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Crank-Nicolson and Legendre spectral collocation methods for a partial integro-differential equation with a singular kernel

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Abstract

In this paper we present an efficient numerical method for the solution of a partial integro-differential equation with a singular kernel. In the time direction, a Crank-Nicolson finite difference scheme is used to approximate the differential term and the product trapezoidal method is employed to treat the integral term. Also for space discretization we apply Legendre spectral collocation method. We discuss the stability and convergence of proposed method and show that the method is unconditionally stable and convergent with order $\mathcal{O}(\tau^{\frac{3}{2}} + N^{-s})$ where τ, N and s are time step size, number of collocation points and regularity of exact solution respectively. We compare the numerical results of proposed method with the results of other schemes in the literature in terms of accuracy, computational order and CPU time to show the efficiency and applicability of it.

Keywords: Partial integro-differential equation, Legendre spectral collocation, Stability, Convergence, Singular kernel.

1 Introduction

Integral equation has been one of the essential tools for various areas of physics and applied mathematics. Many mathematical formulations of physical phenomena contain integro-differential equations.

In this paper we study the numerical solution of following partial integro-differential equation with a weakly singular kernel

$$u_t(x,t) = \gamma u_{xx}(x,t) + \int_0^t (t-s)^{-1/2} u_{xx}(x,s)\,ds, \qquad (x,t) \in \Omega, \tag{1.1}$$

where $\Omega = \{(x,t)| -1 \leq x \leq 1,\ 0 < t \leq T\}$, $\gamma \geq 0$, boundary conditions are

$$u(-1,t) = u(1,t) = 0,$$

and initial condition is

$$u(x,0) = g(x).$$

This type of equations have widely occurred in the mathematical modeling of various physical phenomena, such as heat conduction in materials with memory, combined conduction, convection and radiation problems [11, 16], phenomena associated with linear viscoelastic mechanics

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[4, 17]. The integral term in (1.1) represents the viscosity part of the equation and γ is a Newtonian contribution to the viscosity.
Analytic solutions of most partial differential and integro-differential equations can not be obtained explicitly. So many authors have resorted to numerical solution strategies based on convergence and stability analysis [6, 7, 9]. In the literature, the numerical solution of ordinary integro-differential equations is considered by many authors, for example see [3, 12, 18]. Partial integro-differential equations also have been studied in some papers. For Eq. (1.1), a quasi wavelet based numerical method is given in [14], Cubic B-splines collocation method in [10], the finite difference procedures based on the forward Euler explicit scheme, the backward Euler implicit technique, the Crank-Nicolson implicit formula and Crandall's implicit method are presented in [5], space-time spectral method in [8], a finite difference scheme of order $\mathcal{O}(\tau^{\frac{3}{2}}+h^2)$ in [20] and a compact difference scheme of order $\mathcal{O}(\tau^{\frac{3}{2}}+h^4)$ in [15]. Authors of [13] proposed a numerical method for the fourth-order integro-differential equations using Chebyshev cardinal functions.
The aim of this paper is to introduce an efficient numerical method for the solution of partial integro-differential equation with a singular kernel (1.1). In the time direction, a Crank-Nicolson finite difference scheme is used to approximate the differential term and the product trapezoidal method is employed to treat the integral term. Also for space discretization we apply Legendre spectral collocation method. We prove that the proposed method is unconditionally stable and convergent using energy method. The convergence order of method is $\mathcal{O}(\tau^{\frac{3}{2}}+N^{-s})$. We compare the numerical results of proposed method with the results of other schemes in the literature in terms of accuracy, computational order and CPU time and show that the new method is more efficient.
This article is outlined as follows. In Sections 2, 3, we first give some preliminary and then propose an implicit spectral method for the equation (1.1). In Section 4, the stability analysis of method is studied in detail and the convergence analysis is given in Section 5. In Section 6, the results of numerical experiments are reported to confirm the good accuracy and efficiency of the proposed scheme. Finally we make some concluding remarks in Section 7.

2 The Legendre spectral scheme

Let $\Omega=(-1,1)$. For positive integer number M, let $\tau=\frac{T}{M}$ be the step size of time variable, t, so we have

$$t_n=n\tau,\quad n=0,1,...,M.$$

Also for any integer N let $\mathbb{P}_N$ be the space of algebraic polynomial of degree at most N, $\mathbb{P}_0^N=\{p\in\mathbb{P}_N \mid p(-1)=p(1)=0\}$ and $L_n(x)$ be the Legendre polynomial of degree n which is defined by

$$P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}\left[\left(x^2-1\right)^n\right],\quad n\geq 0.$$

If the nodes and weights of the Gauss-Lobatto integration formula related to Legendre weight are denoted by $\{x_i, w_i\}_{i=0}^N$, then $\{x_i\}_{i=0}^N$ are the zeros of $\left(1-x^2\right)L'_N(x)$ and [19]

$$w_i=\frac{2}{N(N+1)}\frac{1}{\left[L_N(x_i)\right]^2},\quad 0\leq i\leq N.$$

With the above quadrature nodes and weights we have

$$\int_{-1}^{1} p(x)dx = \sum_{i=0}^{N} p(x_i)w_i, \quad \forall\ p(x) \in \mathbb{P}_{2N-1}.$$

Let $L^2(\Omega)$ be the space of measurable functions for which $\|u\|_0 = \left(\int_{-1}^{1} |u^2(x)|\,dx\right)^{1/2}$ is finite. The inner product in this space is defined as $\langle \varphi, \psi\rangle = \int_{-1}^{1} \varphi(x)\psi(x)\,dx$. Also for any integer $s \geq 0$ we denote the Sobolev space $H^s(\Omega)$ as the space of those functions of $L^2(\Omega)$ for which the norm $\|u\|_s = \left(\sum_{k=0}^{s} \|u^{(k)}\|_0^2\right)^{1/2}$ is finite. Finally we define the space $H_0^1(\Omega) = \{u \in H^1(\Omega) \mid u(-1) = u(1) = 0\}$ and set $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$. For simplicity we put $\|u\| = \|u\|_0$.

