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Full exceptional collections on the Lagrangian Grassmannians $LG(4, 8)$ and $LG(5, 10)$

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ABSTRACT

We construct full exceptional collections in the derived categories of coherent sheaves on the Lagrangian Grassmannians $LG(4, 8)$ and $LG(5, 10)$. The construction is radically different from all the previously considered homogeneous spaces in that one has to use homogeneous bundles associated with reducible representations of the parabolic subgroup.

0. Introduction

For a smooth projective variety X over a field k let us denote by $D^b(X)$ the bounded derived category of coherent sheaves on X. Starting with the seminal works [1,2] on $D^b(\mathbb{P}^n)$, the techniques involving derived categories of coherent sheaves have been applied to a variety of problems in algebraic geometry (see e.g., [3,4]). However, there are still some open problems in which not much progress has been made since the 80s. Among them is the problem of describing $D^b(X)$ in the case when X is a homogeneous variety. The method of Beilinson in [1] was generalized by Kapranov to the case of quadrics and to partial flag varieties for series A_n (see [5]). Furthermore, it was realized that the relevant structure is that of a *full exceptional collection*, a notion that can be formulated for an arbitrary triangulated category (see [6] and Section 1). The interest in the categories $D^b(X)$ and the question of the existence of full exceptional collections in these categories is also motivated by their appearance in mirror symmetry via the notion of D-branes (see [7]). One of the related problems is the conjecture of Dubrovin [8] stating that if X has semisimple quantum cohomology then $D^b(X)$ admits a full exceptional collection.

It has been conjectured long ago that for every projective homogeneous variety X of a semisimple algebraic group the category $D^b(X)$ admits a full exceptional collection (of vector bundles). However, the only homogeneous varieties of simple groups for which this is known (other than quadrics and partial flag varieties for series A_n) are as follows:

(i) the isotropic Grassmannian of 2-dimensional planes in a symplectic $2n$-dimensional space (see [9]),
(ii) the isotropic Grassmannian of 2-dimensional planes in an orthogonal $2n + 1$-dimensional space (see [9]),
(iii) the full flag variety for the symplectic and the orthogonal groups (see [10]),
(iv) the isotropic Grassmannians of a 6-dimensional symplectic space (see [10]),
(v) the isotropic Grassmannian of 5-dimensional planes in a 10-dimensional orthogonal space and a certain Grassmannian for type G_2 (see [11]).

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In the case of the Cayley plane, the minimal homogeneous variety for E_6, an exceptional collection of 27 vector bundles, that is conjectured to be full, was constructed in [12].

In the present paper we construct full exceptional collections of vector bundles in the derived categories of coherent sheaves of the Lagrangian Grassmannians LG(4, 8) and LG(5, 10), see Theorems 4.1, 4.3 and 5.5. Although this is still only a small step toward the general conjecture, this case is radically different from all the previously known cases of classical type in that we have to consider homogeneous bundles corresponding to reducible representations of the isotropy group. Namely, the new exceptional bundles are constructed as successive extensions of appropriate Schur functors of the universal quotient bundle. This construction of exceptional objects was recently generalized by Alexander Kuznetsov and the first author to a more general setup (in preparation), and we expect to obtain in this way full exceptional collections on all isotropic Grassmannians (symplectic and orthogonal).

Checking that the collections we construct are full is done in both cases using induction and certain partial isotropic flag varieties. However, the computations turn out to be quite involved. It would be very nice to find a more conceptual proof (cf. Remark in Section 1).

The paper is organized as follows. After recalling basic definitions in Section 1 we compute in Section 2 some Ext-spaces between equivariant vector bundles on $\mathrm{LG}(n, 2n)$ using Bott's theorem. Section 3 contains a construction of exceptional bundles on $\mathrm{LG}(n, 2n)$ as extensions between certain Schur functors of the universal quotient bundle. One of these bundles is used in Section 4 to give a full exceptional collection on LG(4, 8). In Section 5 the case of LG(5, 10) is considered. In this case, to get a full exceptional collection one has to construct one more exceptional bundle as a successive extension of certain Schur functors.

1. Basic definitions

We always work over a fixed ground field k that we assume to be algebraically closed of characteristic zero.

Definition. An *exceptional collection* in a triangulated category $\mathcal{D}$ is a collection of objects $E_1, \ldots, E_n$ satisfying the following vanishing conditions:

$$\mathrm{Hom}^*_{\mathcal{D}}(E_j, E_i) = 0 \quad \text{for } i < j, \qquad \mathrm{Hom}^{\neq 0}_{\mathcal{D}}(E_i, E_i) = 0, \qquad \mathrm{Hom}^0_{\mathcal{D}}(E_i, E_i) = k.$$

Definition. A full triangulated subcategory $\mathcal{C} \subset \mathcal{D}$ is called *admissible* if the inclusion functor $\mathcal{C} \to \mathcal{D}$ admits left and right adjoint functors $\mathcal{D} \to \mathcal{C}$.

It is well known that the triangulated subcategory generated by an exceptional collection is admissible (see [13], Thm. 3.2). For a subcategory $\mathcal{C} \subset \mathcal{D}$ one defines the *right orthogonal* $\mathcal{C}^{\perp} \subset \mathcal{D}$ as the full subcategory given by

$$\mathcal{C}^{\perp} = \{A \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(\mathcal{C}, A) = 0\}.$$

It is known that if $\mathcal{C}$ is admissible then $\mathcal{C}^{\perp}$ is also admissible and $\mathcal{C}^{\perp}$ is equivalent to the Verdier quotient $\mathcal{D}/\mathcal{C}$.

Definition. An exceptional collection $(E_1, \ldots, E_n)$ in a triangulated category $\mathcal{D}$ is called *full* if the triangulated subcategory generated by $(E_1, \ldots, E_n)$ is the whole $\mathcal{D}$.

An exceptional collection is full if and only if $(E_1, \ldots, E_n)^{\perp} = 0$.

Remark. It seems plausible that an exceptional collection $(E_1, \ldots, E_n)$ in $\mathcal{D} = \mathrm{D}^b(X)$ such that classes of E_i generate the Grothendieck group $K_0(X)$, is automatically full. Since the category $(E_1, \ldots, E_n)^{\perp}$ is admissible, in the case when all integer cohomology classes on X are algebraic, this would follow from the Nonvanishing conjecture of Kuznetsov (see [14], Conjecture 9.1 and Corollary 9.3) that a nonzero admissible subcategory should have nonzero Hochschild homology.

Let us also recall the definition of the mutation operation. For an exceptional pair (A, B) in a triangulated category $\mathcal{D}$, the *right mutation* is a pair (B, R_BA), where R_BA is defined by the triangle

$$\cdots \longrightarrow R_BA[-1] \longrightarrow A \longrightarrow \mathrm{Hom}^{\bullet}_{\mathcal{D}}(A, B)^* \otimes B \longrightarrow R_BA \longrightarrow \cdots.$$

The pair (B, R_BA) is again exceptional.

2. Applications of Bott's theorem in the case of Lagrangian Grassmannians

Let V be a symplectic vector space of dimension $2n$. Consider the Largangian Grassmannian $\mathrm{LG}(V)$ of V (we also use the notation $\mathrm{LG}(n, 2n)$). We have the basic exact sequence of vector bundles on $\mathrm{LG}(V)$

$$0 \to \mathcal{U} \to V \otimes \mathcal{O} \to Q \to 0 \tag{2.1}$$

where $\mathcal{U} = Q^*$ is the tautological subbundle, and Q is the tautological quotient-bundle. We set $\mathcal{O}(1) = \wedge^n Q$. This is an ample generator of the Picard group of $\mathrm{LG}(V)$. It is well known that the canonical line bundle on $\mathrm{LG}(V)$ is isomorphic to $\mathcal{O}(-n-1)$.

The variety $\mathrm{LG}(V)$ is a homogeneous space for the symplectic group $\mathrm{Sp}(V) = \mathrm{Sp}(2n)$. Namely, it can be identified with $\mathrm{Sp}(2n)/P$, where P is the maximal parabolic associated with the simple root α_n. Here we use the standard numbering of the vertices in the Dynkin diagram D_n as in [15]. Recall that the semisimple part of P is naturally identified with $\mathrm{GL}(n)$. Thus, to every representation of $\mathrm{GL}(n)$ one can associate a homogeneous vector bundle on $\mathrm{LG}(V)$. This correspondence is compatible with tensor products and the standard representation of $\mathrm{GL}(n)$ corresponds to Q. For our purposes it will be convenient to identify the maximal torus of $\mathrm{Sp}(2n)$ with that of $\mathrm{GL}(n) \subset P$. One can easily check that under this identification the half-sum of all the positive roots of $\mathrm{Sp}(2n)$ is equal to

$$\rho = n\epsilon_1 + (n-1)\epsilon_2 + \cdots + \epsilon_n,$$

where (ϵ_i) is the standard basis of the weight lattice corresponding to $\mathrm{GL}(n)$. Note that with respect to this basis the roots of $\mathrm{Sp}(2n)$ are $\pm\epsilon_i$ and $\pm\epsilon_i \pm \epsilon_j$. Thus, a weight $x_1\epsilon_1 + \cdots + x_n\epsilon_n$ is singular for $\mathrm{Sp}(2n)$ if and only if either there exists i such that $x_i = 0$, or there exist $i \neq j$ such that $x_i = \pm x_j$. The Weyl group W of $\mathrm{Sp}(2n)$ is the semidirect product of S_n and $\mathbb{Z}_2^n$ acting by permutations and sign changes $x_i \mapsto -x_i$. A weight $x_1\epsilon_1 + \cdots + x_n\epsilon_n$ is dominant for $\mathrm{Sp}(2n)$ if and only if $x_1 \geq x_2 \geq \cdots \geq x_n \geq 0$.

For a dominant weight $\lambda = (a_1, \ldots, a_n)$ of $\mathrm{GL}(n)$ (where $a_1 \geq a_2 \geq \cdots \geq a_n$), let S^λ denote the corresponding Schur functor (sometimes we omit the tail of zeros in λ). Note that by definition, $S^{(a_1+1,\ldots,a_n+1)} = \det \otimes S^{(a_1,\ldots,a_n)}$. Hence,

$$S^{(a_1+1,\ldots,a_n+1)}Q \simeq S^{(a_1,\ldots,a_n)}Q(1).$$

Our main computational tool is Bott's theorem on cohomology of homogeneous vector bundles. In the case of the Lagrangian Grassmannian $\mathrm{LG}(V)$ it states the following.

Theorem 2.1 (*Theorem* IV′ *of [16]*).
1. *If $\lambda + \rho$ is singular then $H^*(\mathrm{LG}(V), S^\lambda Q) = 0$;*
2. *if $\lambda + \rho$ is non-singular and $w \in W$ is an element of length ℓ such that $\mu = w(\lambda + \rho) - \rho$ is dominant for $\mathrm{Sp}(2n)$, then $H^i(\mathrm{LG}(V), S^\lambda Q) = 0$ for $i \neq \ell$ and $H^\ell(\mathrm{LG}(V), S^\lambda Q)$ is an irreducible representation of $\mathrm{Sp}(2n)$ with the highest weight μ.*

Below we will often abbreviate $H^*(\mathrm{LG}(V), ?)$ to $H^*(?)$.

Lemma 2.2. *One has*
(i) $H^*(\mathcal{O}(i)) = 0$ for $i \in [-n, -1]$; $H^{>0}(\mathcal{O}) = 0$ and $H^0(\mathcal{O}) = k$.
(ii) $H^*(\wedge^k Q(i)) = 0$ for $k \in [1, n-1]$ and $i \in [-n-1, -1]$. Also, for $k \in [1, n-1]$ one has $H^{>0}(\wedge^k Q) = 0$ and $H^0(\wedge^k Q)$ is an irreducible representation of $\mathrm{Sp}(2n)$ with the highest weight $((1)^k, (0)^{n-k})$ (k 1's).

Proof. (i) We have in this case $\lambda + \rho = (n+i, \ldots, 1+i)$ which is singular of $i \in [-n, -1]$. For $i = 0$ we have $\lambda + \rho = \rho$.
(ii) The bundle $\wedge^k Q$ corresponds to the weight $((1)^k, (0)^{n-k})$. Thus, $\wedge^k Q(i)$ corresponds to $\lambda = ((1+i)^k, (i)^{n-k})$, so $\lambda + \rho = (n+1+i, \ldots, n-k+2+i, n-k+i, \ldots, 1+i)$. In the case when $i \in [-n-1, -n-2+k]$ or $i \in [-n+k, -1]$ one of the coordinates is zero. On the other hand, for $i = -n-1+k$ the sum of the kth and $(k+1)$st coordinates is zero. Hence, $\lambda + \rho$ is singular for $i \in [-n-1, -1]$. □

When computing the Ext-groups on $\mathrm{LG}(V)$ between the bundles of the form $S^\lambda Q$ it is useful to observe that

$$(S^{(a_1,\ldots,a_n)}Q)^* \simeq S^{(a_1-a_n, a_1-a_{n-1},\ldots,0)}(-a_1).$$

To compute the tensor products of the Schur functors we use Littlewood–Richardson rule.

Lemma 2.3. *Assume that $n \geq 3$.*
(i) One has $\mathrm{Hom}^*(\wedge^k Q, \wedge^l Q(i)) = 0$ for $i \in [-n, -1]$ and $k, l \in [0, n-2]$. Also, $\mathrm{Hom}^*(\wedge^k Q, \wedge^l Q) = 0$ for $k, l \in [0, n-2]$ and $k > l$. All the bundles $\wedge^k Q$ are exceptional.
(ii) For $k < n$ one has $\mathrm{Hom}^*(\wedge^k Q, \wedge^{k+1} Q) = V$ (concentrated in degree 0). Furthermore, the natural map

$$Q \to \underline{\mathrm{Hom}}(\wedge^k Q, \wedge^{k+1} Q)$$

induces an isomorphism on H^0.

Proof. (i) Recall that

$$\wedge^k Q^* = \wedge^{n-k} Q(-1) = S^{((1)^{n-k},(0)^k)}Q(-1).$$

Therefore, for $k > l$, $k + l \neq n$, the tensor product $\wedge^k Q^* \otimes \wedge^l Q \simeq \wedge^{n-k} Q \otimes \wedge^l Q(-1)$ decomposes into direct summands of the form $S^\lambda Q$ with $\lambda = ((1)^a, (0)^b, (-1)^c)$, where $b > 0$ and $c > 0$. It is easy to see that in this case $\lambda + ((i)^n) + \rho$ will be singular for $i \in [-n, 0]$. Furthermore, even if $k + l = n$ but $l < k < n-1$, we claim that the weights $\lambda + ((i)^n) + \rho$ will still be singular for $i \in [-n, 0]$. Indeed, this follows easily from the fact that $\lambda = ((1)^a, (0)^b, (-1)^c)$ with $c > 0$, and either $b > 0$ or $c > 1$ or $a > 1$. Hence, $\mathrm{Hom}^*(\wedge^k Q, \wedge^l Q(i)) = 0$, where $i \in [-n, 0]$, $n > k > l \geq 0$ and $(k, l) \neq (n-1, 1)$. Using Serre duality we deduce the needed vanishing for the case $k < l$. In the case when $k = l$ the tensor product $\wedge^k Q^* \otimes \wedge^k Q \simeq \wedge^{n-k} Q \otimes \wedge^k Q(-1)$ will contain exactly one summand isomorphic to $\mathcal{O}$, and the other summands of the same form as above with $c > 0$. The same argument as before shows that $\mathrm{Hom}^*(\wedge^k Q, \wedge^k Q(i)) = 0$ for $i \in [-n, -1]$ and that $\mathrm{Hom}^*(\wedge^k Q, \wedge^k Q) = k$ (concentrated in degree 0).

