int_0^{+\infty} f(y) j_\mu(xy)\ y^{2\mu+1}\ dy, \quad \mu \geq -\frac{1}{2}, \tag{2.6}$$

where j_μ denotes the normalized spherical Bessel function

$$j_\mu(x) = 2^\mu\Gamma(\mu+1)\frac{J_\mu(x)}{x^\mu} = \Gamma(\mu+1)\sum_{n=0}^{+\infty}\frac{(-1)^n(x/2)^{2n}}{n!\ \Gamma(n+\mu+1)}, \tag{2.7}$$

and J_μ is the classical Bessel function (see, Watson [17]). Note that the Hankel transform (2.6) can be rewritten as:

$$\mathcal{H}_\mu(f)(x) = \frac{1}{2^\mu\Gamma(\mu+1)}\int_0^{+\infty} f(y)\mathcal{K}_\mu(x,y)\ y^{2\mu+1}\ dy,$$

where $\mathcal{K}_\mu(x,y)$ is the unique solution of the modified Bessel's equation

$$\begin{cases} \mathcal{L}_\mu\ \mathcal{K}_\mu(.,y) = -y^2\mathcal{K}_\mu(.,y), \\ \mathcal{K}_\mu(0,y) = 1,\ \frac{d}{dx}\mathcal{K}_\mu(0,y) = 0. \end{cases} \tag{2.8}$$

and where $\mathcal{L}_\mu$ is the Bessel operator given by

$$\mathcal{L}_\mu = \frac{d^2}{dx^2} + \frac{2\mu+1}{x}\frac{d}{dx}.$$

Another interesting variant of the Hankel transform is the following:

$$\mathcal{H}_\mu = e^{i(\mu+1)\frac{\pi}{2}} e^{i\frac{\pi}{4}(\mathcal{L}_\mu - x^2)}$$

In [1, 15], the authors gave an uncertainty principle for the Hankel transform. They proved the following uncertainty inequalities:

$$\int_0^{+\infty} x^2|f(x)|^2x^{2\mu+1}\ dx. \int_0^{+\infty} \lambda^2|\mathcal{H}_\mu(f)(\lambda)|^2\lambda^{2\mu+1}\ d\lambda \geq (\mu+1)^2\left(\int_0^{+\infty}|f(x)|^2x^{2\mu+1}\ dx\right)^2,$$

where f is a square integrable function on $]0,+\infty[$ with respect to the measure $d\omega_\mu(x) = x^{2\mu+1}\ dx$.

• The fractional Hankel transformation is a generalization of the conventional Hankel transform. It is given by:

$$\mathcal{H}_\mu^\alpha f(x) = \begin{cases} 2c_{\mu,\alpha}\displaystyle\int_0^{+\infty} e^{-\frac{i}{2}(x^2+y^2)\cot(\alpha)}\ j_\mu\left(\frac{xy}{\sin(\alpha)}\right) f(y)y^{2\mu+1}\ dy, & (2n-1)\pi < \alpha < (2n+1)\pi, \\ f(x), & \alpha = 2n\pi, \\ f(-x), & \alpha = (2n+1)\pi, \end{cases}$$

where, for $(2n-1)\pi < \alpha < (2n+1)\pi,\ c_{\mu,\alpha} = \dfrac{e^{i(\mu+1)(\hat{\alpha}\pi/2-(\alpha-2n\pi))}}{\Gamma(\mu+1)(2|\sin(\alpha)|)^{\mu+1}}$.

Knowing that, for $\alpha \in \mathbb{R}\backslash\pi\mathbb{Z}$, the kernel $\mathcal{K}_{\mu,\alpha}(x,y) = e^{-\frac{i}{2}(x^2+y^2)\cot(\alpha)}\ j_\mu(xy/\sin(\alpha))$ of the fractional Hankel transform is the unique solution of:

$$\begin{cases} \Delta_{\mu,\alpha}\ \mathcal{K}_{\mu,\alpha}(.,y) = -\frac{y^2}{\sin^2(\alpha)}\mathcal{K}_{\mu,\alpha}(.,y), \\ \mathcal{K}_{\mu,\alpha}(0,y) = e^{-\frac{i}{2}\cot(\alpha)y^2},\ \frac{d}{dx}\mathcal{K}_{\mu,\alpha}(0,y) = 0, \end{cases} \tag{2.9}$$

where $\Delta_{\mu,\alpha}$ is the differential operator given by:

$$\Delta_{\mu,\alpha} = \frac{d^2}{dx^2} + \left(\frac{2\mu+1}{x} + 2i\cot(\alpha)x\right)\frac{d}{dx} + 2i(\mu+1)\cot(\alpha) - \cot^2(\alpha)x^2.$$

The exponential form for the fractional Hankel transform is the following:

$$\mathcal{H}_\mu^\alpha = e^{i\alpha(\mu+1)} e^{i\frac{\alpha}{2}(\mathcal{L}_\mu - x^2)}$$

At the end of this paper we derive the following uncertainty principle for the fractional Hankel transform:

$$\int_0^{+\infty}(x-a)^2|\mathcal{H}_\mu^\alpha f(x)|^2x^{2\mu+1}\ dx. \int_0^{+\infty}(x-b)^2|\mathcal{H}_\mu^\beta(f)(x)|^2x^{2\mu+1}\ dx \geq \sin^2(\alpha-\beta)(\mu+1)^2\left(\int_0^{+\infty}|f(x)|^2x^{2\mu+1}\ dx\right)^2.$$

2.3 The Heisenberg's inequality for the Dunkl transform

The Dunkl transform on the real line is both an extension of the Hankel transform to the whole real line and a generalization of the Fourier transform. It was introduced by Dunkl in [7], where already many basic properties were established. Dunkl's results were completed and extended later on by de Jeu in [4]. The Dunkl transform of a function $f \in L^1(\mathbb{R}, |x|^{2\mu+1}dx)$ is given by

$$D_\mu f(\lambda) = \frac{1}{2^{\mu+1}\Gamma(\mu+1)}\int_{-\infty}^{+\infty} f(x)E_\mu(-i\lambda x)|x|^{2\mu+1}dy, \quad \mu \geq -\frac{1}{2}, \tag{2.10}$$

where

$$E_\mu(z) = j_\mu(iz) + \frac{z}{2(\mu+1)}j_{\mu+1}(iz), \tag{2.11}$$

is the one-dimensional Dunkl kernel [6]. It is well known that the functions $E_\mu(\lambda.)$ is, for $\mu \geq -1/2$ the unique solution of the initial value problem

$$\begin{cases} \Lambda_\mu f = \lambda f, \\ f(0) = 1, \end{cases} \tag{2.12}$$

where

$$\Lambda_\mu f(x) = \frac{d}{dx} f(x) + \frac{2\mu+1}{x}\left[\frac{f(x)-f(-x)}{2}\right],$$

is the Dunkl operator with parameter μ associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$ (see[5]). The exponential form of the Dunkl transform is the following:

$$D_\mu = e^{i(\mu+1)\frac{\pi}{2}} e^{i\frac{\pi}{4}(\Lambda_\mu^2 - x^2)}$$

In [15], the authors derived a Heisenberg-Weyl type inequality for the Dunkl transform on the real line. They obtained, for functions f in the appropriate space, the following inequalities

$$var_\mu(f)\ var_\mu(D_\mu(f)) \geq \left((\mu+\frac{1}{2})\left(\|f_e\|_{2,\mu}^2 - \|f_o\|_{2,\mu}^2\right) + \frac{1}{2}\right)^2, \tag{2.13}$$

where

$$var_\mu(f) = \|(x - \langle xf, f\rangle_\mu)\, f\|_{2,\mu}^2 \quad \text{and} \quad var_\mu(D_\mu(f)) = \|(\Lambda_\mu - \langle \Lambda_\mu f, f\rangle_\mu)\, f\|_{2,\mu}^2,$$

and where $f_o(x) = \frac{f(x)-f(-x)}{2}$ and $f_e(x) = \frac{f(x)+f(-x)}{2}$ are the odd and the even parts of f respectively. Here, $\langle .,.\rangle_\mu$ denotes the $L^2(\mathbb{R}, |x|^{2\mu+1}dx)$ inner product and $\|.\|_{2,\mu}$ its associated norm.

3 Fractional Dunkl transform in $L^1_\mu(\mathbb{R})$

Notation: We denote by
- $C_0(\mathbb{R})$ the space of continuous functions on $\mathbb{R}$ which vanish at infinity.
- $C^m(\mathbb{R})$ (resp. $C^m_c(\mathbb{R})$), the space of C^m-functions on $\mathbb{R}$ (resp. with compact support).
- $\mathcal{S}(\mathbb{R})$ the space of C^∞-functions on $\mathbb{R}$ which are rapidly decreasing with their derivatives.
- $L^p_\mu(\mathbb{R})$ the space of measurable functions on $\mathbb{R}$ such that

$$\begin{aligned} \|f\|_{\mu,p} &= \left(\int_{-\infty}^{+\infty} |f(y)|^p |y|^{2\mu+1}\, dy\right)^{\frac{1}{p}} < +\infty, \quad \text{if } 1 \leq p < +\infty, \\ \|f\|_{\mu,\infty} &= \operatorname*{ess\,sup}_{x\in\mathbb{R}} |f(x)| < \infty. \end{aligned}$$

To describe the harmonic analysis in our setting we begin by introducing the differential-difference operator Λ^α_μ defined for $f \in C^1(\mathbb{R})$ by:

$$\begin{aligned} \Lambda^\alpha_\mu f(x) &:= \frac{d}{dx} f(x) + \frac{2\mu+1}{x}\left[\frac{f(x)-f(-x)}{2}\right] + i\cot(\alpha) x f(x) \\ &= \Lambda_\mu f(x) + i\cot(\alpha) x f(x), \end{aligned}$$

where $\alpha \in \mathbb{R}\backslash\pi\mathbb{Z}$ and $\mu \geq -1/2$.
In the case $\mu = -1/2$, Λ^α_μ is reduced to the operator:
- $\dfrac{d}{dx}$ (when $\alpha = \pi/2$) which is closely related to the FT (2.2).
- $\dfrac{d}{dx} + i\cot(\alpha)x$, which is closely related to the fractional Fourier transform (2.5).

If we consider the case $\mu > -1/2$:
- Λ^α_μ coincides, for $\alpha = \pi/2$, with the Dunkl operator Λ_μ which is closely related to the Dunkl transform (2.12).

The restriction to the even subspace $\mathcal{C}^2(\mathbb{R})^e = \left\{f \in \mathcal{C}^2(\mathbb{R}) :\ f(x) = f(-x)\right\}$ of the square $(\Lambda^\alpha_\mu)^2$ is:
- The Bessel operator $\mathcal{L}_\mu$ (when $\alpha = \pi/2$) which is closely related to the Hankel transform (2.8).
- The operator $\Delta_{\mu,\alpha}$ which is closely related to the fractional Hankel transform (2.9).

One purpose of this paper is to provide that Λ^α_μ is closely related to a fractional Dunkl transform on the real line. We begin with the following Proposition:

Proposition 3.1 *Let $\alpha \in \mathbb{R}\backslash\pi\mathbb{Z}$.*
(1) *The operator Λ_μ^α is related to the Dunkl operator Λ_μ by*

$$e^{\frac{i}{2}\cot(\alpha)x^2} \circ \Lambda_\mu^\alpha \circ e^{-\frac{i}{2}\cot(\alpha)x^2} = \Lambda_\mu.$$

(2) *For $y \in \mathbb{C}$, the differential-difference equation*

$$\begin{cases} \Lambda_\mu^\alpha f = \frac{iy}{\sin(\alpha)} f, \\ f(0) = e^{-\frac{i}{2}y^2\cot(\alpha)}, \end{cases}$$

has a unique analytic solution given by

$$K_{\mu,\alpha}(x,y) \quad = \quad e^{-\frac{i}{2}(x^2+y^2)\cot(\alpha)} E_\mu(ixy/\sin(\alpha))$$

where E_μ is the Dunkl-kernel given by (2.11).

Proof.
(1) Let f be a differentiable function on $\mathbb{R}$ and let g be the function defined by $g(x) = e^{-\frac{i}{2}x^2\cot(\alpha)}f(x)$. An easy calculation shows that $\Lambda_\mu^\alpha g(x) = e^{-\frac{i}{2}x^2\cot(\alpha)}\Lambda_\mu f(x)$. Then $e^{\frac{i}{2}x^2\cot(\alpha)}\Lambda_\mu^\alpha g(x) = \Lambda_\mu f(x)$.
(2) The second assertion follows from the first together with (2.12).

We now summarize some properties of the kernel $K_{\mu,\alpha}(x,y)$.

Proposition 3.2 *Let $\alpha \in \mathbb{R}\backslash\pi\mathbb{Z}$.*
(1) *For each $x \in \mathbb{R}$ and $y \in \mathbb{C}$, we have the integral representation*

$$K_{\mu,\alpha}(x,y) \quad = \quad \frac{\Gamma(\mu+1)}{\sqrt{\pi}\Gamma(\mu+1/2)} e^{-\frac{i}{2}(x^2+y^2)\cot(\alpha)} \int_{-1}^{1} e^{\frac{ixyt}{\sin(\alpha)}}(1-t^2)^{\mu-1/2}(1+t)\ dt. \tag{3.1}$$

In particular, we have

$$\forall x \in \mathbb{R}, \quad \forall y \in \mathbb{R}, \quad |K_{\mu,\alpha}(x,y)| \leq 1. \tag{3.2}$$

(2) *There exists $a(\mu,\alpha) > 0$ such that for all x and $y \in \mathbb{R}$, we have*

$$|K_{\mu,\alpha}(x,y)| \leq a(\mu,\alpha)\ \min(1, |xy|^{-(\mu+1/2)}). \tag{3.3}$$

Proof.
(1) (3.1) is a direct application of the Bochner-type representation for the Dunkl-kernel E_μ (see [13] or [14]):

$$E_\mu(\lambda x) = \frac{\Gamma(\mu+1)}{\sqrt{\pi}\Gamma(\mu+1/2)} \int_{-1}^{1} e^{\lambda xt}(1-t^2)^{\mu-1/2}(1+t)\ dt. \tag{3.4}$$

(2) We recall, from [17], the asymptotic expansions for Bessel function $J_\mu(z)$ as $|z| \to \infty$:

$$J_\mu(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\mu\pi}{2} - \frac{\pi}{4}\right), \quad |\arg(z)| < \pi.$$

Then

$$K_{\mu,\alpha}(x,y) = O\left(|xy|^{-(\mu+\frac{1}{2})}\right) \quad \text{for} \quad |xy| \to \infty. \tag{3.5}$$

Thus, (3.2) and (3.5) determine our choice of $a(\mu,\alpha)$.