The following lemmas are needed in analysis of proposed method.

Lemma 2.1. *[2] For any continuous function v, let $\mathcal{I}_N v \in \mathbb{P}_N$ denotes its interpolant at the points $x_i,\ i = 0, 1, ..., N$, then*

$$\|v - \mathcal{I}_N v\|_{\mu} \leq C\|v\|_{\sigma_1} N^{2\mu-\sigma_1}, \quad \sigma_1 > \frac{1}{2},\ \ 0 \leq \mu \leq \sigma_1.$$

Lemma 2.2. *(Gronwall's inequality [21]) Suppose that the discrete function*

$$\{w^n \mid n = 0, 1, ..., N,\ N\tau = T\},$$

satisfies the inequality

$$w^n \leq A + \tau \sum_{l=1}^{n} B_l w^l,$$

where A and $B_l,\ l = 0, 1, 2, ..., N$ are nonnegative constants. Then

$$\max_{1\leq n\leq N} |w^n| \leq A \exp\left(2\tau \sum_{l=1}^{n} B_l\right),$$

where τ is sufficiently small such that $\tau. \max_{1\leq l\leq N} B_l \leq \frac{1}{2}$.

Lemma 2.3. *[20] Let $\mathcal{I}(f, t) = \int_0^t (t - s)^{-1/2} f(s)\,ds$, then we can write*

$$\mathcal{I}(f, t_{n+1/2}) = \frac{1}{2}\left[\mathcal{I}(f, t_n) + \mathcal{I}(f, t_{n+1})\right] + O\left(\tau^2 t_{n+1}^{-3/2}\right), \quad n \geq 0.$$

We use the following product trapezoidal method to approximate $\mathcal{I}(f, t)$ [15, 20]

$$\mathcal{I}(f, t_n) = A_n f(t_0) + \sum_{p=0}^{n} \beta_p f(t_{n-p}) + O(\tau^{3/2}), \quad 1 \leq n \leq N, \tag{2.1}$$

in which

$$\begin{aligned}
A_n &= 2\left[t_n^{1/2} - \frac{1}{\tau}\int_{t_n}^{t_{n+1}} \theta^{1/2} d\theta\right], \quad \beta_0 = \frac{2}{\tau}\int_0^{t_1} \theta^{1/2} d\theta + \frac{4\sqrt{\tau}}{3}\beta \\
\beta_1 &= \frac{2}{\tau}\left[\int_{t_1}^{t_2} \theta^{1/2} d\theta - \int_{t_0}^{t_1} \theta^{1/2} d\theta\right] - \frac{4\sqrt{\tau}}{3}\beta, \\
\beta_p &= \frac{2}{\tau}\left[\int_{t_p}^{t_{p+1}} \theta^{1/2} d\theta - \int_{t_{p-1}}^{t_p} \theta^{1/2} d\theta\right], \quad p \geq 2,
\end{aligned} \tag{2.2}$$

where β is a nonnegative constant and is dependent of τ and h.

Lemma 2.4. *[20] Let M be a positive integer and $\{a_n\}_{n=0}^{\infty}$ be a sequence of real numbers with the following properties*

$$a_n \geqslant 0, \quad a_{n+1} - a_n \leqslant 0, \quad a_{n+1} - 2a_n + a_{n-1} \geqslant 0. \tag{2.3}$$

Then for each vector $(V_1, V_2, ..., V_M)$ with M real entries

$$\sum_{n=0}^{M-1}\left(\sum_{p=0}^{n} a_p V_{n+1-p}\right) V_{n+1} \geqslant 0. \tag{2.4}$$

Lemma 2.5. *[20] Let β satisfies*

$$\frac{-3\sqrt{3}+8\sqrt{2}-6}{3} \leqslant \beta \leqslant 4 - 12\sqrt{3} + 12\sqrt{2}, \tag{2.5}$$

then the sequence $\{\beta_p\}_{p=0}^{\infty}$ defined by (2.2) satisfies (2.3).

3 Proposed method

We denote $u_j^n = u(x_j, t_n)$, $\bar{u}_j^n = \frac{1}{2}\left(u_j^n + u_j^{n+1}\right)$. Note that we can write

$$u_t(x_j, t_{n+1/2}) = \frac{1}{\tau}\left(u_j^{n+1} - u_j^n\right) + \mathcal{O}\left(\tau^2\right), \tag{3.1}$$

and

$$u(x_j, t_{n+1/2}) = \bar{u}_j^n + \mathcal{O}\left(\tau^2\right). \tag{3.2}$$

Considering Eq. (1.1) at point $(x_j, t_{n+1/2})$ results

$$u_t(x_j, t_{n+1/2}) = \gamma u_{xx}(x_j, t_{n+1/2}) + \mathcal{I}(u_{xx}, t_{n+1/2}). \tag{3.3}$$

