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The assessment on the lifetime performance index of products with Gompertz distribution based on the progressive type I interval censored sample

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A B S T R A C T

It is a very important topic in the manufacturing industry to evaluate whether the quality of products adhered to the desire level. The larger-the-better process capability index (PCI) C_L is frequently used to measure the performance of lifetimes of products. Gompertz distribution has many applications in the lifetime data analysis. The maximum likelihood estimator for C_L is utilized to develop a hypothesis testing procedure with respect to a lower specification limit based on the progressive type I interval censored sample. Finally, two practical examples are given to demonstrate the application of our proposed testing procedure to assess whether the lifetime performance reached the desire level.

1. Introduction

In recent years, consumers are in the pursuit of more stringent product quality requirements for many high-tech products such as tablet, mobile phones, etc. In practice, many researchers have developed a variety of methods to assess the quality of the product and process capability indices had been widely used to evaluate the process performance to the continuous improvement of quality and productivity. For measuring the quality of products, process capability indices (PCIs) including C_p, C_{pk}, C_{pm} and C_{pmk} had been widely used to monitor the target-the-better type quality characteristics (see Montgomery [1] for more examples and details). The lifetime of products is expected to be a larger-the-better type quality characteristic, the index C_L is considered to assess the performance of lifetime following a Gompertz distribution by developing the hypothesis testing procedure by which purchasers could then employ it to determine whether the lifetime of electronic components reached to the required level. For exponential distribution lifetime, Tong et al. [2] constructed the uniformly minimum variance unbiased estimator (UMVUE) of C_L and built a hypothesis testing procedure for the complete sample. Laumen and Cramer [3] proposed the Inference for the lifetime performance index with progressively Type-II censored samples from gamma distributions. Feng et al. [4] used the least squares method to estimate the parameters of the Makeham distribution. Torres [5] used the least squares, maximum likelihood and moment methods to estimate the parameters of the shifted Gompertz distribution. Jodrá [6] gave a closed-form expression for the quantile function of the Gompertz–Makeham distribution.

In practice, not all lifetimes of all products on test can be observed due to the time limitation or other restrictions on money, material resources or experimental difficulties. In such cases, some incomplete data could be observed, such as progressive censoring data (see Balakrishnan and Cramer [7], Balakrishnan and Aggarwala [8], Aggarwala [9], Wu et al. [10], Wu et al. [11], Sanjel and Balakrishnan [12], Lee et al. [13], and Wu [14]). This paper is focusing on the progressive type I

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interval sample which is described briefly as follows: Suppose n items of product are placed on a life test at the starting time 0 of experiment. Set $(t_1, \ldots, t_m)$ to be the pre-specified inspection time points for m time intervals, where t_m is time where the experiment is terminated. Let $(p_1, \ldots, p_m)$ be the pre-specified progressive removal probability of the remaining survival units at time t_i, $i = 1, \ldots, m$, where $p_m = 1$. In the ith inspection time interval $(t_{i-1}, t_i]$, the number of failure items X_i is observed and then removed $R_i = [n - \sum_{j=1}^{i-1} X_j - \sum_{j=1}^{i-1} R_j]p_i$ items randomly from the remaining $n - \sum_{j=1}^{i-1} X_j - \sum_{j=1}^{i-1} R_j$ survival items, where $[w]$ represents the largest integer smaller than or equal to w, $i = 1, \ldots, m$. The progressive type-I interval censoring to allow the removal of surviving items during the life test is very convenient for the quality personnel to conduct for life test. For this type of censoring sample, Wu and Lin [15] utilized the maximum likelihood estimator (MLE) for C_L to develop a hypothesis testing procedure for the one-parameter exponential distribution. Wu and Lin [16] presented a testing procedure for C_L when the lifetimes of product follow a Weibull distribution under progressive type I interval censoring. There are many applications of Gompertz distribution in the lifetime data analysis (See Johnson et al. [17]). This distribution is applicable in the theory of extreme-order statistics and it is used as a good fit to data from clinical trials in medical science (see Moustafa and Ramadan [18]). Lee et al. [19] utilized the UMVUE for C_L to develop a testing procedure to assess the performance of lifetime products from Gompertz distribution under failure-censored sampling. In this paper, our research goal is focusing on the estimation of C_L for Gompertz distribution and propose a hypothesis testing procedure to assess the lifetime performance for products with this lifetime distribution.

The rest of this paper is organized as follows: In Section 2, the lifetime performance index for products with Gompertz lifetimes is addressed and the mathematical increasing relationship between the lifetime performance index and conforming rate is discussed. In Section 3, the MLE of the lifetime performance index is derived and its asymptotic normal distribution is obtained. A hypothesis testing procedure for the lifetime performance index is developed in Section 4 and the power analysis of the proposed testing procedure is also investigated. In Section 5, two practical examples are given to illustrate the proposed testing procedure. At last, the conclusion is made on the proposed testing procedures in Section 6.

2. The lifetime performance index and the relationship to conforming rate

Suppose that the lifetime (U) of products has a Gompertz distribution with the probability density function (pdf) and the cumulative distribution function (cdf) as follows:

$$f_U(u) = \lambda \exp\left[\beta u - \frac{\lambda}{\beta}\left(e^{\beta u} - 1\right)\right], u > 0, \beta > 0, \lambda > 0, \tag{1}$$

and

$$F_U(u) = 1 - e^{-\frac{\lambda}{\beta}\left(e^{\beta u} - 1\right)}, u > 0, \beta > 0, \lambda > 0. \tag{2}$$

The failure rate function is defined as

$$h_U(u) = \frac{f_U(u)}{1 - F_U(u)} = \lambda \exp(\beta u). \tag{3}$$

The Gompertz distribution can address the force of mortality or failure rate increases exponentially over time. It describes human mortality quite accurately.

Let $Y = \frac{1}{\beta}\left(e^{\beta U} - 1\right)$, $\beta > 0$. Then this new random variable Y has one-parameter exponential distribution and its pdf and cdf are given as follows:

$$f_Y(y) = \lambda e^{-\lambda y}, y > 0, \lambda > 0 \tag{4}$$

and

$$F_Y(y) = 1 - \exp(-\lambda y), \ y > 0, \lambda > 0, \tag{5}$$

where λ is the scale parameter and the failure rate function is defined as

$$r_Y(y) = \frac{f_y(y)}{1 - F_y(y)} = \lambda. \tag{6}$$

The lifetime of products is a larger-the-better type quality characteristic since products with longer lifetime tend to be more competitive in these emerging markets. Suppose consumers desire the lifetime to exceed L unit times. Montgomery [1] developed a process capability index C_L to measure the capability for larger-the-better type quality characteristics as follows:

$$C_L = \frac{\mu - L}{\sigma}, \tag{7}$$

where μ represents the process mean, σ denotes the process standard deviation, and L is the known pre-specified lower specification limit. This lifetime performance index is usually used to evaluate the performance of lifetime of products.