Definition 3.1 *For $0 < |\alpha| < \pi$, we define the fractional Dunkl transform of a function f in $L_\mu^1(\mathbb{R})$ by:*

$$D_\mu^\alpha f(x) = A_\alpha \int_{-\infty}^{+\infty} f(y) K_{\mu,\alpha}(x,y)|y|^{2\mu+1} dy, \tag{3.6}$$

where

$$A_\alpha = \frac{e^{i(\mu+1)(\hat{\alpha}\pi/2-\alpha)}}{\Gamma(\mu+1)(2|\sin(\alpha)|)^{\mu+1}} \quad \textit{and} \quad \hat{\alpha} := \mathit{sgn}(\sin(\alpha)).$$

3.1 Case $\alpha = 0$ or $\alpha = \pi$.

In order to define D_μ^0 or D_μ^π, we need another integral representation for the fractional Dunkl transform D_μ^α. We begin by the following lemmas

Lemma 3.1 *Consider for any $a \in \mathbb{C}$ such that $\Re a > 0$, the function f_a defined by*

$$f_a(y) = e^{-ay^2} E_\mu(ixy),$$

where $x \in \mathbb{R}$. Then

$$D_\mu f_a(\xi) \quad = \quad \frac{e^{-\frac{1}{4a}(x^2+\xi^2)}}{(2a)^{\mu+1}} E_\mu(x\xi/2a).$$

Proof. Since $f_a \in L_\mu^1(\mathbb{R})$, it follows that

$$D_\mu f_a(\xi) = \frac{1}{2^{\mu+1}\Gamma(\mu+1)} \int_{-\infty}^{+\infty} e^{-ay^2} E_\mu(ixy) E_\mu(-i\xi y)|y|^{2\mu+1} dy.$$

By (2.11),

$$\begin{aligned} E_\mu(ixy)E_\mu(-i\xi y) \quad &= \quad j_\mu(xy)j_\mu(\xi y) + \frac{x\xi y^2}{4(\mu+1)^2} j_{\mu+1}(xy)j_{\mu+1}(\xi y) \\ &+ \quad \frac{ixy}{2(\mu+1)} j_{\mu+1}(xy)j_\mu(\xi y) - \frac{i\xi y}{2(\mu+1)} j_\mu(xy)j_{\mu+1}(\xi y). \end{aligned}$$

Then

$$D_\mu f_a(\xi) \quad = \quad \frac{1}{2^\mu \Gamma(\mu+1)}\ B_\mu + \frac{x\xi}{2^{\mu+2}(\mu+1)\Gamma(\mu+2)}\ B_{\mu+1},$$

where

$$\begin{aligned} B_\mu \quad &= \quad \int_0^{+\infty} e^{-ay^2} j_\mu(xy)j_\mu(\xi y)\ y^{2\mu+1}\ dy = \frac{1}{2}\int_0^{+\infty} e^{-ay} j_\mu(x\sqrt{y})j_\mu(\xi\sqrt{y})\ y^\mu\ dy \\ &= \quad \frac{2^{2\mu-1}\Gamma^2(\mu+1)}{(x\xi)^\mu} \int_0^{+\infty} e^{-ay} J_\mu(x\sqrt{y})J_\mu(\xi\sqrt{y})\ dy. \end{aligned}$$

To compute B_μ, we need the following formulas (see 7.4.21 (4) in [9])

$$\int_0^{+\infty} e^{-\delta y} J_\nu(2r\sqrt{y})J_\nu(2s\sqrt{y})\ dy = \frac{e^{-(1/\delta)(s^2+r^2)}}{\delta} I_\nu(2rs/\delta),$$

where

$$I_\nu(z) \quad = \quad e^{-i\pi\nu/2} J_\nu(iz) = (z/2)^\nu \sum_{n=0}^{+\infty} \frac{(z/2)^{2n}}{n!\ \Gamma(n+\nu+1)},$$

$|\arg(\delta)| < \pi/2$ and $\nu \geq 0$. Let us take $\delta = a,\ r = x/2$ and $s = \xi/2$, then

$$B_\mu \quad = \quad \frac{2^{2\mu-1}\Gamma^2(\mu+1)}{(x\xi)^\mu} \frac{e^{-\frac{x^2}{4a}}\ e^{-\frac{\xi^2}{4a}}}{a} I_\mu(x\xi/2a) = \frac{2^\mu\Gamma(\mu+1)}{(2a)^{\mu+1}}\ e^{-\frac{1}{4a}(x^2+\xi^2)} j_\mu(ix\xi/2a).$$

Lemma 3.2 *Let $c, d \in \mathbb{C}$ such that $\Re c > 0$. For all $x, y \in \mathbb{R}$, we have*

$$\left| e^{-cx^2} E_\mu(dxy) \right| \leq\ e^{-\frac{(\Re d)^2 y^2}{4\Re c}}.$$

Proof. By (3.4), it follows that

$$|E_\mu(dxy)| \leq e^{|\Re d||x||y|}. \tag{3.7}$$

Then

$$\left|e^{-cx^2} E_\mu(dxy)\right| \quad \leq \quad e^{-\Re c x^2 + |\Re d||x||y|}.$$

On the other hand, we have,

$$\sup_{r\geq 0}(-\Re c\, r^2 + |\Re d||y|r) = -\frac{(\Re d)^2 y^2}{4\Re c}.$$

Which gives the result.

Theorem 3.1 *Let $0 < |\alpha| < \pi$ such that $|\alpha| \neq \pi/2$. For $f \in L^1_\mu(\mathbb{R}) \cap L^2_\mu(\mathbb{R})$ with $D_\mu f \in L^1_\mu(\mathbb{R})$, we have:*
(1)

$$D^\alpha_\mu f(x) \quad = \quad \frac{1}{\Gamma(\mu+1)} \left(\frac{e^{-i\alpha}}{2\cos\alpha}\right)^{\mu+1} \int_{-\infty}^{+\infty} e^{\frac{i}{2}(x^2+y^2)\tan\alpha}\; E_\mu\left(\frac{ixy}{\cos(\alpha)}\right) D_\mu f(y)|y|^{2\mu+1}\; dy. \tag{3.8}$$

(2)

$$\begin{aligned} \lim_{\alpha\to 0^+} D^\alpha_\mu f(x) \quad &= \quad \lim_{\alpha\to 0^-} D^\alpha_\mu f(x) = f(x),\;\; a.e. \\ \lim_{\alpha\to \pi^+} D^\alpha_\mu f(x) \quad &= \quad \lim_{\alpha\to \pi^-} D^\alpha_\mu f(x) = f(-x),\;\; a.e \end{aligned}$$

Proof.
(1) For any $a > 0$, define

$$F_a(x) = \int_{-\infty}^{+\infty} f(y) g_a(y) |y|^{2\mu+1}\; dy,$$

where $g_a(y) = e^{-(a+\frac{i}{2}\cot(\alpha))y^2} E_\mu(ixy/\sin(\alpha))$.
From (3.7), we deduce that $|g_a(y)| \leq 1$. Then $|f(y)g_a(y)| \leq |f(y)|$, so we can apply the dominated convergence theorem to get

$$\lim_{a\to 0} F_a(x) = A^{-1}_\alpha\; e^{\frac{i}{2}x^2\cot(\alpha)} D^\alpha_\mu f(x). \tag{3.9}$$

Using Lemma 3.1 with $a \leftrightarrow a + \frac{i}{2}\cot(\alpha)$, one can schow

$$\begin{aligned} D_\mu f_a(\xi) \quad &= \quad \frac{1}{(2a + i\cot(\alpha))^{\mu+1}}\; e^{-\frac{x^2}{4a\sin^2(\alpha)+i\sin(2\alpha)}}\; e^{-\frac{\xi^2}{4a+2i\cot(\alpha)}} \\ &\times \quad E_\mu\left(\frac{x\xi}{2a\sin(\alpha) + i\cos(\alpha)}\right). \end{aligned}$$

Now applying the Parseval formula for the Dunkl transform (see [4]) together with Lemma 3.1, we obtain

$$\begin{aligned} F_a(x) \quad &= \quad \frac{1}{(2a + i\cot(\alpha))^{\mu+1}}\; e^{-\frac{x^2}{4a\sin^2(\alpha)+i\sin 2\alpha}} \\ &\times \quad \int_{-\infty}^{+\infty} e^{-\frac{\xi^2}{4a+2i\cot(\alpha)}}\; E_\mu\left(\frac{x\xi}{2a\sin(\alpha)+i\cos(\alpha)}\right) D_\mu f(-\xi)\; |\xi|^{2\mu+1}\; d\xi. \end{aligned}$$

Applying Lemma 3.2 with $c = \frac{1}{4a+2i\cot(\alpha)}$ and $d = \frac{1}{2a\sin(\alpha)+i\cos(\alpha)}$, then

$$\Re c = \frac{a\sin^2(\alpha)}{4a^2\sin^2(\alpha) + \cos^2(\alpha)} \quad \text{and} \quad \Re d = \frac{2a\sin(\alpha)}{4a^2\sin^2(\alpha)+\cos^2(\alpha)}.$$

Therefore

$$\begin{aligned} \left| e^{-\frac{\xi^2}{4a+2i\cot(\alpha)}}\; E_\mu\left(\frac{x\xi}{2a\sin(\alpha)+i\cos(\alpha)}\right) D_\mu f(-\xi)\right| \quad &\leq \quad e^{\frac{ax^2}{4a^2\sin^2(\alpha)+\cos^2(\alpha)}}\; |D_\mu f(-\xi)| \\ &\leq \quad B_x |D_\mu f(-\xi)|. \end{aligned}$$

where $B_x = \sup_{a\in]0,1]} e^{\frac{a \ |x|^2}{4a^2\sin^2(\alpha)+\cos^2(\alpha)}}$. The function $\xi \mapsto D_k f(-\xi)$ is in $L^1_\mu(\mathbb{R})$, then the dominated convergence theorem implies

$$\lim_{a\to 0} F_a(x) = \frac{e^{\frac{ix^2}{\sin(2\alpha)}}}{(i\cot(\alpha))^{\mu+1}} \int_{-\infty}^{+\infty} e^{\frac{i}{2}\xi^2\tan\alpha}\, E_\mu(-ix\xi/\cos(\alpha)) D_\mu f(-\xi)|\xi|^{2\mu+1}\, d\xi. \tag{3.10}$$

Hence, (3.9) and (3.10) gives after simplification

$$D^\alpha_\mu f(x) \quad = \quad \frac{1}{\Gamma(\mu+1)}\left(\frac{e^{-i\alpha}}{2\cos\alpha}\right)^{\mu+\frac{1}{2}} \int_{-\infty}^{+\infty} e^{\frac{i}{2}(x^2+y^2)\tan\alpha}\, E_\mu\left(\frac{-ixy}{\cos(\alpha)}\right) D_\mu f(-y)|y|^{2\mu+1}\, dy. \tag{3.11}$$

Finally, if we make the change of variables $u=-y$ in (3.11), then we find (3.8).
(2) Follows from (3.8) together with the dominated convergence theorem and the inversion formula for the Dunkl transform (see Theorem 4.20 in [4]).
From the above theorem, we extend the definition (3.6) to $\alpha = 0$ or $\alpha=\pi$ as follows

Definition 3.2 *For $f \in L^1_\mu(\mathbb{R})$, define*
(1) $D^0_\mu f(x) = f(x)$,
(2) $D^\pi_\mu f(x) = f(-x)$.

3.2 Case $\alpha \in \mathbb{R}$.

We make the following definition.

Definition 3.3 *For $n \in \mathbb{Z}$ and $f \in L^1_\mu(\mathbb{R})$, define*
(1) $D^{2n\pi}_\mu f(x) = f(x)$,
(2) $D^{(2n+1)\pi}_\mu f(x) = f(-x)$,
(3) $D^{\alpha+2n\pi}_\mu f(x) = D^\alpha_\mu f(x),\ \alpha \in \mathbb{R}$.

Note that if $(2n-1)\pi < \alpha < (2n+1)\pi$ and $n \in \mathbb{Z}$,

$$D^\alpha_\mu f(x) \quad = \quad \frac{e^{i(\mu+1)(\hat{\alpha}\pi/2-(\alpha-2n\pi))}}{\Gamma(\mu+1)(2|\sin(\alpha)|)^{\mu+1}} \int_{-\infty}^{+\infty} f(y) K_{\mu,\alpha}(x,y)|y|^{2\mu+1} dy, \tag{3.12}$$

where $\hat{\alpha} = \mathrm{sgn}(\sin(\alpha))$

Particular case

- When $\alpha = -\pi/2,\ D^\alpha_\mu$ reduces to the Dunkl transform D_μ.
- If f is an even function then $D^\alpha_\mu f$ coincides with the fractional Hankel transform $\mathcal{H}^\alpha_\mu f$ of f.
- When $\mu = -1/2,\ D^\alpha_\mu$ coincides with the fractional Fourier transform $\mathcal{F}^\alpha$.

4 Basic properties of D^α_μ.

In this section, we discuss basic properties of D^α_μ for general $\alpha \in \mathbb{R}$.

4.1 Riemann-Lebesgue lemma and the reversibility property.

Theorem 4.1 *Let $\alpha \in \mathbb{R}$.*
(1) *Here suppose $\alpha \notin \pi\mathbb{Z}$. For all $f \in L^1_\mu(\mathbb{R})$, $D^\alpha_\mu f$ belongs to $\mathcal{C}_0(\mathbb{R}^N)$ and verifies*

$$\|D^\alpha_\mu f\|_\infty \leq \frac{1}{\Gamma(\mu+1)(2|\sin(\alpha)|)^{\mu+1}}\ \|f\|_1.$$

(2) *For all $f \in L^1_\mu(\mathbb{R})$ with $D^\alpha_\mu f \in L^1_\mu(\mathbb{R})$,*

$$(D^{-\alpha}_\mu \circ D^\alpha_\mu) f = f,\ \ a.e,$$

and

$$(D^\alpha_\mu \circ D^{-\alpha}_\mu) f = f,\ \ a.e.$$

(3) D_μ^α *is a one-to-one and onto mapping from* $\mathcal{S}(\mathbb{R})$ *into* $\mathcal{S}(\mathbb{R})$. *Moreover,*

$$(D_\mu^\alpha)^{-1} f = D_k^{-\alpha} f, \quad f \in \mathcal{S}(\mathbb{R}). \tag{4.1}$$

Proof.
(1) The first statement follows immediately from Riemann-Lebesgue lemma for the Dunkl transform (see [4], Corollary 4.7).
(2) Since D_μ^α is periodic in α with period 2π, we can assume that $\alpha \in (-\pi, \pi]$. We see immediately that

$$\begin{aligned} D_\mu^0 \circ D_\mu^0 f &= f, \\ D_\mu^\pi \circ D_\mu^\pi f &= f. \end{aligned}$$

When $0 < |\alpha| < \pi$, we have

$$\begin{aligned} D_\mu^{-\alpha} \circ D_\mu^\alpha f(x) &= \frac{1}{(4\sin^2(\alpha))^{\mu+1}\Gamma^2(\mu+1)} e^{\frac{i}{2}x^2\cot(\alpha)} \int_{-\infty}^{+\infty} E_\mu(-ixy/\sin(\alpha)) \\ &\times \left(\int_{-\infty}^{+\infty} e^{-\frac{i}{2}z^2\cot(\alpha)} f(z) E_\mu(iyz/\sin(\alpha))\, |z|^{2\mu+1} dz \right) |y|^{2\mu+1}\, dy. \end{aligned}$$

By the change of variables $u = \frac{y}{\sin(\alpha)}$, we obtain

$$\begin{aligned} D_\mu^{-\alpha} \circ D_\mu^\alpha f(x) &= e^{\frac{i}{2}x^2\cot(\alpha)} \frac{1}{4^{\mu+1}\Gamma^2(\mu+1)} \int_{-\infty}^{+\infty} E_\mu(-ixu) \\ &\times \left(\int_{-\infty}^{+\infty} e^{-\frac{i}{2}z^2\cot(\alpha)} f(z) E_\mu(iuz) |z|^{2\mu+1}\, dz \right) |u|^{2\mu+1} du \\ &= e^{\frac{i}{2}x^2\cot(\alpha)} D_\mu \left(D_\mu \left[e^{-\frac{i}{2}z^2\cot(\alpha)} f(-z) \right] \right)(x), \\ &= f(x),\ a.\ e. \end{aligned}$$

The last equality follows from the inversion formula for the Dunkl transform.
(3) That $D_\mu^\alpha : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R})$ is an homeomorphism follows from [4], Corollary 4.22, and the fact that the mapping M_λ defined by

$$(M_\lambda f)(x) = e^{\frac{i}{2}\lambda x^2} f\ ,\ f \in \mathcal{S}(\mathbb{R})$$

is an automorphism on $\mathcal{S}(\mathbb{R})$ for each $\lambda \in \mathbb{R}$. The statement $(D_\mu^\alpha)^{-1} = D_\mu^{-\alpha}$ follows from part (2).