Using (2.1), (3.1), (3.3) and Lemma 2.3 we obtain

$$\frac{1}{\tau}\left(u_j^{n+1} - u_j^n\right) = \gamma\left(\bar{u}_{xx}\right)_j^n +$$

$$\frac{1}{2}\left(A_n\left(u_{xx}\right)_j^0+\sum_{p=0}^{n}\beta_p\left(u_{xx}\right)_j^{n-p}+A_{n+1}\left(u_{xx}\right)_j^0+\sum_{p=0}^{n+1}\beta_p\left(u_{xx}\right)_j^{n-p+1}\right)+r_j^n, \tag{3.4}$$

where

$$\left|r_j^n\right|\le C\left(\tau^{\frac{3}{2}}+\tau^2+\tau^2 t_n^{-3/2}\right). \tag{3.5}$$

Now we propose the implicit spectral method for the solution of Eq. (1.1) as follows. At any time-level we look for a function $U^{n+1}\in\mathbb{P}_0^N$ such that

$$\begin{aligned}\frac{1}{\tau}\left(U_j^{n+1}-U_j^n\right)&=\frac{\gamma}{2}\left\{(U_{xx})_j^n+(U_{xx})_j^{n+1}\right\}+\tilde{A}_n(U_{xx})_j^0+\\&\frac{1}{2}\sum_{p=0}^{n}\beta_p\left\{(U_{xx})_j^{n-p}+(U_{xx})_j^{n-p+1}\right\},\\&j=2,3,...,N-1,\qquad n=0,1,2,...,M-1,\end{aligned} \tag{3.6}$$

where $\tilde{A}_n=\frac{A_n+A_{n+1}+\beta_{n+1}}{2}$ and $U^0=\mathcal{I}_N u_0$ in which $\mathcal{I}_N$ is the interpolation operator at the Legendre Gauss-Lobatto points. For analysis of method (3.6), it is convenient to state it in variational form. To this end, we first introduce the discrete inner product

$$\langle\varphi,\psi\rangle_N=\sum_{i=0}^{N}\varphi(x_i)\psi(x_i)w_i, \tag{3.7}$$

which satisfies

$$\langle\varphi,\psi\rangle_N=\langle\varphi,\psi\rangle\quad\text{for all }\varphi,\psi\text{ in which }\varphi.\psi\in\mathbb{P}_{2N-1}. \tag{3.8}$$

It is shown in [2] that the discrete norm $\|v\|_N=\{\langle v,v\rangle_N\}^{\frac{1}{2}}$ is equivalent to norm $\|v\|$, i.e.

$$\|v\|\le\|v\|_N\le\sqrt{2}\|v\|,\quad\text{for all }\ v\in\mathbb{P}_N. \tag{3.9}$$

Now the proposed method (3.6) can be restated as follows:
Find $U^{n+1}\in\mathbb{P}_0^N$ for $n=0,1,2,...,M-1$ such that

$$\begin{aligned}\frac{1}{\tau}\langle U^{n+1}-U^n,v\rangle_N&=\frac{\gamma}{2}\Big\langle(U_{xx})^n+(U_{xx})^{n+1},v\Big\rangle_N+\tilde{A}_n\Big\langle(U_{xx})^0,v\Big\rangle_N+\\&\frac{1}{2}\sum_{p=0}^{n}\beta_p\Big\langle(U_{xx})^{n-p}+(U_{xx})^{n-p+1},v\Big\rangle_N,\ \text{for }v\in\mathbb{P}_0^N.\end{aligned} \tag{3.10}$$

4 Stability of proposed method

In this section we show that the proposed scheme is unconditionally stable.

Theorem 4.1. *Let $u^0\in H^2(\Omega)$, then the implicit spectral scheme (3.10) is unconditionally stable and we have*

$$\left\|U^{k+1}\right\|_N^2\le C^*\left(\left\|U^0\right\|_N^2+\left\|(U_{xx})^0\right\|_N^2\right),\quad k=0,1,...,M-1.$$

Proof. Taking $v = 2\bar{U}^n$ in (3.10) and summing up n from 0 to m we obtain

$$\frac{1}{\tau}\sum_{n=0}^{m}\left\langle U^{n+1}-U^{n},2\bar{U}^{n}\right\rangle_N = \frac{\gamma}{2}\sum_{n=0}^{m}\Big\langle (U_{xx})^{n}+(U_{xx})^{n+1},2\bar{U}^{n}\Big\rangle_N + \sum_{n=0}^{m}\tilde{A}_n\Big\langle (U_{xx})^{0},2\bar{U}^{n}\Big\rangle_N +$$

$$\frac{1}{2}\sum_{n=0}^{m}\sum_{p=0}^{n}\beta_p\Big\langle (U_{xx})^{n-p}+(U_{xx})^{n-p+1},2\bar{U}^{n}\Big\rangle_N. \quad (4.1)$$

For the left hand side of (4.1) we have

$$\frac{1}{\tau}\sum_{n=0}^{m}\left\langle U^{n+1}-U^{n},2\bar{U}^{n}\right\rangle_N = \frac{1}{\tau}\sum_{n-0}^{m}\left(\left\|U^{n+1}\right\|_N^2-\left\|U^{n}\right\|_N^2\right) = \frac{1}{\tau}\left(\left\|U^{m+1}\right\|_N^2-\left\|U^{0}\right\|_N^2\right). \quad (4.2)$$