The mean and standard deviation of the lifetime of products following the distribution defined in Eq. (4) are given by $\mu = E(Y) = \frac{1}{\lambda}$ and $\sigma = \sqrt{Var(Y)} = \frac{1}{\lambda}$. Then the capability index C_L can be expressed as

$$C_L = \frac{\mu - L}{\sigma} = \frac{\frac{1}{\lambda} - L}{\frac{1}{\lambda}} = 1 - \lambda L \tag{8}$$

Observed that when $1/\lambda > L$, we have the index $C_L > 0$; when $1/\lambda < L$, we have the index $C_L < 0$. It is also observed that the smaller the failure rate λ the larger the lifetime performance index C_L. Therefore, the lifetime performance index C_L can accurately assess the performance of lifetime of products.

An item of product is regarded as a conforming one if its lifetime U exceeds the pre-specified lower specification limit L_U. Since $Y = \frac{1}{\beta}\left(e^{\beta U} - 1\right)$ is an increasing function of U, then $L = \frac{1}{\beta}\left(e^{\beta L_U} - 1\right)$ can be regarded as the lower specification limit for new lifetime variable Y for this item to be a conforming one (i.e. $Y \geq L$). The conforming rate is calculated as

$$P_r = P(U \geq L_U) = P(Y \geq L) = \exp(-\lambda L) = \exp(C_L - 1), -\infty < C_L < 1. \tag{9}$$

Apparently, the conforming rate P_r is an increasing function of the lifetime performance index C_L. The values of C_L and the corresponding conforming rates P_r are listed in Table 1. From Table 1, it can be seen that if P_r is desired to exceed 0.904837, then C_L must be required to exceed 0.9.

3. Maximum likelihood estimator of the lifetime performance index

Consider the incomplete sample $X_1, \ldots, X_m$ under progressive type I interval censoring observed at pre-set inspection times $t_1, \ldots, t_m$ with $R_1, \ldots, R_m$ survival units randomly removed from the remaining survival units under the removal percentages $p_1, \ldots, p_m$ at inspection times $t_1, \ldots, t_m$. The likelihood function based on progressive type I interval censored sample $X_1, \ldots, X_m$ is (see for example Casella and Berger [20])

$$\begin{aligned} L(\lambda) &\propto \prod_{i=1}^{m} \left(F_U(t_i) - F_U(t_{i-1})\right)^{X_i} \left(1 - F_U(t_i)\right)^{R_i} \\ &= \prod_{i=1}^{m} \left(1 - e^{-\lambda(y_i - y_{i-1})}\right)^{X_i} \left(e^{-\lambda(R_i y_i + X_i y_{i-1})}\right), \end{aligned} \tag{10}$$

where $y_i = \frac{1}{\beta}\left(e^{\beta t_i} - 1\right)$.

The log-likelihood function is

$$\ln L(\lambda) = \sum_{i=1}^{m} X_i \ln\left(1 - e^{-\lambda(y_i - y_{i-1})}\right) - \lambda \sum_{i=1}^{m} (R_i y_i + X_i y_{i-1}). \tag{11}$$

Differentiating the log-likelihood function with respect to parameter λ and equating it to zero, we can obtain the log-likelihood equation as follows:

$$\frac{d}{d\lambda} \ln L(\lambda) = \sum_{i=1}^{m} X_i \frac{(y_i - y_{i-1})\, e^{-\lambda(y_i - y_{i-1})}}{1 - e^{-\lambda(y_i - y_{i-1})}} - \sum_{i=1}^{m} (R_i\, y_i + X_i\, y_{i-1}). \tag{12}$$

The MLE of λ can be obtained by solving Eq. (12) numerically and denoted by $\hat{\lambda}$.

It is known that the distribution of MLE is an asymptotic normal distribution. The Fisher's information is $I(\lambda) = -E[\frac{d^2 \ln L(\lambda)}{d\lambda^2}]$ From Eq. (12), we have

$$\frac{d^2}{d\lambda^2} \ln L(\lambda) = \sum_{i=1}^{m} X_i \frac{-(y_i - y_{i-1})^2\, e^{-\lambda(y_i - y_{i-1})}}{\left(1 - e^{-\lambda(y_i - y_{i-1})}\right)^2} \tag{13}$$

From Wu and Lin [15], we have

$$X_i | X_{i-1}, \ldots, X_1, R_{i-1}, \ldots, R_1 \sim \text{Binomial}\left(n - \sum_{i=1}^{i-1} X_i - \sum_{i=1}^{i-1} R_i, q_i\right), \tag{14}$$

where

$$q_i = \frac{F(t_i) - F(t_{i-1})}{1 - F(t_{i-1})} = \frac{\exp(-\frac{\lambda}{\beta}(e^{\beta t_{i-1}} - 1)) - \exp(-\frac{\lambda}{\beta}(e^{\beta t_i} - 1))}{\exp(-\frac{\lambda}{\beta}\left(e^{\beta t_{i-1}} - 1\right))} = 1 - \exp(-\frac{\lambda}{\beta}(e^{\beta t_i} - e^{\beta t_{i-1}})), \quad i = 1, \ldots, m.$$

Table 1
The lifetime performance index C_L and its corresponding conforming rates P_r.

C_L	P_r	C_L	P_r	C_L	P_r
$-\infty$	0.000000	−0.125	0.324652	0.550	0.637628
−3.000	0.018316	0.000	0.367879	0.575	0.653770
−2.750	0.023518	0.125	0.416862	0.600	0.670320
−2.500	0.030197	0.150	0.427415	0.625	0.687289
−2.250	0.038774	0.175	0.438235	0.650	0.704688
−2.125	0.043937	0.200	0.449329	0.675	0.722527
−2.000	0.049787	0.225	0.460704	0.700	0.740818
−1.750	0.063928	0.250	0.472367	0.725	0.759572
−1.500	0.082085	0.275	0.484325	0.750	0.778801
−1.250	0.105399	0.300	0.496585	0.775	0.798516
−1.125	0.119433	0.325	0.509156	0.800	0.818731
−1.000	0.135335	0.350	0.522046	0.825	0.839457
−0.750	0.173774	0.375	0.535261	0.850	0.860708
−0.500	0.223130	0.400	0.548812	0.875	0.882497
−0.250	0.286505	0.425	0.562705	0.900	0.904837
−0.225	0.293758	0.450	0.576950	0.925	0.927743
−0.200	0.301194	0.475	0.591555	0.950	0.951229
−0.175	0.308819	0.500	0.606531	0.975	0.975310
−0.150	0.316637	0.525	0.621885	1.000	1.000000