4.2 An index additivity property

We begin by following lemmas:

Lemma 4.1 *Let* $\epsilon > 0,\ \alpha,\ \beta \in \mathbb{R}\backslash\pi\mathbb{Z}$ *and* $(x, z) \in \mathbb{R}^2$. *Then*

$$\begin{aligned} \int_{-\infty}^{+\infty} e^{-\epsilon y^2} K_{\mu,\alpha}(x,y) K_{\mu,\beta}(y,z) |y|^{2\mu+1}\, dy &= \frac{2^{\mu+1}\Gamma(\mu+1)}{c_{\alpha,\beta}(\epsilon)} \exp\left(-\frac{i}{2}(x^2\cot(\alpha) + z^2\cot(\beta)) \right) \\ &\times \exp\left(-(r_1(\epsilon)x^2 + r_2(\epsilon)z^2) \right)\ E_\mu\left(-r_3(\epsilon)xz \right), \end{aligned}$$

where

$$r_1(\epsilon) = \frac{1}{4\epsilon\sin^2(\alpha) + 2i\sin(\alpha)(\sin(\alpha+\beta)/\sin(\beta))}, \quad r_2(\epsilon) = \frac{1}{4\epsilon\sin^2(\beta) + 2i\sin(\beta)(\sin(\alpha+\beta)/\sin(\alpha))},$$

$$r_3(\epsilon) = \frac{1}{2\epsilon\sin(\alpha)\sin(\beta) + i\sin(\alpha+\beta)} \quad \textit{and} \quad c_{\alpha,\beta}(\epsilon) = \left(2\epsilon + i\frac{\sin(\alpha+\beta)}{\sin(\alpha)\sin(\beta)} \right)^{\mu+1}.$$

Proof. Replacing $K_{\mu,\alpha}(x,y)$ and $K_{\mu,\beta}(y,z)$ by their definitions, we get

$$\int_{-\infty}^{+\infty} e^{-\epsilon y^2} K_{\mu,\alpha}(x,y)K_{\mu,\beta}(y,z)|y|^{2\mu+1}\,dy = \exp\left(-\frac{i}{2}(x^2\cot(\alpha)+z^2\cot(\beta))\right)$$

$$\times \int_{-\infty}^{+\infty} e^{-(\epsilon+\frac{i}{2}(\cot(\alpha)+\cot(\beta)))y^2} E_\mu\left(\frac{ixy}{\sin(\alpha)}\right) E_\mu\left(\frac{iyz}{\sin(\beta)}\right)|y|^{2\mu+1}\,dy.$$

The desired result follows from Lemma 3.1

Lemma 4.2 *Let α,β be in $\mathbb{R}\backslash\pi\mathbb{Z}$ such that $\alpha+\beta\in\mathbb{R}\backslash\pi\mathbb{Z}$ and let f be in $L^1_\mu(\mathbb{R})$ with $D^\beta_\mu f\in L^1_\mu(\mathbb{R})$. Then*

$$\int_{-\infty}^{+\infty} K_{\mu,\alpha}(x,y)\left(\int_{-\infty}^{+\infty} f(z)K_{\mu,\beta}(y,z)|z|^{2\mu+1}dz\right)|y|^{2\mu+1}dy = 2^{\mu+1}\Gamma(\mu+1)\left|\frac{\sin(\alpha)\sin(\beta)}{\sin(\alpha+\beta)}\right|^{\mu+1}$$

$$\times\quad e^{-\frac{i\pi}{2}(\mu+1)\tilde{a}(\alpha,\beta)}\int_{-\infty}^{+\infty} f(z)\ K_{\mu,\alpha+\beta}(x,z)|z|^{2\mu+1}\,dz,$$

where

$$\hat{a}(\alpha,\beta) = sgn\left(\frac{\sin(\alpha+\beta)}{\sin(\alpha)\sin(\beta)}\right).$$

Proof. For any positive number ϵ, we define the function I_ϵ on $\mathbb{R}$ by

$$I_\epsilon(x) = \int_{-\infty}^{+\infty} e^{-\epsilon y^2} K_{\mu,\alpha}(x,y)\left(\int_{-\infty}^{+\infty} f(z)K_{\mu,\beta}(y,z)|z|^{2\mu+1}dz\right)|y|^{2\mu+1}dy.$$

Since $D^\beta_\mu f\in L^1_\mu(\mathbb{R})$, it follows from the dominated convergence theorem that

$$\lim_{\epsilon\to 0} I_\epsilon(x) = \int_{-\infty}^{+\infty} K_{\mu,\alpha}(x,y)\left(\int_{-\infty}^{+\infty} f(z)K_{\mu,\beta}(y,z)|z|^{2\mu+1}dz\right)|y|^{2\mu+1}dy.$$

Using Fubini's Theorem and Lemma 4.1, we obtain

$$\begin{aligned} I_\epsilon(x) &= \frac{2^{\mu+1}\Gamma(\mu+1)}{c_{\alpha,\beta}(\epsilon)}\ e^{-(\frac{i}{2}\cot(\alpha)+r_1(\epsilon))x^2} \\ &\times \int_{-\infty}^{+\infty} e^{-(\frac{i}{2}\cot(\beta)+r_2(\epsilon))z^2} f(z)E_\mu\left(-r_3(\epsilon)xz\right)|z|^{2\mu+1}\,dz. \end{aligned}$$

Clearly

$$\begin{aligned} \lim_{\epsilon\to 0} c_{\alpha,\beta}(\epsilon) &= \left|\frac{\sin(\alpha+\beta)}{\sin(\alpha)\sin(\beta)}\right|^{\mu+1} e^{\frac{i\pi}{2}(\mu+1)\hat{a}(\alpha,\beta)}, \\ \lim_{\epsilon\to 0} e^{-(\frac{i}{2}\cot(\alpha)+r_1(\epsilon))x^2} &= e^{-\frac{i}{2}x^2\cot(\alpha+\beta)}, \\ \lim_{\epsilon\to 0} e^{-(\frac{i}{2}\cot(\beta)+r_2(\epsilon))z^2} f(z)E_\mu\left(-r_3(\epsilon)xz\right) &= e^{-\frac{i}{2}z^2\cot(\alpha+\beta)} f(z)E_\mu(ixz/\sin(\alpha+\beta)). \end{aligned}$$

Applying again Lemma 3.2 with $c=\frac{i}{2}\cot(\beta)+r_2(\epsilon)$ and $d=-r_3(\epsilon)$, then

$$\Re c = \frac{\epsilon\sin^2(\alpha)}{4\epsilon^2\sin^2(\alpha)\sin^2(\beta)+\sin^2(\alpha+\beta)} \quad\text{and}\quad \Re d = -\frac{2\epsilon\sin(\alpha)\sin(\beta)}{4\epsilon^2\sin^2(\alpha)\sin^2(\beta)+\sin^2(\alpha+\beta)}.$$

Therefore

$$\begin{aligned} \left|e^{-(\frac{i}{2}\cot(\beta)+r_2(\epsilon))z^2} f(z)E_\mu\left(-r_3(\epsilon)xz\right)\right| &\leq e^{\frac{\epsilon\sin^2(\beta)x^2}{4\epsilon^2\sin^2(\alpha)\sin^2(\beta)+\sin^2(\alpha+\beta)}}\ |f(z)| \\ &\leq C_x|f(z)|, \end{aligned}$$

where $C_x = \sup\limits_{\epsilon\in]0,1]} e^{\frac{\epsilon x^2\sin^2(\beta)}{4\epsilon^2\sin^2(\alpha)\sin^2(\beta)+\sin^2(\alpha+\beta)}}$. Thus, the dominated convergence theorem leads to

$$\begin{aligned} \lim_{\epsilon\to 0} I_\epsilon(x) &= 2^{\mu+1}\Gamma(\mu+1)\left|\frac{\sin(\alpha)\sin(\beta)}{\sin(\alpha+\beta)}\right|^{\mu+1} e^{-\frac{i\pi}{2}(\mu+1)\hat{a}(\alpha,\beta)} e^{-\frac{i}{2}x^2\cot(\alpha+\beta)} \\ &\times \int_{-\infty}^{+\infty} f(z)\ e^{-\frac{i}{2}z^2\cot(\alpha+\beta)} E_\mu(ixz/\sin(\alpha+\beta))\ |z|^{2\mu+1}\,dz. \end{aligned}$$

This completes the proof.

Lemma 4.3 *Let α, β be in $]-\pi, 0[\cup]0, \pi[$. Then*

$$\left(\hat{\alpha}+\hat{\beta}-\hat{a}(\alpha,\beta)\right)\frac{\pi}{2}-(\alpha+\beta)=sgn(\sin(\alpha+\beta))\frac{\pi}{2}-(\alpha+\beta-2r\pi),$$

where

$$r=\begin{cases}1, & \text{if } \alpha+\beta\in]\pi,2\pi[,\\ 0, & \text{if } \alpha+\beta\in]-\pi,0[\cup]0,\pi[,\\ -1, & \text{if } \alpha+\beta\in]-2\pi,-\pi[.\end{cases}$$

Proof. is obviously.

Theorem 4.2 *Let α, β be in $\mathbb{R}$ and let f be in $L^1_\mu(\mathbb{R})$ with $D^\beta_\mu f \in L^1_\mu(\mathbb{R})$. Then*

$$D^\alpha_\mu \circ D^\beta_\mu(f) = D^{\alpha+\beta}_\mu(f).$$

with equality a. e when $\alpha+\beta \in \pi\mathbb{Z}$.

Proof. Since D^α_μ is periodic in α with period 2π, we can assume that α and β are in $(-\pi, \pi]$.
We shall divide the proof into five steps.
Step I Suppose that $\alpha \in]-\pi, 0[\cup]0, \pi[$ and $\beta = \pi$. By the change of variables $u = -y$, we obtain

$$\begin{aligned} D^\alpha_\mu(D^\pi_\mu f)(x) &= A_\alpha \int_{-\infty}^{+\infty} f(-y)K_{\mu,\alpha}(x,y)|y|^{2\mu+1}\,dy \\ &= A_\alpha \int_{-\infty}^{+\infty} f(y)K_{\mu,\alpha}(-x,y)|y|^{2\mu+1}\,dy \\ &= D^\pi_\mu(D^\alpha_\mu f)(x). \end{aligned}$$

An easy calculation shows that

$$\operatorname{sgn}(\sin(\alpha+\pi))\frac{\pi}{2}-(\alpha+\pi-2r\pi)=A_\alpha,$$

where

$$r=\begin{cases}0, & \text{if } \alpha\in]-\pi,0[,\\ 1, & \text{if } \alpha\in]0,\pi[.\end{cases}$$

Since $K_{\mu,\alpha}(-x,y) = K_{\mu,\alpha+\pi}(x,y)$, definition 3.12 implies that

$$(D^\alpha_\mu \circ D^\pi_\mu)f = D^{\alpha+\pi}_\mu f. \tag{4.2}$$

Step II Suppose that $\alpha \in]-\pi, 0[\cup]0, \pi[$ and $\beta = \pi-\alpha$. As $-\alpha \in]-\pi, 0[\cup]0, \pi[$, (4.2) implies that $D^{\pi-\alpha}_\mu = D^{-\alpha}_\mu \circ D^\pi_\mu$. Therefore, in view of Theorem 4.1,

$$\begin{aligned} \left(D^\alpha_\mu \circ D^{\pi-\alpha}_\mu\right)(f) &= \left(D^\alpha_\mu \circ D^{-\alpha}_\mu\right)(D^\pi_\mu f) \\ &= D^\pi_\mu f,\ \text{a.e.} \end{aligned}$$

Step III Suppose that $\alpha \in]-\pi, 0[\cup]0, \pi[$ and $\beta = -\alpha$. This is exactly statement (2) of Theorem 4.1.
Step IV Suppose that $\alpha \in]-\pi, 0[\cup]0, \pi[$ and $\beta = -\pi-\alpha$. The proof of this is similar to step I.
Step V. Suppose that $\alpha, \beta \in]-\pi, 0[\cup]0, \pi[$ and $\alpha+\beta \notin \{-\pi, 0, \pi\}$. In this case

$$D^\alpha_\mu D^\beta_\mu f(x) = \frac{e^{i(\mu+1)((\hat{\alpha}+\hat{\beta})\frac{\pi}{2}-(\alpha+\beta))}}{\Gamma^2(\mu+1)(4|\sin\alpha\sin\beta|)^{\mu+1}}\,A(x),$$

where

$$A(x) = \int_{-\infty}^{+\infty} K_{\mu,\alpha}(x,y)\left(\int_{-\infty}^{+\infty} f(z)K_{\mu,\beta}(y,z)|z|^{2\mu+1}dz\right)|y|^{2\mu+1}dy.$$

By Lemmas 4.2 and 4.3 and Definition 3.3, we have

$$\begin{aligned} D^\alpha_\mu D^\beta_\mu f(x) &= \frac{e^{i(\mu+1)((\hat{\alpha}+\hat{\beta}-\hat{a}(\alpha,\beta))\frac{\pi}{2}-(\alpha+\beta))}}{\Gamma(\mu+1)|2\sin(\alpha+\beta)|^{\mu+1}}\int_{-\infty}^{+\infty} f(z)K_{\mu,\alpha+\beta}(x,z)|z|^{2\mu+1}dz \\ &= \frac{e^{i(\mu+1)(\operatorname{sgn}(\sin(\alpha+\beta))\frac{\pi}{2}-(\alpha+\beta-2r\pi))}}{\Gamma(\mu+1)|2\sin(\alpha+\beta)|^{\mu+1}}\int_{-\infty}^{+\infty} f(z)K_{\mu,\alpha+\beta}(x,z)|z|^{2\mu+1}dz \\ &= D^{\alpha+\beta}_\mu f(x). \end{aligned}$$

4.3 Operational formulae

Proposition 4.1 *Let $f \in \mathcal{S}(\mathbb{R})$ and $\alpha \in \mathbb{R}\backslash\pi\mathbb{Z}$. Then*
(1) $D_\mu^\alpha(yf(y))(x) = -i\sin(\alpha)\Lambda_\mu^\alpha(D_\mu^\alpha f)(x)$.
(2) $xD_\mu^\alpha(f)(x) = i\sin(\alpha)D_\mu^\alpha\left(\Lambda_\mu^{-\alpha}f\right)(x)$.