(ii) The tensor product $\wedge^k Q^* \otimes \wedge^{k+1} Q \simeq \wedge^{n-k} Q \otimes \wedge^{k+1} Q(-1)$ decomposes into the direct sum of Q and of summands of the form $S^\lambda Q$ with $\lambda = ((1)^a, (0)^b, (-1)^c)$, where $c > 0$. In the latter case the weight $\lambda + \rho$ is singular, so these summands do not contribute to cohomology. □

Next, for $k \in [1, n-3]$ consider the vector bundle $R_k := S^{(2,(1)^k)}Q$, so that we have a direct sum decomposition

$$Q \otimes \wedge^{k+1} Q = \wedge^{k+2} Q \oplus R_k.$$

One can check that R_k itself is not exceptional but in the next section we are going to construct a related exceptional bundle on $\mathrm{LG}(V)$.

Lemma 2.4. *For $1 \le k \le n-3$, $0 \le l \le n-2$ and $-n \le i \le -1$ one has*

$$\mathrm{Hom}^*(\wedge^l Q, R_k(i)) = \mathrm{Hom}^*(R_k, \wedge^l Q(i)) = 0.$$

Furthermore, for $l > k+1$ one has $\mathrm{Hom}^*(\wedge^l Q, R_k) = 0$, while for $l < k$ one has $\mathrm{Hom}^*(R_k, \wedge^l Q) = 0$.

Proof. By Littlewood–Richardson rule, the tensor product $\wedge^l Q^* \otimes R_k = \wedge^{n-l} Q \otimes R_k(-1)$ decomposes into direct summands of the form S^λ, where λ has one of the following types:
(i) $\lambda = (1, (0)^{k+n-l}, (-1)^{l-k-1})$, provided $l \ge k+1$ (note that $k+n-l \ge 3$);
(ii) $\lambda = ((1)^a, (0)^b, (-1)^c)$, where $1 \le a \le k+1$, $a+b \ge k+1$, $a+b+c = n$, $2a+b = k+n-l+2$;
(iii) $\lambda = (2, (1)^a, (0)^b, (-1)^c)$, where $a \le k$, $a+b \ge k$, $a+b+c = n-1$, $2a+b = k+n-l-1$.

In case (i) the weight $\lambda + ((i)^n) + \rho$ will be singular for $i \in [-n-1, -1]$. In the case $l > k+1$ it will also be singular for $i = 0$. Next, let us consider case (ii). If $b > 0$ then the weight $\lambda + ((i)^n) + \rho$ will be singular for $i \in [-n-1, -1]$ and if in addition $c > 0$ then it will be also singular for $i = 0$. Note that the case $c = 0$ occurs only when $l \le k+1$. In the case $b = 0$ we should have $a = k+1$, so $2 \le a \le n-2$, which implies that $\lambda + ((i)^n) + \rho$ is singular for $i \in [-n-1, 0]$. Finally, let us consider case (iii). If $a > 0$, $b > 0$ and $c > 0$ then the weight $\lambda + ((i)^n) + \rho$ will be singular for $i \in [-n-1, 0]$. The case $c = 0$ can occur only when $l \le k$. In the case $b = 0$ we should have $a = k$, so $c = n-k-1 \ge 2$ which implies that the above weight is still singular for $i \in [-n-1, 0]$. In the case $a = 0$ we have $b = k+n-l-1 \ge 2$, so we deduce that the above weight will be singular for $i \in [-n, 0]$. Note that the case $a = 0$ can occur only for $l \ge k$.

The above analysis shows the vanishing of $\mathrm{Hom}^*(\wedge^l Q, R_k(i))$ for $i \in [-n, -1]$, as well as vanishing of $\mathrm{Hom}^*(\wedge^l Q, R_k)$ for $l > k+1$ and of $\mathrm{Hom}^*(\wedge^l Q, R_k(-n-1))$ for $l < k$. Applying Serre duality we deduce the remaining assertions. □

Note that in the above lemma we have skipped the calculation of $\mathrm{Hom}^*(R_k, \wedge^l Q)$ and $\mathrm{Hom}(\wedge^l Q, R_k)$ for $l = k$ and $l = k+1$. This will be done in the following lemma, where we also prove a number of other auxiliary statements. Let us consider a natural map $f : V \otimes \wedge^{k+1} Q \to R_k$ induced by the projection $Q \otimes \wedge^{k+1} Q \to R$ and the map $V \otimes \mathcal{O} \to Q$.

Lemma 2.5. *Assume $1 \le k \le n-3$. Then one has*
(i) $\mathrm{Hom}^{>0}(\wedge^{k+1} Q, R_k) = 0$ *and* $\mathrm{Hom}^0(\wedge^{k+1} Q, R_k) = V$. *The map f induces an isomorphism on* $\mathrm{Hom}^*(\wedge^{k+1} Q, ?)$.
(ii) *One has* $\mathrm{Hom}^{>0}(\wedge^k Q, R_k) = 0$. *Also, the natural map $Q \otimes Q \to \underline{\mathrm{Hom}}(\wedge^k Q, R_k)$ induces an isomorphism on H^0, so that* $\mathrm{Hom}^0(Q, R_k) \simeq V \otimes V/k$.
(iii) $\mathrm{Hom}^1(R_k, \wedge^k Q) = k$, $\mathrm{Hom}^1(R_k, \wedge^{k+1} Q) = V$, $\mathrm{Hom}^{\ne 1}(R_k, \wedge^k Q) = \mathrm{Hom}^{\ne 1}(R_k, \wedge^{k+1} Q) = 0$. *The natural map $S^2Q^* \to \underline{\mathrm{Hom}}(R_k, \wedge^k Q)$ induces an isomorphism on H^1.*
(iv) $\mathrm{Hom}^{>1}(R_k, R_k) = 0$, $\mathrm{Hom}^0(R_k, R_k) = k$, $\mathrm{Hom}^1(R_k, R_k) = V \otimes V/k$.
(v) $H^*(Q^* \otimes S^2Q^*) = 0$.
(vi) $H^i(Q^* \otimes Q \otimes S^2Q^*) = 0$ *for $i \ne 1$.*

Proof. (i) By Littlewood–Richardson rule we have

$$\wedge^{k+1} Q^* \otimes R_k \simeq \wedge^{n-k-1} Q(-1) \otimes S^{(2,(1)^k)}Q \simeq Q \oplus \cdots$$

where the remaining summands correspond to highest weights $\lambda = (a_1, \ldots, a_n)$ such that $a_n = -1$. For such λ the weight $\lambda + \rho$ is singular, hence these summands do not contribute to cohomology. Thus, the unique embedding of Q into $\underline{\mathrm{Hom}}(\wedge^{k+1} Q, R_k)$ induces an isomorphism on cohomology. This immediately implies the result (recall that $H^*(Q) = V$ by Lemma 2.2).
(ii) Applying Littlewood–Richardson rule again we find

$$\wedge^k Q^* \otimes R_k \simeq S^2Q \oplus \wedge^2 Q \oplus \cdots$$

where the remaining summands correspond to highest weights $\lambda = (a_1, \ldots, a_n)$ with $a_n = -1$. The sum of the first two terms is exactly the image of the natural embedding $Q \otimes Q \to \underline{\mathrm{Hom}}(Q, R)$.
(iii) We have

$$R_k^* \otimes \wedge^k Q \simeq S^2Q^* \oplus \cdots$$

where all the remaining summands correspond to highest weights $\lambda = (a_1, \ldots, a_n)$ such that either $a_n = -1$ or $(a_{n-1}, a_n) = (-1, -2)$. In both cases $\lambda + \rho$ is singular, hence these summands do not contribute to cohomology. for $S^2Q^* =$

$S^{((0)^{n-1},-2)}Q$ one has $\lambda+\rho=(n,\dots,2,-1)$. Hence, applying a simple reflection we get exactly ρ. This means that only H^1 is nonzero, and it is 1-dimensional.

Similarly,

$$R_k^*\otimes\wedge^{k+1}Q\simeq S^{(1,(0)^{n-2},-2)}Q\oplus\cdots$$

where the remaining summands have singular $\lambda+\rho$. For $\lambda=(1,(0)^{n-2},-2)$ we have $\lambda+\rho=(n+1,\dots,2,-1)$. This differs by a single reflection from $\rho+(1,(0)^{n-1})$. Hence only H^1 is nonzero and $H^1(R_k^*\otimes\wedge^{k+1}Q)\simeq V$.

(iv) We have

$$R_k^*\otimes R_k\simeq S^{((2)^{n-2},1,0)}Q(-2)\otimes S^{(2,1)}Q\simeq S^{(2,(0)^{n-2},-2)}Q\oplus S^{(1,1,(0)^{n-3},-2)}Q\oplus\mathcal{O}\oplus\cdots,$$

where the remaining terms do not contribute to cohomology. The first two terms contribute only to H^1. Namely, the corresponding weights $\lambda+\rho$ differ by a single reflection from $\rho+(2,(0)^{n-1})$ and $\rho+(1,1,(0)^{n-2})$, respectively.

(v) We have

$$Q^*\otimes S^2Q^*\simeq S^3Q^*\oplus S^{(2,1)}Q^*\simeq S^{((0)^{n-1},3)}Q\oplus S^{((0)^{n-2},-1,-2)}Q.$$

In both cases $\lambda+\rho$ is singular.

(vi) We have

$$Q\otimes Q^*\otimes S^2Q^*=Q\otimes S^3Q^*\oplus Q\otimes S^{(2,1)}Q^*\simeq(S^{((0)^{n-1},-2)}Q)^{\oplus 2}\oplus\cdots,$$

where the remaining summands do not contribute to cohomology. For the first summand we have $\lambda+\rho=(n,\dots,2,-1)$ which is obtained by applying a simple reflection to a dominant weight. Hence, the cohomology is concentrated in degree 1. □

3. A family of exceptional vector bundles on LG(V)

Let us fix $k\in[1,n-3]$. The natural map $f:V\otimes\wedge^{k+1}Q\to Q\otimes\wedge^{k+1}Q$ is surjective, so we obtain an exact sequence of vector bundles

$$0\to S_k\to V\otimes\wedge^{k+1}Q\xrightarrow{f}R_k\to 0.\tag{3.1}$$

Using the composite nature of f we also get an exact sequence

$$0\to Q^*\otimes\wedge^{k+1}Q\to S_k\to\wedge^{k+2}Q\to 0.\tag{3.2}$$

We have a natural embedding of vector bundles

$$\wedge^kQ\hookrightarrow\underline{\mathrm{Hom}}(Q,\wedge^{k+1}Q)=Q^*\otimes\wedge^{k+1}Q\hookrightarrow S_k.$$

Now we define E_k to be the quotient $S_k/\wedge^kQ$, so that we have an exact sequence

$$0\to\wedge^kQ\to S_k\to E_k\to 0.\tag{3.3}$$

Lemma 3.1. *The exact sequence (3.3) splits canonically, so we have $S_k\simeq\wedge^kQ\oplus E_k$. Furthermore, the bundles $\wedge^kQ$ and E_k are orthogonal to each other, i.e.,*

$$\mathrm{Hom}^*(\wedge^kQ,E_k)=\mathrm{Hom}^*(E_k,\wedge^kQ)=0.$$

Proof. First, we claim that $\mathrm{Hom}^0(S_k,\wedge^kQ)=k$ and $\mathrm{Hom}^i(S_k,\wedge^kQ)=0$ for $i\neq 0$. Indeed, this follows immediately from the exact sequence (3.1) and from Lemma 2.5(iii) since $\mathrm{Hom}^*(\wedge^{k+1}Q,\wedge^kQ)=0$ by Lemma 2.3. Next, using the vanishing of $\mathrm{Hom}^*(\wedge^{k+2}Q,\wedge^kQ)$ and the exact sequence (3.2) we see that the embedding $Q^*\otimes\wedge^{k+1}Q\hookrightarrow S_k$ induces an isomorphism on $\mathrm{Hom}^*(?,\wedge^kQ)$. Hence, the nonzero morphism $S_k\to\wedge^kQ$ restricts to the nonzero morphism $Q^*\otimes\wedge^{k+1}Q\to\wedge^kQ$, unique up to scalar. The latter morphism is proportional to the natural contraction operation. Hence, its restriction to $\wedge^kQ\subset Q^*\otimes\wedge^{k+1}Q$ is nonzero. Therefore, we get a splitting of (3.3). The vanishing of $\mathrm{Hom}^*(E_k,\wedge^kQ)$ also follows. On the other hand, from the exact sequence (3.1), using Lemma 2.5(ii) we get $\mathrm{Hom}^0(\wedge^kQ,S_k)=k$ and $\mathrm{Hom}^i(\wedge^kQ,S_k)=0$ for $i\neq 0$. This implies that $\mathrm{Hom}^*(\wedge^kQ,E_k)=0$. □

By the above lemma we have a unique morphism $S_k\to\wedge^kQ$ extending the identity morphism from $\wedge^kQ\subset S_k$. Pushing forward the extension given by (3.1) under this morphism we get an extension

$$0\to\wedge^kQ\to F_k\to R_k\to 0.\tag{3.4}$$

Furthermore, we also get an exact sequence

$$0\to E_k\to V\otimes\wedge^{k+1}Q\to F_k\to 0.\tag{3.5}$$

Theorem 3.2. *Let $k \in [1, n-3]$. The bundle F_k is the unique nontrivial extension of R_k by $\wedge^k Q$. The bundles E_k and F_k are exceptional, and F_k is the right mutation of E_k through $\wedge^{k+1} Q$. Also, one has $F_k^*(1) \simeq E_{n-2-k}$.*

Proof. *Step* 1. $\mathrm{Hom}^*(\wedge^{k+1} Q, E_k) = \mathrm{Hom}^*(F_k, \wedge^k Q) = 0$. Indeed, the first vanishing follows immediately from the exact sequence (3.1). The second vanishing follows from the exact sequence (3.5) since $\mathrm{Hom}^*(\wedge^{k+1} Q, \wedge^k Q) = 0$ and $\mathrm{Hom}^*(E_k, \wedge^k Q) = 0$ by Lemma 3.1.

Step 2. F_k is a nontrivial extension of R_k by $\wedge^k Q$ (recall that by Lemma 2.5(iii) there is a unique such extension). Indeed, otherwise we would have a surjective map $F_k \to \wedge^k Q$ which is impossible by Step 1.