From (14), we have

$$\begin{aligned} E(X_i) &= E(X_i \mid X_{i-1}, \ldots, X_1, R_{i-1}, \ldots, R_1) \\ &= nq_i \prod_{j=1}^{i-1} \left(1-p_j\right)\left(1-q_j\right), \quad i = 1, \ldots, m. \end{aligned} \tag{15}$$

Therefore, the Fisher's information number will be

$$I(\lambda) = -E[\frac{d^2 \ln L(\lambda)}{d\lambda^2}] = \frac{n}{\lambda^2} \sum_{i=1}^{m} \frac{\ln 2\,(1-q_i)}{q_i} \prod_{j=1}^{i-1} \left(1-p_j\right) \prod_{k=1}^{i} (1-q_k)\,. \tag{16}$$

Then we have $\hat{\lambda} \xrightarrow[m\to\infty]{d} \mathrm{N}\left(\lambda, I^{-1}(\lambda)\right)$.

For the convenience of inspection, equal interval lengths $t_i - t_{i-1} = t$, $i = 1, \ldots, m$ are considered, i.e. $t_i = it$, $i = 1, \ldots, m$, then we have $q_i = 1 - \exp(-\frac{\lambda}{\beta}(e^{i\beta t} - e^{(i-1)\beta t})) = 1 - \exp(-\frac{\lambda}{\beta} e^{(i-1)\beta t}(e^{\beta t} - 1))$. Eq. (12) can be simplified as

$$\frac{d}{d\lambda} \ln L(\lambda) = \sum_{i=1}^{m} X_i \frac{\ln(1-q_i)(1-q_i)}{q_i} - \sum_{i=1}^{m} \frac{1}{\beta} \left(R_i \left(e^{i\beta t} - 1\right) + X_i \left(e^{(i-1)\beta t} - 1\right)\right) \equiv 0 \tag{17}$$

Solving the above log-likelihood equation numerically, we can obtain the MLE of λ as $\hat{\lambda}$. Furthermore, under equal removal percentages i.e. $p_1 = \cdots = p_{m-1} = p$, the asymptotic variance of $\hat{\lambda}$ can be obtained as $V\left(\hat{\lambda}\right) = I^{-1}(\lambda)$, where

$$I(\lambda) = \frac{n}{\lambda^2} \sum_{i=1}^{m} \frac{\ln 2\,(1-q_i)}{q_i} (1-p)^{i-1} \prod_{k=1}^{i} (1-q_k)\,, \quad \text{where } q_i = 1 - \exp(-\frac{\lambda}{\beta} e^{(i-1)\beta t}(e^{\beta t} - 1)).$$

By the property of the invariance of MLE, the MLE of C_L can be obtained as

$$\hat{C}_L = 1 - \hat{\lambda} L. \tag{18}$$

Since we have $\hat{\lambda} \xrightarrow[m\to\infty]{d} \mathrm{N}\left(\lambda, I^{-1}(\lambda)\right)$. then we can show that

$$\hat{C}_L = 1 - \hat{\lambda} L \xrightarrow[m\to\infty]{d} N\left(C_L, V(\hat{C}_L)\right), \tag{19}$$

where $V(\hat{C}_L) = L^2 V(\hat{\lambda})$.

Hence, the MLE $\hat{C}_L$ is an asymptotic unbiased estimator of C_L. The estimate of the asymptotic variance of $\hat{C}_L$ is obtained as

$$\hat{V}(\hat{C}_L) = L^2 \hat{V}(\hat{\lambda}) = L^2 I^{-1}(\hat{\lambda}) \tag{20}$$

4. Testing procedure algorithm for the lifetime performance index

In this section, a statistical testing procedure is proposed to assess whether the lifetime performance index achieves the pre-specified desired level c_0. If the lifetime performance index C_L is greater than the desired level c_0, the production process is regarded as being capable. The statistical hypothesis procedure is presented as follows:

Set up the null hypothesis and alternative hypothesis as

H_0: $C_L \le c_0$ (the process is not capable) v.s. H_a: $C_L > c_0$ (the process is capable). We utilize the MLE of C_L as the testing statistic and the critical value C_L^0 is determined as follows:

$$P\left(\hat{C}_L > C_L^0 \,|C_L \le c_0\right) = P\left(1-\hat{\lambda}L > C_L^0 \middle| \lambda \ge \frac{1-c_0}{L}\right) = P\left(\hat{\lambda} < \frac{1-C_L^0}{L} \middle| \lambda \ge \frac{1-c_0}{L}\right)$$
$$= P\left(Z < \left(\frac{1-C_L^0}{L} - \lambda\right)/\sqrt{g(\lambda)} \middle| \lambda \ge \frac{1-c_0}{L}\right) \le \alpha,$$

where $g(\lambda) = V\left(\hat{\lambda}\right), Z = \frac{\hat{\lambda}-\lambda}{\sqrt{g(\lambda)}} \xrightarrow[m\to\infty]{d} \mathrm{N}(0,1)$.

Set $\sup_{\lambda \ge \frac{1-c_0}{L}} P\left(Z < \left(\frac{1-C_L^0}{L} - \lambda\right)/\sqrt{g(\lambda)}\right) = \alpha$

The supremum of $P\left(Z < \left(\frac{1-C_L^0}{L} - \lambda\right)/\sqrt{g(\lambda)}\right)$ occurred at $\lambda = \lambda_0 = \frac{1-c_0}{L}$ since $\sqrt{g(\lambda)}$ is an increasing function of λ when $\lambda \ge \frac{1-c_0}{L}$. Thus, we have

$$P\left(Z < \left(\frac{1-C_L^0}{L} - \lambda_0\right)/\sqrt{g(\lambda_0)} \middle| \lambda_0 = \frac{1-c_0}{L}\right) = \alpha,$$

$\Rightarrow \left(\frac{1-C_L^0}{L} - \lambda_0\right)/\sqrt{g(\lambda_0)} = Z_{1-\alpha}$ where $\lambda_0 = \frac{1-c_0}{L}$ and $Z_{1-\alpha}$ represents the upper αth percentile of a standard normal distribution.

Thus, the critical values can be determined as

$$C_L^0 = 1 - L\left(\lambda_0 + Z_{1-\alpha}\sqrt{g(\lambda_0)}\right), \tag{21}$$

where $\lambda_0 = \frac{1-c_0}{L}$. The rejection region for this test is $\left\{\hat{C}_L \middle| \hat{C}_L > C_L^0\right\}$.