Proof.
(1) Since D_μ^α is periodic in α with period 2π, we can assume that $0 < |\alpha| < \pi$.
Let us rewrite equality (3.6) in the form

$$D_\mu^\alpha(f) = f_1 f_2,$$

where

$$f_1(x) = e^{-\frac{i}{2}\cot(\alpha)x^2}, \quad f_2(x) = b_\alpha D_\mu\left[e^{-\frac{i}{2}\cot(\alpha)y^2}f(y)\right]\left(-\frac{x}{\sin(\alpha)}\right) \quad \text{and} \quad b_\alpha = \frac{e^{i(\mu+1)(\hat{\alpha}\pi/2-\alpha)}}{(|\sin(\alpha)|)^{\mu+1}}.$$

The product rule of the Dunkl operators Λ_μ, gives

$$\Lambda_\mu(D_\mu^\alpha f) = f_1\Lambda_\mu(f_2) + f_2\Lambda_\mu(f_1).$$

By virtue of Corollary 2.11 in [7], we deduce

$$\Lambda_\mu(f_2)(x) = \frac{ib_\alpha}{\sin(\alpha)}D_\mu\left[ye^{-\frac{i}{2}\cot(\alpha)y^2}f(y)\right]\left(-\frac{x}{\sin(\alpha)}\right).$$

Taking into account of $\Lambda_\mu(f_1)(x) = -ix\cot(\alpha)f_1(x)$, we can easily prove

$$\Lambda_\mu(D_\mu^\alpha f)(x) = -ix\cot(\alpha)D_\mu^\alpha f(x) + \frac{i}{\sin(\alpha)}D_\mu^\alpha(yf(y))(x).$$

This complete the proof.
(2) It suffices to assume that $0 < |\alpha| < \pi$. From Lemma 2.9 in [7], we deduce

$$\begin{aligned}
\frac{ix}{\sin(\alpha)}\left(D_\mu^\alpha f\right)(x) &= A_\alpha\int_{-\infty}^{+\infty}\Lambda_\mu^\alpha(K_{\mu,\alpha}(x,.))f(y)|y|^{2\mu+1}\,dy \\
&= -A_\alpha\int_{-\infty}^{+\infty}K_{\mu,\alpha}(x,y)\overline{\Lambda_\mu^\alpha\overline{f}(y)}\,|y|^{2\mu+1}\,dy \\
&= -A_\alpha\int_{-\infty}^{+\infty}K_{\mu,\alpha}(x,y)(\Lambda_\mu^{-\alpha}f)(y)\,|y|^{2\mu+1}\,dy \\
&= -D_\mu^\alpha\left(\Lambda_\mu^{-\alpha}f\right)(x).
\end{aligned}$$

5 Fractional Dunkl transform in $L_\mu^2(\mathbb{R})$

In this section, we discuss the extension of the fractional Dunkl transform D_μ^α to $L_\mu^2(\mathbb{R})$.

5.1 Plancherel theorem

Proposition 5.1 *Let f and g be in $L_\mu^1(\mathbb{R})$ and $\alpha \in \mathbb{R}\backslash\pi\mathbb{Z}$. Then*

$$\int_{-\infty}^{+\infty}D_\mu^\alpha f(x)\overline{g(x)}|x|^{2\mu+1}\,dx = \int_{-\infty}^{+\infty}f(x)\overline{D_\mu^{-\alpha}g(x)}|x|^{2\mu+1}\,dx.$$

Proof. Let f and $g \in L_\mu^1(\mathbb{R})$. As D_μ^α is periodic in α with period 2π, we suppose $0 < |\alpha| < \pi$. Using Fubini's theorem we write

$$\begin{aligned}
\int_{-\infty}^{+\infty}D_\mu^\alpha f(x)\overline{g(x)}|x|^{2\mu+1}\,dx &= A_\alpha\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}K_{\mu,\alpha}(x,y)f(y)|y|^{2\mu+1}\,dy\right)\overline{g(x)}|x|^{2\mu+1}\,dx, \\
&= \int_{-\infty}^{+\infty}f(y)\overline{A_{-\alpha}\int_{-\infty}^{+\infty}g(x)K_{\mu,\alpha}(x,y)|x|^{2\mu+1}\,dx}|y|^{2\mu+1}\,dy, \\
&= \int_{-\infty}^{+\infty}f(y)\overline{D_\mu^{-\alpha}g(y)}|y|^{2\mu+1}\,dy.
\end{aligned}$$

This complete the proof.

Corollary 5.1 *Let $f \in \mathcal{S}(\mathbb{R})$ and $\alpha \in \mathbb{R}$. Then*

$$\left\|D_\mu^\alpha f\right\|_{2,\mu} = \|f\|_{2,\mu}.$$

Proof. It is easy to check that Corollary 5.1 holds for $\alpha = 0$ and $\alpha = \pi$. Now let $0 < |\alpha| < \pi$ and $f \in \mathcal{S}(\mathbb{R})$. By Proposition 5.1 and Theorem 4.1, (3), we have

$$\begin{aligned}
\left\|D_\mu^\alpha f\right\|_{2,\mu} &= \int_{-\infty}^{+\infty} D_\mu^\alpha f(x)\overline{D_\mu^\alpha f(x)}|x|^{2\mu+1}\,dx,\\
&= \int_{-\infty}^{+\infty} f(x)\overline{D_\mu^{-\alpha} D_\mu^\alpha f(x)}|x|^{2\mu+1}\,dx,\\
&= \|f\|_{2,\mu}.
\end{aligned}$$

Theorem 5.1 *Let $\alpha \in \mathbb{R}$.*
(1) If $f \in L_\mu^1(\mathbb{R}) \cap L_\mu^2(\mathbb{R})$, then $D_\mu^\alpha f \in L_\mu^2(\mathbb{R})$ and $\left\|D_\mu^\alpha f\right\|_{2,\mu} = \|f\|_{2,\mu}$.
(2) The fractional Dunkl transform D_μ^α have a unique extension to an unitary operator on $L_\mu^2(\mathbb{R})$. More precisely if the extension is also denoted by $f \to D_k^\alpha f$, then D_k^α is a unitary operator on $L_\mu^2(\mathbb{R})$ with inverse $(D_k^\alpha)^{-1} = D_k^{-\alpha}$.

Proof. It suffices to assume that $0 < |\alpha| < \pi$. From Corollary 5.1 and the density of $\mathcal{S}(\mathbb{R})$ in $L_\mu^2(\mathbb{R})$, we deduce the existence of a unique continuous operator $\hat{D}_k^\alpha$ on $L_\mu^2(\mathbb{R})$ that coincides with D_μ^α on $\mathcal{S}(\mathbb{R})$. If $f, g \in \mathcal{S}(\mathbb{R})$ then

$$\begin{aligned}
\int_{-\infty}^{+\infty} \hat{D}_k^\alpha f(x)\overline{g(x)}|x|^{2\mu+1}\,dx &= \int_{-\infty}^{+\infty} D_\mu^\alpha f(x)\overline{g(x)}|x|^{2\mu+1}\,dx\\
&= \int_{-\infty}^{+\infty} f(x)\overline{D_\mu^{-\alpha} g(x)}|x|^{2\mu+1}\,dx\\
&= \int_{-\infty}^{+\infty} f(x)\overline{\hat{D}_k^{-\alpha} g(x)}|x|^{2\mu+1}\,dx.
\end{aligned}$$

Let $f, g \in L_\mu^2(\mathbb{R})$. By the density of $\mathcal{S}(\mathbb{R})$ in $L_\mu^2(\mathbb{R})$, we conclude that

$$\int_{-\infty}^{+\infty} \hat{D}_k^\alpha f(x)\overline{g(x)}|x|^{2\mu+1}\,dx = \int_{-\infty}^{+\infty} f(x)\overline{\hat{D}_\mu^{-\alpha} g(x)}|x|^{2\mu+1}\,dx. \tag{5.1}$$

Now, if $f \in L_\mu^1(\mathbb{R}) \cap L_\mu^2(\mathbb{R})$ and $g \in \mathcal{S}(\mathbb{R})$, then

$$\begin{aligned}
\int_{-\infty}^{+\infty} D_\mu^\alpha f(x)\overline{g(x)}|x|^{2\mu+1}\,dx &= \int_{-\infty}^{+\infty} f(x)\overline{D_\mu^{-\alpha} g(x)}|x|^{2\mu+1}\,dx\\
&= \int_{-\infty}^{+\infty} f(x)\overline{\hat{D}_\mu^{-\alpha} g(x)}|x|^{2\mu+1}\,dx\\
&= \int_{-\infty}^{+\infty} \hat{D}_\mu^\alpha f(x)\overline{g(x)}|x|^{2\mu+1}\,dx.
\end{aligned}$$

Hence $D_\mu^\alpha f = \hat{D}_\mu^\alpha f$, a.e, which proves the first statement in part (1). The second statement of part (1) follows from Corollary 5.1. Part (2) follows from part (1), Corollary 5.1 and Theorem 4.1, (2).

As a consequence of the previous Theorem and of Theorem 4.2, we state the following Corollary.

Corollary 5.2 *For each $f \in L_\mu^2(\mathbb{R})$ and $\alpha, \beta \in \mathbb{R}$, we have*

$$D_\mu^\alpha \circ D_\mu^\beta(f) = D_\mu^{\alpha+\beta}(f).$$

5.2 Eigenfunctions of the operator D_μ^α

In this subsection, we introduce the following operators on $\mathcal{S}(\mathbb{R})$ by:

$$T_\mu f(x) = 2^{-1/2}\left[xf(x) - \Lambda_\mu f(x)\right], \quad T_\mu^* f(x) = 2^{-1/2}\left[xf(x) + \Lambda_\mu f(x)\right]$$

and

$$\mathbb{H}_\mu = T_\mu^* T_\mu + T_\mu T_\mu^*.$$

After an integration by parts, it is easy to check that T_μ^* is the adjoint of T_μ in $L^2_\mu(\mathbb{R})$. More precisely: if f and g are in $\mathcal{S}(\mathbb{R})$, then

$$\langle T_\mu f, g\rangle_\mu = \langle f, T_\mu^* g\rangle_\mu. \tag{5.2}$$

In the next proposition, we discuss intertwining properties of D_μ^α with T_μ, T_μ^* and $\mathbb{H}_\mu$.

Proposition 5.2 *The following relations hold:*
(1) $D_\mu^\alpha \circ T_\mu = e^{i\alpha}\left(T_\mu \circ D_\mu^\alpha\right)$ *on* $\mathcal{S}(\mathbb{R})$.
(2) $D_\mu^\alpha \circ T_\mu^* = e^{-i\alpha}\left(T_\mu^* \circ D_\mu^\alpha\right)$ *on* $\mathcal{S}(\mathbb{R})$.
(3) $D_\mu^\alpha \circ \mathbb{H}_\mu = \mathbb{H}_\mu \circ D_\mu^\alpha$ *on* $\mathcal{S}(\mathbb{R})$.

Proof.
(1) Clearly, $D_\mu^0 \circ T_\mu = T_\mu = T_\mu \circ D_\mu^0$ and $D_\mu^\pi \circ T_\mu = e^{i\pi}\, T_\mu \circ D_\mu^\pi$.
Now let $0 < |\alpha| < \pi$. From the relation $\sqrt{2}\, T_\mu = x - \Lambda_\mu = (-ie^{i\alpha}/\sin(\alpha))x - \Lambda_\mu^{-\alpha}$, we deduce

$$\sqrt{2} D_\mu^\alpha \circ T_\mu = -\frac{ie^{i\alpha}}{\sin(\alpha)} D_\mu^\alpha \circ x - D_\mu^\alpha \circ \Lambda_\mu^{-\alpha}.$$

By Proposition 4.1, it follows that

$$\begin{aligned}\sqrt{2} D_\mu^\alpha \circ T_\mu &= -e^{i\alpha}\Lambda_\mu^\alpha \circ D_\mu^\alpha + \frac{i}{\sin(\alpha)}\, x \circ D_\mu^\alpha \\ &= e^{i\alpha}(x - \Lambda_\mu) \circ D_\mu^\alpha = \sqrt{2}\, e^{i\alpha} T_\mu \circ D_\mu^\alpha.\end{aligned}$$

(2) follows by taking adjoints in (1)

Remark 5.1 *By induction, one can show:*

$$D_\mu^\alpha \circ T_\mu^n = e^{in\alpha}\left(T_\mu^n \circ D_\mu^\alpha\right) \quad \textit{and} \quad D_\mu^\alpha \circ T_\mu^{*n} = e^{-in\alpha}\left(T_\mu^{*n} \circ D_\mu^\alpha\right) \;\; \textit{on } \mathcal{S}(\mathbb{R}).$$

The following commutator identities are useful in the sequel.