Step 3. E_k is isomorphic to $F_{n-2-k}^*(1)$. We have $Q^* \otimes \wedge^{k+1} Q \simeq \wedge^k Q \oplus R_{n-k-2}^*(1)$. Therefore, from the exact sequence (3.2) we get an exact sequence

$$0 \to R_{n-2-k}^*(1) \to E_k \to \wedge^{k+2} Q \to 0.$$

We claim that it does not split. Indeed, otherwise we would get an inclusion $\wedge^{k+2} Q \hookrightarrow E_k$ which is impossible since $\mathrm{Hom}(\wedge^{k+1} Q, \wedge^{k+2} Q) \neq 0$ but $\mathrm{Hom}(\wedge^{k+1} Q, E_k) = 0$. Comparing this with the extension (3.4) for $n-2-k$ instead of k we get the result.

Step 4. The natural map

$$H^0(Q \otimes Q) \otimes H^1(S^2 Q^*) \to H^1(Q \otimes Q \otimes S^2 Q^*)$$

is an isomorphism. Indeed, it is easy to check using Bott's theorem that both sides are isomorphic to $V^{\otimes 2}/k$, so it is enough to check surjectivity. Therefore, it suffices to check surjectivity of the maps

$$H^0(Q) \otimes H^1(S^2 Q^*) \to H^1(Q \otimes S^2 Q^*) \quad \text{and}$$
$$H^0(Q) \otimes H^1(Q \otimes S^2 Q^*) \to H^1(Q \otimes Q \otimes S^2 Q^*).$$

Using the exact sequence (2.1) we deduce this from the vanishing of $H^2(Q^* \otimes S^2 Q^*)$ and $H^2(Q^* \otimes Q \otimes S^2 Q^*)$ (see Lemma 2.5(v), (vi)).

Step 5. The composition map

$$\mathrm{Hom}^0(\wedge^k Q, R_k) \otimes \mathrm{Hom}^1(R_k, \wedge^k Q) \to \mathrm{Hom}^1(R_k, R_k)$$

is an isomorphism. Note that by Lemma 2.5(ii)–(iv), both sides are isomorphic to $V^{\otimes 2}/k = S^2 V \oplus \wedge^2 V/k$, so it is enough to check surjectivity. Let us define the natural morphisms

$$\alpha : S^2 Q^* \to R_k^* \otimes \wedge^k Q,$$
$$\beta : Q \otimes Q \to \wedge^k Q^* \otimes R_k,$$

as follows. Consider the Koszul complex for the symmetric algebra $S^* Q$

$$0 \to \wedge^{k+2} Q \xrightarrow{d_1} Q \otimes \wedge^{k+1} Q \xrightarrow{d_2} S^2 Q \otimes \wedge^k Q \to \cdots.$$

Then R_k can be identified with the image of d_2 (or cokernel of d_1). In particular, we have a natural embedding $R_k \to S^2 Q \otimes \wedge^k Q$ which induces α by duality. On the other hand, the natural projection $Q \otimes \wedge^{k+1} Q \to R_k$ gives rise to the composed map

$$Q \otimes Q \otimes \wedge^k Q \xrightarrow{\mathrm{id}_Q \otimes \mu_k} Q \otimes \wedge^{k+1} Q \to R_k$$

where $\mu_k : Q \otimes \wedge^k Q \to \wedge^{k+1} Q$ is given by the exterior product. The map β is obtained from the above map by duality. The morphisms α and β can be combined into a map

$$\gamma : S^2 Q^* \otimes Q \otimes Q \xrightarrow{\alpha \otimes \beta} R_k^* \otimes \wedge^k Q \otimes \wedge^k Q^* \otimes R_k \to R_k^* \otimes R_k,$$

where the last arrow is induced by the trace map on $\wedge^k Q$. By Step 4, it remains to check that the maps α, β and γ induce isomorphisms on cohomology. In fact, we are going to prove that all these maps are embeddings of a direct summand by constructing the maps p_α, p_β and p_γ in the opposite direction such that $p_\alpha \circ \alpha$, $p_\beta \circ \beta$ and $p_\gamma \circ \gamma$ are proportional to identity. To this end we use the Koszul complex for the exterior algebra $\wedge^* Q$

$$\cdots \to S^2 Q \otimes \wedge^k Q \xrightarrow{\delta_2} Q \otimes \wedge^{k+1} Q \xrightarrow{\delta_1} \wedge^{k+2} Q \to 0.$$

We can identify R_k with the kernel of δ_1 (or image of δ_2). Hence, we have natural map $S^2 Q \otimes \wedge^k Q \to R_k$. By duality this corresponds to a map

$$p_\alpha : R_k^* \otimes \wedge^k Q \to S^2 Q^*.$$

On the other hand, we have a natural embedding

$$R_k \to Q \otimes \wedge^{k+1} Q \to Q \otimes Q \otimes \wedge^k Q$$

that gives rise to a map $p_\beta : \wedge^k Q^* \otimes R_k \to Q \otimes Q$. Combining p_α and p_β we obtain a map

$$p_\gamma : R_k^* \otimes R_k \to R_k^* \otimes \wedge^k Q \otimes \wedge^k Q^* \otimes R_k \to S^2 Q^* \otimes Q \otimes Q.$$

A routine calculation proves our claim about the compositions $p_\alpha \circ \alpha$, $p_\beta \circ \beta$ and $p_\gamma \circ \gamma$.

Step 6. Now we can prove that F_k is exceptional (and hence, E_k is also exceptional by Step 3). Applying the functor $\mathrm{Hom}^*(F_k, ?)$ to the exact sequence (3.4) and using Step 1 we get isomorphisms $\mathrm{Hom}^i(F_k, F_k) \simeq \mathrm{Hom}^i(F_k, R_k)$. Next, applying the functor $\mathrm{Hom}^*(?.R_k)$ to the same sequence we get a long exact sequence

$$\cdots \to \mathrm{Hom}^{i-1}(R_k, \wedge^k Q) \to \mathrm{Hom}^i(R_k, R_k) \to \mathrm{Hom}^i(F_k, R_k) \to \mathrm{Hom}^i(R_k, \wedge^k Q) \to \cdots.$$

It remains to apply Lemma 2.5(iii) and Step 5 to conclude that $\mathrm{Hom}^i(F_k, R_k) = 0$ for $i > 0$ and $\mathrm{Hom}^0(F_k, R_k) = k$.

Step 7. To check that F_k is the right mutation of E_k through $\wedge^{k+1} Q$ it remains to prove that $\mathrm{Hom}^i(E_k, \wedge^{k+1} Q) = 0$ for $i \neq 0$ and $\mathrm{Hom}^0(E_k, \wedge^{k+1} Q)$. Applying the functor $\mathrm{Hom}^*(?, \wedge^{k+1} Q)$ to the sequence (3.1) we get by Lemma 2.5(iii) an exact sequence

$$0 \to V \to \mathrm{Hom}^0(S_k, \wedge^{k+1} Q) \to V \to 0$$

along with the vanishing of $\mathrm{Hom}^{>0}(S_k, \wedge^{k+1} Q)$. Since $S_k = \wedge^k Q \oplus E_k$, the assertion follows. □

We are going to compute some Hom-spaces involving the bundles E_k that we will need later.

Lemma 3.3. *Assume that $l \in [0, n-2]$ and $k \in [1, n-3]$. Then for $i \in [-n, -1]$ one has*

$$\mathrm{Hom}^*(\wedge^l Q, E_k(i)) = \mathrm{Hom}^*(E_k, \wedge^l Q(i)) = 0.$$

For $l > k$ one has $\mathrm{Hom}^*(\wedge^l Q, E_k) = 0$, while for $l < k$ one has $\mathrm{Hom}(E_k, \wedge^l Q) = 0$ (recall that for $l = k$ both these spaces vanish by Lemma 3.1).

Proof. It is enough to check similar assertions with S_k instead of E_k. Using the exact sequence (3.1) we reduce the required vanishing for $i \in [-1, -n]$ to Lemmas 2.3(i) and 2.4. To prove the remaining vanishings we use in addition the fact that $\mathrm{Hom}^*(\wedge^{k+1} Q, S_k) = 0$ that follows from Lemma 2.5(i). □

4. The case of LG(4, 8)

Now let us assume that V is 8-dimensional. Let $E = E_1$.

Theorem 4.1. *The following collection on* $\mathrm{LG}(4, 8)$ *is exceptional:*

$$(\mathcal{O}, E, Q, \wedge^2 Q, \mathcal{O}(1), Q(1), \wedge^2 Q(1), \ldots, \mathcal{O}(4), Q(4), \wedge^2 Q(4)).$$

Proof. We already know that all these bundles are exceptional. The required orthogonality conditions follow from Lemmas 2.3, 3.1 and 3.3. □

Lemma 4.2. *Let $\mathcal{C} \subset \mathrm{D}^b(\mathrm{LG}(4, 8))$ be the triangulated subcategory generated by the exceptional collection in Theorem 4.1. Then the following bundles belong to $\mathcal{C}$:*

(i) $Q^*(j), j = 0, \ldots, 4$;
(ii) $S^2 Q(j), j = 0, \ldots, 4$;
(iii) $Q \otimes \wedge^2 Q(j), j = 0, \ldots, 3$;
(iv) $Q \otimes Q^*(j), j = 1, \ldots, 4$.

Proof. *Step* 1. $Q^*(j), S^2 Q^*(j) \in \mathcal{C}$ for $j = 0, \ldots, 4$. Indeed, the fact that $Q^*(j) \in \mathcal{C}$ follows immediately from (2.1). Similarly, the assertion for $S^2 Q^*(j)$ follows from the exact sequence

$$0 \to S^2 Q^* \to S^2 V \otimes \mathcal{O} \to V \otimes Q \to \wedge^2 Q \to 0 \tag{4.1}$$

obtained from (2.1).

Step 2. $Q \otimes Q(j) \in \mathcal{C}$ for $j = 1, 2, 3, 4$. This follows from the exact sequence

$$0 \to \wedge^2 Q^* \to V \otimes Q^* \to S^2 V \otimes \mathcal{O} \to S^2 Q \to 0, \tag{4.2}$$

dual to (4.1), since $\wedge^2 Q^* = \wedge^2 Q(-1)$ and $Q^*(j) \in \mathcal{C}$ by Step 1.

Step 3. $\wedge^2 Q \otimes \wedge^2 Q(2) \in \mathcal{C}$. It follows from the basic sequence (2.1) that $\wedge^4 V \otimes \mathcal{O}(3)$ has a filtration with the consecutive quotients $\mathcal{O}(4)$, $Q^* \otimes Q^*(4)$, $\wedge^2 Q \otimes \wedge^2 Q(2)$, $Q \otimes Q(2)$ and $\mathcal{O}(2)$. All of them except for $\wedge^2 Q \otimes \wedge^2 Q(2)$ belong to $\mathcal{C}$, by Steps 1 and 2. This implies the assertion.

Step 4. $Q^* \otimes Q(j) \in \mathcal{C}$ for $j = 1, 2, 3, 4$. Indeed, tensoring (2.1) with Q we get an exact sequence

$$0 \to Q^* \otimes Q \to V \otimes Q \to Q \otimes Q \to 0,$$

so the assertion follows from Step 2.

Step 5. $Q \otimes \wedge^2 Q \in \mathcal{C}$. First, observe that $S_1 = Q \oplus E \in \mathcal{C}$. Now the exact sequence (3.1) shows that $R_1 \in \mathcal{C}$. But $Q \otimes \wedge^2 Q = \wedge^3 Q \oplus S^{(2,1)}Q = Q^*(1) \oplus R_1$, so it is in $\mathcal{C}$ (recall that $Q^*(1) \in \mathcal{C}$ by Step 1).

Step 6. $Q \otimes \wedge^2 Q(j-1)$, $Q \otimes S^2 Q(j) \in \mathcal{C}$ for $j = 1, 2, 3, 4$. Consider the exact sequence

$$0 \to Q \otimes \wedge^2 Q \to V \otimes Q^* \otimes Q(1) \to S^2 V \otimes Q(1) \to Q \otimes S^2 Q(1) \to 0$$

obtained by tensoring (4.2) with $Q(1)$. Using Steps 4 and 5 we deduce that $Q \otimes S^2 Q(1) \in \mathcal{C}$. Note that the subcategory $\mathcal{C}$ is admissible, so it is closed under passing to direct summands. Since $Q \otimes S^2 Q(1) = S^3 Q(1) \oplus S^{(2,1)} Q(1)$, we derive that $S^{(2,1)}Q(1) \in \mathcal{C}$. This implies that $Q \otimes \wedge^2 Q(1) = Q^*(2) \oplus S^{(2,1)}Q(1) \in \mathcal{C}$ (where $Q^*(2) \in \mathcal{C}$ by Step 1). Now we tensor the above exact sequence by $\mathcal{O}(1)$ and iterate the above argument.

Step 7. $Q \otimes S^3 Q(2) \in \mathcal{C}$. Consider the exact sequence

$$0 \to Q(-1) \to V \otimes \wedge^2 Q(-1) \to S^2 V \otimes Q^* \to S^3 V \otimes \mathcal{O} \to S^3 Q \to 0 \tag{4.3}$$

obtained from (2.1). Tensoring it with $Q(2)$ and using Steps 2, 4 and 6 we deduce the assertion.

Step 8. $S^2 Q \otimes S^2 Q(2) \in \mathcal{C}$. We have $S^2 Q \otimes S^2 Q(2) = Q \otimes S^3 Q(2) \oplus S^{(2,2)} Q(2)$. Hence, by Step 7, it is enough to check that $S^{(2,2)}Q(2) \in \mathcal{C}$. But $S^{(2,2)}Q(2)$ is a direct summand in $\wedge^2 Q \otimes \wedge^2 Q(2)$, so the assertion follows from Step 3.

Step 9. $S^4 Q(1) \in \mathcal{C}$. This follows immediately from the exact sequence

$$0 \to \mathcal{O} \to V \otimes Q \to S^2 V \otimes \wedge^2 Q \to S^3 V \otimes Q^*(1) \to S^4 V \otimes \mathcal{O}(1) \to S^4 Q(1) \to 0$$

deduced from (2.1).

Step 10. $\wedge^2 Q \otimes S^2 Q(1) \in \mathcal{C}$. Consider the exact sequence

$$0 \to \wedge^2 Q^* \to \wedge^2 V \otimes \mathcal{O} \to V \otimes Q \to S^2 Q \to 0$$

deduced from (2.1). Tensoring it with $S^2 Q(2)$ we get the exact sequence

$$0 \to \wedge^2 Q \otimes S^2 Q(1) \to \wedge^2 V \otimes S^2 Q(2) \to V \otimes Q \otimes S^2 Q(2) \to S^2 Q \otimes S^2 Q(2) \to 0.$$

Here all the nonzero terms except for the first one belong to $\mathcal{C}$ by Steps 2, 6 and 8, so the assertion follows.