For the first term in the right-hand side, since $(U_{xx})^n.\,\bar{U}^n \in \mathbb{P}_0^{N-1}$, using (3.8) we have

$$\frac{\gamma}{2}\sum_{n=0}^{m}\Big\langle (U_{xx})^{n}+(U_{xx})^{n+1},2\bar{U}^{n}\Big\rangle_N = \frac{\gamma}{2}\sum_{n=0}^{m}\int_I 2\left\{(U_{xx})^{n}+(U_{xx})^{n+1}\right\}\bar{U}^{n} =$$

$$-\frac{\gamma}{2}\sum_{n=0}^{m}\int_I 2\left\{(U_{x})^{n}+(U_{x})^{n+1}\right\}\bar{U}_x^{n} = -2\gamma\sum_{n=0}^{m}\left\langle \bar{U}_x^n,\bar{U}_x^n\right\rangle_N = -2\gamma\sum_{n=0}^{m}\left\|(\bar{U}_x)^n\right\|_N^2 \le 0 \quad (4.3)$$

For the second term in the right-hand side we can write

$$\sum_{n=0}^{m}\tilde{A}_n\Big\langle (U_{xx})^{0},2\bar{U}^{n}\Big\rangle_N \le \frac{1}{2}\left(\sum_{n=0}^{m}\tilde{A}_n\right)\left\|(U_{xx})^0\right\|_N^2 + \frac{1}{2}\sum_{n=0}^{m}\left\|2\bar{U}^n\right\|_N^2. \quad (4.4)$$

It is shown in [15] that

$$\sum_{n=0}^{m}\tilde{A}_n \le C\tau^{\frac{1}{2}},$$

so

$$\sum_{n=0}^{m}\tilde{A}_n\Big\langle (U_{xx})^{0},2\bar{U}^{n}\Big\rangle_N \le C\tau^{\frac{1}{2}}\left\|(U_{xx})^0\right\|_N^2 + \frac{1}{2}\sum_{n=0}^{m}\left\|2\bar{U}^n\right\|_N^2. \quad (4.5)$$

Also similar to (4.3) we can write

$$\frac{1}{2}\sum_{n=0}^{m}\sum_{p=0}^{n}\beta_p\Big\langle (U_{xx})^{n-p}+(U_{xx})^{n-p+1},2\bar{U}^{n}\Big\rangle_N = \sum_{n=0}^{m}\sum_{p=0}^{n}\beta_p\Big\langle (\bar{U}_{xx})^{n-p},2\bar{U}^{n}\Big\rangle_N =$$

$$-\sum_{n=0}^{m}\sum_{p=0}^{n}\beta_p\Big\langle (\bar{U}_{x})^{n-p},2(\bar{U}_x)^{n}\Big\rangle_N.$$

So from Lemma 2.4 we can write

$$\frac{1}{2}\sum_{n=0}^{m}\sum_{p=0}^{n}\beta_p\Big\langle (U_{xx})^{n-p}+(U_{xx})^{n-p+1},2\bar{U}^{n}\Big\rangle_N \le 0. \quad (4.6)$$

Using (4.1)-(4.6) we obtain

$$\frac{1}{\tau}\left(\left\|U^{m}\right\|_N^2-\left\|U^{0}\right\|_N^2\right) \le C\tau^{\frac{1}{2}}\left\|(U_{xx})^0\right\|_N^2 + \sum_{n=0}^{m}\left\|2\bar{U}^n\right\|_N^2. \quad (4.7)$$

Regarding to

$$\left\|2\bar{U}^n\right\|_N^2 = \left\|U^{n+1}+U^n\right\|_N^2 \le 2\left(\left\|U^{n+1}\right\|_N^2 + \left\|U^n\right\|_N^2\right),$$

we can write (4.7) as follows

$$\left\|U^{m+1}\right\|_N^2 \le \left\|U^0\right\|_N^2 + C\tau^{\frac{3}{2}}\left\|(U_{xx})^0\right\|_N^2 + 4\tau\sum_{n=0}^{m+1}\left\|U^n\right\|_N^2.$$

Now using Gronwall Lemma 2.2, for sufficiently small values of τ, we obtain

$$\begin{aligned}\left\|U^{m+1}\right\|_N^2 &\le \left(\left\|U^0\right\|_N^2 + C\tau^{\frac{3}{2}}\left\|(U_{xx})^0\right\|_N^2\right)e^{8\tau(m+1)}\\ &\le C^*\left(\left\|U^0\right\|_N^2 + \left\|(U_{xx})^0\right\|_N^2\right),\end{aligned}$$

where

$$C^* = e^{8T}\max\left\{1, C\tau^{\frac{3}{2}}\right\}.$$

□

5 Convergence of proposed method

We first introduce the following operators

$$\mathbf{P}_N : H \to \mathbb{P}_N,\ \langle v - \mathbf{P}_N v, \varphi\rangle = 0,\ \text{ for all } \varphi \in \mathbb{P}_N, \tag{5.1}$$

$$\Pi_N : V \to \mathbb{P}_N^0,\ \ \langle(\Pi_N v)_{xx}, \varphi\rangle = \langle v_{xx}, \varphi\rangle,\ \text{for all } \varphi \in \mathbb{P}_N^0. \tag{5.2}$$

Lemma 5.1. *[2] The following estimates hold*

$$\|v - \mathbf{P}_N v\|_\mu \le C\|v\|_\sigma N^{3\mu/2-\sigma},\quad \sigma \ge 0,\ \ 0 \le \mu \le 1,$$

$$\|v - \Pi_N v\|_\mu \le C\|v\|_\sigma N^{\mu-\sigma},\quad \sigma \ge 1,\ \ 0 \le \mu \le 1.$$

For any function $v \in V$ we will use the following notation

$$\langle E(v), \varphi\rangle = \langle v, \varphi\rangle_N - \langle v, \varphi\rangle,\ \text{ for all } \varphi \in \mathbb{P}_N, \tag{5.3}$$

then it can be shown that [1]

$$|\langle E(v), \varphi\rangle| \leqslant C\left\{\|v - \mathbf{P}_{N-1}v\| + \|v - \mathcal{I}_N v\|\right\}\|\varphi\|,\ \text{ for all } \varphi \in \mathbb{P}_N. \tag{5.4}$$