Proposition 5.3 *Let* $n \in \mathbb{N}$. *Then*
(1)

$$[T_\mu^*, T_\mu] = 1 + (2\mu+1)s, \quad \textit{where} \quad sf(x) = f(-x).$$

(2)

$$[T_\mu^*, T_\mu^n] = \begin{cases} 2rT_\mu^{2r-1}; & n = 2r, \\ T_\mu^{2r} \circ ((2r+1) + (2\mu+1)s); & n = 2r+1. \end{cases} \tag{5.3}$$

(3)

$$[T_\mu^{*n}, T_\mu] = \begin{cases} 2rT_\mu^{*(2r-1)}; & n = 2r, \\ T_\mu^{*(2r)} \circ ((2r+1) + (2\mu+1)s); & n = 2r+1. \end{cases}$$

Proof.
(1) A simple calculation shows that

$$\begin{aligned} 2[T_\mu^*, T_\mu] &= (x+\Lambda_\mu)(x-\Lambda_\mu) - (x-\Lambda_\mu)(x+\Lambda_\mu) \\ &= 2[\Lambda_\mu, x] = 2(1 + (2\mu+1)s), \quad \text{where} \quad sf(x) = f(-x). \end{aligned}$$

(2) and (3) follow from the preceding formula by induction on n

Now let $h_0^\mu(x) = \frac{1}{\sqrt{\Gamma(\mu+1)}} e^{-x^2/2}$ be the standard Gaussian on $\mathbb{R}$. Then we have:

Proposition 5.4 *Let $n \in \mathbb{N}$. Then*
(1) $T_\mu^* h_0^\mu = 0$.
(2) $D_\mu^\alpha h_0^\mu = h_0^\mu$.
(3) $T_\mu^{*n} T_\mu^n h_0^\mu = \dfrac{[\frac{n}{2}]!\ 2^n\ \Gamma(\mu + [\frac{n+1}{2}] + 1)}{\Gamma(\mu+1)}\ h_0^\mu$, *where $[x]$ denotes the greatest integer function.*

Proof.
(1) Since h_0^μ is even, $\sqrt{2} T_\mu^*(h_0^\mu) = x h_0^\mu + \frac{d}{dx} h_0^\mu = 0$.
(2) It is clear that $D_\mu^0 h_0^\mu = h_0^\mu$ and $D_\mu^\pi h_0^\mu = h_0^\mu$. When $0 < |\alpha| < \pi$,

$$\begin{aligned} \sqrt{\Gamma(\mu+1)} D_\mu^\alpha h_0^\mu(x) &= \frac{e^{i(\mu+1)(\hat{\alpha}\pi/2-\alpha)} e^{-\frac{i}{2}x^2 \cot(\alpha)}}{|\sin(\alpha)|^{\mu+1}} \\ &\times \frac{1}{2^{\mu+1}\Gamma(\mu+1)} \int_{-\infty}^{+\infty} e^{-(\frac{1}{2}+\frac{i}{2}\cot(\alpha))y^2} E_\mu\left(\frac{ixy}{\sin(\alpha)}\right) |y|^{2\mu+1}\ dy. \end{aligned}$$

Using Lemma 3.1 with $a \leftrightarrow \frac{1}{2} + \frac{i}{2}\cot(\alpha)$, $\xi = 0$ and $x \leftrightarrow \frac{x}{\sin(\alpha)}$, one can schow

$$\begin{aligned} \frac{1}{2^{\mu+1}\Gamma(\mu+1)} \int_{-\infty}^{+\infty} e^{-(\frac{1}{2}+\frac{i}{2}\cot(\alpha))y^2} E_\mu\left(\frac{ixy}{\sin(\alpha)}\right) |y|^{2\mu+1}\ dy &= \frac{e^{-\frac{x^2}{2(1+i\cot(\alpha))\sin^2(\alpha)}}}{(1+i\cot(\alpha))^{\mu+1}} \\ &= \frac{e^{-(\frac{1}{2}-\frac{i}{2}\cot(\alpha))x^2}}{A_\alpha}. \end{aligned}$$

Hence, $D_\mu^\alpha h_0^\mu(x) = h_0^\mu(x)$.
(3) Using the fact that $T_\mu^* h_0^\mu = 0$, we have

$$\begin{aligned} T_\mu^{*(n+1)} T_\mu^{n+1} h_0^\mu &= T_\mu^{*n}\left(T_\mu^* T_\mu^{n+1}\right) h_0^\mu \\ &= T_\mu^{*n}\left([T_\mu^*, T_\mu^{n+1}] - T_\mu^{n+1} T_\mu^*\right) h_0^\mu \\ &= T_\mu^{*n} [T_\mu^*, T_\mu^{n+1}] h_0^\mu. \end{aligned}$$

By (5.3), we get

$$[T_\mu^*, T_\mu^{n+1}] h_0^\mu = \begin{cases} (n+1) T_\mu^n h_0^\mu, & \text{if } n \text{ is even,} \\ (2\mu+n+2) T_\mu^n h_0^\mu, & \text{if } n \text{ is odd.} \end{cases}$$

Hence,

$$T_\mu^{*(n+1)} T_\mu^{n+1} h_0^\mu = \begin{cases} (n+1) T_\mu^{*n} T_\mu^n h_0^\mu, & \text{if } n \text{ is even,} \\ (2\mu+n+2) T_\mu^{*n} T_\mu^n h_0^\mu, & \text{if } n \text{ is odd.} \end{cases}$$

Consequently, the assertion now follows by induction on n.

Definition 5.1 *We define the nth generalized Hermite function h_n^μ by:*

$$h_0^\mu(x) = \frac{1}{\sqrt{\Gamma(\mu+1)}} e^{-\frac{x^2}{2}} \quad \textit{and for} \quad n \geq 1 \quad h_n^\mu(x) = c_n T_\mu^n h_0^\mu(x),$$

where

$$c_n = \sqrt{\frac{\Gamma(\mu+1)}{[\frac{n}{2}]!\ 2^n\ \Gamma(\mu + [\frac{n+1}{2}] + 1)}}.$$

We collect some properties of the one-dimensional generalized Hermite functions:

Proposition 5.5 *The one-dimensional generalized Hermite functions satisfy:*
(1) $T_\mu h_n^\mu = \frac{c_n}{c_{n+1}} h_{n+1}^\mu$.
(2) $T_\mu^* h_n^\mu = \frac{c_{n-1}}{c_n} h_{n-1}^\mu$.
(3) $\mathbb{H}_\mu h_n^\mu = 2(\mu+n+1) h_n^\mu$.

Remark 5.2 *In view of the previous proposition, we call T_μ and T_μ^* the creation operator and annihilation operator, respectively, for the one-dimensional generalized Hermite functions $h_n^\mu,\ n = 0, 1, 2, \ldots$.*
Part (3) of the above proposition says that h_n^μ is an eigenfunction of the generalized Hermite operator $\mathbb{H}_\mu$ corresponding to the eigenvalue $2(n + \mu + 1)$.

Definition 5.2 *The function $H_n^\mu(x) = e^{\frac{x^2}{2}} h_n^\mu(x)$ is a polynomial of degree n, called the nth one-dimensional generalized Hermite polynomial.*

From Proposition 5.5, we get the following proposition

Proposition 5.6
(1) $\frac{\sqrt{2}c_n}{c_{n+1}} H_{n+1}^\mu(x) = 2xH_n^\mu(x) - \left(H_n^\mu\right)'(x) - \frac{2\mu+1}{x} \frac{H_n^\mu(x) - H_n^\mu(-x)}{2}$.
(2) $\frac{\sqrt{2}c_{n-1}}{c_n} H_{n-1}^\mu(x) = \left(H_n^\mu\right)'(x) + \frac{2\mu+1}{x} \frac{H_n^\mu(x) - H_n^\mu(-x)}{2}$.
(3) $\frac{\sqrt{2}c_n}{c_{n+1}} H_{n+1}^\mu(x) = 2xH_n^\mu(x) - H_{n-1}^\mu(x)$.

Remark 5.3 *By induction we see that:*
• *Every polynomial of degree $\le n$ on $\mathbb{R}$ is a linear combination of one-dimensional generalized Hermite polynomial of degree $\le n$.*

The aim of this subsection is the following theorem:

Theorem 5.2 *The generalized Hermite functions $\{h_n^\mu\}_{n=0}^\infty$ are a basis of eigenfunctions of the fractional Dunkl transform D_μ^α on $L_\mu^2(\mathbb{R})$, satisfying*

$$D_\mu^\alpha h_n^\mu(x) = e^{in\alpha} h_n^\mu(x). \tag{5.4}$$

Proof. We begin our proof by showing that the family $\{h_n^\mu\}_{n=0}^\infty$ is an orthonormal basis of $L_\mu^2(\mathbb{R})$. Let m and n be non negative integers such that $m < n$. By (5.2) and part (3) of Proposition 5.4, we get

$$\begin{aligned} \langle h_m^\mu, h_n^\mu \rangle_\mu &= c_m c_n \left\langle T_\mu^m h_0^\mu, T_\mu^n h_0^\mu \right\rangle_\mu \\ &= c_m c_n \left\langle T_\mu^{*m} T_\mu^m h_0^\mu, T_\mu^{n-m} h_0^\mu \right\rangle_\mu \\ &= \frac{c_n}{c_m} \left\langle h_0^\mu, T_\mu^{n-m} h_0^\mu \right\rangle_\mu \\ &= \frac{c_n}{c_m} \left\langle T_\mu^{*(n-m)} h_0^\mu, h_0^\mu \right\rangle_\mu \\ &= 0. \end{aligned}$$

Also

$$\begin{aligned} \langle h_n^\mu, h_n^\mu \rangle_\mu &= c_n^2 \left\langle T_\mu^n h_0^\mu, T_\mu^n h_0^\mu \right\rangle_\mu \\ &= c_n^2 \left\langle T_\mu^{*n} T_\mu^n h_0^\mu, h_0^\mu \right\rangle_\mu \\ &= \|h_0^\mu\|_{2,\mu}^2 \\ &= 1. \end{aligned}$$

For completeness, let f be any function in $L_\mu^2(\mathbb{R})$ such that $\langle f, h_n^\mu \rangle_\mu = 0$ for all non negative integers n. Then, for all polynomials p, we get, by Remark 5.3,

$$\langle f, e^{-\frac{x^2}{2}} p \rangle_\mu = 0. \tag{5.5}$$

Our object is to show that (5.5) implies $D_\mu \left[f e^{-\frac{y^2}{2}} \right] = 0$ and therefore by the injectivity of the Dunkl transform, $f(y)e^{-\frac{y^2}{2}} = 0$, a.e., and hence $f = 0$. By (2.10) and (2.11), we have

$$\begin{aligned} 2^{\mu+1}\Gamma(\mu+1) D_\mu \left[f e^{-\frac{y^2}{2}} \right](x) &= \int_{-\infty}^{+\infty} f(y) e^{-\frac{y^2}{2}} j_\mu(xy) |y|^{2\mu+1}\, dy \\ &+ \frac{ix}{2(\mu+1)} \int_{-\infty}^{+\infty} f(y) e^{-\frac{y^2}{2}} j_{\mu+1}(xy) |y|^{2\mu+2}\, dy. \end{aligned}$$

Noting that

$$\int_{-\infty}^{+\infty} f(y)e^{-\frac{y^2}{2}} j_\mu(xy)|y|^{2\mu+1}\, dy \quad = \quad \Gamma(\mu+1)\int_{-\infty}^{+\infty}\sum_{n=0}^{+\infty}\frac{(-1)^n(x/2)^{2n}}{n!\;\Gamma(n+\mu+1)} f(y)e^{-\frac{y^2}{2}} y^{2n}|y|^{2\mu+1}\, dy.$$

By the Schwarz inequality

$$\begin{aligned}\int_{-\infty}^{+\infty} |f(y)|e^{-\frac{y^2}{2}} y^{2n}|y|^{2\mu+1} \quad dy &\le \|f\|_{2,\mu}\left(\int_{-\infty}^{+\infty} e^{-y^2} y^{4n}|y|^{2\mu+1}\, dy\right)^{1/2} \\ &\le \quad \|f\|_{2,\mu}\sqrt{\Gamma(2n+\mu+1)}.\end{aligned}$$

Then

$$\sum_{n=0}^{+\infty}\frac{(x/2)^{2n}}{n!\;\Gamma(n+\mu+1)}\int_{-\infty}^{+\infty} |f(y)|e^{-\frac{y^2}{2}} y^{2n}|y|^{2\mu+1}\, dy < \infty.$$

Hence, using (5.5)

$$\begin{aligned}\int_{-\infty}^{+\infty} f(y)e^{-\frac{y^2}{2}} j_\mu(xy)|y|^{2\mu+1}\, dy \quad &= \quad \Gamma(\mu+1)\sum_{n=0}^{+\infty}\frac{(-1)^n(x/2)^{2n}}{n!\;\Gamma(n+\mu+1)}\int_{-\infty}^{+\infty} f(y)e^{-\frac{y^2}{2}} y^{2n}|y|^{2\mu+1}\, dy \\ &= \quad 0.\end{aligned}$$

Similarly, one can show

$$\int_{-\infty}^{+\infty} f(y)e^{-\frac{y^2}{2}} j_{\mu+1}(xy)|y|^{2\mu+1}\, dy = 0.$$

Finally, it remains to prove (5.4). Since $D_\mu^\alpha h_0^\mu = h_0^\mu$, Remark 5.1 and Definition 5.1 gives the desired result.

Corollary 5.3 *The family of operators $\{D_\mu^\alpha\}_{\alpha\in\mathbb{R}}$ is a $\mathcal{C}_0$-group of unitary operators on $L_\mu^2(\mathbb{R})$.*

Proof. From Corollary 5.2, we deduce that the family $\{D_\mu^\alpha\}_{\alpha\in\mathbb{R}}$ satisfies the algebraic properties of a group:

$$D_\mu^0 = I, \quad D_\mu^\alpha \circ D_\mu^\beta = D_\mu^{\alpha+\beta} = D_\mu^\beta \circ D_\mu^\alpha\ ; \quad \alpha,\ \beta \in \mathbb{R}.$$

For the strong continuity, assume that $f \in L_\mu^2(\mathbb{R})$. Then we can expand f in the orthonormal basis $\{h_n^\mu\}_{n=0}^\infty$ as follows:

$$f = \sum_{n=0}^{\infty}\langle f, h_n^\mu\rangle_\mu\ h_n^\mu = \sum_{n=0}^{\infty}\hat{f}_n\ h_n^\mu,$$

where

$$\hat{f}_n = \langle f, h_n^\mu\rangle_\mu = \int_{-\infty}^{+\infty} f(x)\overline{h_n^\mu(x)}|x|^{2\mu+1}dx.$$

By Theorem 5.2, we can write

$$D_\mu^\alpha f = \sum_{n=0}^{+\infty} e^{in\alpha}\langle f, h_n^\mu\rangle_\mu\ h_n^\mu$$

and therefore

$$\left\|D_\mu^\alpha f - f\right\|_{2,\mu}^2 = \sum_{n=0}^{+\infty}|e^{in\alpha}-1|^2\,|\langle f, h_n^\mu\rangle_\mu|^2\,.$$

Finally, we can interchange limits and sum to get:

$$\lim_{\alpha\to 0}\left\|D_\mu^\alpha f - f\right\|_{2,\mu}^2 = 0.$$

5.3 The generator of the $\mathcal{C}_0$-group $\{D_\mu^\alpha\}_{\alpha\in\mathbb{R}}$

The infinitesimal generator G of the $\mathcal{C}_0$-group $\{D_\mu^\alpha\}_{\alpha\in\mathbb{R}}$ is defined by

$$\begin{array}{rcl} G : D(G) \subseteq L^2_\mu(\mathbb{R}) & \longrightarrow & L^2_\mu(\mathbb{R}), \\ f & \longmapsto & Gf \end{array}$$

where

$$\begin{array}{rcl} D(G) & = & \left\{ f \in L^2_\mu(\mathbb{R}) : \lim\limits_{\alpha\to 0}(1/\alpha)[D_\mu^\alpha f - f] \in L^2_\mu(\mathbb{R}) \right\}, \\ Gf & = & \lim\limits_{\alpha\to 0}(1/\alpha)[D_\mu^\alpha f - f], \quad f \in D(G). \end{array}$$

Since $\{D_\mu^\alpha\}_{\alpha\in\mathbb{R}}$ is unitary, it follows from Stone's Theorem ([10], p. 32) that iG is self-adjoint.