Step 11. Finally, we are going to deduce that $Q \otimes Q \in \mathcal{C}$. Tensoring (4.3) by $Q(1)$ we get an exact sequence

$$0 \to Q \otimes Q \to V \otimes Q \otimes \wedge^2 Q \to S^2 V \otimes Q^* \otimes Q(1) \to S^3 V \otimes Q(1) \to Q \otimes S^3 Q(1) \to 0.$$

All the nonzero terms except for the first and the last belong to $\mathcal{C}$ by Steps 4 and 5. Thus, it is enough to check that $Q \otimes S^3 Q(1) \in \mathcal{C}$. We have $Q \otimes S^3 Q(1) = S^4 Q(1) \oplus S^{(3,1)} Q(1)$. It remains to observe that $S^4 Q(1) \in \mathcal{C}$ by Step 9, while $S^{(3,1)}Q(1) \in \mathcal{C}$ as a direct summand of $\wedge^2 Q \otimes S^2 Q(1)$ which is in $\mathcal{C}$ by Step 10. □

Theorem 4.3. *The exceptional collection on* LG(4, 8) *considered in Theorem 4.1 is full.*

Proof. Recall that Q is dual to the universal subbundle $\mathcal{U} = \mathcal{U}_4 \subset V \otimes \mathcal{O}$. Taking the dual of the collection in question we obtain the collection

$$(\wedge^2 \mathcal{U}_4(-4), \mathcal{U}_4(-4), \mathcal{O}(-4), \ldots, \wedge^2 \mathcal{U}_4(-1), \mathcal{U}_4(-1), \mathcal{O}(-1), \wedge^2 \mathcal{U}_4, \mathcal{U}, E^*, \mathcal{O}) \tag{4.4}$$

that generates the admissible triangulated subcategory $\mathcal{C}^* \subset \mathrm{D}^b(\mathrm{LG}(4, 8))$. It is enough to check that $\mathcal{C}^* = \mathrm{D}^b(\mathrm{LG}(4, 8))$. Consider the diagram where p and π are natural projections:

$$\begin{tikzcd} & \mathrm{F}_{1,4,8} \arrow[dl, "\pi"'] \arrow[dr, "p"] & \\ \mathbb{P}^7 & & LG(4,8) \end{tikzcd}$$

Here $\mathrm{F}_{1,4,8}$ is the partial flag variety consisting of pairs $(l \subset U)$, where l is a line in a Lagrangian subspace $U \subset V$. The variety $\mathrm{F}_{1,4,8}$ is naturally embedded into the product $\mathbb{P}^7 \times \mathrm{LG}(4, 8)$. Let us denote by $i : \mathrm{F}_{1,4,8} \hookrightarrow \mathbb{P}^7 \times \mathrm{LG}(4, 8)$ the natural embedding. Consider the fiber $\pi^{-1}(x)$ over a point x in $\mathbb{P}^7$. The variety $\pi^{-1}(x)$ is isomorphic to the Lagrangian Grassmannian LG(3, 6).

There is a rank three vector bundle $\mathcal{U}_3$ on $F_{1,4,8}$ such that its restriction to any fiber $\pi^{-1}(x)$ is isomorphic to the universal bundle over this fiber. Recall that the derived category of coherent sheaves on $\pi^{-1}(x)$ has a full exceptional collection:

$$(\mathcal{U}_3|_{\pi^{-1}(x)} \otimes \mathcal{O}_\pi(-3), \mathcal{O}_{\pi^{-1}(x)}(-3), \mathcal{U}_3|_{\pi^{-1}(x)} \otimes \mathcal{O}_\pi(-2), \mathcal{O}_{\pi^{-1}(x)}(-2), \dots, \mathcal{U}_3|_{\pi^{-1}(x)}, \mathcal{O}). \tag{4.5}$$

Here $\mathcal{O}_\pi(-1)$ is a line bundle that is isomorphic to $\det \mathcal{U}_3$. Therefore, by Theorem 3.1 of [10], the category $D^b(F_{1,4,8})$ has a semiorthogonal decomposition:

$$D^b(F_{1,4,8}) = \langle \pi^* D^b(\mathbb{P}^7) \otimes \mathcal{U}_3 \otimes \mathcal{O}_\pi(-3), \pi^* D^b(\mathbb{P}^7) \otimes \mathcal{O}_\pi(-3), \dots, \pi^* D^b(\mathbb{P}^7) \otimes \mathcal{U}_3, \pi^* D^b(\mathbb{P}^7)\rangle. \tag{4.6}$$

There is a short exact sequence of vector bundles on $F_{1,4,8}$:

$$0 \to \pi^*\mathcal{O}(-1) \to p^*\mathcal{U}_4 \to \mathcal{U}_3 \to 0. \tag{4.7}$$

Taking determinants we get an isomorphism of line bundles $p^*\mathcal{O}(-1) = \pi^*\mathcal{O}(-1) \otimes \mathcal{O}_\pi(-1)$. Therefore, we can replace $\mathcal{O}_\pi(i)$ by $p^*\mathcal{O}(i)$ in the above semiorthogonal decomposition. Thus, to prove the statement it is sufficient to show that all the subcategories

$$p_*(\pi^* D^b(\mathbb{P}^7) \otimes \mathcal{U}_3) \otimes \mathcal{O}(j), p_*(\pi^* D^b(\mathbb{P}^7)) \otimes \mathcal{O}(j), \quad \text{for } j = 0, \dots, -3$$

belong to $\mathcal{C}^*$.

The functor $p_*\pi^* : D^b(\mathbb{P}^7) \to D^b(LG(4, 8))$ can be computed using the Koszul resolution of the sheaf $i_*\mathcal{O}_{F_{1,4,8}}$ on $\mathbb{P}^7 \times LG(4, 8)$:

$$0 \to \pi^*\mathcal{O}(-4) \otimes p^*\mathcal{O}(-1) \to \cdots \to \pi^*\mathcal{O}(-2) \otimes \wedge^2 p^*\mathcal{U}_4 \to \pi^*\mathcal{O}(-1) \otimes p^*\mathcal{U}_4 \to \mathcal{O} \to i_*\mathcal{O}_{F_{1,4,8}} \to 0. \tag{4.8}$$

Using this resolution we immediately check the inclusion

$$p_*(\pi^* D^b(\mathbb{P}^7)) \otimes \mathcal{O}(j) \subset \langle \mathcal{O}(j-1), \wedge^3 \mathcal{U}_4(j) = \mathcal{U}_4^*(j-1), \wedge^2 \mathcal{U}_4(j), \mathcal{U}_4(j), \mathcal{O}(j)\rangle.$$

By Lemma 4.2(i), for $j = -3, \dots, 0$ the right-hand side belongs to $\mathcal{C}^*$.

Next, using the sequence (4.7) we see that to prove the inclusion $p_*(\pi^* D^b(\mathbb{P}^7) \otimes \mathcal{U}_3) \otimes \mathcal{O}(j) \subset \mathcal{C}^*$ it is enough to check that

$$\langle \mathcal{U}_4(j-1), \mathcal{U}_4 \otimes \mathcal{U}_4^*(j-1), \mathcal{U}_4 \otimes \wedge^2 \mathcal{U}_4(j), \mathcal{U}_4 \otimes \mathcal{U}_4(j), \mathcal{U}_4(j)\rangle \subset \mathcal{C}^*$$

for $j = -3, \dots, 0$. It remains to apply Lemma 4.2 (and dualize). □

Another version of the proof. We can simplify computations in the above argument by using a different semiorthogonal decomposition of $D^b(F_{1,4,8})$:

$$D^b(F_{1,4,8}) = \langle \pi^* D^b(\mathbb{P}^7) \otimes \mathcal{U}_3 \otimes p^*\mathcal{O}(-3), \pi^* D^b(\mathbb{P}^7) \otimes p^*\mathcal{O}(-3), \dots, \pi^* D^b(\mathbb{P}^7), \pi^* D^b(\mathbb{P}^7) \otimes \mathcal{U}_3^*\rangle.$$

The restriction of this decomposition to the fiber $\pi^{-1}(x) \simeq LG(3, 6)$ is the exceptional collection obtained from collection (4.5) by the right mutation of $\mathcal{U}_3|_{\pi^{-1}(x)}$ through $\mathcal{O}$. In the same way as above we check that

$$p_*(\pi^* D^b(\mathbb{P}^7)) \otimes \mathcal{O}(j) \subset \mathcal{C}^* \quad \text{for } j = -3, \dots, 0,$$
$$p_*(\pi^* D^b(\mathbb{P}^7) \otimes \mathcal{U}_3) \otimes \mathcal{O}(j) \subset \mathcal{C}^* \quad \text{for } j = -1, -2, -3, \quad \text{and}$$
$$p_*(\pi^*\mathcal{O}(i) \otimes \mathcal{U}_3^*) \in \mathcal{C}^* \quad \text{for } i = -6, \dots, 0.$$

The point is that this will only require using (easy) Steps 1, 2, 4–6 of Lemma 4.2. Thus, if we consider the semiorthogonal decomposition

$$D^b(F_{1,4,8}) = \langle \mathcal{A}, \langle \pi^*\mathcal{O}(1) \otimes \mathcal{U}_3^*\rangle\rangle,$$

where $\mathcal{A} = \langle \pi^*\mathcal{O}(1) \otimes \mathcal{U}_3^*\rangle^\perp$, then $p_*\mathcal{A} \in \mathcal{C}^*$. By adjointness, it follows that for an object $E \in D^b(LG(4, 8))$ such that $\mathrm{Hom}(E, \mathcal{C}^*) = 0$, one has $p^*E \in \langle \pi^*\mathcal{O}(1) \otimes \mathcal{U}_3^*\rangle$, i.e., $p^*E \simeq V^\bullet \otimes \pi^*\mathcal{O}(1) \otimes \mathcal{U}_3^*$ for a graded vector space $V^\bullet$. Hence, $E \simeq V^\bullet \otimes p_*(\pi^*\mathcal{O}(1) \otimes \mathcal{U}_3^*)$. Finally, using resolution (4.8) and the dual of sequence (4.7) one can compute that

$$p_*(\pi^*\mathcal{O}(1) \otimes \mathcal{U}_3^*) \simeq \wedge^2 \mathcal{U}_4^* \simeq \wedge^2 \mathcal{U}_4(1).$$

Thus, $E \simeq V^\bullet \otimes \wedge^2 \mathcal{U}_4(1)$. But $\mathrm{Hom}^*(\wedge^2 \mathcal{U}_4(1), \wedge^2 \mathcal{U}_4(-4)) \neq 0$ by Serre duality, so the condition $\mathrm{Hom}^*(E, \mathcal{C}^*) = 0$ implies that $V^\bullet = 0$. □

5. The case of LG(5, 10)

In this section we assume that $n = 5$ (so V is 10-dimensional). It turns out that in this case the exceptional bundles constructed so far do not generate the entire derived category $D^b(\mathrm{LG}(5, 10))$. We are going to construct another exceptional bundle on LG(5, 10) starting from the bundle $T = S^{(3,1,1)}Q$. Let us denote by ω_i the ith fundamental weight of the root system C_5. For a dominant weight λ we denote by $V(\lambda)$ the corresponding irreducible representation of $\mathrm{Sp}(10)$ (for example, $V(\omega_1) = V$, $V(\omega_2) = \wedge^2 V/k$, $V(2\omega_1) = S^2V$).

Lemma 5.1. *Assume that* $n = 5$.

(i) $\mathrm{Hom}^*(\wedge^i Q, T(j)) = 0$ *for* $i \in [0, 3]$, $j \in [-5, -1]$. *Also,* $\mathrm{Hom}^*(T, \mathcal{O}) = 0$.
(ii) $\mathrm{Hom}^*(R_1, T(j)) = 0$ *for* $j \in [-5, -1]$.
(iii) $\mathrm{Hom}^*(T, T(-3)) = 0$.
(iv) $\mathrm{Hom}^i(T, T) = 0$ *for* $i > 2$, $\mathrm{Hom}^2(T, T) = V(2\omega_1 + \omega_2) \oplus V(\omega_1 + \omega_3)$, $\mathrm{Hom}^1(T, T) = V^{\otimes 2}/k \oplus S^2V$, $\mathrm{Hom}^0(T, T) = k$.
(v) $\mathrm{Hom}^i(\wedge^3 Q, T) = 0$ *for* $i > 0$ *and* $\mathrm{Hom}^0(\wedge^3 Q, T) = S^2V$. *Also,* $\mathrm{Hom}^i(T, \wedge^3 Q) = 0$ *for* $i \neq 1, 2$, $\mathrm{Hom}^1(T, \wedge^3 Q) = k$ *and* $\mathrm{Hom}^2(T, \wedge^3 Q) = \wedge^2 V/k$.
(vi) $\mathrm{Hom}^i(\wedge^2 Q, T) = 0$ *for* $i > 0$ *and* $\mathrm{Hom}^0(\wedge^2 Q, T) = V(3\omega_1) \oplus V(\omega_1 + \omega_2)$. *Also,* $\mathrm{Hom}^i(T, \wedge^2 Q) = 0$ *for* $i \neq 2$ *and* $\mathrm{Hom}^2(T, \wedge^2 Q) = V$.
(vii) $\mathrm{Hom}^i(Q, T) = 0$ *for* $i > 0$ *and* $\mathrm{Hom}^0(Q, T) = V(2\omega_1 + \omega_2) \oplus V(\omega_1 + \omega_3)$. *Also,* $\mathrm{Hom}^i(T, Q) = 0$ *for* $i \neq 2$ *and* $\mathrm{Hom}^2(T, Q) = k$.
(viii) $\mathrm{Hom}^i(R_1, T) = 0$ *for* $i > 1$, $\mathrm{Hom}^1(R_1, T) = V(2\omega_1 + \omega_2) \oplus V(\omega_1 + \omega_3)$ *and* $\mathrm{Hom}^0(R_1, T) = V^{\otimes 2}/k$. *Also,* $\mathrm{Hom}^i(T, R_1) = 0$ *for* $i \neq 1, 2$, $\mathrm{Hom}^1(T, R_1) = k$ *and* $\mathrm{Hom}^2(T, R_1) = V^{\otimes 2}/k$.
(ix) $\mathrm{Hom}^*(T, S^2Q^*) = 0$.
(x) $\mathrm{Hom}^i(T, S^3Q^*) = 0$ *for* $i \neq 4$.
(xi) $\mathrm{Hom}^i(T, R_3) = 0$ *for* $i \neq 1$.
(xii) $\mathrm{Hom}^i(T, \wedge^3 Q \otimes Q^*) = 0$ *for* $i \neq 2$.
(xiii) $\mathrm{Hom}^4(T, \wedge^3 Q \otimes \wedge^2 Q^*) = 0$.

The proof is a straightforward application of Bott's theorem. By part (viii) of the above lemma, we have a canonical nonsplit extension of vector bundles

$$0 \to R_1 \to P \to T \to 0. \tag{5.1}$$

Lemma 5.2. (i) *The map* $\mathrm{Hom}^1(R_1, Q) \to \mathrm{Hom}^2(T, Q)$ *induced by* (5.1) *is an isomorphism.*
(ii) *The map* $\mathrm{Hom}^1(R_1, \wedge^2 Q) \to \mathrm{Hom}^2(T, \wedge^2 Q)$ *induced by* (5.1) *is an isomorphism.*
(iii) *The map* $\mathrm{Hom}^1(R_1, R_1) \to \mathrm{Hom}^2(T, R_1)$ *induced by* (5.1) *is an isomorphism.*
(iv) *The map* $\mathrm{Hom}^1(R_1, T) \to \mathrm{Hom}^2(T, T)$ *induced by* (5.1) *is an isomorphism, while the map* $\mathrm{Hom}^0(R_1, T) \to \mathrm{Hom}^1(T, T)$ *is injective.*
(v) *One has* $\mathrm{Hom}^*(P, Q) = \mathrm{Hom}^*(P, \wedge^2 Q) = \mathrm{Hom}^*(P, R_1) = \mathrm{Hom}^{>1}(P, P) = 0$ *and* $\mathrm{Hom}^1(P, P) = S^2V$, $\mathrm{Hom}^0(P, P) = k$. *Also,* $\mathrm{Hom}^i(P, \wedge^3 Q) = 0$ *for* $i \neq 1$ *and* $\mathrm{Hom}^1(P, \wedge^3 Q) = k$.