The proposed testing procedure computational algorithm about for C_L is organized as follows:

Algorithm:

Step 1: Set $\alpha = 0.1$ and give a known lower specification L_U for lifetime U. Then the lower specification for the new lifetime Y is $L = \frac{1}{\beta}\left(e^{\beta L_U} - 1\right)$. Observe the progressive type I interval censored sample $X_1, \ldots, X_m$ at the pre-set times $t_1, \ldots, t_m$ with censoring schemes of $R_1, \ldots, R_m$ from the Gompertz distribution,

Step 2: Determine the required level c_0 to achieve a pre-assigned conforming rate P_r from Table 1. Then the testing null hypothesis H_0: $C_L \le c_0$ and the alternative hypothesis H_a: $C_L > c_0$ are constructed, and c_0 is the target value.

Step 3: Obtain the MLE of λ numerically denoted as $\hat{\lambda}$ and then calculate the value of test statistic $\hat{C}_L = 1 - \hat{\lambda}L$.

Step 4: For the level of significance of α, we can calculate the critical value $C_L^0 = 1 - L\left(\lambda_0 + Z_{1-\alpha}\sqrt{g(\lambda_0)}\right)$, where $\lambda_0 = \frac{1-c_0}{L}$.

Step 5: The decision rule is to conclude that the lifetime performance index of the products meets the required level if $\hat{C}_L > C_L^0$.

Moreover, the power of this statistical test is calculated as follows:

The power $h(c_1)$ of the test at this point $C_L = c_1 > c_0$ is

$$h(c_1) = P\left(\hat{C}_L > C_L^0 \,|c_1 = 1 - \lambda_1 L\right)$$
$$= P\left(1 - \hat{\lambda}L > 1 - L\left(\lambda_0 + Z_{1-\alpha}\sqrt{g(\lambda_0)}\right) \middle| \lambda_1 = \frac{1-c_1}{L}\right)$$
$$= P\left(\hat{\lambda} < \left(\lambda_0 + Z_{1-\alpha}\sqrt{g(\lambda_0)}\right) \middle| \lambda_1 = \frac{1-c_1}{L}\right)$$

Table 2
The values of $h(c_1)$ at $\alpha = 0.01$ for $c_1 = 0.75(0.025)0.80, 0.815(0.01)0.855, 0.885, 0.925$, $m = 5(1)8$, $n = 60(20)100$ and $p = 0.05\,(0.025)\,0.1$ under $L = 0.05$, $T = 0.5$, and $c_0 = 0.8$.

c_1												
m	n	p	0.750	0.775	0.800	0.815	0.825	0.835	0.845	0.855	0.885	0.925
5	60	0.050	0.00049	0.00216	0.01000	0.02507	0.04568	0.08162	0.14149	0.23498	0.71499	0.99935
		0.075	0.00052	0.00224	0.01000	0.02457	0.04425	0.07828	0.13472	0.22287	0.68858	0.99889
		0.100	0.00056	0.00232	0.01000	0.02410	0.04290	0.07515	0.12837	0.21146	0.66201	0.99818
	80	0.050	0.00023	0.00150	0.01000	0.03003	0.06032	0.11598	0.21006	0.35199	0.88251	0.99999
		0.075	0.00025	0.00157	0.01000	0.02935	0.05823	0.11091	0.19981	0.33470	0.86394	0.99999
		0.100	0.00027	0.00164	0.01000	0.02871	0.05625	0.10614	0.19016	0.31825	0.84409	0.99997
	100	0.050	0.00011	0.00108	0.01000	0.03505	0.07606	0.15385	0.28414	0.46885	0.95850	1.00000
		0.075	0.00013	0.00114	0.01000	0.03418	0.07323	0.14688	0.27047	0.44776	0.94871	1.00000
		0.100	0.00014	0.00120	0.01000	0.03335	0.07056	0.14032	0.25752	0.42742	0.93752	1.00000
6	60	0.050	0.00045	0.00206	0.01000	0.02587	0.04812	0.08760	0.15417	0.25848	0.76638	0.99984
		0.075	0.00049	0.00215	0.01000	0.02526	0.04634	0.08337	0.14550	0.24295	0.73646	0.99966
		0.100	0.00053	0.00223	0.01000	0.02469	0.04468	0.07946	0.13749	0.22855	0.70603	0.99932
	80	0.050	0.00020	0.00141	0.01000	0.03118	0.06406	0.12545	0.22991	0.38621	0.91624	1.00000
		0.075	0.00023	0.00149	0.01000	0.03034	0.06143	0.11899	0.21681	0.36440	0.89778	1.00000
		0.100	0.00025	0.00156	0.01000	0.02956	0.05898	0.11301	0.20464	0.34387	0.87743	0.99999
	100	0.050	0.00010	0.00101	0.01000	0.03657	0.08126	0.16720	0.31106	0.51073	0.97476	1.00000
		0.075	0.00011	0.00107	0.01000	0.03550	0.07768	0.15830	0.29371	0.48474	0.96636	1.00000
		0.100	0.00013	0.00113	0.01000	0.03448	0.07436	0.15005	0.27747	0.45984	0.95626	1.00000
7	60	0.050	0.00043	0.00200	0.01000	0.02641	0.04981	0.09185	0.16339	0.27577	0.80097	0.99995
		0.075	0.00047	0.00209	0.01000	0.02571	0.04773	0.08686	0.15306	0.25727	0.76858	0.99987
		0.100	0.00051	0.00219	0.01000	0.02506	0.04582	0.08231	0.14367	0.24033	0.73530	0.99970
	80	0.050	0.00019	0.00136	0.01000	0.03196	0.06670	0.13231	0.24447	0.41125	0.93635	1.00000
		0.075	0.00021	0.00144	0.01000	0.03100	0.06361	0.12464	0.22888	0.38557	0.91833	1.00000
		0.100	0.00024	0.00152	0.01000	0.03010	0.06078	0.11764	0.21460	0.36161	0.89788	1.00000
	100	0.050	0.00009	0.00096	0.01000	0.03763	0.08499	0.17694	0.33080	0.54087	0.98315	1.00000
		0.075	0.00010	0.00103	0.01000	0.03638	0.08077	0.16637	0.31027	0.51079	0.97587	1.00000
		0.100	0.00012	0.00110	0.01000	0.03522	0.07690	0.15670	0.29127	0.48213	0.96667	1.00000
8	60	0.050	0.00042	0.00197	0.01000	0.02676	0.05094	0.09480	0.16995	0.28828	0.82466	0.99998
		0.075	0.00046	0.00207	0.01000	0.02599	0.04863	0.08920	0.15827	0.26730	0.79045	0.99994
		0.100	0.00050	0.00217	0.01000	0.02529	0.04653	0.08415	0.14777	0.24830	0.75500	0.99984
	80	0.050	0.00019	0.00133	0.01000	0.03249	0.06852	0.13717	0.25498	0.42939	0.94887	1.00000
		0.075	0.00021	0.00142	0.01000	0.03142	0.06507	0.12852	0.23733	0.40049	0.93132	1.00000
		0.100	0.00023	0.00150	0.01000	0.03044	0.06193	0.12072	0.22134	0.37376	0.91089	1.00000
	100	0.050	0.00009	0.00093	0.01000	0.03835	0.08760	0.18392	0.34509	0.56250	0.98782	1.00000
		0.075	0.00010	0.00100	0.01000	0.03696	0.08286	0.17197	0.32191	0.52907	0.98136	1.00000
		0.100	0.00012	0.00108	0.01000	0.03568	0.07857	0.16117	0.30068	0.49738	0.97283	1.00000