Theorem 5.3 *$-iG$ is a self-adjoint extension of the operator $\frac{1}{2}\mathbb{H}_\mu - (\mu+1)$.*

Proof. Note firstly that $D(\frac{1}{2}\mathbb{H}_\mu - (\mu+1)) = D(\mathbb{H}_\mu) = \mathcal{S}(\mathbb{R})$ and $\mathbb{H}_\mu$ is symmetric. We will show that $-iG$ is an extension of $\frac{1}{2}\mathbb{H}_\mu - (\mu+1)$. Let $f \in \mathcal{S}(\mathbb{R})$, the inversion formula for the Dunkl transform (see Theorem 4.20 in [4]):

$$f(x) = \frac{1}{2^{\mu+1}\Gamma(\mu+1)} \int_{-\infty}^{+\infty} D_\mu f(y) E_\mu(ixy)|y|^{2\mu+1}\, dy$$

together with (3.8) implies

$$\begin{array}{rcl} \dfrac{D_\mu^\alpha f(x) - f(x)}{\alpha} & = & \dfrac{r_1(\alpha)}{2^{\mu+1}\Gamma(\mu+1)} \displaystyle\int_{-\infty}^{+\infty} e^{\frac{i}{2}(x^2+y^2)\tan(\alpha)} E_\mu\left(\frac{ixy}{\cos(\alpha)}\right) D_\mu f(y)|y|^{2\mu+1}\, dy \\ & + & \dfrac{1}{2^{\mu+1}\Gamma(\mu+1)} \displaystyle\int_{-\infty}^{+\infty} r_2(\alpha,x,y) D_\mu f(y)|y|^{2\mu+1}\, dy, \end{array}$$

where

$$r_1(\alpha) = \frac{\left(\frac{e^{-i\alpha}}{\cos(\alpha)}\right)^{\mu+1} - 1}{\alpha} \quad \text{and} \quad r_2(\alpha,x,y) = \frac{e^{\frac{i}{2}(x^2+y^2)\tan(\alpha)} E_\mu\left(\frac{ixy}{\cos(\alpha)}\right) - 1}{\alpha}.$$

A limiting argument using the dominated convergence theorem allows us to

$$\begin{array}{rcl} \lim\limits_{\alpha\to 0} \dfrac{D_\mu^\alpha f(x) - f(x)}{\alpha} & = & -\dfrac{i(\mu+1)}{2^{\mu+1}\Gamma(\mu+1)} \displaystyle\int_{-\infty}^{+\infty} D_\mu f(y) E_\mu(ixy)|y|^{2\mu+1}\, dy \\ & + & \dfrac{1}{2^{\mu+1}\Gamma(\mu+1)} \displaystyle\int_{-\infty}^{+\infty} \dfrac{i}{2}(x^2+y^2) D_\mu f(y)|y|^{2\mu+1}\, dy \\ & = & -i(\mu+1)f(x) + \dfrac{i}{2}\left(x^2 f(x) + D_\mu\left[y^2 D_\mu f(y)\right](-x)\right). \end{array}$$

From Corollary 2.11 in [7], we deduce

$$-y^2 D_\mu f(y) = D_\mu[\Lambda_\mu^2 f](y),$$

Therefore

$$\begin{array}{rcl} -D_\mu\left[y^2 D_\mu f(y)\right](-x) & = & D_\mu^2[\Lambda_\mu^2 f(y)](-x) \\ & = & \Lambda_\mu f(x). \end{array}$$

Finally, $f \in D(G)$ and $Gf = i\left(\frac{1}{2}\mathbb{H}_\mu - (\mu+1)\right) f$.

Corollary 5.4 *The exponential form for the fractional Dunkl transform is:*

$$D_\mu^\alpha = e^{i\alpha(\mu+1)} e^{i\frac{\alpha}{2}(\Lambda_\mu^2 - x^2)}.$$

6 Heisenberg inequality for D^{α}_{μ}.

Throughout this section, Q denote the multiplication operator on $L^2_\mu(\mathbb{R})$ defined by $Qf(x) = xf(x)$: its domain is

$$D(Q) = \left\{ f \in L^2_\mu(\mathbb{R}) : \ xf \in L^2_\mu(\mathbb{R}) \right\}.$$

Definition 6.1 *Let $\mu \geq -1/2$ and $\alpha \in \mathbb{R}$. We define the generalized Sobolev spaces $H_2^{\mu,\alpha}(\mathbb{R})$ as follows:*

$$H_2^{\mu,\alpha}(\mathbb{R}) := \left\{ f \in L^2_\mu(\mathbb{R}) : \ xD^\alpha_\mu f \in L^2_\mu(\mathbb{R}) \right\}.$$

We provide this space with the norm

$$\|f|H_2^{\mu,\alpha}(\mathbb{R})\|_{2,\mu} := \left(\|f\|^2_{2,\mu} + \|xD^\alpha_\mu f\|^2_{2,\mu}\right)^{1/2}.$$

Note that $H_2^{\mu,0}(\mathbb{R}) = H_2^{\mu,\pi}(\mathbb{R}) = D(Q)$.

Proposition 6.1 *The following properties holds.*
(1) The fractional Dunkl transform D^α_μ is a unitary isomorphism from $H_2^{\mu,\alpha}(\mathbb{R})$ to $L^2(\mathbb{R}, \ dm_\mu(x))$ where $dm_\mu(x) = (1+x^2)|x|^{2\mu+1}\,dx$. In particular, $H_2^{\mu,\alpha}(\mathbb{R})$ is a Hilbert space with the inner product given by

$$\langle f, g\rangle_{m_\mu} = \int_{-\infty}^{+\infty} D^\alpha_\mu(f)(x)\overline{D^\alpha_\mu(g)(x)}\ dm_\mu(x).$$

(2) $\mathcal{S}(\mathbb{R})$ is a dense subspace of $H_2^{\mu,\alpha}(\mathbb{R})$.
(3) For $\alpha \in \mathbb{R}\backslash\pi\mathbb{Z}$, the operator $\Lambda^{-\alpha}_\mu$ extends canonically to $H_2^{\mu,\alpha}(\mathbb{R})$ by setting

$$\Lambda^{-\alpha}_\mu f := \frac{-i}{\sin(\alpha)} D^{-\alpha}_\mu \left[y.D^\alpha_\mu(f)\right] \quad \text{for } f \in H_2^{\mu,\alpha}(\mathbb{R}).$$

(4) For $\alpha \in \mathbb{R}\backslash\pi\mathbb{Z}$, the operator $i\Lambda^{-\alpha}_\mu$ is symmetric on $L^2_\mu(\mathbb{R})$ with domain $D(i\Lambda^{-\alpha}_\mu) = D(\Lambda^{-\alpha}_\mu) = H_2^{\mu,\alpha}(\mathbb{R})$.
(5) Let α and $\beta \in \mathbb{R}\backslash\pi\mathbb{Z}$. For all $f \in H_2^{\mu,\alpha}(\mathbb{R}) \cap H_2^{\mu,\beta}(\mathbb{R}) \cap D(Q)$, we have

$$\Lambda^{-\alpha}_\mu f = \Lambda^{-\beta}_\mu f + \frac{i\sin(\alpha-\beta)}{\sin(\alpha)\sin(\beta)}\ xf. \tag{6.1}$$

Proof.
(1) This is clear from Definition 6.1 and the fact that D^α_μ is an unitary operator on $L^2_\mu(\mathbb{R})$.
(2) This follows easily from (1) and the fact that $\mathcal{C}^\infty_c(\mathbb{R})$ is dense in $L^2(\mathbb{R}, \ dm_\mu(x))$.
(3) Follows from Proposition 4.1, (2).
(4) Follows from (5.1).
(5) To see this we first note that the equality (6.1) holds when $f \in \mathcal{S}(\mathbb{R})$. Now, if $f \in H_2^{\mu,\alpha}(\mathbb{R}) \cap H_2^{\mu,\beta}(\mathbb{R}) \cap D(Q)$ and $g \in \mathcal{S}(\mathbb{R})$, then

$$\begin{aligned}
\left\langle i\Lambda^{-\alpha}_\mu f, g\right\rangle_\mu &= \left\langle f, i\Lambda^{-\alpha}_\mu g\right\rangle_\mu \\
&= \left\langle f, i\Lambda^{-\beta}_\mu g\right\rangle_\mu - \frac{\sin(\alpha-\beta)}{\sin(\alpha)\sin(\beta)}\langle f, xg\rangle_\mu \\
&= \left\langle i\Lambda^{-\beta}_\mu f - \frac{\sin(\alpha-\beta)}{\sin(\alpha)\sin(\beta)} xf, g\right\rangle_\mu.
\end{aligned}$$

By the density of $\mathcal{S}(\mathbb{R})$ in $L^2_\mu(\mathbb{R})$, we obtain the desired result.

Proposition 6.2 *Let $\alpha \in \mathbb{R}\backslash\pi\mathbb{Z}$ and suppose $-1/2 \leq \mu < 0$, then*

$$H_2^{\mu,\alpha}(\mathbb{R}) \hookrightarrow \mathcal{C}_0(\mathbb{R}).$$

Proof. Let f be in $H_2^{\mu,\alpha}(\mathbb{R})$ with $-1/2 \leq \mu < 0$.
We have

$$\int_{-\infty}^{+\infty} |D^\alpha_\mu(f)(x)||x|^{2\mu+1}\ dx = \int_{-\infty}^{+\infty} (1+x^2)^{-\frac{1}{2}}(1+x^2)^{\frac{1}{2}}|D^\alpha_\mu(f)(x)||x|^{2\mu+1}\ dx.$$

Using Cauchy-Schwartz inequality we deduce that

$$\begin{aligned}\left\|D_\mu^\alpha f\right\|_{1,\mu} &\leq \left(\int_{-\infty}^{+\infty}\frac{|x|^{2\mu+1}}{1+x^2}dx\right)^{\frac{1}{2}}\|f|H_2^{\mu,\alpha}(\mathbb{R})\|_{2,\mu}\\ &= \sqrt{\frac{\pi}{\sin(-\pi\mu)}}\ \|f|H_2^{\mu,\alpha}(\mathbb{R})\|_{2,\mu}. \qquad (6.2)\end{aligned}$$

Hence $D_\mu^\alpha f$ belongs to $L^1_\mu(\mathbb{R})$ and therefore $D_\mu^\alpha f \in L^1_\mu(\mathbb{R})\cap L^2_\mu(\mathbb{R})$. Thus from Theorem 5.1,(2), we have

$$f(x) = A_{-\alpha}\int_{-\infty}^{+\infty} D_\mu^\alpha f(y)K_{\mu,-\alpha}(x,y)|y|^{2\mu+1}\ dy, \quad \text{a. e.}$$

We identify f with the second member, then we deduce that f belongs to $\mathcal{C}_0(\mathbb{R})$ and using (6.2) we show that the injection of $H_2^{\mu,\alpha}(\mathbb{R})$ into $\mathcal{C}_0(\mathbb{R})$ is continuous.

Lemma 6.1 *Let $\alpha\in\mathbb{R}\backslash\pi\mathbb{Z}$, $\mu\geq 0$, $x\in\mathbb{R}\backslash\{0\}$ and $f\in H_2^{\mu,\alpha}(\mathbb{R})$. There exists $c_1 = c_1(\mu) > 0$ such that*

$$\int_{-1/|x|}^{1/|x|}|D_\mu^\alpha f(y)|\ |y|^{2\mu+1}\ dy \leq c_1\ \|f|H_2^{\mu,\alpha}(\mathbb{R})\|_{2,\mu}\begin{cases}|x|^{-\mu} & \text{if } \mu>0,\\ |\ln|x||^{\frac{1}{2}} & \text{if } \mu=0\end{cases} \qquad (6.3)$$

$$\int_{\{|y|\geq\frac{1}{|x|}\}}|D_\mu^\alpha f(y)|\ |y|^{\mu+\frac{1}{2}}\ dy \leq c_1\sqrt{|x|}\ \|f|H_2^{\mu,\alpha}(\mathbb{R})\|_{2,\mu}. \qquad (6.4)$$

Proof. By the Cauchy-Schwartz inequality, we deduce that:

$$\begin{aligned}\int_{\{|y|\geq\frac{1}{|x|}\}}|D_\mu^\alpha f(y)|\ |y|^{\mu+\frac{1}{2}}\ dy &\leq \left(\int_{\{|y|\geq\frac{1}{|x|}\}}(1+y^2)|D_\mu^\alpha f(y)|^2\ |y|^{2\mu+1}\ dy\right)^{\frac{1}{2}}\left(\int_{\{|y|\geq\frac{1}{|x|}\}}\frac{dy}{1+y^2}\right)^{\frac{1}{2}}\\ &\leq \sqrt{\varphi(x)}\ \|f|H_2^{\mu,\alpha}(\mathbb{R})\|_{2,\mu},\end{aligned}$$

and

$$\begin{aligned}\int_{-1/|x|}^{1/|x|}|D_\mu^\alpha f(y)|\ |y|^{2\mu+1}\ dy &\leq \left(\int_{-1/|x|}^{1/|x|}(1+y^2)|D_\mu^\alpha f(y)|^2\ |y|^{2\mu+1}\ dy\right)^{\frac{1}{2}}\left(\int_{-1/|x|}^{1/|x|}\frac{|y|^{2\mu+1}}{1+y^2}\ dy\right)^{\frac{1}{2}}\\ &\leq \sqrt{\psi(x)}\ \|f|H_2^{\mu,\alpha}(\mathbb{R})\|_{2,\mu},\end{aligned}$$

where

$$\begin{aligned}\varphi(x) &= \int_{\{|y|\geq\frac{1}{|x|}\}}\frac{dy}{1+y^2} = 2\arctan(|x|),\\ \psi(x) &= \int_{-1/|x|}^{1/|x|}\frac{|y|^{2\mu+1}}{1+y^2}\ dy = \frac{2}{|x|^{2\mu}}\int_0^1\frac{y^{2\mu+1}}{x^2+y^2}\ dy.\end{aligned}$$

We therefore obtain (6.3) and (6.4) and complete the proof of the Lemma.

Proposition 6.3 *Let $\alpha\in\mathbb{R}\backslash\pi\mathbb{Z}$, $\mu\geq 0$ and $f\in H_2^{\mu,\alpha}(\mathbb{R})$. Then there exists a function $\psi\in\mathcal{C}(\mathbb{R}\backslash\{0\})$ such that $f(x)=\psi(x)$, a. e and for all $x\in\mathbb{R}\backslash\{0\}$,*

$$|\psi(x)|\leq c\ \|f|H_2^{\mu,\alpha}(\mathbb{R})\|_{2,\mu}\begin{cases}|x|^{-\mu} & \text{if } \mu>0,\\ |\ln|x||^{\frac{1}{2}} & \text{if } \mu=0,\end{cases}$$

where $c = c(\mu,\alpha) > 0$.