Proof. (i) We have to check that the natural map

$$\mathrm{Hom}^1(R_1, Q) \otimes \mathrm{Hom}^1(T, R_1) \to \mathrm{Hom}^2(T, Q)$$

is an isomorphism. Note that both sides are 1-dimensional (see Lemmas 2.5(iii) and 5.1(vii), (viii)), so it is enough to check that this map is nonzero. We have natural embeddings $S^2Q^* \to R_1^* \otimes Q$ and $S^2Q^* \to T^* \otimes R_1$ inducing isomorphisms on H^1. Let us consider the induced map

$$\alpha : S^2Q^* \otimes S^2Q^* \to T^* \otimes Q.$$

Note that

$$S^2Q^* \otimes S^2Q^* = S^4Q^* \oplus S^{(2,2)}Q^* \oplus S^{(3,1)}Q^*,$$

where the first two terms have zero cohomology while the last term has 1-dimensional H^2. Thus, it is enough to check that the restriction of α to $S^{(3,1)}Q^*$ is nonzero and that the natural map

$$H^1(S^2Q^*) \otimes H^1(S^2Q^*) \to H^2(S^2Q^* \otimes S^2Q^*)$$

between 1-dimensional spaces is nonzero. Let us start by splitting the exact sequence (4.1) into two short exact sequences

$$0 \to S^2Q^* \to S^2V \otimes \mathcal{O} \to K \to 0 \tag{5.2}$$

$$0 \to K \to V \otimes Q \to \wedge^2 Q \to 0. \tag{5.3}$$

Then (5.2) induces the surjections $H^0(K) \to H^1(S^2Q^*)$ and $H^1(K \otimes S^2Q^*) \to H^2(S^2Q^* \otimes S^2Q^*)$ (by the vanishing of $H^1(\mathcal{O})$ and $H^2(S^2Q^*)$). Hence, it is enough to check that the natural map

$$H^0(K) \otimes H^1(S^2Q^*) \to H^1(K \otimes S^2Q^*)$$

is an isomorphism (note that both sides are isomorphic to $S^2V \oplus k$). Now the sequence (5.3) induces embeddings $H^0(K) \to V \otimes H^0(Q)$ and $H^1(K \otimes S^2Q^*) \to V \otimes H^1(Q \otimes S^2Q^*)$ (by the vanishing of $H^0(\wedge^2 Q \otimes S^2Q^*)$). Hence, we are reduce to proving that the map

$$H^0(Q) \otimes H^1(S^2Q^*) \to H^1(Q \otimes S^2Q^*)$$

is an isomorphism. But this follows from the exact sequence (2.1) and the vanishing of $H^*(Q^* \otimes S^2Q^*)$.

It remains to check that the restriction of α to $S^{3,1}Q^* \subset S^2Q^* \otimes S^2Q^*$ is nonzero (where we can just think of Q as a vector space). Let us view T (resp., R_1) as the image of the Koszul differential $S^2Q \otimes \wedge^3 Q \to S^3Q \otimes \wedge^2 Q$ (resp., $Q \otimes \wedge^2 Q \to S^2Q \otimes Q$). Then the embedding $S^2Q^* \hookrightarrow T^* \otimes R_1$ corresponds to the composed map

$$S^2Q^* \otimes T \to S^2Q^* \otimes S^3Q \otimes \wedge^2 Q \to Q \otimes \wedge^2 Q \to R_1, \tag{5.4}$$

where the second arrow is induced by the natural map $S^2Q^* \otimes S^3Q \to Q$. On the other hand, the embedding $S^2Q^* \hookrightarrow R_1^* \otimes Q$ corresponds to the natural map $S^2Q^* \otimes R_1 \to Q$ induced by the embedding $R_1 \to S^2Q \otimes Q$. Thus, α corresponds to the composed map

$$\alpha' : S^2Q^* \otimes S^2Q^* \otimes T \to S^2Q^* \otimes S^2Q^* \otimes S^3Q \otimes \wedge^2 Q \to S^2Q^* \otimes Q \otimes \wedge^2 Q \to S^2Q^* \otimes S^2Q \otimes Q \to Q,$$

where the third arrow is induced by the Koszul differential. Let us choose a basis $e_1, \ldots, e_n$ for Q and define an element $t \in T$ by

$$t = e_4^2 e_1 \otimes (e_2 \wedge e_3) + e_4^2 e_2 \otimes (e_3 \wedge e_1) + e_4^2 e_3 \otimes (e_1 \wedge e_2),$$

where we view T as a subbundle in $S^3Q \otimes \wedge^2 Q$. Then one can compute the induced functional on $S^2Q^* \otimes S^2Q^*$

$$x \mapsto \langle \alpha'(x \otimes t), e_3 \rangle = 2\langle x, (e_0e_2) \wedge (e_0e_1) \rangle,$$

where $x \in S^2Q^* \otimes S^2Q^*$. Now we observe that $S^{(3,1)}Q$ can be identified with the image of $\wedge^2(S^2Q)$ under the natural map $\beta : S^2Q \otimes S^2Q \to S^3Q \otimes Q$ given by

$$\beta(f \otimes (v_1v_2)) = (fv_1) \otimes v_2 + (fv_2) \otimes v_1.$$

Finally we compute that

$$\beta((e_0e_2) \wedge (e_0e_1)) = (e_4^2e_2) \otimes e_1 - (e_4^2e_1) \otimes e_2 \neq 0,$$

which finishes the proof.

(ii) Since both the source and the target are isomorphic to V, it is enough to check surjectivity. Furthermore, it suffices to prove that the composition map

$$\mathrm{Hom}^1(T, R_1) \otimes \mathrm{Hom}^1(R_1, Q) \otimes \mathrm{Hom}^0(Q, \wedge^2 Q) \to \mathrm{Hom}^2(T, \wedge^2 Q)$$

is surjective. By part (i), this reduces to surjectivity of the composition map

$$\mathrm{Hom}^2(T, Q) \otimes \mathrm{Hom}^0(Q, \wedge^2 Q) \to \mathrm{Hom}^2(T, \wedge^2 Q).$$

Looking at the exact sequence (5.3), we see that this would follow from the vanishing of $\mathrm{Hom}^3(T, K)$. But this vanishing follows from the exact sequence (5.2) since $\mathrm{Hom}^*(T, \mathcal{O}) = \mathrm{Hom}^*(T, S^2Q^*) = 0$ (see Lemma 5.1(i), (ix)).

(iii) Both the source and the target are isomorphic to $V^{\otimes 2}/k$ (see Lemmas 2.5(iv) and 5.1(viii)), so it suffices to check surjectivity. By part (ii), it is enough to prove that the map

$$\mathrm{Hom}^2(T, \wedge^2 Q) \otimes V \to \mathrm{Hom}^2(T, R_1) \tag{5.5}$$

is surjective. Let us first check that $S^2V \subset \mathrm{Hom}^2(T, R_1)$ is in the image. The exact sequence (2.1) induces a long exact sequence

$$\cdots \to \mathrm{Hom}^2(T, \wedge^2 Q \otimes Q^*) \to \mathrm{Hom}^2(T, \wedge^2 Q) \otimes V \xrightarrow{f} \mathrm{Hom}^2(T, \wedge^2 Q \otimes Q) \to \cdots$$

Using Bott's theorem one can check that $\mathrm{Hom}^2(T, \wedge^2 Q \otimes Q^*)$ does not contain any factors isomorphic to S^2V, so the restriction of f to S^2V is an embedding. On the other hand,

$$\mathrm{Hom}^2(T, \wedge^2 Q \otimes Q) = \mathrm{Hom}^2(T, R_1) \oplus \mathrm{Hom}^2(T, \wedge^3 Q),$$

where the second factor is $\wedge^2 V/k$, hence, S^2V projects nontrivially to $\mathrm{Hom}^2(T, R_1)$. It remains to check that $\wedge^2 V/k \subset \mathrm{Hom}^2(T, R_1)$ is in the image of the map (5.5). It suffices to prove that it is in the image of the map

$$\mathrm{Hom}^2(T, Q \otimes Q) \otimes V \to \mathrm{Hom}^2(T, R_1),$$

or even

$$\mathrm{Hom}^2(T, Q) \otimes H^0(\wedge^2 Q) \to \mathrm{Hom}^2(T, R_1).$$

We have a natural map

$$\gamma : S^2Q^* \otimes \wedge^2 Q^* \to T^* \otimes Q,$$

such that its composition with the embedding $T^* \otimes Q \hookrightarrow S^2Q^* \otimes \wedge^3 Q^* \otimes Q$ (induced by the surjection $S^2Q \otimes \wedge^3 Q \to T$) is the identity map on S^2Q^* tensored with the natural embedding $\wedge^2 Q^* \to \wedge^3 Q^* \otimes Q$. Note that this implies that γ itself is an embedding. Hence, γ induces an isomorphism on H^2. Next, we claim that the composition map

$$H^2(S^2Q^* \otimes \wedge^2 Q^*) \otimes H^0(\wedge^2 Q) \to H^2(S^2Q^* \otimes \wedge^2 Q^* \otimes \wedge^2 Q)$$

is surjective. Indeed, it is enough to check this with $H^0(\wedge^2 Q)$ replaced by $\wedge^2 V$. Then the exact sequence

$$0 \to S^2Q^* \to V \otimes Q^* \to \wedge^2 V \otimes \mathcal{O} \to \wedge^2 Q \to 0$$

shows that this follows from the vanishing of $H^3(S^2Q^* \otimes \wedge^2 Q^* \otimes Q^*)$ and $H^4(S^2Q^* \otimes \wedge^2 Q^* \otimes S^2Q^*)$, both of which are easily checked using Bott's theorem. Now it remains to prove that the composed map

$$S^2Q^* \otimes \wedge^2 Q^* \otimes \wedge^2 Q \xrightarrow{\gamma\otimes \mathrm{id}} T^* \otimes Q \otimes \wedge^2 Q \to T^* \otimes R_1$$

induces an embedding on H^2. It is enough to prove that the kernel of this map is S^2Q^*. Using the embedding of T^* into $S^2Q^* \otimes \wedge^3 Q^*$ this reduces to checking that the composition of the natural maps

$$\wedge^2 Q^* \otimes \wedge^2 Q \to \wedge^3 Q^* \otimes Q \otimes \wedge^2 Q \to \wedge^3 Q^* \otimes R_1$$

has $\mathcal{O}$ as a kernel. Replacing this map by its composition with the embedding $\wedge^3 Q^* \otimes R_1 \hookrightarrow \wedge^3 Q^* \otimes S^2Q \otimes Q$ we see that it is enough to prove the following fact from linear algebra. Suppose we have a linear map $A : \wedge^2 Q \to \wedge^2 Q$ such that the induced map

$$\wedge^3 Q \to Q \otimes \wedge^2 Q \xrightarrow{\mathrm{id}\otimes A} Q \otimes \wedge^2 Q \xrightarrow{d} S^2Q \otimes Q$$

is zero, where d is Koszul differential. Then A is proportional to identity. To prove this statement we recall that the kernel of d is exactly $\wedge^3 Q \subset Q \otimes \wedge^2 Q$. Thus, the condition on A is that $\mathrm{id}_Q \otimes A$ preserves $\wedge^3 Q \subset Q \otimes \wedge^2 Q$. Let us fix some basis (e_i) of Q and let $\partial_i : \wedge^3 Q \to \wedge^2 Q$ be the odd partial derivatives corresponding to the dual basis of Q^*. Consider

$$e_1 \otimes A(e_2 \wedge e_3) + e_2 \otimes A(e_3 \wedge e_1) + e_3 \otimes A(e_1 \wedge e_2) = \eta \in \wedge^3 Q \subset Q \otimes \wedge^2 Q.$$

Contracting with e_3^* in the first factor of the tensor product $Q \otimes \wedge^2$ we obtain $A(e_1 \wedge e_2) = \partial_3 \eta$. Hence, $\partial_3 A(e_1 \wedge e_2) = \partial_3^2 \eta = 0$. In a similar way $\partial_i A(e_1 \wedge e_2) = 0$ for $i > 2$. It follows that $A(e_1 \wedge e_2)$ is proportional to $e_1 \wedge e_2$. Thus, for every pair of elements $x, y \in Q$, $A(x \wedge y)$ is proportional to $x \wedge y$. This implies that A is proportional to identity.

(iv) We have $\mathrm{Hom}^1(R_1, T) \simeq \mathrm{Hom}^2(T, T)$ (see Lemma 5.1(iv), (viii)), so for the first assertion it is enough to check the surjectivity. By part (ii), it suffices to check that the map

$$\mathrm{Hom}^2(T, \wedge^2 Q) \otimes \mathrm{Hom}^0(\wedge^2 Q, T) \to \mathrm{Hom}^2(T, T)$$

is surjective. Furthermore, it is enough to prove that the map

$$\mathrm{Hom}^2(T, \wedge^2 Q) \otimes \mathrm{Hom}^0(\wedge^2 Q, \wedge^3 Q) \otimes \mathrm{Hom}^0(\wedge^3 Q, T) \to \mathrm{Hom}^2(T, T)$$

is surjective. We are going to do this in two steps: first, we will check that the map

$$\mathrm{Hom}^2(T, \wedge^2 Q) \otimes V \to \mathrm{Hom}^2(T, \wedge^3 Q) \tag{5.6}$$

is surjective, and then we will show the surjectivity of

$$\mathrm{Hom}^2(T, \wedge^3 Q) \otimes \mathrm{Hom}^0(\wedge^3 Q, T) \to \mathrm{Hom}^2(T, T). \tag{5.7}$$

From the exact sequence (2.1) we get the following long exact sequence

$$0 \to S^3Q^* \to S^3V \otimes \mathcal{O} \to S^2V \otimes Q \to V \otimes \wedge^2 Q \to \wedge^3 Q \to 0.$$

Thus, the surjectivity of (5.6) follows from the vanishing of $\mathrm{Hom}^3(T, Q)$, $\mathrm{Hom}^4(T, \mathcal{O})$ and $\mathrm{Hom}^5(T, S^3Q^*)$ (see Lemma 5.1(i), (vii), (x)). To deal with (5.7) we use the natural embedding $S^2Q \to \wedge^3 Q^* \otimes T$ inducing an isomorphism on H^0. Note also that since $\wedge^3 Q \otimes S^2Q \simeq T \oplus R_3$, Lemma 5.1(xi) implies that the projection $T^* \otimes \wedge^3 Q \otimes S^2Q \to T^* \otimes T$ induces an isomorphism on H^2. Thus, we are reduced to showing the surjectivity of

$$\mathrm{Hom}^2(T, \wedge^3 Q) \otimes H^0(S^2Q) \to \mathrm{Hom}^2(T, \wedge^3 Q \otimes S^2Q).$$

It suffices to prove the surjectivity of the maps

$$\mathrm{Hom}^2(T, \wedge^3 Q) \otimes V \to \mathrm{Hom}^2(T, \wedge^3 Q \otimes Q),$$
$$\mathrm{Hom}^2(T, \wedge^3 Q \otimes Q) \otimes V \to \mathrm{Hom}^2(T, \wedge^3 Q \otimes S^2Q).$$

The exact sequence (2.1) shows that the surjectivity of the first map follows from the vanishing of $\mathrm{Hom}^3(T, \wedge^3 Q \otimes Q^*)$ (see Lemma 5.1(xii)). Similarly, for the second map we use the exact sequence

$$0 \to \wedge^2 Q^* \to \wedge^2 V \otimes \mathcal{O} \to V \otimes Q \to S^2Q \to 0$$

along with the vanishing of $\mathrm{Hom}^3(T, \wedge^3 Q)$ and $\mathrm{Hom}^4(T, \wedge^3 Q \otimes \wedge^2 Q^*)$ (see Lemma 5.1(v), (xiii)).