$$
\begin{aligned}
&= P\left(\frac{\hat{\lambda}-\lambda_1}{\sqrt{g(\lambda_1)}} < \frac{\left(\lambda_0 + Z_{1-\alpha}\sqrt{g(\lambda_0)} - \lambda_1\right)}{\sqrt{g(\lambda_1)}} \,\middle|\, \lambda_1 = \frac{1-c_1}{L}\right) \\
&= \Phi\left(\frac{\lambda_0 - \lambda_1 + Z_{1-\alpha}\sqrt{g(\lambda_0)}}{\sqrt{g(\lambda_1)}}\right),
\end{aligned}
\tag{22}
$$

where $\Phi(\cdot)$ is the cdf for the standard normal distribution, $\lambda_0 = \frac{1-c_0}{L}$ and $\lambda_1 = \frac{1-c_1}{L}$.

In addition, the level $100(1-\alpha)\%$ lower confidence bound for C_L can be derived as follows:

From Eq. (19) and using Slusky's Theorem (Casella and Berger [20]), we have $Z = \frac{\hat{C}_L - C_L}{\sqrt{\hat{V}(\hat{C}_L)}} \xrightarrow[m\to\infty]{d} N(0,1)$, where $\hat{V}\left(\hat{C}_L\right)$ is defined in Eq. (20). Use $Z = \frac{\hat{C}_L - C_L}{\sqrt{\hat{V}(\hat{C}_L)}}$ as the pivotal quantity, then we can obtain the lower confidence bound for C_L as $\hat{C}_L - Z_\alpha\sqrt{\hat{V}\left(\hat{C}_L\right)}$.

The powers for testing $H_0\colon C_L \le 0.8$ given in Eq. (22) are tabulated in Tables 2–4 at $\alpha = 0.01, 0.05, 0.1$ respectively for $c_1 = 0.75(0.025)0.80, 0.815(0.01)0.855, 0.885, 0.925$, $m = 5(1)8$, $n = 60(20)100$ and $p = 0.05\,(0.025)\,0.1$ under $L_U = 0.0495$ and then $L = 0.05$, $T = 0.5$. The powers are also plotted in Figs. 1–4 for some typical cases. From Tables 2–4 and Figs. 1–4, we have the following findings:

(1) the power $h(c_1)$ is an increasing function of n for fixed $m = 5$, $p = 0.05$ and $\alpha = 0.05$ as shown in Fig. 1 (Other combinations of m, p and α also have the same pattern); (2) the power $h(c_1)$ is an increasing function of m for fixed $n = 60$, $p = 0.05$ and $\alpha = 0.05$ as shown in Fig. 2 (Other combinations of n, p and α also have the same pattern) (3) the power $h(c_1)$ is an increasing function of p for fixed $n = 60$, $m = 5$ and $\alpha = 0.05$ as shown in Fig. 3 (Other combinations of n, m and α

Table 3
The values of $h(c_1)$ at $\alpha = 0.05$ for $c_1 = 0.75(0.025)0.80, 0.815(0.01)0.855, 0.885, 0.925$, $m = 5(1)8$, $n = 60(20)100$ and $p = 0.05\,(0.025)\,0.1$ under $L = 0.05$, $T = 0.5$, and $c_0 = 0.8$.

			c_1									
m	n	p	0.750	0.775	0.800	0.815	0.825	0.835	0.845	0.855	0.885	0.925
5	60	0.050	0.00323	0.01283	0.05000	0.10815	0.17469	0.27123	0.40026	0.55481	0.94237	1.00000
		0.075	0.00344	0.01322	0.05000	0.10651	0.17068	0.26356	0.38799	0.53836	0.93267	0.99999
		0.100	0.00365	0.01362	0.05000	0.10493	0.16684	0.25622	0.37618	0.52231	0.92210	0.99999
	80	0.050	0.00169	0.00954	0.05000	0.12344	0.21196	0.34085	0.50590	0.68453	0.98590	1.00000
		0.075	0.00183	0.00990	0.05000	0.12133	0.20669	0.33091	0.49104	0.66708	0.98224	1.00000
		0.100	0.00197	0.01026	0.05000	0.11931	0.20164	0.32137	0.47661	0.64978	0.97793	1.00000
	100	0.050	0.00092	0.00727	0.05000	0.13810	0.24836	0.40716	0.59887	0.78295	0.99693	1.00000
		0.075	0.00101	0.00760	0.05000	0.13554	0.24186	0.39528	0.58255	0.76655	0.99581	1.00000
		0.100	0.00111	0.00792	0.05000	0.13308	0.23563	0.38383	0.56656	0.74996	0.99439	1.00000
6	60	0.050	0.00296	0.01225	0.05000	0.11104	0.18206	0.28586	0.42431	0.58756	0.96003	1.00000
		0.075	0.00319	0.01271	0.05000	0.10903	0.17709	0.27631	0.40912	0.56761	0.95079	1.00000
		0.100	0.00342	0.01315	0.05000	0.10713	0.17240	0.26729	0.39466	0.54828	0.94037	1.00000
	80	0.050	0.00151	0.00900	0.05000	0.12725	0.22191	0.36018	0.53533	0.71907	0.99190	1.00000
		0.075	0.00166	0.00941	0.05000	0.12466	0.21536	0.34785	0.51720	0.69861	0.98896	1.00000
		0.100	0.00181	0.00982	0.05000	0.12221	0.20917	0.33615	0.49973	0.67833	0.98531	1.00000
	100	0.050	0.00081	0.00679	0.05000	0.14284	0.26082	0.43049	0.63118	0.81492	0.99856	1.00000
		0.075	0.00090	0.00715	0.05000	0.13968	0.25274	0.41585	0.61163	0.79641	0.99781	1.00000
		0.100	0.00100	0.00752	0.05000	0.13668	0.24510	0.40187	0.59255	0.77760	0.99678	1.00000
7	60	0.050	0.00281	0.01190	0.05000	0.11303	0.18726	0.29635	0.44165	0.61093	0.97027	1.00000
		0.075	0.00305	0.01240	0.05000	0.11072	0.18149	0.28520	0.42397	0.58806	0.96150	1.00000
		0.100	0.00330	0.01288	0.05000	0.10855	0.17610	0.27477	0.40728	0.56601	0.95132	1.00000
	80	0.050	0.00141	0.00868	0.05000	0.12991	0.22900	0.37409	0.55640	0.74308	0.99483	1.00000
		0.075	0.00157	0.00912	0.05000	0.12692	0.22137	0.35972	0.53550	0.72020	0.99241	1.00000
		0.100	0.00173	0.00956	0.05000	0.12412	0.21424	0.34621	0.51550	0.69754	0.98923	1.00000
	100	0.050	0.00074	0.00650	0.05000	0.14616	0.26974	0.44727	0.65400	0.83642	0.99922	1.00000
		0.075	0.00084	0.00689	0.05000	0.14250	0.26032	0.43026	0.63176	0.81631	0.99869	1.00000
		0.100	0.00094	0.00728	0.05000	0.13907	0.25151	0.41417	0.61016	0.79579	0.99790	1.00000
8	60	0.050	0.00273	0.01169	0.05000	0.11439	0.19089	0.30381	0.45410	0.62768	0.97653	1.00000
		0.075	0.00298	0.01222	0.05000	0.11184	0.18446	0.29133	0.43434	0.60237	0.96817	1.00000
		0.100	0.00324	0.01273	0.05000	0.10947	0.17852	0.27979	0.41586	0.57812	0.95820	1.00000
	80	0.050	0.00136	0.00848	0.05000	0.13175	0.23401	0.38405	0.57148	0.75997	0.99640	1.00000
		0.075	0.00151	0.00895	0.05000	0.12844	0.22549	0.36798	0.54829	0.73513	0.99433	1.00000
		0.100	0.00168	0.00941	0.05000	0.12536	0.21761	0.35302	0.52625	0.71056	0.99149	1.00000
	100	0.050	0.00071	0.00632	0.05000	0.14849	0.27608	0.45928	0.67019	0.85117	0.99952	1.00000
		0.075	0.00081	0.00673	0.05000	0.14442	0.26556	0.44032	0.64576	0.82981	0.99912	1.00000
		0.100	0.00091	0.00714	0.05000	0.14065	0.25580	0.42252	0.62213	0.80795	0.99848	1.00000