We denote

$$\mathcal{I}^n(f) = \frac{1}{2}\sum_{p=0}^{n}\beta_p\left[f(t_{n-p}) + f(t_{n-p+1})\right].$$

Let u be the exact solution, we put

$$e^n = u^n - U^n,\quad \tilde{e}^n = \Pi_N u^n - U^n.$$

Theorem 5.2. *If u be the exact solution of Eq. (1.1), $u^0 \in H^{s+2}(\Omega)$, $u_t \in C(0,T;H^s(\Omega))$, $u \in C(0,T;H^{s+2}(\Omega))$, then we have*

$$\|e^n\| \leqslant C\left(N^{-s} + \tau^{\frac{3}{2}}\right). \tag{5.5}$$

Proof. Let $\tilde{u} = \Pi_N u$. Using (2.1), (3.1)- (3.2), (3.4)- (3.5), (5.2) and Lemma 5.3, $\tilde{u}$ satisfies

$$\begin{aligned}
\frac{1}{\tau}\langle \tilde{u}^{n+1} - \tilde{u}^n, v\rangle_N &= \gamma\langle (\bar{\tilde{u}}^n)_{xx}, v\rangle_N + \tilde{A}_n\langle (\tilde{u}^0)_{xx}, v\rangle_N + \\
&\left\langle \tilde{u}_t^{n+\frac{1}{2}} - u_t^{n+\frac{1}{2}}, v\right\rangle + \left\langle E\left(\tilde{u}_t^{n+\frac{1}{2}}\right), v\right\rangle + \langle \mathcal{I}^n(\tilde{u}_{xx}^n), v\rangle_N \\
&-\left\langle \mathcal{I}\left(\tilde{u}_{xx}, t_{n+\frac{1}{2}}\right) - \mathcal{I}\left(u_{xx}, t_{n+\frac{1}{2}}\right), v\right\rangle - \left\langle E\left(\mathcal{I}\left(\tilde{u}_{xx}, t_{n+\frac{1}{2}}\right)\right), v\right\rangle \\
&+\left\langle r^{n+\frac{1}{2}}, v\right\rangle, \text{ for all } v \in \mathbb{P}_N^0,
\end{aligned} \tag{5.6}$$

in which

$$\left\|r^{n+\frac{1}{2}}\right\| \leq C\left(\tau^{\frac{3}{2}} + \tau^2 + \tau^2 t_{n+1}^{-3/2}\right).$$

Subtracting (3.10) from (5.6), gives

$$\begin{aligned}
\frac{1}{\tau}\langle \tilde{e}^{n+1} - \tilde{e}^n, v\rangle_N &= \gamma\langle (\bar{\tilde{e}}^n)_{xx}, v\rangle_N + \tilde{A}_n\langle (\tilde{e}^0)_{xx}, v\rangle_N + \\
&\left\langle \tilde{u}_t^{n+\frac{1}{2}} - u_t^{n+\frac{1}{2}}, v\right\rangle + \left\langle E\left(\tilde{u}_t^{n+\frac{1}{2}}\right), v\right\rangle + \langle \mathcal{I}^n(\tilde{e}_{xx}^n), v\rangle_N \\
-\left\langle \mathcal{I}\left(\tilde{u}_{xx}, t_{n+\frac{1}{2}}\right) - \mathcal{I}\left(u_{xx}, t_{n+\frac{1}{2}}\right), v\right\rangle &- \left\langle E\left(\mathcal{I}\left(\tilde{u}_{xx}, t_{n+\frac{1}{2}}\right)\right), v\right\rangle + \left\langle r^{n+\frac{1}{2}}, v\right\rangle.
\end{aligned} \tag{5.7}$$

Putting $v = 2\bar{\tilde{e}}^n$ and summing up n from 0 to m, (5.7) leads to

$$\begin{aligned}
\|\tilde{e}^{m+1}\|_N^2 - \|\tilde{e}^0\|_N^2 &= -2\tau\gamma\sum_{n=0}^{m}\|(\bar{\tilde{e}}^n)_x\|_N + \tau\sum_{n=0}^{m}\tilde{A}_n\langle (\tilde{e}^0)_{xx}, \bar{\tilde{e}}^n\rangle_N + \\
&\tau\sum_{n=0}^{m}\left\langle \tilde{u}_t^{n+\frac{1}{2}} - u_t^{n+\frac{1}{2}}, 2\bar{\tilde{e}}^n\right\rangle + \tau\sum_{n=0}^{m}\left\langle E\left(\tilde{u}_t^{n+\frac{1}{2}}\right), 2\bar{\tilde{e}}^n\right\rangle \\
&+\tau\sum_{n=0}^{m}\langle \mathcal{I}^n(\tilde{e}_{xx}^n), 2\bar{\tilde{e}}^n\rangle_N - \tau\sum_{n=0}^{m}\left\langle \mathcal{I}\left(\tilde{u}_{xx}, t_{n+\frac{1}{2}}\right) - \mathcal{I}\left(u_{xx}, t_{n+\frac{1}{2}}\right), 2\bar{\tilde{e}}^n\right\rangle \\
&-\tau\sum_{n=0}^{m}\left\langle E\left(\mathcal{I}\left(\tilde{u}_{xx}, t_{n+\frac{1}{2}}\right)\right), 2\bar{\tilde{e}}^n\right\rangle + \tau\sum_{n=0}^{m}\left\langle r^{n+\frac{1}{2}}, 2\bar{\tilde{e}}^n\right\rangle.
\end{aligned} \tag{5.8}$$