Now let us prove the injectivity of the map $\mathrm{Hom}^0(R_1, T) \to \mathrm{Hom}^1(T, T)$. We have a natural embedding $S^2Q \to R_1^* \otimes T$ inducing isomorphism on H^0 and an embedding $S^2Q^* \to T^* \otimes R_1$ inducing isomorphism on H^1. We claim that the composed map

$$S^2Q \otimes S^2Q^* \to T^* \otimes T \tag{5.8}$$

induces an embedding on $S^{(2,0,0,0,-2)}Q \subset S^2Q \otimes S^2Q^*$. To prove this we can replace Q by a vector space with a basis $e_1, \dots, e_5$. Let $e_1^*, \dots, e_5^*$ be the dual basis of Q^*. It is enough to check that the lowest weight vector $e_1^2 \otimes (e_5^*)^2$ maps to a nonzero element of $T^* \otimes T$ under (5.8). By definition, this endomorphism of T is the composition of the map

$$T \to S^3Q \otimes \wedge^2 Q \xrightarrow{\partial_5^2 \otimes \mathrm{id}} Q \otimes \wedge^2 Q \to R_1$$

with the map

$$R_1 \to Q \otimes \wedge^2 Q \xrightarrow{e_1^2} S^2Q \otimes Q \otimes \wedge^2 Q \to S^3Q \otimes \wedge^2 Q \to T.$$

Viewing T as a direct summand of $S^2Q \otimes \wedge^3 Q$ we obtain from the first (resp., second) map a map $f : S^2Q \otimes \wedge^3 Q \to R_1$ (resp., $g : R_1 \to S^2Q \otimes \wedge^3 Q$). Identifying R_1 with $Q \otimes \wedge^2 Q / \wedge^3 Q$ we can write

$$f(t \otimes (x \wedge y \wedge z)) = \partial_5^2(tx) \otimes (y \wedge z) + \partial_5^2(ty) \otimes (z \wedge x) + \partial_5^2(tz) \otimes (x \wedge y) \mod \wedge^3 Q,$$
$$g(x \otimes (y \wedge z) \mod \wedge^3 Q) = 2(e_1x) \otimes (e_1 \wedge y \wedge z) + (e_1y) \otimes (e_1 \wedge x \wedge z) + (e_1z) \otimes (e_1 \wedge y \wedge x),$$

where $t \in S^2Q$ and $x, y, z \in Q$ (for appropriate rescaling of g). Hence,

$$gf((e_4e_5) \otimes (e_1 \wedge e_2 \wedge e_5)) = 2g(e_4 \otimes (e_1 \wedge e_2) \mod \wedge^3 Q) = 2e_1^2 \otimes (e_1 \wedge e_4 \wedge e_2) \neq 0.$$

Thus, the map (5.8) induces an embedding on H^1. So we are reduced to checking that the natural map

$$H^0(S^2Q) \otimes H^1(S^2Q^*) \to H^1(S^2Q \otimes S^2Q^*)$$

is an isomorphism. Since both sides are isomorphic to S^2V, it is enough to prove surjectivity. The exact sequence (4.2) shows that this follows from the vanishing of $H^2(Q^* \otimes S^2Q^*)$ and $H^3(\wedge^2 Q^* \otimes S^2Q^*)$, which can be checked using Bott's theorem.
(v) The vanishing of $\mathrm{Hom}^*(P, Q)$, $\mathrm{Hom}^*(P, \wedge^2 Q)$, $\mathrm{Hom}^*(P, R_1)$ follow from directly from parts (i)-(iv) along with the computation of the relevant spaces in Lemmas 2.5 and 5.1. Similarly, we derive that $\mathrm{Hom}^0(P, T) = k$, $\mathrm{Hom}^1(P, T) = S^2V$ and $\mathrm{Hom}^i(P, T) = 0$ for $i > 1$. Now one computes $\mathrm{Hom}^*(P, P)$ by applying the functor $\mathrm{Hom}(P, ?)$ to the exact sequence (5.1) and using the vanishing of $\mathrm{Hom}^*(P, R_1)$. To compute $\mathrm{Hom}^*(P, \wedge^3 Q)$ it remains to check that the map

$$\mathrm{Hom}^1(R_1, \wedge^3 Q) \to \mathrm{Hom}^2(T, \wedge^3 Q)$$

induced by (5.1) is an isomorphism. Since both sides are isomorphic to $\wedge^2 V/k$, it is enough to prove surjectivity. But this follows immediately from part (ii) along with the surjectivity of the map (5.6) proved in part (iv). □

By part (v) of the above lemma, we have a canonical nonsplit extension of vector bundles

$$0 \to \wedge^3 Q \to G \to P \to 0. \tag{5.9}$$

Theorem 5.3. *The vector bundle G is exceptional and* $\mathrm{Hom}^*(G, \wedge^3 Q) = 0$.

Proof. First, applying the functor $\mathrm{Hom}(?, \wedge^3 Q)$ to the sequence (5.9) and using Lemma 5.2(v) we find that $\mathrm{Hom}^*(G, \wedge^3 Q) = 0$. Next, applying the functor $\mathrm{Hom}(G, ?)$ to this sequence we derive isomorphisms $\mathrm{Hom}^i(G, G) \simeq \mathrm{Hom}^i(G, P)$. Recall that $\mathrm{Hom}^*(\wedge^3 Q, R_1) = 0$ by Lemma 2.4. Hence, applying the functor $\mathrm{Hom}(\wedge^3 Q, ?)$ to the sequence (5.1) and using Lemma 5.1(v) we obtain that $\mathrm{Hom}^i(\wedge^3 Q, P) = 0$ for $i > 0$ and $\mathrm{Hom}^0(\wedge^3 Q, P) = S^2V$. Thus, using the sequence (5.9) again along with the computation of $\mathrm{Hom}^*(P, P)$ (see Lemma 5.2(v)) we see that it is enough to check that the natural map

$$\mathrm{Hom}^0(\wedge^3 Q, P) \otimes \mathrm{Hom}^1(P, \wedge^3 Q) \to \mathrm{Hom}^1(P, P)$$

is an isomorphism. Since $\mathrm{Hom}^*(P, R_1) = \mathrm{Hom}^*(\wedge^3 Q, R_1) = 0$ (see Lemma 5.2(v)), the exact sequence (5.1) gives an isomorphism of the above map with

$$\mathrm{Hom}^0(\wedge^3 Q, T) \to \mathrm{Hom}^1(P, T)$$

induced by a nonzero element in $\mathrm{Hom}^1(P, \wedge^3 Q)$. Since the natural map $\mathrm{Hom}^1(T, \wedge^3 Q) \to \mathrm{Hom}^1(P, \wedge^3 Q)$ is an isomorphism (as we have seen in the proof of Lemma 5.2(v)), the above map factors as the composition of the map

$$\mathrm{Hom}^0(\wedge^3 Q, T) \xrightarrow{f} \mathrm{Hom}^1(T, T)$$

induced by a nonzero element in $\mathrm{Hom}^1(T, \wedge^3 Q)$ followed by the map h in the exact sequence

$$0 \to \mathrm{Hom}^0(R_1, T) \xrightarrow{g} \mathrm{Hom}^1(T, T) \xrightarrow{h} \mathrm{Hom}^1(P, T) \to 0.$$

Thus, it is enough to check that the images of the maps f and g are complementary in $\mathrm{Hom}^1(T, T)$. Since $\mathrm{Hom}^0(R_1, T) = V^{\otimes 2}/k$, $\mathrm{Hom}^0(\wedge^3 Q, T) = S^2 V$, while $\mathrm{Hom}^1(T, T) = V^{\otimes 2}/k \oplus S^2 V$ (see Lemma 5.1(iv), (v), (viii)), it suffices to prove that the images of $S^2 V$ under f and g have trivial intersection. Note that we have a natural embedding $S^2 Q \to R_1^* \otimes T$ (resp., $S^2 Q \to \wedge^3 Q^* \otimes T$) inducing an embedding of $S^2 V$ into $\mathrm{Hom}^0(R_1, T)$ (resp., into $\mathrm{Hom}^0(\wedge^3 Q, T)$). On the other hand, a nonzero element in $\mathrm{Hom}^1(T, \wedge^3 Q)$ is the image of the nonzero element in $H^1(S^2 Q^*)$ with respect to the embedding $S^2 Q^* \to T^* \otimes \wedge^3 Q$. Furthermore, we have seen in the end of the proof of Lemma 5.2(iv) that the natural map $H^0(S^2 Q) \otimes H^1(S^2 Q^*) \to H^1(S^2 Q \otimes S^2 Q^*)$ is an isomorphism. Thus, it is enough to prove that the natural maps

$$\alpha : S^2 Q^* \otimes S^2 Q \to (T^* \otimes \wedge^3 Q) \otimes (\wedge^3 Q^* \otimes T) \to T^* \otimes T \quad \text{and}$$
$$\beta : S^2 Q^* \otimes S^2 Q \to (T^* \otimes R_1) \otimes (R_1^* \otimes T) \to T^* \otimes T$$

induce linearly independent maps on H^1. In fact, since $H^1(S^2 Q^* \otimes S^2 Q)$ comes from the summand $S^{(2,0,0,0,-2)} Q \subset S^2 Q^* \otimes S^2 Q$, generated by the lowest weight vector $v = (e_5^*)^2 \otimes e_1^2$ (where (e_i) is the basis of Q), it suffices to check that $\alpha(v)$ and $\beta(v)$ are not proportional in $T^* \otimes T$. Recall that in the proof of Lemma 5.1(iv) we have constructed the maps $f : S^2 Q \otimes \wedge^3 Q \to R_1$ and $g : R_1 \to S^2 Q \otimes \wedge^3 Q$ such that gf is a multiple of the composition

$$S^2 Q \otimes \wedge^3 Q \to T \xrightarrow{\beta(v)} T \to S^2 Q \otimes \wedge^3 Q.$$

On the other hand, $\alpha(v)$ is given by the following composition

$$T \to S^2 Q \otimes \wedge^3 Q \xrightarrow{\partial_5^2} \wedge^3 Q \xrightarrow{e_1^2} S^2 Q \otimes \wedge^3 Q \to T.$$

Let us denote by $\pi : S^2 Q \otimes \wedge^3 Q \to S^2 Q \otimes \wedge^3 Q$ the projection with the image T, given by

$$\pi(ab \otimes (x \wedge y \wedge z)) = \frac{3}{5} ab \otimes (x \wedge y \wedge z) + (ax \otimes (b \wedge y \wedge z) + bx \otimes (a \wedge y \wedge z) + c.p.(x, y, z)),$$

where $a, b, x, y, z, \in Q$, the omitted terms $c.p.(x, y, z)$ are obtained by cyclically permuting x, y, z. Then we are reduced to checking that gf is not proportional to the composition

$$h : S^2 Q \otimes \wedge^3 Q \xrightarrow{\pi} S^2 Q \otimes \wedge^3 Q \xrightarrow{\partial_5^2} \wedge^3 Q \xrightarrow{e_1^2} S^2 Q \otimes \wedge^3 Q \xrightarrow{\pi} S^2 Q \otimes \wedge^3 Q.$$

To this end we compute

$$\begin{aligned}
\frac{1}{2} gf(e_4 e_5 \otimes (e_2 \wedge e_3 \wedge e_5)) &= f(e_4 \otimes (e_2 \wedge e_3)) \\
&= 2e_1 e_4 \otimes (e_1 \wedge e_2 \wedge e_3) - e_1 e_2 \otimes (e_1 \wedge e_3 \wedge e_4) + e_1 e_3 \otimes (e_1 \wedge e_2 \wedge e_4), \\
\frac{25}{2} h(e_4 e_5 \otimes (e_2 \wedge e_3 \wedge e_5)) &= 3e_1^2 \otimes (e_2 \wedge e_3 \wedge e_4) + 2e_1 e_4 \otimes (e_1 \wedge e_2 \wedge e_3) \\
&\quad + 2e_1 e_2 \otimes (e_1 \wedge e_3 \wedge e_4) - 2e_1 e_3 \otimes (e_1 \wedge e_2 \wedge e_4),
\end{aligned}$$

which are clearly not proportional. □

Lemma 5.4. *On* $\mathrm{LG}(5, 10)$ *one has* $\mathrm{Hom}^*(R_1, R_1(i)) = 0$ *for* $i \in [-5, -1]$.

Proof. The proof is similar to that of Lemma 2.5(iv) and is left to the reader. □

Theorem 5.5. *Let us consider the following two blocks:*

$$\mathcal{A} = (\mathcal{O}, Q, \wedge^2 Q, F_1, \wedge^3 Q, G) \quad \textit{and} \quad \mathcal{B} = (\mathcal{O}, Q, \wedge^2 Q, F_1, \wedge^3 Q).$$

Then $(\mathcal{A}, \mathcal{B}(1), \mathcal{B}(2), \mathcal{A}(3), \mathcal{B}(4), \mathcal{B}(5))$ *is a full exceptional collection in* $\mathrm{D}^b(\mathrm{LG}(5, 10))$.

Proof. The required semiorthogonality conditions not involving G follow from the fact that F_1 is the right mutation of E_1 through $\wedge^2 Q$ and from Lemmas 2.3, 3.1, 3.3 and 5.4. Using Serre duality and sequences (5.1) and (5.9) we can reduce all the remaining semiorthogonality conditions to Lemmas 5.1 and 5.2 and Theorem 5.3 (for $\mathrm{Hom}^*(G(3), G) = 0$ we need in addition the vanishing of $\mathrm{Hom}^*(\wedge^3 Q(3), \wedge^3 Q)$ and $\mathrm{Hom}^*(R_1(3), R_1)$ that follows from Lemmas 2.3 and 5.4).