also have the same pattern); (4) From Fig. 4, the power $h(c_1)$ is an increasing function of α for any combinations of n, m and p; (5) From Figs. 1–4, the power $h(c_1)$ increases when the value of c_1 increases for any combinations of n, m, p and α.

5. Two practical examples

To illustrate our proposed testing procedure, two practical examples are given as follows:

Example 1 (*Simulated Data*)**.** The following are the failure times (years) of electric products generated by computer software R of size $n = 60$ from a Gompertz distribution with pdf as Eq. (1) under $\lambda = 1$ and $\beta = 2$. The generation of the Gompertz is illustrated as below: if M is a random number following the distribution of *Uniform*$(0, 1)$, then $\ln(-\frac{\beta}{\lambda}\ln M + 1)/\beta$ follows a Gompertz distribution.

0.0141, 0.0294, 0.0351, 0.0542, 0.0551, 0.0882, 0.1103, 0.1244, 0.1317, 0.1332,
0.1393, 0.1733, 0.1994, 0.2303, 0.2315, 0.2696, 0.2854, 0.3051, 0.3104, 0.3167,
0.3250, 0.3399, 0.3479, 0.3692, 0.3705, 0.3909, 0.3979, 0.4123, 0.4320, 0.4393,
0.4640, 0.4877, 0.5091, 0.5115, 0.5234, 0.5337, 0.5342, 0.5928, 0.5992, 0.6099,
0.6129, 0.6589, 0.6765, 0.6775, 0.6867, 0.6956, 0.7145, 0.7186, 0.7451, 0.7614,
0.7726, 0.9043, 0.9408, 0.9598, 0.9736, 1.0432, 1.0433, 1.0492, 1.1972, 1.3161

We create a progressive type I censored sample based on this data. Let the number of inspections $m = 8$ with the equal length of inspection interval $t = 0.1$ (years) and the pre-specified removal percentages of the remaining survival units given by $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 1)$. Now we can start to do the above testing procedure about C_L as follows:

Table 4
The values of $h(c_1)$ at $\alpha = 0.1$ for $c_1 = 0.75(0.025)0.80, 0.815(0.01)0.855, 0.885, 0.925$, $m = 5(1)8$, $n = 60(20)100$ and $p = 0.05\ (0.025)\ 0.1$ under $L = 0.05$, $T = 0.5$, and $c_0 = 0.8$.