Now we obtain a bound for each term in (5.8). Denote $\|e^q\| = \max\limits_{0\leq n\leq N}\|e^n\|$. It is shown in [15] that

$$\tau\sum_{n=0}^{m}\tilde{A}_n \leq 2\tau^{\frac{3}{2}}T,$$

using

$$\left\|\tilde{e}_{xx}^{0}\right\| = \left\|\tilde{u}_{xx}^{0} - U_{xx}^{0}\right\| = \left\|\tilde{u}_{xx}^{0} - \mathcal{I}_N u_{xx}^{0}\right\| \leq CN^{-s}\left\|u^0\right\|_{s+2},$$

and Cauchy-Schwarz inequality, we have

$$\tau \sum_{n=0}^{m} \left\langle \tilde{A}_n \left(\tilde{e}^0\right)_{xx}, 2\bar{\tilde{e}}^n \right\rangle_N \leq \tau \sum_{n=0}^{m} \tilde{A}_n \left\|\left(\tilde{e}^0\right)_{xx}\right\|_N \left\|2\bar{\tilde{e}}^n\right\|_N$$

$$\leq 2\left\|\left(\tilde{e}^0\right)_{xx}\right\| \left\|\tilde{e}^q\right\| \tau \sum_{n=0}^{m} \tilde{A}_n \leq C_1 N^{-s} \left\|\tilde{e}^q\right\|. \tag{5.9}$$

Also using Lemma 2.4

$$\tau \sum_{n=0}^{m} \left\langle \mathcal{I}^n\left(\tilde{e}_{xx}^n\right), 2\bar{\tilde{e}}^n \right\rangle_N = \frac{\tau}{2} \sum_{n=0}^{m} \sum_{p=0}^{n} \beta_p \left\langle \left(\tilde{e}_{xx}\right)^{n-p} + \left(\tilde{e}_{xx}\right)^{n-p+1}, 2\bar{\tilde{e}}^n \right\rangle_N$$

$$= -\tau \sum_{n=0}^{m} \sum_{p=0}^{n} \beta_p \left\langle \bar{\tilde{e}}_x^{n-p}, 2\bar{\tilde{e}}_x^n \right\rangle_N \leq 0. \tag{5.10}$$

For other terms we have

$$\tau \sum_{n=0}^{m} \left\langle \tilde{u}_t^{n+\frac{1}{2}} - u_t^{n+\frac{1}{2}}, 2\bar{\tilde{e}}^n \right\rangle \leq \tau \sum_{n=0}^{m} \left\| \tilde{u}_t^{n+\frac{1}{2}} - u_t^{n+\frac{1}{2}} \right\| \left\|2\bar{\tilde{e}}^n\right\|$$

$$\leq C_2^* \left\|\tilde{e}^q\right\| N^{-s} \sum_{n=0}^{m} \tau \leq C_2 \left\|\tilde{e}^q\right\| N^{-s}, \tag{5.11}$$

$$\tau \sum_{n=0}^{m} \left\langle E\left(\tilde{u}_t^{n+\frac{1}{2}}\right), 2\bar{\tilde{e}}^n \right\rangle \leqslant$$

$$C_3^{**} \tau \sum_{n=0}^{m} \left\{ \left\| \tilde{u}_t^{n+\frac{1}{2}} - \mathbf{P}_{N-1} \tilde{u}_t^{n+\frac{1}{2}} \right\| + \left\| \tilde{u}_t^{n+\frac{1}{2}} - \mathcal{I}_N \tilde{u}_t^{n+\frac{1}{2}} \right\| \right\} \left\|\bar{\tilde{e}}^n\right\| \leqslant$$

$$C_3^* \left\|\tilde{e}^q\right\| N^{-s} \sum_{n=0}^{m} \tau \leq C_3 \left\|\tilde{e}^q\right\| N^{-s}, \tag{5.12}$$

$$\tau \sum_{n=0}^{m} \left\langle r^{n+\frac{1}{2}}, 2\bar{\tilde{e}}^n \right\rangle \leq \tau \sum_{n=0}^{m} \left\| r^{n+\frac{1}{2}} \right\| \left\|2\bar{\tilde{e}}^n\right\| \leq \left\|\tilde{e}^q\right\| \tau \sum_{n=0}^{m} \left\| r^{n+\frac{1}{2}} \right\|$$

$$\leq C_4^* \left\|\tilde{e}^q\right\| \tau \sum_{n=0}^{m} \left(\tau^{\frac{3}{2}} + \tau^2 + \tau^2 t_{n+1}^{-3/2} \right)$$

$$= C_4^* \|\tilde{e}^q\| \tau^{\frac{3}{2}} \sum_{n=0}^{m} \left(\tau + \tau^{\frac{3}{2}} + (n+1)^{-3/2}\right) \leq C_4 \|\tilde{e}^q\| \tau^{\frac{3}{2}}, \tag{5.13}$$

$$\tau \sum_{n=0}^{m} \left\langle \mathcal{I}\left(\tilde{u}_{xx} - u_{xx}, t_{n+\frac{1}{2}}\right), 2\bar{\tilde{e}}^n \right\rangle =$$

$$2\tau \sum_{n=0}^{m} \int\limits_{0}^{t_{n+\frac{1}{2}}} \left(t_{n+\frac{1}{2}} - s\right)^{-\frac{1}{2}} \int\limits_{-1}^{1} \left(\tilde{u}_{xx}(x,s) - u_{xx}(x,s)\right) \bar{\tilde{e}}(x, t_n) dx\, ds$$