Now let us prove that our exceptional collection is full. Following the method of proof of Theorem 4.3 (involving the partial isotropic flag manifold $\mathrm{F}_{1,5,10}$ and the relative analog of our collection for $\mathrm{LG}(4, 8)$) one can reduce this to checking that the subcategory $\mathcal{C}$ generated by our exceptional collection contains the subcategories

$$\mathcal{P} \otimes \mathcal{O}(j), \mathcal{P} \otimes Q(j), \mathcal{P} \otimes \wedge^2 Q(j), \mathcal{P} \otimes Q \otimes Q, \mathcal{P} \otimes Q \otimes \wedge^2 Q,$$

where $j = 0, \ldots, 4$ and $\mathcal{P} = \langle \mathcal{O}, Q, \wedge^2 Q, \wedge^3 Q, Q^*(1), \mathcal{O}(1)\rangle$. This gives the following list of objects that have to be in $\mathcal{C}$:

(i) $\mathcal{O}(j), Q(j), \wedge^2 Q(j)$ for $j = 0, \ldots, 5$;
(ii) $Q \otimes Q(j), \wedge^3 Q(j), Q \otimes \wedge^2 Q(j), Q \otimes \wedge^3 Q(j), \wedge^2 Q \otimes \wedge^2 Q(j), \wedge^2 Q \otimes \wedge^3 Q(j)$ for $j = 0, \ldots, 4$;
(iii) $Q^*(j), Q^* \otimes Q(j), Q^* \otimes \wedge^2 Q(j)$ for $j = 1, \ldots, 5$;
(iv) $Q \otimes Q \otimes Q, Q \otimes Q \otimes \wedge^2 Q, Q \otimes Q \otimes \wedge^3 Q, Q^* \otimes Q \otimes Q(1), Q \otimes \wedge^2 Q \otimes \wedge^2 Q, Q \otimes \wedge^2 Q \otimes \wedge^3 Q, Q^* \otimes Q \otimes \wedge^2 Q(1)$.

The fact that all these objects belong to $\mathcal{C}$ follows from Lemmas 5.6–5.9, 5.9 and 5.13. □

In the following lemmas we often use the fact that $\mathcal{C}$ is closed under direct summands (since it is an admissible subcategory). Also, by a resolution of S^nQ we mean the exact sequence

$$\cdots \to \wedge^2 Q^* \otimes S^{n-2}V \otimes \mathcal{O} \to Q^* \otimes S^{n-1}V \otimes \mathcal{O} \to S^nV \otimes \mathcal{O} \to S^nQ \to 0.$$

By the standard filtration of $\wedge^k(V \otimes \mathcal{O})$ we mean the filtration associated with exact sequence (2.1). This filtration has vector bundles $\wedge^i Q^* \otimes \wedge^{k-i} Q$ as consecutive quotients. Recall also that $\wedge^5 Q = \mathcal{O}(1)$, so we have isomorphisms $\wedge^i Q^*(1) \simeq \wedge^{5-i} Q$.

Lemma 5.6. (i) *For $j = 0, \ldots, 5$ the following objects are in $\mathcal{C}$: $\mathcal{O}(j), Q(j), \wedge^2 Q(j), \wedge^3 Q(j), Q \otimes \wedge^2 Q(j), Q^*(j), Q^* \otimes \wedge^2 Q(j), S^2Q^*(j)$.*
(ii) *For $j = 1, \ldots, 5$ the following objects are in $\mathcal{C}$: $Q^* \otimes Q^*(j), Q^* \otimes Q(j), Q \otimes Q(j), S^nQ(j)$ for $n \geq 2$.*
(iii) *For $j = 0, \ldots, 4$ one has $Q \otimes \wedge^3 Q(j) \in \mathcal{C}$ and $Q^* \otimes \wedge^3 Q(j) \in \mathcal{C}$.*
(iv) *For $j = 1, \ldots, 4$ one has $\wedge^3 Q \otimes \wedge^2 Q(j-1) = \wedge^2 Q^* \otimes \wedge^2 Q(j) \in \mathcal{C}$ and $S^2Q \otimes \wedge^2 Q(j) \in \mathcal{C}$.*
(v) *For $j = 1, \ldots, 5$ and for $n \geq 2$ one has $Q \otimes S^nQ(j) \in \mathcal{C}$ and $Q^* \otimes S^nQ(j) \in \mathcal{C}$.*
(vi) *For $j = 1, \ldots, 5$ the following objects are in $\mathcal{C}$: $Q \otimes Q \otimes Q(j), Q^* \otimes Q \otimes Q(j), Q^* \otimes Q^* \otimes Q(j)$ and $Q^* \otimes Q^* \otimes Q^*(j)$.*

Proof. (i) To check the assertion for $Q \otimes \wedge^2 Q(j)$ we observe that $R_1 = S^{2,1}Q$ is contained in $\langle Q, F_1\rangle$ as follows from exact sequence (3.4). This implies that $Q \otimes \wedge^2 Q(j) = \wedge^3 Q(j) \oplus S^{2,1}Q(j)$ belongs to $\mathcal{C}$ for $j = 0, \ldots, 5$.

The assertions for $Q^*(j)$ and $Q^* \otimes \wedge^2 Q(j)$ follow from the sequence (2.1). The assertion for $S^2Q^*(j)$ follows by considering the dual sequence to the resolution of S^2Q.
(ii) Use the decomposition $Q^* \otimes Q^*(j) = S^2Q^*(j) \oplus \wedge^2 Q^*(j) = S^2Q^*(j) \oplus \wedge^3 Q(j-1)$ and (i). Then use sequence (2.1). For $S^nQ(j)$ the assertion is checked using part (i) and the resolution of S^nQ.
(iii) To prove the assertion for $Q \otimes \wedge^3 Q(j)$ use the isomorphism $Q \otimes \wedge^3 Q(j) \equiv Q \otimes \wedge^2 Q^*(j+1)$ and consider the standard filtration of $\wedge^3(V \otimes \mathcal{O})$ tensored with $\mathcal{O}(j+1)$ (and then use part (i)). For the second assertion use sequence (2.1).
(iv) To check that $\wedge^2 Q^* \otimes \wedge^2 Q(j) \in \mathcal{C}$ use the standard filtration of $\wedge^4(V \otimes \mathcal{O})$ tensored with $\mathcal{O}(j)$. Next, to derive that $S^2Q \otimes \wedge^2 Q(j) \in \mathcal{C}$ use resolution of S^2Q.
(v) For $Q \otimes S^nQ(j)$ use the resolution for S^nQ tensored with $Q(j)$ and parts (i), (ii) and (iii). For $Q^* \otimes S^nQ(j)$ use sequence (2.1) and part (ii).
(vi) The assertion for $Q \otimes Q \otimes Q(j)$ follows from the decomposition $Q \otimes Q \otimes Q(j) = Q \otimes \wedge^2 Q(j) \oplus Q \otimes S^2Q(j)$ and parts (i) and (v). The rest follows using sequence (2.1) and part (ii). □

Lemma 5.7. (i) *One has $S^{3,1,1}Q \in \mathcal{C}$ and $S^{3,1,1}Q(3) \in \mathcal{C}$.*
(ii) *One has $S^2Q \otimes \wedge^3 Q \in \mathcal{C}$ and $S^2Q \otimes \wedge^3 Q(3) \in \mathcal{C}$.*
(iii) *One has $\wedge^2 Q^* \otimes \wedge^3 Q \in \mathcal{C}$ and $\wedge^2 Q^* \otimes \wedge^3 Q(3) \in \mathcal{C}$.*

Proof. (i) First, exact sequence (5.9) shows that $P, P(3) \in \mathcal{C}$. Next, exact sequence (5.1) shows that $T, T(3) \in \mathcal{C}$, where $T = S^{3,1,1}Q$.
(ii) Since we have the decomposition

$$S^2Q \otimes \wedge^3 Q = S^{3,1,1}Q \oplus S^{2,1,1,1}Q,$$

part (i) shows that it is enough to check the similar assertion for $S^{2,1,1,1}Q$. But $S^{2,1,1,1}Q$ is a direct summand in $Q \otimes \wedge^4 Q = Q \otimes Q^*(1)$, so the statement follows from Lemma 5.6(ii).
(iii) This follows from (ii) using resolution for S^2Q and Lemma 5.6(iii). □

Lemma 5.8. (i) *For $j = 1, 2, 3$, $\wedge^3 Q \otimes \wedge^3 Q(j) \in \mathcal{C}$ if and only if $\wedge^2 Q \otimes \wedge^2 Q(j) \in \mathcal{C}$.*
(ii) *For $j = 1, \ldots, 4$, $\wedge^2 Q \otimes \wedge^2 Q(j) \in \mathcal{C}$ if and only if $S^2Q \otimes S^2Q(j) \in \mathcal{C}$.*
(iii) *For $j = 1, \ldots, 4$, the following conditions are equivalent:*
(1) *$\wedge^2 Q^* \otimes \wedge^3 Q(j-1) \in \mathcal{C}$;*
(2) *$\wedge^2 Q^* \otimes S^2Q(j) \in \mathcal{C}$;*
(3) *$Q \otimes Q \otimes \wedge^2 Q(j) \in \mathcal{C}$.*

Proof. (i) Use the standard filtration of $\wedge^5(V\otimes\mathcal{O})$ tensored with $\mathcal{O}(j+1)$ and Lemma 5.6(ii).
(ii) Use the decompositions

$$S^2Q\otimes S^2Q=Q\otimes S^3Q\oplus S^{2,2}Q,\qquad \wedge^2Q\otimes\wedge^2Q=Q\otimes\wedge^3Q\oplus S^{2,2}Q$$

and Lemma 5.6(iii), (v).
(iii) First, the equivalence of (1) and (2) follows by considering the resolution of S^2Q and using Lemma 5.6(iii). Next, we observe that

$$\wedge^2Q^*\otimes Q\otimes Q(j)=\wedge^2Q^*\otimes\wedge^2Q(j)\oplus\wedge^2Q^*\otimes S^2Q(j)$$

and that $\wedge^2Q^*\otimes\wedge^2Q(j)\in\mathcal{C}$ for $j=1,\dots,4$ by Lemma 5.6(iv). Therefore, (2) is equivalent to $\wedge^2Q^*\otimes Q\otimes Q(j)\in\mathcal{C}$. On the other hand, sequence (2.1) and Lemma 5.6(i) imply that in condition (3) we can replace $Q\otimes Q\otimes\wedge^2Q(j)$ with $Q^*\otimes Q\otimes\wedge^2Q(j)$. Now the equivalence of (2) and (3) follows by considering the standard filtration of $\wedge^3(V\otimes\mathcal{O})$ tensored with $Q(j)$ and using Lemma 5.6(i), (iii). □

Lemma 5.9. (i) *For $j=0,1,2,3$ one has $\wedge^3Q\otimes\wedge^3Q(j-1)\in\mathcal{C}$, $S^2Q\otimes\wedge^3Q(j)\in\mathcal{C}$ and $S^2Q\otimes S^2Q(j+1)\in\mathcal{C}$.*
(ii) *For $j=1,2,3,4$ one has $\wedge^2Q\otimes Q\otimes Q(j)\in\mathcal{C}$ and $\wedge^2Q\otimes Q^*\otimes Q(j)\in\mathcal{C}$.*
(iii) *One has $\wedge^2Q\otimes\wedge^2Q(j)\in\mathcal{C}$, $\wedge^3Q\otimes\wedge^3Q(j-1)\in\mathcal{C}$ for $j=1,\dots,4$, and $S^2Q\otimes\wedge^3Q(j)\in\mathcal{C}$ for $j=0,\dots,4$.*
(iv) *One has $\wedge^3Q\otimes Q\otimes Q(j)\in\mathcal{C}$ and $\wedge^3Q\otimes Q^*\otimes Q(j)\in\mathcal{C}$ for $j=0,1,2,3$.*
(v) *One has $\wedge^3Q\otimes\wedge^2Q\otimes Q(j)\in\mathcal{C}$ for $j=1,2$.*

Proof. (i) For $j=0$ and $j=3$ the first assertion follows from Lemma 5.7(iii). By Lemma 5.8(iii) this implies that $S^2Q\otimes\wedge^3Q\in\mathcal{C}$ and $S^2Q\otimes\wedge^3Q(3)\in\mathcal{C}$. Next, using the resolution for S^2Q and Lemma 5.6(v) we obtain $S^2Q\otimes S^2Q(1)\in\mathcal{C}$ and $S^2Q\otimes S^2Q(4)\in\mathcal{C}$. By Lemma 5.8(ii), this implies that $\wedge^2Q\otimes\wedge^2Q(1)\in\mathcal{C}$ and $\wedge^2Q\otimes\wedge^2Q(4)\in\mathcal{C}$. By Lemma 5.8(i), it follows that $\wedge^3Q\otimes\wedge^3Q(1)\in\mathcal{C}$, which also leads to $S^2Q\otimes\wedge^3Q(2)\in\mathcal{C}$ and $S^2Q\otimes S^2Q(3)\in\mathcal{C}$ as before.

On the other hand, combining Lemma 5.8(i) with Lemma 5.7(iii) we also get $\wedge^2Q\otimes\wedge^2Q(2)\in\mathcal{C}$. By Lemma 5.8(ii), this implies that $S^2Q\otimes S^2Q(2)\in\mathcal{C}$. Considering the resolution for S^2Q this leads to $S^2Q\otimes\wedge^3Q(1)\in\mathcal{C}$ and $\wedge^3Q\otimes\wedge^3Q\in\mathcal{C}$ as before.
(ii) The first assertion immediately follows from (i) and from Lemma 5.8(iii). The second follows from the first using sequence (2.1).
(iii) This follows from (i), (ii) and Lemma 5.8(i).
(iv) Using sequence (2.1) and Lemma 5.6(iii) we see that it is enough to show that $\wedge^3Q\otimes Q\otimes Q(j)\in\mathcal{C}$. To this end we use the decomposition

$$\wedge^3Q\otimes Q\otimes Q(j)=\wedge^3Q\otimes\wedge^2Q(j)\oplus\wedge^3Q\otimes S^2Q(j),$$

part (iii) and Lemma 5.6(iv).
(v) We start with the isomorphism $\wedge^3Q\otimes\wedge^2Q\otimes Q(j)\simeq\wedge^2Q^*\otimes\wedge^2Q\otimes Q(j+1)$. Now the assertion follows by considering the standard filtration of $\wedge^4(V\otimes\mathcal{O})$ tensored with $Q(j+1)$ and using parts (ii), (iv) and Lemma 5.8(ii). □

Lemma 5.10. (i) *For $j=1,2,3$ one has $\wedge^2Q\otimes\wedge^2Q\otimes Q(j)\in\mathcal{C}$.*
(ii) *One has $\wedge^2Q\otimes\wedge^2Q\otimes\wedge^2Q(2)\in\mathcal{C}$.*
(iii) *One has $\wedge^2Q\otimes\wedge^2Q\otimes S^2Q(2)\in\mathcal{C}$.*
(iv) *One has $\wedge^3Q\otimes\wedge^2Q(4)\in\mathcal{C}$.*

Proof. (i) Suppose first that $j=1$. Then considering the filtration of $\wedge^5(V\otimes\mathcal{O})\otimes Q(2)$ and using Lemma 5.6(vi), as well as the fact that $Q^*\otimes Q^*\otimes\wedge^2Q(3)\in\mathcal{C}$ (which is a consequence of Lemma 5.9(ii)), we reduce ourselves to showing that $\wedge^3Q\otimes\wedge^3Q\otimes Q(1)\in\mathcal{C}$. Now the isomorphism $\wedge^3Q\otimes\wedge^3Q\otimes Q(1)\simeq\wedge^3Q\otimes\wedge^2Q^*\otimes Q(2)$ and the standard filtration of $\wedge^3Q\otimes\wedge^3(V\otimes\mathcal{O})(2)$ show that it is enough to check that the following objects are in $\mathcal{C}$:

$$\wedge^3Q\otimes\wedge^2Q\otimes Q^*(2),\qquad\wedge^3Q\otimes\wedge^3Q(2),\qquad\wedge^3Q\otimes\wedge^3Q^*(2).$$

For the second and the third this follows from Lemma 5.9(iii) and Lemma 5.6(iv), respectively. For the first object this follows from Lemmas 5.6(iv) and 5.9(v) using (2.1).