c_1												
m	n	p	0.750	0.775	0.800	0.815	0.825	0.835	0.845	0.855	0.885	0.925
5	60	0.050	0.00779	0.02875	0.10000	0.19729	0.29643	0.42445	0.57338	0.72458	0.98265	1.00000
		0.075	0.00826	0.02956	0.10000	0.19478	0.29092	0.41516	0.56060	0.71026	0.97889	1.00000
		0.100	0.00875	0.03036	0.10000	0.19236	0.28561	0.40617	0.54811	0.69602	0.97460	1.00000
	80	0.050	0.00432	0.02210	0.10000	0.21979	0.34513	0.50337	0.67453	0.82615	0.99685	1.00000
		0.075	0.00465	0.02286	0.10000	0.21668	0.33828	0.49231	0.66077	0.81319	0.99583	1.00000
		0.100	0.00499	0.02362	0.10000	0.21367	0.33167	0.48157	0.64719	0.80004	0.99456	1.00000
	100	0.050	0.00249	0.01737	0.10000	0.24078	0.39031	0.57288	0.75430	0.89267	0.99948	1.00000
		0.075	0.00271	0.01806	0.10000	0.23710	0.38227	0.56062	0.74069	0.88206	0.99924	1.00000
		0.100	0.00295	0.01876	0.10000	0.23354	0.37451	0.54864	0.72709	0.87106	0.99893	1.00000
6	60	0.050	0.00717	0.02753	0.10000	0.20188	0.30690	0.44264	0.59887	0.75323	0.98919	1.00000
		0.075	0.00768	0.02845	0.10000	0.19882	0.30012	0.43123	0.58341	0.73649	0.98597	1.00000
		0.100	0.00820	0.02936	0.10000	0.19590	0.29367	0.42030	0.56841	0.71986	0.98212	1.00000
	80	0.050	0.00388	0.02094	0.10000	0.22560	0.35836	0.52522	0.70194	0.85164	0.99841	1.00000
		0.075	0.00423	0.02179	0.10000	0.22179	0.34994	0.51175	0.68564	0.83712	0.99769	1.00000
		0.100	0.00460	0.02265	0.10000	0.21816	0.34190	0.49877	0.66961	0.82230	0.99674	1.00000
	100	0.050	0.00219	0.01629	0.10000	0.24775	0.40597	0.59721	0.78122	0.91302	0.99979	1.00000
		0.075	0.00242	0.01707	0.10000	0.24323	0.39611	0.58241	0.76546	0.90167	0.99966	1.00000
		0.100	0.00267	0.01785	0.10000	0.23892	0.38669	0.56805	0.74973	0.88975	0.99946	1.00000
7	60	0.050	0.00680	0.02676	0.10000	0.20508	0.31434	0.45566	0.61704	0.77319	0.99264	1.00000
		0.075	0.00735	0.02777	0.10000	0.20156	0.30648	0.44244	0.59932	0.75448	0.98982	1.00000
		0.100	0.00790	0.02876	0.10000	0.19823	0.29908	0.42991	0.58224	0.73591	0.98632	1.00000
	80	0.050	0.00363	0.02020	0.10000	0.22969	0.36779	0.54083	0.72118	0.86875	0.99909	1.00000
		0.075	0.00400	0.02114	0.10000	0.22529	0.35804	0.52531	0.70279	0.85303	0.99856	1.00000
		0.100	0.00438	0.02207	0.10000	0.22114	0.34882	0.51047	0.68475	0.83693	0.99779	1.00000
	100	0.050	0.00201	0.01561	0.10000	0.25268	0.41716	0.61447	0.79973	0.92612	0.99990	1.00000
		0.075	0.00226	0.01646	0.10000	0.24746	0.40575	0.59753	0.78226	0.91426	0.99982	1.00000
		0.100	0.00252	0.01731	0.10000	0.24253	0.39494	0.58121	0.76482	0.90169	0.99968	1.00000
8	60	0.050	0.00659	0.02628	0.10000	0.20730	0.31959	0.46497	0.63005	0.78728	0.99459	1.00000
		0.075	0.00716	0.02736	0.10000	0.20340	0.31085	0.45025	0.61045	0.76696	0.99208	1.00000
		0.100	0.00774	0.02841	0.10000	0.19975	0.30269	0.43641	0.59167	0.74684	0.98883	1.00000
	80	0.050	0.00348	0.01974	0.10000	0.23256	0.37450	0.55199	0.73482	0.88051	0.99942	1.00000
		0.075	0.00386	0.02074	0.10000	0.22768	0.36364	0.53477	0.71469	0.86385	0.99900	1.00000
		0.100	0.00426	0.02172	0.10000	0.22311	0.35347	0.51843	0.69504	0.84673	0.99836	1.00000
	100	0.050	0.00191	0.01518	0.10000	0.25615	0.42514	0.62677	0.81266	0.93483	0.99994	1.00000
		0.075	0.00216	0.01608	0.10000	0.25035	0.41244	0.60807	0.79381	0.92260	0.99989	1.00000
		0.100	0.00243	0.01698	0.10000	0.24492	0.40052	0.59016	0.77501	0.90953	0.99978	1.00000

Step 1: Set $\alpha = 0.1$ and the lower lifetime limit $L_U = 0.033829$. Then the lower lifetime limit $L = \frac{1}{\beta}\left(e^{\beta L_U} - 1\right) = 0.035$. Observe the progressive type I interval censored sample $(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8) = (6,6,3,8,3,3,4,1)$ at the pre-set times $(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) = (0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8)$ with censoring schemes of $(R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8) = (2,7,2,3,4,2,1,5)$.

Step 2: If the conforming rate P_r of products is required to exceed 0.818731, then the lifetime performance index target value c_0 should be taken as 0.8 from Table 1. Thus, the testing null hypothesis $H_0\colon C_L \le 0.8$ and the alternative hypothesis $H_a\colon C_L > 0.8$ are constructed.

Step 3: Obtain the MLE of λ numerically as $\hat{\lambda} = 0.9018836$ and then calculate the value of test statistic $\hat{C}_L = 1 - \hat{\lambda}L = 1 - 0.9018836(0.035) = 0.9684341$.

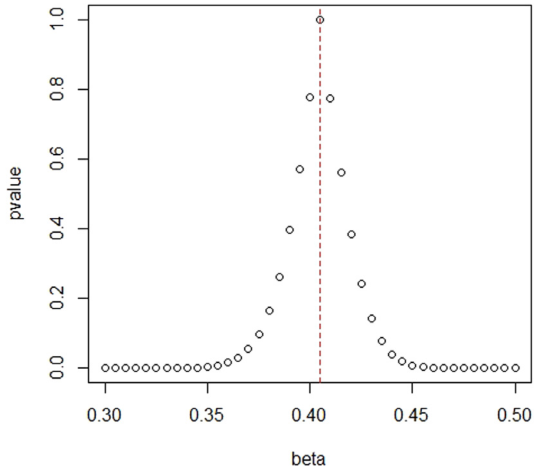
Step 4: For the level of significance $\alpha = 0.1$, we can calculate the critical value $C_L^0 = 1 - L\left(\lambda_0 + Z_{1-\alpha}\sqrt{g(\lambda_0)}\right) = 0.846455682$.

Step 5: Since $\hat{C}_L = 0.9684341 > C_L^0 = 0.846455682$ we concluded to reject the null hypothesis $H_0\colon C_L \le 0.8$ and concluded that the lifetime performance index of electric product is significantly adhered to the desired level 0.8.

Example 2. The tumor-free time (days) of $n = 30$ rats which are injected by an identical amount of tumor cells and are fed by unsaturated fat diets for 200 days were observed in Lee and Wang [21]. The data are given as follows:

Tumor-free time (days) for each of 30 rats

60	63	63	63	66	66	66	68	70	70
77	77	84	91	91	94	98	101	105	108
109	112	112	115	126	143	153	161	164	178



Using the Gini statistic (see Gill and Gastwirth [22]), the plot of Gini test p-value against various values of β from 0.3 to 0.50 is given in the above figure. It shows that the value of $\beta = 0.405$ gives the largest p-value 0.9998514 which strongly indicated the fit of Gompertz distribution. Therefore, the value of β is determined as 0.405. Let the number of inspections $m = 5$ with the equal length of inspection interval $t = 35$ (days) and the pre-specified removal percentages of the remaining survival units given by $(p_1, p_2, p_3, p_4, p_5) = (0.15, 0.15, 0.15, 0.15, 1)$. Based on this setup, we create a progressive type I censored sample from the relief time data. Now we can start to do the proposed testing procedure about C_L as follows:

Step 1: Set $\alpha = 0.05$ and the lower lifetime limit for relief time is $L_U = 7.024468$ days. Then the lower lifetime limit $L = \frac{1}{\beta}\left(e^{\beta L_U} - 1\right) = 40$. Observe the progressive type I interval censored sample $(X_1, X_2, X_3, X_4, X_5) = (0, 7, 8, 5, 2)$ at the pre-set times $(t_1, t_2, t_3, t_4, t_5) = (35,70,105,140,175)$ with censoring schemes of $(R_1, R_2, R_3, R_4, R_5) = (4, 3, 0, 0, 1)$.