$$\leq \tau \sum_{n=0}^{m} \int\limits_{0}^{t_{n+\frac{1}{2}}} \left(t_{n+\frac{1}{2}} - s\right)^{-\frac{1}{2}} \left(\|\tilde{u}_{xx}(.,s) - u_{xx}(.,s)\| \left\|2\bar{\tilde{e}}^n\right\|\right) ds$$

$$\leq C_5^* \tau \|u\|_{L^\infty(0,T;H^{s+2})} N^{-s} \|\tilde{e}^q\| \sum_{n=0}^{m} \int\limits_{0}^{t_{n+\frac{1}{2}}} \left(t_{n+\frac{1}{2}} - s\right)^{-\frac{1}{2}} ds \leq C_5 \|\tilde{e}^q\| N^{-s}, \tag{5.14}$$

$$\tau \sum_{n=0}^{m} \left\langle E\left(\mathcal{I}\left(u_{xx}, t_{n+\frac{1}{2}}\right)\right), 2\bar{\tilde{e}}^n \right\rangle \leq C_6^* \tau \sum_{n=0}^{m} \left\{ \left\| \mathcal{I}\left(u_{xx}, t_{n+\frac{1}{2}}\right) - \mathbf{P}_{N-1}\mathcal{I}\left(u_{xx}, t_{n+\frac{1}{2}}\right) \right\| + \right.$$

$$\left. \left\| \mathcal{I}\left(u_{xx}, t_{n+\frac{1}{2}}\right) - \mathcal{I}_N \mathcal{I}\left(u_{xx}, t_{n+\frac{1}{2}}\right) \right\| \right\} \left\|\bar{\tilde{e}}^n\right\| \leq C_6^{**} \tau \sum_{n=0}^{m} \left\{ \left\| \mathcal{I}\left(u_{xx}, t_{n+\frac{1}{2}}\right) \right\|_s N^{-s} \left\|\bar{\tilde{e}}^n\right\| \right\}$$

$$\leq C_6^{***} \tau \sum_{n=0}^{m} \left\{ \|u\|_{L^\infty(0,T;H^{s+2})} N^{-s} \left\|\bar{\tilde{e}}^n\right\| \right\} \leq C_6 \|\tilde{e}^q\| N^{-s}. \tag{5.15}$$

Now (5.8)- (5.15) gives

$$\|\tilde{e}^q\|_N^2 \leq \left\|\tilde{e}^0\right\|_N^2 + C^{**} \left(N^{-s} + \tau^{\frac{3}{2}}\right) \|\tilde{e}^q\|. \tag{5.16}$$

using $\left\|\tilde{e}^0\right\|_N^2 \leq C^* \|\tilde{e}^q\| N^{-s}$, (5.16) gives

$$\|\tilde{e}^q\| \leq C \left(N^{-s} + \tau^{-\frac{3}{2}}\right). \tag{5.17}$$

Now we obtain

$$\|e^n\| = \|u^n - \tilde{u} + \tilde{u}^n - U^n\| \leq C \left(N^{-s} + \tau^{\frac{3}{2}}\right). \tag{5.18}$$

□

6 Numerical results

In this section we present the numerical results of the proposed method on several test problems. We tested the accuracy and stability of the method described in this paper by performing the mentioned scheme for different values of h and τ. We performed our computations using **Matlab** 7 software on a Pentium IV, 2800 MHz CPU machine with 2 Gbyte of memory. We calculate the computational order of method presented in this article for time variable with the following formula

$$\mathrm{C-order} = \log_2\left(\frac{\|L_\infty(2\tau, N)\|}{\|L_\infty(\tau, N)\|}\right).$$

6.1 Test problem 1

We consider Eq. (1.1) with $\gamma = 0$, i.e

$$u_t(\xi,t) = \int_0^t (t-s)^{-1/2} u_{\xi\xi}(\xi,s)\,ds, \quad 0 \le \xi \le 1 \tag{6.1}$$

and the following initial condition

$$u(\xi,0) = \sin(\pi\xi). \tag{6.2}$$

The exact solution of this problem is [5, 15, 20]

$$u(\xi,t) = M\left(\pi^{5/2} t^{3/2}\right)\sin(\pi\xi), \tag{6.3}$$

where

$$M(z) = \sum_{n=0}^{\infty} (-1)^n \Gamma\left(\frac{3}{2}n+1\right)^{-1} z^n.$$

Using $\xi = \frac{x}{2} + \frac{1}{2}$ we first transform (6.1)-(6.3) to the following problem

$$\begin{cases} u_t(x,t) = 4\int_0^t (t-s)^{-1/2} u_{xx}(x,s)\,ds, \quad -1 \le x \le 1, \\ u(x,0) = \sin\left(\frac{\pi x}{2} + \frac{\pi}{2}\right), \\ u(x,t) = M\left(\pi^{5/2} t^{3/2}\right)\sin\left(\frac{\pi x}{2} + \frac{\pi}{2}\right). \end{cases}$$

We put $T = 1$ and $N = 10$ and compare the results of present method with the compact finite difference scheme developed in [15] in Tables 1,2.