Now consider the case $j=2$ or $j=3$. By sequence (2.1) and Lemma 5.9(iii), it is enough to prove that $\wedge^2Q\otimes\wedge^2Q\otimes Q^*(j)\in\mathcal{C}$. Now the standard filtration of $\wedge^2Q\otimes\wedge^3(V\otimes\mathcal{O})(j)$ shows that it is enough to check that the following objects are in $\mathcal{C}$:

$$\wedge^2Q\otimes\wedge^3Q(j),\qquad\wedge^2Q\otimes\wedge^3Q^*(j),\qquad\wedge^2Q\otimes\wedge^2Q^*\otimes Q(j).$$

But this follows from Lemmas 5.6(iv), 5.9(iii) and 5.9(v), respectively.
(ii) First, considering the standard filtration of $\wedge^5(V\otimes\mathcal{O})\otimes\wedge^2Q(3)$, we reduce ourselves to showing that the following objects are in $\mathcal{C}$:

$$\wedge^3Q\otimes\wedge^3Q\otimes\wedge^2Q(2),\qquad Q\otimes Q\otimes\wedge^2Q(2),\qquad Q^*\otimes Q^*\otimes\wedge^2Q(4).$$

For the second and the third this follows from Lemma 5.9(ii). Now using the isomorphism $\wedge^3 Q \otimes \wedge^3 Q \otimes \wedge^2 Q(2) \simeq \wedge^3 Q \otimes \wedge^2 Q^* \otimes \wedge^2 Q(3)$ and the standard filtration of $\wedge^3 Q \otimes \wedge^4(V \otimes \mathcal{O})(3)$ we are led to showing that the following objects are in $\mathcal{C}$:

$$\wedge^3 Q \otimes \wedge^3 Q \otimes Q^*(3), \qquad \wedge^3 Q \otimes \wedge^2 Q \otimes Q(2), \qquad \wedge^3 Q \otimes Q(2), \qquad \wedge^3 Q \otimes Q^*(4).$$

For the second object this follows from Lemma 5.9(v), while for the last two it follows from Lemma 5.6(iii). Thus, it remains to check that $\wedge^3 Q \otimes \wedge^3 Q \otimes Q^*(3) \in \mathcal{C}$. Using the standard filtration of $\wedge^5(V \otimes \mathcal{O}) \otimes Q^*(4)$ we see that it is enough to verify that the following objects are in $\mathcal{C}$:

$$\wedge^2 Q \otimes \wedge^2 Q \otimes Q^*(3), \qquad Q \otimes Q \otimes Q^*(3), \qquad Q^*(3), \qquad Q^*(5), \qquad Q^* \otimes Q^* \otimes Q^*(5).$$

For the second and the last object this follows from Lemma 5.6(vi). On the other hand, using (2.1), part (i) and Lemma 5.9(iii) we see that $\wedge^2 Q \otimes \wedge^2 Q \otimes Q^*(3) \in \mathcal{C}$.
(iii) First, using the resolution for S^2Q we reduce the problem to showing that $\wedge^2 Q \otimes \wedge^2 Q \otimes \wedge^2 Q^*(2) \in \mathcal{C}$ (here we also use part (i), sequence (2.1) and Lemma 5.9(iii)). Next, the standard filtration of $\wedge^2 Q \otimes \wedge^4(V \otimes \mathcal{O})(2)$ shows that it is enough to check that the following objects are in $\mathcal{C}$:

$$\wedge^2 Q \otimes \wedge^3 Q \otimes Q^*(2), \qquad \wedge^2 Q \otimes \wedge^2 Q \otimes Q(1), \qquad \wedge^2 Q \otimes Q^*(3), \qquad \wedge^2 Q \otimes Q(1).$$

For the last two objects this follows from Lemma 5.6(i). For the second object the assertion follows from part (i). Finally, to check that $\wedge^2 Q \otimes \wedge^3 Q \otimes Q^*(2) \in \mathcal{C}$ we use sequence (2.1), Lemma 5.9(v) and Lemma 5.6(iv).
(iv) Let us start with the decomposition

$$\wedge^3 Q \otimes \wedge^2 Q(4) = \mathcal{O}(5) \oplus S^{2,1,1,1}Q(4) \oplus S^{2,2,1}Q(4).$$

Now observe that $S^{2,1,1,1}Q(4)$ is a direct summand in $Q \otimes \wedge^4 Q(4) = Q \otimes Q^*(5)$ which is in $\mathcal{C}$ by Lemma 5.6(ii), while $S^{2,2,1}Q(4)$ is a direct summand in $S^2Q \otimes S^2Q \otimes Q(4)$. Using the resolution of S^2Q we reduce ourselves to checking that the following objects are in $\mathcal{C}$:

$$S^2Q \otimes \wedge^2 Q^* \otimes Q(4), \qquad S^2Q \otimes Q(4), \qquad S^2Q \otimes Q^* \otimes Q(4).$$

For the second object this follows from Lemma 5.6(v). Using (2.1) we can replace the third object by $S^2Q \otimes Q \otimes Q(4) = S^2Q \otimes \wedge^2 Q(4) \oplus S^2Q \otimes S^2Q(4)$ which is in $\mathcal{C}$ by Lemmas 5.6(iv) and 5.9(i). Next, we use the isomorphism $S^2Q \otimes \wedge^2 Q^* \otimes Q(4) \simeq S^2Q \otimes \wedge^3 Q \otimes Q(3)$ and the resolution of S^2Q to reduce the problem to showing that the following objects are in $\mathcal{C}$:

$$\wedge^2 Q^* \otimes \wedge^3 Q \otimes Q(3), \qquad \wedge^3 Q \otimes Q(3), \qquad \wedge^3 Q \otimes Q^* \otimes Q(3).$$

The second and third objects are in $\mathcal{C}$ by Lemmas 5.6(iii) and 5.9(iv), respectively. For the first object we use the isomorphism $\wedge^2 Q^* \otimes \wedge^3 Q \otimes Q(3) \simeq \wedge^3 Q \otimes \wedge^3 Q \otimes Q(2)$ and the standard filtration of $\wedge^5(V \otimes \mathcal{O}) \otimes Q(3)$ to reduce ourselves to proving that the following objects are in $\mathcal{C}$:

$$\wedge^2 Q \otimes \wedge^2 Q \otimes Q(2), \qquad Q \otimes Q \otimes Q(2), \qquad Q^* \otimes Q^* \otimes Q(4).$$

For the first object this follows from (i), and for the second and the third—from Lemma 5.6(vi). □

Lemma 5.11. (i) *One has* $S^3Q \otimes S^3Q(2) \in \mathcal{C}$.
(ii) *One has* $\wedge^2 Q \otimes \wedge^2 Q \in \mathcal{C}$.
(iii) *One has* $Q \otimes Q \in \mathcal{C}$.

Proof. (i) Consider the decomposition

$$S^3Q \otimes S^3Q(2) = S^6Q(2) \oplus S^{5,1}Q(2) \oplus S^{4,2}Q(2) \oplus S^{3,3}Q(2).$$

By Lemma 5.6(ii), we have $S^6Q(2) \in \mathcal{C}$. Next, we observe that $S^{5,1}Q(2)$ is a direct summand in $S^4Q \otimes \wedge^2 Q(2)$ and use the resolution of S^4Q to deduce that this object is in $\mathcal{C}$ from the inclusions $Q \otimes \wedge^2 Q(1) \in \mathcal{C}, Q^* \otimes \wedge^2 Q(2) \in \mathcal{C}, \wedge^2 Q^* \otimes \wedge^2 Q(2) \in \mathcal{C}, \wedge^3 Q^* \otimes \wedge^2 Q(2) \in \mathcal{C}$, that follow from Lemmas 5.6(i), 5.6(iv) and 5.9(iii). Finally, we note that $S^{4,2}Q(2) \oplus S^{3,3}Q(2)$ is a direct summand in

$$\wedge^2 Q \otimes \wedge^2 Q \otimes Q \otimes Q(2) = \wedge^2 Q \otimes \wedge^2 Q \otimes \wedge^2 Q(2) \oplus \wedge^2 Q \otimes \wedge^2 Q \otimes S^2Q(2)$$

which is in $\mathcal{C}$ by Lemma 5.10(ii), (iii).
(ii) We use the isomorphism $\wedge^2 Q \otimes \wedge^2 Q \simeq \wedge^3 Q^* \otimes \wedge^3 Q^*(2)$ and then use the resolution of S^3Q twice to relate this to $S^3Q \otimes S^3Q(2)$ which is in $\mathcal{C}$ by part (i). It remains to check that the objects that appear in between, namely,

$$S^3Q(2), \qquad S^3Q \otimes Q^*(2), \qquad S^3Q \otimes \wedge^2 Q^*(2), \qquad \wedge^3 Q^*(2), \qquad Q^* \otimes \wedge^3 Q^*(2), \qquad \wedge^2 Q^* \otimes \wedge^3 Q^*(2),$$

are all in $\mathcal{C}$. For the last three objects this follows from Lemma 5.6(i), (iv), while for the first three one has to use the resolution of S^3Q to reduce to the objects we have already dealt with.
(iii) The standard filtration of $\wedge^5(V \otimes \mathcal{O})(1)$ reduces the problem to showing that $\wedge^2 Q \otimes \wedge^2 Q$ and $\wedge^3 Q \otimes \wedge^3 Q$ are in $\mathcal{C}$ (where we also use Lemma 5.6(ii)). It remains to apply part (ii) and Lemma 5.9(iii). □

Lemma 5.12. (i) *One has* $S^3Q \otimes S^2Q(1) \in \mathcal{C}$.
(ii) *One has* $\wedge^2 Q \otimes S^2Q \in \mathcal{C}$.
(iii) *One has* $\wedge^2 Q \otimes Q \otimes Q \in \mathcal{C}$.
(iv) *One has* $\wedge^3 Q \otimes \wedge^2 Q \otimes Q \in \mathcal{C}$.

Proof. (i) Consider the decomposition

$$S^3Q \otimes S^2Q(1) = S^5Q(1) \oplus S^{4,1}Q(1) \oplus S^{3,2}Q(1).$$

By Lemma 5.6(ii), we have $S^5Q(1) \in \mathcal{C}$. On the other hand, $S^{4,1}Q(1)$ is a direct summand in $S^3Q \otimes \wedge^2 Q(1)$. The resolution of S^3Q relates the latter object to $Q^* \otimes \wedge^2 Q(1)$, $\wedge^2 Q^* \otimes \wedge^2 Q(1)$ and $\wedge^3 Q^* \otimes \wedge^2 Q(1)$ which are all in $\mathcal{C}$ (for the last one use Lemma 5.11(ii)). Finally, $S^{3,2}Q(1)$ is a direct summand in $\wedge^2 Q \otimes \wedge^2 Q \otimes Q(1)$ which is in $\mathcal{C}$ by Lemma 5.10(i).
(ii) Using the resolution for S^3Q we can relate $\wedge^2 Q \otimes S^2Q = \wedge^3 Q^* \otimes S^2Q(1)$ with $S^3Q \otimes S^2Q(1)$, which is in $\mathcal{C}$ by part (i). The objects appearing in between, namely, $Q^* \otimes S^2Q(1)$ and $\wedge^2 Q^* \otimes S^2Q(1)$ are in $\mathcal{C}$, by Lemmas 5.6(vi), 5.7(ii).
(iii) Since $\wedge^2 Q \otimes Q \otimes Q = \wedge^2 Q \otimes \wedge^2 Q \oplus \wedge^2 Q \otimes S^2Q$, this follows from part (ii) and Lemma 5.11(ii).
(iv) Considering the filtration of $\wedge^4(V \otimes \mathcal{O}) \otimes Q(1)$ we reduce ourselves to showing that the following objects are in $\mathcal{C}$:

$$\wedge^2 Q \otimes Q \otimes Q, \qquad Q \otimes Q, \qquad Q^* \otimes Q(2), \qquad Q^* \otimes Q \otimes \wedge^3 Q(1).$$

Now the assertion follows from part (iii) and Lemmas 5.11(iii), 5.6(ii) and 5.9(iv). □

Lemma 5.13. (i) *One has* $S^2Q \otimes S^4Q(1) \in \mathcal{C}$.
(ii) *One has* $S^2Q \otimes Q \in \mathcal{C}$.
(iii) *One has* $Q \otimes Q \otimes Q \in \mathcal{C}$.
(iv) *One has* $\wedge^2 Q \otimes \wedge^2 Q \otimes Q \in \mathcal{C}$.

Proof. (i) Consider the decomposition

$$S^2Q \otimes S^4Q(1) = S^6Q(1) \oplus S^{5,1}Q(1).$$

By Lemma 5.6(ii), we have $S^6Q(1) \in \mathcal{C}$. On the other hand, $S^{5,1}Q(1)$ is a direct summand in $\wedge^2 Q \otimes S^4Q(1)$. Using the resolution of S^4Q we reduce the problem to checking that the following objects are in $\mathcal{C}$:

$$Q \otimes \wedge^2 Q, \qquad \wedge^2 Q \otimes \wedge^2 Q, \qquad \wedge^2 Q^* \otimes \wedge^2 Q(1), \qquad Q^* \otimes \wedge^2 Q(1), \qquad \wedge^2 Q(1),$$

which follows from our previous work (for the second object use Lemma 5.11(ii)).
(ii) Tensoring the resolution for S^4Q with $S^2Q(1)$ we get an exact sequence

$$0 \to S^2Q \otimes Q \to V \otimes S^2Q \otimes \wedge^2 Q \to S^2V \otimes S^2Q \otimes \wedge^3 Q \to S^3V \otimes S^2Q \otimes Q^*(1)$$
$$\to S^4V \otimes S^2Q(1) \to S^2Q \otimes S^4Q(1) \to 0.$$

By part (i), one has $S^2Q \otimes S^4Q(1) \in \mathcal{C}$. Next, $S^2Q \otimes Q^*(1)$ and $S^2Q(1)$ are in $\mathcal{C}$ by Lemma 5.6(v), (ii). Finally, $S^2Q \otimes \wedge^2 Q$ and $S^2Q \otimes \wedge^3 Q$ are in $\mathcal{C}$ by Lemmas 5.12(ii) and 5.9(iii). Hence, $S^2Q \otimes Q \in \mathcal{C}$.
(iii) This follows from the decomposition

$$Q \otimes Q \otimes Q = S^2Q \otimes Q \oplus \wedge^2 Q \otimes Q,$$

part (ii) and Lemma 5.6(i).
(iv) This is proved by the same method as the case $j = 1$ of Lemma 5.10(i), using part (iii). □

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