Proof. Let $f\in H_2^{\mu,\alpha}(\mathbb{R})$ and $x\in\mathbb{R}\backslash\{0\}$. From (3.3) we see that

$$\begin{aligned}\int_{-\infty}^{+\infty}|D_\mu^\alpha f(y)K_{\mu,-\alpha}(x,y)|\ |y|^{2\mu+1}\ dy &\leq a(\mu,-\alpha)\int_{-\infty}^{+\infty}|D_\mu^\alpha f(y)|\min\left(1,|xy|^{-(\mu+\frac{1}{2})}\right)\ |y|^{2\mu+1}\ dy\\ &= a(\mu,-\alpha)\int_{-1/|x|}^{1/|x|}|D_\mu^\alpha f(y)|\ |y|^{2\mu+1}\ dy + \frac{a(\mu,-\alpha)}{|x|^{\mu+\frac{1}{2}}}\int_{\{|y|\geq\frac{1}{|x|}\}}|D_\mu^\alpha f(y)|\ |y|^{\mu+\frac{1}{2}}\ dy.\end{aligned}$$

By Lemma 6.1, it follows that there exists $c = c(\mu, \alpha) > 0$ such that

$$\int_{-\infty}^{+\infty} |D_\mu^\alpha f(y) K_{\mu,-\alpha}(x,y)| \; |y|^{2\mu+1} \; dy \quad \leq \quad c \; \|f|H_2^{\mu,\alpha}(\mathbb{R})\|_{2,\mu} \begin{cases} |x|^{-\mu} & \text{if } \mu > 0, \\ |\ln |x||^{\frac{1}{2}} & \text{if } \mu = 0. \end{cases} \tag{6.5}$$

The next step is to show that the function ψ defined on $\mathbb{R}\backslash\{0\}$ by

$$\psi(x) = A_{-\alpha} \int_{-\infty}^{+\infty} D_\mu^\alpha f(y) K_{\mu,-\alpha}(x,y) \; |y|^{2\mu+1} \; dy$$

satisfies the conclusion of the proposition. Since $\mathcal{S}(\mathbb{R})$ is a dense subspace of $H_2^{\mu,\alpha}(\mathbb{R})$, there exists a sequence $(f_n) \subset \mathcal{S}(\mathbb{R})$ such that $\lim\limits_{n\to\infty} f_n = f$ in $H_2^{\mu,\alpha}(\mathbb{R})$. Using (4.1) and (6.5) we obtain

$$|\psi(x) - f_n(x)| \leq c \; \|(f - f_n)|H_2^{\mu,\alpha}(\mathbb{R})\|_{2,\mu} \begin{cases} |x|^{-\mu} & \text{if } \mu > 0, \\ |\ln |x||^{\frac{1}{2}} & \text{if } \mu = 0. \end{cases}$$

Then (f_n) converges locally uniformly on $\mathbb{R}\backslash\{0\}$ to ψ; this means that ψ is continuous on $\mathbb{R}\backslash\{0\}$. On the other hand, the convergence in $H_2^{\mu,\alpha}(\mathbb{R})$ implies the convergence in $L_\mu^2(\mathbb{R})$, we can therefore extract a subsequence (f_{n_k}) that converges almost everywhere to f on $\mathbb{R}$, so that $f(x) = \psi(x)$, *a. e.*

Proposition 6.4 *Let $\alpha \in \mathbb{R}\backslash\pi\mathbb{Z}$ and $f \in D(\Lambda_\mu^{-\alpha}) \cap D(Q)$. Then*

$$\Re \left\langle Qf, \Lambda_\mu^{-\alpha} f \right\rangle_\mu = -\frac{1}{2} \|f\|_{2,\mu}^2 - (\mu + 1/2) \left(\|f_e\|_{2,\mu}^2 - \|f_o\|_{2,\mu}^2 \right). \tag{6.6}$$

Proof.
We begin the proof by showing that

$$\Re \left\langle Q \; \varphi f, \Lambda_\mu^{-\alpha} f \right\rangle_\mu \quad = \quad -\frac{\left\|\sqrt{\varphi} f\right\|_{2,\mu}^2}{2} - (\mu + 1/2) \left(\|\sqrt{\varphi} f_e\|_{2,\mu}^2 - \|\sqrt{\varphi} f_o\|_{2,\mu}^2 \right) - \frac{1}{2} \left\langle |y|\varphi' f, f \right\rangle_\mu, \tag{6.7}$$

where $f \in \mathcal{S}(\mathbb{R})$ and φ is non-negative continuously differentiable function with compact support.
Obviously,

$$\Re \left\langle Q \; \varphi f, \Lambda_\mu^{-\alpha} f \right\rangle_\mu = \Re \left(\left\langle Q \; \varphi f, \Lambda_\mu f \right\rangle_\mu + i \cot(\alpha) \left\langle Q \; \varphi f, Qf \right\rangle_\mu \right) = \Re \left\langle Q \; \varphi f, \Lambda_\mu f \right\rangle_\mu.$$

Since $f \in \mathcal{S}(\mathbb{R})$, it follows that

$$\begin{aligned} \Re \left\langle Q \; \varphi f, \Lambda_\mu f \right\rangle_\mu \quad &= \quad \Re \left(\int_{-\infty}^{+\infty} y\varphi(y) f(y) \overline{f'(y)} |y|^{2\mu+1} \; dy - (\mu + 1/2) \int_{-\infty}^{+\infty} \varphi(y) f(y) \overline{f(-y)} |y|^{2\mu+1} \; dy \right. \\ &+ \quad \left. (\mu + 1/2) \|\sqrt{\varphi} f\|_{2,\mu}^2 \right). \end{aligned}$$

Write $f = u + iv$, where u and v are real-valued functions, we obtain

$$\begin{aligned} \Re \left(\int_{-\infty}^{+\infty} y\varphi(y) f(y) \overline{f'(y)} |y|^{2\mu+1} \; dy \right) \quad &= \quad \frac{1}{2} \int_{-\infty}^{+\infty} y\varphi(y) \left(|f(y)|^2 \right)' |y|^{2\mu+1} \; dy \\ &= \quad \frac{1}{2} \lim_{n\to\infty} \left(\int_{-\infty}^{-\frac{1}{n}} + \int_{\frac{1}{n}}^{+\infty} \right) y\varphi(y) \left(|f(y)|^2 \right)' |y|^{2\mu+1} \; dy. \end{aligned}$$

Integrate by parts and take $n \to \infty$, we get

$$\frac{1}{2} \lim_{n\to\infty} \left(\int_{-\infty}^{-\frac{1}{n}} + \int_{\frac{1}{n}}^{+\infty} \right) y\varphi(y) \left(|f(y)|^2 \right)' |y|^{2\mu+1} \; dy \quad = \quad -(\mu + 1) \|\sqrt{\varphi} f\|_{2,\mu}^2 - \frac{1}{2} \left\langle |y|\varphi' f, f \right\rangle_\mu.$$

Write $f(x) = f_e(x) + f_o(x)$, where $f_e(x) = \frac{1}{2}(f(x) + f(-x))$ and $f_o(x) = \frac{1}{2}(f(x) - f(-x))$, we have

$$\int_{-\infty}^{+\infty} \varphi(y) f(y) \overline{f(-y)} |y|^{2\mu+1} \; dy = \|\sqrt{\varphi} f_e\|_{2,\mu}^2 - \|\sqrt{\varphi} f_o\|_{2,\mu}^2.$$

Finally, we obtain (6.7).
Next, we will extend (6.7) to all functions $f \in D(\Lambda_\mu^{-\alpha}) \cap D(Q)$. For this purpose, let $f \in D(\Lambda_\mu^{-\alpha}) \cap D(Q)$ and $(f_n) \subset \mathcal{S}(\mathbb{R})$ such that $f_n \to f$ in $H_2^{\mu,\alpha}(\mathbb{R})$. Since $D_\mu^{-\alpha}$ is unitary on $L_\mu^2(\mathbb{R})$, then

$$\begin{aligned}\left\|\Lambda_\mu^{-\alpha} f\right\|_{2,\mu} &= \frac{1}{|\sin(\alpha)|}\left\|D_\mu^{-\alpha}(y.D_\mu^\alpha(f))\right\|_{2,\mu} \\ &= \frac{1}{|\sin(\alpha)|}\|y.D_\mu^\alpha(f)\|_{2,\mu} \\ &\leq \frac{1}{|\sin(\alpha)|}\|f|H_2^{\mu,\alpha}(\mathbb{R})\|_{2,\mu}.\end{aligned}$$

Therefore $\Lambda_\mu^{-\alpha} f_n \to \Lambda_\mu^{-\alpha} f$ in $L_\mu^2(\mathbb{R})$. From the inequality

$$\begin{aligned}\|x\varphi f_n - x\varphi f\|_{2,\mu} &\leq \|x\varphi\|_\infty \|f_n - f\|_{2,\mu} \\ &\leq \|x\varphi\|_\infty \|(f_n - f)|H_2^{\mu,\alpha}(\mathbb{R})\|_{2,\mu},\end{aligned}$$

we see that $x\varphi f_n \to x\varphi f$ in $L_\mu^2(\mathbb{R})$. So that

$$\lim_{n\to\infty} \Re\left\langle Q\ \varphi f_n, \Lambda_\mu^{-\alpha} f_n\right\rangle_\mu = \Re\left\langle Q\ \varphi f, \Lambda_\mu^{-\alpha} f\right\rangle_\mu.$$

Similarly one can see

$$\lim_{n\to\infty}\left\langle |y|\varphi' f_n, f_n\right\rangle_\mu = \left\langle |y|\varphi' f, f\right\rangle_\mu.$$

Using the fact that the support of $\sqrt{\varphi}$ is compact, then

$$\begin{aligned}\lim_{n\to\infty}\|\sqrt{\varphi} f_n\|_{2,\mu} &= \|\sqrt{\varphi} f\|_{2,\mu} \\ \lim_{n\to\infty}\|\sqrt{\varphi} f_{n,e}\|_{2,\mu} &= \|\sqrt{\varphi} f_e\|_{2,\mu} \\ \lim_{n\to\infty}\|\sqrt{\varphi} f_{n,o}\|_{2,\mu} &= \|\sqrt{\varphi} f_o\|_{2,\mu}.\end{aligned}$$

We conclude that (6.7) is true for all $f \in D(\Lambda_\mu^{-\alpha}) \cap D(Q)$.
To prove (6.6), let $\varphi \in \mathcal{C}_c^1(\mathbb{R})$ with $\varphi \geq 0$ and $\varphi(0) = 1$. Let φ_n be the function on $\mathbb{R}$ defined by $\varphi_n(x) = \varphi(x/n)$. By the preceding calculation we have

$$\Re\left\langle Q\ \varphi_n f, \Lambda_\mu^{-\alpha} f\right\rangle_\mu = -\frac{\left\|\sqrt{\varphi_n} f\right\|_{2,\mu}^2}{2} - (\mu + 1/2)\left(\|\sqrt{\varphi_n} f_e\|_{2,\mu}^2 - \|\sqrt{\varphi_n} f_o\|_{2,\mu}^2\right) - \frac{1}{2n}\left\langle |y|\varphi'(y/n) f, f\right\rangle_\mu,$$

where $f \in D(\Lambda_\mu^{-\alpha}) \cap D(Q)$. Since $xf\overline{\Lambda_\mu^{-\alpha} f}$ and $|x|.|f|^2$ are in $L_\mu^1(\mathbb{R})$, the dominated convergence theorem can be applied to show that

$$\begin{aligned}\lim_{n\to\infty} \Re\left\langle Q\ \varphi_n f, \Lambda_\mu^{-\alpha} f\right\rangle_\mu &= \Re\left\langle Qf, \Lambda_\mu^{-\alpha} f\right\rangle_\mu, \\ \lim_{n\to\infty}\frac{1}{2n}\left\langle |y|\varphi'(y/n) f, f\right\rangle_\mu &= 0.\end{aligned}$$

The functions f, f_e and f_0 are all in $L_\mu^2(\mathbb{R})$, the dominated convergence theorem can be invoked again to give

$$\lim_{n\to\infty}\frac{\left\|\sqrt{\varphi_n} f\right\|_{2,\mu}^2}{2} + (\mu + 1/2)\left(\|\sqrt{\varphi_n} f_e\|_{2,\mu}^2 - \|\sqrt{\varphi_n} f_o\|_{2,\mu}^2\right) = \frac{\|f\|_{2,\mu}^2}{2} - (\mu + 1/2)\left(\|f_e\|_{2,\mu}^2 - \|f_o\|_{2,\mu}^2\right).$$

Corollary 6.1 *Let $\alpha \in \mathbb{R}\backslash\pi\mathbb{Z}$, $\beta \in \mathbb{R}\backslash\pi\mathbb{Z}$ and $f \in D(\Lambda_\mu^{-\alpha}) \cap D(\Lambda_\mu^{-\beta}) \cap D(Q)$. Then*

$$\Im\left\langle \Lambda_\mu^{-\alpha} f, \Lambda_\mu^{-\beta} f\right\rangle_\mu = \frac{\sin(\alpha - \beta)}{\sin(\alpha)\sin(\beta)}\left(\frac{1}{2}\|f\|_{2,\mu}^2 - (\mu + 1/2)\left(\|f_e\|_{2,\mu}^2 - \|f_o\|_{2,\mu}^2\right)\right).$$

Proof.
By (6.1), we get

$$\left\langle \Lambda_\mu^{-\alpha} f, \Lambda_\mu^{-\beta} f\right\rangle_\mu = \left\|\Lambda_\mu^{-\alpha} f\right\|_{2,\mu}^2 + \frac{i\sin(\alpha - \beta)}{\sin(\alpha)\sin(\beta)}\left\langle \Lambda_\mu^{-\alpha} f, xf\right\rangle_\mu.$$

Then

$$\Im\left\langle \Lambda_\mu^{-\alpha} f, \Lambda_\mu^{-\beta} f\right\rangle_\mu \quad = \quad \frac{\sin(\alpha-\beta)}{\sin(\alpha)\sin(\beta)} \Re\left\langle xf, \Lambda_\mu^{-\alpha} f\right\rangle_\mu,$$

and the desired result is therefore a consequence of Proposition 6.4.