Table 1: Comparison of maximum error for Test problem 1 with $\beta = 0$

	Present method			Method of [15]		
τ	Error	CPU time	C-order	Error	CPU time	C-order
1/20	3.8677×10^{-3}	0.0048	–	8.93117×10^{-2}	0.0080	–
1/40	1.0874×10^{-3}	0.0139	1.8305	3.21568×10^{-2}	0.0186	1.47372
1/80	3.1496×10^{-4}	0.0436	1.7876	1.15715×10^{-2}	0.0621	1.47455
1/160	9.3874×10^{-5}	0.1593	1.7464	4.16970×10^{-3}	0.2304	1.47256
1/320	2.8770×10^{-5}	0.5911	1.7062	1.50417×10^{-3}	0.8783	1.47098
1/640	9.0549×10^{-6}	2.3133	1.6678	5.46742×10^{-4}	3.3257	1.46004
1/1280	2.9198×10^{-6}	9.1880	1.6328	–	–	
1/2560	9.6146×10^{-7}	37.164	1.6026	–	–	

Table 2: Comparison of maximum error for Test problem 1 with $\beta = 0.1$

	Present method			Method of [15]		
τ	Error	CPU time	C-order	Error	CPU time	C-order
1/20	2.9307×10^{-3}	0.0050	–	8.71410×10^{-2}	0.0106	–
1/40	9.2153×10^{-4}	0.0140	1.6691	3.13501×10^{-2}	0.0217	1.47488
1/80	2.7753×10^{-4}	0.0453	1.7314	1.13353×10^{-2}	0.0579	1.46764
1/160	8.3741×10^{-5}	0.1599	1.7286	4.10482×10^{-3}	0.2[illegible]73	1.46544
1/320	2.5677×10^{-5}	0.6062	1.7055	1.48438×10^{-3}	[illegible].8807	1.46745
1/640	8.0424×10^{-6}	2.4593	1.6748	5.3766×10^{-4}	3.3037	1.46510
1/1280	2.5756×10^{-6}	9.5111	1.6427	–	–	
1/2560	8.4213×10^{-7}	41.527	1.6128	–	–	

As we see from Tables 1,2, in comparison with the method of [15], the proposed scheme in this paper has high-order of accuracy and needs to less CPU time. Also numerical results reflect that convergence order is at least $\frac{3}{2}$ in time component.

6.2 Test problem 2

We consider partial integro-differential equation with a weakly singular kernel

$$u_t(x,t) = u_{xx}(x,t) + \int_0^t (t-s)^{-1/2} u(x,s)\, ds, \quad -1 \le x \le 1, \tag{6.4}$$

with the following initial condition

$$u(x,0) = 1 - x^2. \tag{6.5}$$

We put $N = 32$ and use the numerical solution corresponding to $N = 32$ and $\tau = 1/3000$ as reference solution. Tables 3,4 show the errors and computational order of presented method for different values of β, τ and T.

Table 3: Errors and computational order for Test problem 2 at $T = 0.5$

	$\beta = 0$		$\beta = 0.1$	
τ	Error	C-order	Error	C-order
1/30	$3.6[illegible]57 \times 10^{-4}$	–	2.5178×10^{-4}	–
1/60	1.0671×10^{-4}	1.6298	1.0459×10^{-4}	1.2674
1/120	3.1662×10^{-5}	1.7529	4.1802×10^{-5}	1.3231
1/240	$9.499[illegible] \times 10^{-6}$	1.7369	1.5949×10^{-5}	1.3901
1/480	2.8442×10^{-6}	1.7398	5.7492×10^{-6}	1.4720
1/960	8.0062×10^{-6}	1.8288	1.9543×10^{-6}	1.5567

Table 4: Errors and computational order for Test problem 2 at $T = 1$

	$\beta = 0$		$\beta = 0.1$	
τ	Error	C-order	Error	C-order
1/30	9.2773×10^{-4}	–	1.0326×10^{-3}	–
1/60	2.9978×10^{-4}	1.6298	3.3917×10^{-4}	1.6062
1/120	9.8680×10^{-5}	1.6031	1.1291×10^{-4}	1.5869
1/240	3.2732×10^{-5}	1.5921	3.7753×10^{-4}	1.5805
1/480	1.0699×10^{-5}	1.6132	1.2410×10^{-5}	1.6051
1/960	3.2302×10^{-6}	1.7278	3.7609×10^{-5}	1.7224

Tables 3,4, show the high-accuracy of proposed method and the numerical results reflect that convergence order is bout $\frac{3}{2}$ in time component. Figure 1 presents the numerical solution of this problem with $N = 32$, $\tau = 1/500$ and different values of final times.

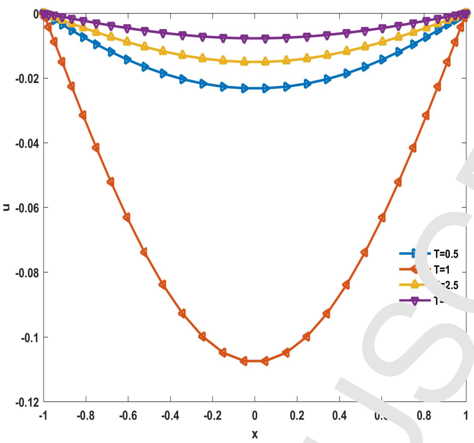

Figure 1: Numerical solution of Test problem 2 with $N = 32$, $\tau = 1/500$ and different values of final times.

7 Conclusion

In this paper we proposed an efficient numerical method for the solution of a partial integro-differential equation with a singular kernel. In the time direction, a Crank-Nicolson finite difference scheme is used to approximate the differential term and the product trapezoidal method is employed to treat the integral term. Also for space discretization we applied Legendre spectral collocation method. We proved that the method is unconditionally stable and convergent with order $\mathcal{O}(\tau^{\frac{3}{2}} + N^{-s})$. We compared our numerical results with analytical solutions and other methods in the literature and showed that the proposed method is efficient in both accuracy and CPU time.

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