Step 2: If the conforming rate P_r of products is desired to exceed 0.904837, then the lifetime performance index target value c_0 should be taken as 0.9 from Table 1. Thus, the testing null hypothesis H_0: $C_L \le 0.9$ and the alternative hypothesis H_a: $C_L > 0.9$ are constructed.

Step 3: Obtain the MLE of λ numerically as $\hat{\lambda} = 0.00107$ and then calculate the value of test statistic $\hat{C}_L = 1 - \hat{\lambda}L = 1 - 0.00107(40) = 0.9572$.

Step 4: For the level of significance $\alpha = 0.05$, we can calculate the critical value $C_L^0 = 1 - L\left(\lambda_0 + Z_{1-\alpha}\sqrt{g(\lambda_0)}\right) = 0.9304834$.

Step 5: Since $\hat{C}_L = 0.9572 > C_L^0 = 0.9304834$, we concluded to reject the null hypothesis H_0: $C_L \le 0.8$. Thus, we can conclude that the lifetime performance index of product does meet the required level.

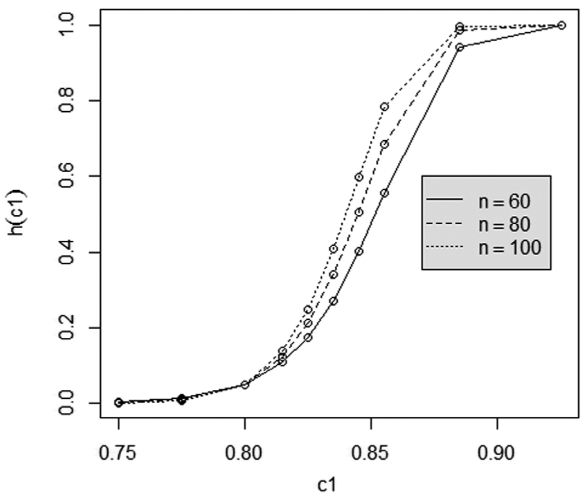


Fig. 1. Power function for the test at $\alpha = 0.05$ under $m = 5$ and $p = 0.05$.

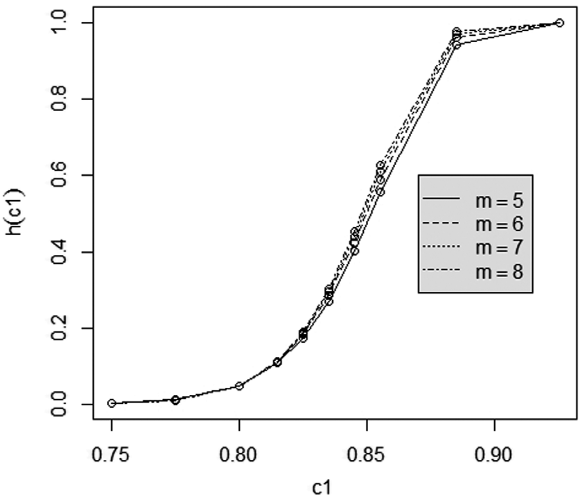


Fig. 2. Power function for the test at $\alpha = 0.05$ under $n = 60$ and $p = 0.05$.

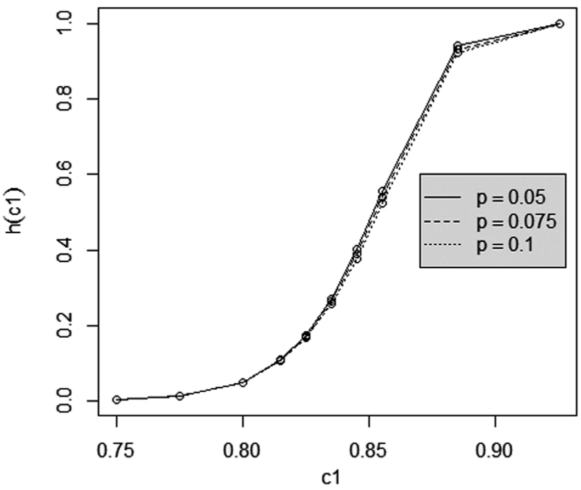


Fig. 3. Power function for the test at $\alpha = 0.05$ under $n = 60$ and $m = 5$.

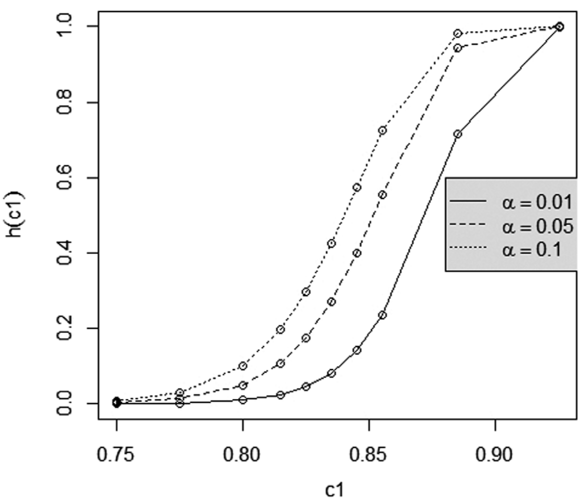


Fig. 4. Power function for the test under $n = 60$, $m = 5$ and $p = 0.05$.

6. Conclusion

The inference about process capability indices is a very important issue in these days since manufacturers frequently utilize them to evaluate the lifetime performance of many products. For the convenience of inspection and some uncontrollable experimental factors, the progressive type I interval censored sample is observed. Therefore in this research, we developed a testing procedure by utilizing the MLE of C_L to test whether the lifetime performance index C_L for products with Gompertz distribution meets the desire level under progressive type I interval censoring. The power analysis under various values of n, m, p and α is analyzed and discussed. At last, two examples are given to demonstrate the testing procedure to assess whether the lifetime performance index reaches the desire target level based on the progressive type I interval censored sample.

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