Definition 6.2 *Let $f \in D(Q)$ with $\|f\|_{2,\mu} = 1$. We define the μ-variance of f by*

$$var_\mu(f) = \|xf\|_{2,\mu}^2 - \langle xf, f\rangle_\mu^2 = \|(x - \langle xf, f\rangle_\mu)f\|_{2,\mu}^2. \tag{6.8}$$

Proposition 6.5 *Let $\alpha \in \mathbb{R}\backslash\pi\mathbb{Z}$ and $f \in D(\Lambda_\mu^{-\alpha}) \cap D(Q)$ with $\|f\|_{2,\mu} = 1$. Then*

$$var_\mu(D_\mu^\alpha(f)) = \sin^2(\alpha) \left\|(\Lambda_\mu^{-\alpha} - \langle \Lambda_\mu^{-\alpha} f, f\rangle_\mu) f\right\|_{2,\mu}^2.$$

Proof.
By (6.8) and Theorem 5.1, we have

$$\begin{aligned}
\mathrm{var}_\mu(D_\mu^\alpha(f)) &= \left\|xD_\mu^\alpha(f)\right\|_{2,\mu}^2 - \left\langle xD_\mu^\alpha(f), D_\mu^\alpha(f)\right\rangle_\mu^2 \\
&= \left\|i\sin(\alpha) D_\mu^\alpha(\Lambda_\mu^{-\alpha} f)\right\|_{2,\mu}^2 - \left\langle i\sin(\alpha) D_\mu^\alpha(\Lambda_\mu^{-\alpha} f), D_\mu^\alpha(f)\right\rangle_\mu^2 \\
&= \sin^2(\alpha)\left(\left\|D_\mu^\alpha(\Lambda_\mu^{-\alpha} f)\right\|_{2,\mu}^2 + \left\langle D_\mu^\alpha(\Lambda_\mu^{-\alpha} f), D_\mu^\alpha(f)\right\rangle_\mu^2\right) \\
&= \sin^2(\alpha)\left(\left\|\Lambda_\mu^{-\alpha} f\right\|_{2,\mu}^2 + \left\langle \Lambda_\mu^{-\alpha} f, f\right\rangle_\mu^2\right) \\
&= \sin^2(\alpha)\left\|(\Lambda_\mu^{-\alpha} - \langle \Lambda_\mu^{-\alpha} f, f\rangle_\mu) f\right\|_{2,\mu}^2.
\end{aligned}$$

Theorem 6.1 *Let $\alpha \in \mathbb{R}\backslash\pi\mathbb{Z}$, $\beta \in \mathbb{R}$ and $f \in H_2^{\mu,\alpha}(\mathbb{R}) \cap H_2^{\mu,\beta}(\mathbb{R}) \cap D(Q)$ with $\|f\|_2 = 1$. Then*

$$var_\mu(D_\mu^\alpha(f))\ var_\mu(D_\mu^\beta(f)) \geq \sin^2(\alpha-\beta)\left(\left(\mu+\frac{1}{2}\right)\left(\|f_e\|_{2,\mu}^2 - \|f_o\|_{2,\mu}^2\right) + \frac{1}{2}\right)^2. \tag{6.9}$$

Moreover, equality holds if and only if $f(x) = \lambda\ e^{m\frac{x^2}{2}} E_\mu(ax)$ for some a, m and $\lambda \in \mathbb{C}$.

Proof.
• **Case** $\alpha, \beta \in \mathbb{R}\backslash\pi\mathbb{Z}$.
By Proposition 6.5 and the Schwarz inequality, it follows that

$$\begin{aligned}
\frac{\mathrm{var}_\mu(D_\mu^\alpha(f))\ \mathrm{var}_\mu(D_\mu^\beta(f))}{\sin^2(\alpha)\sin^2(\beta)} &= \left\|(\Lambda_\mu^{-\alpha} - \langle \Lambda_\mu^{-\alpha} f, f\rangle_\mu) f\right\|_{2,\mu}^2 \left\|(\Lambda_\mu^{-\beta} - \langle \Lambda_\mu^{-\beta} f, f\rangle_\mu) f\right\|_{2,\mu}^2 \\
&\geq \left|\left\langle(\Lambda_\mu^{-\alpha} - \langle \Lambda_\mu^{-\alpha} f, f\rangle_\mu) f, (\Lambda_\mu^{-\beta} - \langle \Lambda_\mu^{-\beta} f, f\rangle_\mu) f\right\rangle\right|^2 \\
&\geq \left|\Im\left(\langle \Lambda_\mu^{-\alpha} f, \Lambda_\mu^{-\beta} f\rangle - \langle \Lambda_\mu^{-\alpha} f, f\rangle \overline{\langle \Lambda_\mu^{-\beta} f, f\rangle}\right)\right|^2.
\end{aligned} \tag{6.10}$$

Since the operators $\Lambda_\mu^{-\alpha}$ and $\Lambda_\mu^{-\beta}$ are skew symmetric, we conclude that $\langle \Lambda_\mu^{-\alpha} f, f\rangle \overline{\langle \Lambda_\mu^{-\beta} f, f\rangle}$ is real and therefore the inequality (6.10) becomes

$$\begin{aligned}
\frac{\mathrm{var}_\mu(D_\mu^\alpha(f))\ \mathrm{var}_\mu(D_\mu^\beta(f))}{\sin^2(\alpha)\sin^2(\beta)} &\geq \left|\Im\langle \Lambda_\mu^{-\alpha} f, \Lambda_\mu^{-\beta} f\rangle\right|^2 \\
&= \frac{\sin^2(\alpha-\beta)}{\sin^2(\alpha)\sin^2(\beta)}\left(\frac{1}{2}\|f\|_{2,\mu}^2 - (\mu+1/2)\left(\|f_e\|_{2,\mu}^2 - \|f_o\|_{2,\mu}^2\right)\right)^2,
\end{aligned}$$

where the last inequality follows from Corollary 6.1.
To show when (6.9) holds with equality, we use the Cauchy-Schwartz inequality in (6.10) and taking the imaginary part. Hence, (6.9) holds with equality if and only if

$$ic\left(\Lambda_\mu^{-\alpha} f - \langle \Lambda_\mu^{-\alpha} f, f\rangle_\mu f\right) = \left(\Lambda_\mu^{-\beta} f - \langle \Lambda_\mu^{-\beta} f, f\rangle_\mu f\right) \quad a.\ e, \tag{6.11}$$

for some $c \in \mathbb{R}$. By (6.1), (6.11) becomes

$$(ic-1)\Lambda_\mu^{-\beta} f = \left[(ic-1)\langle \Lambda_\mu^{-\beta} f, f\rangle - c\,\frac{\sin(\alpha-\beta)}{\sin(\alpha)\sin(\beta)}\langle xf, f\rangle\right] f + c\,\frac{\sin(\alpha-\beta)}{\sin(\alpha)\sin(\beta)}\,xf.$$

Thus we are lead to solve the equation

$$\Lambda_\mu^{-\beta} f = (a-bx)f, \tag{6.12}$$

where

$$a = \langle \Lambda_\mu^{-\beta} f, f\rangle_\mu + \frac{c}{1-ic}\,\frac{\sin(\alpha-\beta)}{\sin(\alpha)\sin(\beta)}\langle xf, f\rangle_\mu \quad \text{and} \quad b = \frac{c}{1-ic}\,\frac{\sin(\alpha-\beta)}{\sin(\alpha)\sin(\beta)}.$$

First we shall show that for each solution $f \in H_2^{\mu,\beta}(\mathbb{R})$ of (6.12), there exists a function $\psi \in \mathcal{C}^\infty(\mathbb{R}\backslash\{0\})$ such that:

$$f = \psi \quad a.\ e \quad \text{and} \quad \Lambda_\mu^{-\beta}\psi = \Lambda_\mu \psi - i\cot(\beta)x\psi \quad \text{on } \mathbb{R}\backslash\{0\}. \tag{6.13}$$

Let $f \in H_2^{\mu,\beta}(\mathbb{R})$ and $(f_n) \subset \mathcal{S}(\mathbb{R})$ such that $f_n \to f$ in $H_2^{\mu,\beta}(\mathbb{R})$. Let $0 < \epsilon < |x|$. Clearly

$$\begin{aligned}
\int_\epsilon^{|x|} \Lambda_\mu^{-\beta} f(y)|y|^{2\mu+1}\,dy &= \lim_{n\to\infty}\int_\epsilon^{|x|} \Lambda_\mu^{-\beta} f_n(y)|y|^{2\mu+1}\,dy\\
&= \lim_{n\to\infty}\left(\int_\epsilon^{|x|}\Lambda_\mu f_n(y)|y|^{2\mu+1}\,dy - i\cot(\beta)\int_\epsilon^{|x|} yf_n(y)|y|^{2\mu+1}\,dy\right)\\
&= \lim_{n\to\infty}\left(|x|^{2\mu+1}f_n(|x|) - \epsilon^{2\mu+1}f_n(\epsilon)\right) - (\mu+1/2)\int_\epsilon^{|x|}(f(y)+f(-y))|y|^{2\mu}\,dy\\
&- \ i\cot(\beta)\int_\epsilon^{|x|} yf(y)|y|^{2\mu+1}\,dy.
\end{aligned} \tag{6.14}$$

For $x \in \mathbb{R}\backslash\{0\}$, define

$$\psi(x) = A_{-\beta}\int_{-\infty}^{+\infty} D_\mu^\beta f(y)K_{-\beta}(x,y)\ |y|^{2\mu+1}\,dy.$$

The fact that ψ is well defined on $\mathbb{R}\backslash\{0\}$, belongs to $\mathcal{C}(\mathbb{R}\backslash\{0\})$, the sequence (f_n) converges locally uniformly on $\mathbb{R}\backslash\{0\}$ to ψ and $f = \psi$ $a.\ e$ can be shown by the same argument that was used in the proof of Proposition 6.3. Using these facts, we can write (6.14) in the form

$$\begin{aligned}
\int_\epsilon^{|x|}\Lambda_\mu^{-\beta}\psi(y)|y|^{2\mu+1}\,dy &= |x|^{2\mu+1}\psi(|x|) - \epsilon^{2\mu+1}\psi(\epsilon) - (\mu+1/2)\int_\epsilon^{|x|}(\psi(y)+\psi(-y))|y|^{2\mu}\,dy\\
&- \ i\cot(\beta)\int_\epsilon^{|x|} y\psi(y)|y|^{2\mu+1}\,dy,
\end{aligned}$$

which implies (6.13) and according to (6.12), the function ψ is a solution of the differential-difference equation

$$\Lambda_\mu\psi = (a+mx)\psi,$$

where $m = i\cot(\beta) - b$. Let $F(x) = e^{-m\frac{x^2}{2}}\psi(x)$. An application of the product rule of the Dunkl operators Λ_μ shows that F is a solution of the following differential-difference equation

$$\Lambda_\mu F = aF. \tag{6.15}$$

From the decomposition in the form $F = F_e + F_o$ where F_e is even and F_o is odd, equation (6.15) is equivalent to the following system:

$$F_o^{'}(x) + (2\mu+1)\frac{F_0(x)}{x} = aF_e(x); \quad F_e^{'}(x) = aF_o(x). \tag{6.16}$$

Clearly $a \neq 0$, because for $a = 0$, F_e is of the form $\lambda\,|x|^{-(2\mu+1)}$ which contradicts Proposition 6.3, and therefore, $F_e^{'}$ is a solution of the modified Bessel equation

$$y^{''} + \frac{2\mu+1}{x}y^{'} - a^2y = 0 \quad \text{on} \quad \mathbb{R}\backslash\{0\}. \tag{6.17}$$

We recall from [17] that the general solution of the modified Bessel's equation (6.17) of order μ, is

$$y(x)=\lambda_1\ j_\mu(iax)+\lambda_2\ |x|^{-2\mu}j_{-\mu}(iax)\ ;\quad \mu\neq 0,\pm1,\pm2,\dots \tag{6.18}$$

When μ is an integer, a general solution is also given by

$$y(x)=\lambda_1\ j_\mu(iax)+\lambda_2\ |x|^{-\mu}Y_\mu(ia|x|)\ ;\quad \mu=0,\pm1,\pm2,\dots \tag{6.19}$$

where Y_μ is the Bessel function of the second kind.
An examination of the asymptotic behavior of ψ as $x\to 0$ (see Proposition 6.3) together with (6.18) and (6.19) show that

$$\psi_e(x)=\lambda_1\ e^{m\frac{x^2}{2}}\ j_\mu(iax). \tag{6.20}$$

Again, by (6.15) we have

$$\psi_o(x)=\frac{\lambda_1}{a}\ e^{m\frac{x^2}{2}}\frac{d}{dx}\left(j_\mu(iax)\right)=\lambda_1\ e^{m\frac{x^2}{2}}\frac{ax}{2(\mu+1)}j_{\mu+1}(iax) \tag{6.21}$$

Combining (6.20) and (6.21) gives

$$\psi(x)=\lambda_1\ e^{m\frac{x^2}{2}}E_\mu(ax). \tag{6.22}$$

• **Case** $\alpha\in\mathbb{R}\backslash\pi\mathbb{Z}$ and $\beta\in\pi\mathbb{Z}$.
Let $\beta\in\pi\mathbb{Z}$ and $f\in H_2^{\mu,\alpha}(\mathbb{R})\cap D(Q)$. It is easy to see that

$$\mathrm{var}_\mu(D_\mu^\beta(f))=\mathrm{var}_\mu(f).$$

In view of the Cauchy-Schwartz inequality and Proposition 6.4, we have

$$\begin{aligned}\frac{\mathrm{var}_\mu(D_\mu^\alpha(f))\ \mathrm{var}_\mu(D_\mu^\beta(f))}{\sin^2(\alpha)} &= \|(x-\langle xf,f\rangle_\mu)f\|_{2,\mu}^2\left\|(\Lambda_\mu^{-\alpha}-\langle\Lambda_\mu^{-\alpha}f,f\rangle_\mu)f\right\|_{2,\mu}^2\\ &\geq \left|\left\langle(x-\langle xf,f\rangle_\mu)f,(\Lambda_\mu^{-\alpha}-\langle\Lambda_\mu^{-\alpha}f,f\rangle_\mu)f\right\rangle\right|^2\\ &\geq \left|\Re\left(\langle xf,\Lambda_\mu^{-\alpha}f\rangle\right)\right|^2=\left((\mu+1/2)\left(\|f_e\|_{2,\mu}^2-\|f_o\|_{2,\mu}^2\right)+1/2\right)^2.\end{aligned} \tag{6.23}$$

In a similar fashion, as in the first case, it may be shown that (6.23) holds with equality if and only if

$$f(x)=\lambda e^{m\frac{x^2}{2}}E_\mu(ax)$$

where λ,a,m are an appropriate constant in $\mathbb{R}$.

Corollary 6.2 *Let $\alpha\in\mathbb{R}\backslash\pi\mathbb{Z}$, $\beta\in\mathbb{R}$ and $f\in H_2^{\mu,\alpha}(\mathbb{R})\cap H_2^{\mu,\beta}(\mathbb{R})\cap D(Q)$ with $f(x)=f(-x)$. Then*

$$var_\mu(\mathcal{H}_\mu^\alpha(f))\ var_\mu(\mathcal{H}_\mu^\beta(f))\geq\sin^2(\alpha-\beta)\ (\mu+1)^2\|f\|_{2,\mu}^4.$$

Moreover, equality holds if and only if $f(x)=\lambda\ e^{m\frac{x^2}{2}}j_\mu(ax)$ for some a,m and $\lambda\in\mathbb{C}$.

Proof. is a direct consequence of the previous Theorem.

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