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The spectral distance in the Moyal plane

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ABSTRACT

We study the noncommutative geometry of the Moyal plane from a metric point of view. Starting from a non-compact spectral triple based on the Moyal deformation $\mathcal{A}$ of the algebra of Schwartz functions on $\mathbb{R}^2$, we explicitly compute Connes' spectral distance between the pure states of $\mathcal{A}$ corresponding to eigenfunctions of the quantum harmonic oscillator. For other pure states, we provide a lower bound to the spectral distance, and show that the latest is not always finite. As a consequence, we show that the spectral triple (Gayral et al. (2004) [17]) is not a spectral metric space in the sense of Bellissard et al. (2010) [19]. This motivates the study of truncations of the spectral triple, based on $M_n(\mathbb{C})$ with arbitrary $n \in \mathbb{N}$, which turn out to be compact quantum metric spaces in the sense of Rieffel. Finally the distance is explicitly computed for $n = 2$.

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1. Introduction

Mainly motivated by quantum mechanics, where physical quantities are no longer functions on a manifold as in classical mechanics but are now elements of a noncommutative operator algebra, noncommutative geometry [1] aims at describing "spaces" in terms of algebras $\mathcal{A}$ rather than as sets of points. Many examples of noncommutative spaces are known (e.g. almost commutative manifolds [2], fuzzy spaces [3], deformations for actions of $\mathbb{R}^n$ [4], and Drinfel'd–Jimbo [5,6], Connes–Landi [7] and Connes–Dubois–Violette [8] deformations), but little work has been done regarding their metric aspect. Let us recall that in Connes theory a natural distance [9] on the state space $\mathcal{S}(\mathcal{A})$ of $\mathcal{A}$ is obtained via the construction of a *spectral triple* (cf. Section 2.2), the latest providing a noncommutative generalization of many tools of differential geometry [10]. In this paper we focus on the associated distance, whose definition is recalled in Definition 3.1, from now on called the *spectral distance*.

On a (finite dimensional, complete) Riemannian spin manifold M, the spectral distance d between pure states of the commutative algebra $C_0^\infty(M)$ of smooth functions vanishing at infinity coincides with the geodesic distance between the corresponding points, while d between non-pure states coincides with the Wasserstein distance of order 1 between the corresponding probability distributions [11]. In the noncommutative case, the meaning of the spectral distance is still obscure, essentially due to a lack of examples: d has been explicitly computed only for finite dimensional noncommutative algebras (e.g. [12,13]) and almost commutative geometries [14,15] where it exhibits some interesting links with the Carnot–Carathéodory distance in sub-Riemannian geometry [16]. For shedding more light on the spectral distance in a noncommutative framework, a natural idea is to investigate the motion of a quantum particle along a line which, from

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a mathematical point of view, amounts to studying the well known Moyal plane. An associated spectral triple has been proposed in [17], built around the algebra $\mathcal{A}$ of Schwartz functions on $\mathbb{R}^2$ equipped with the Moyal product $\star$. Although this spectral triple is an isospectral deformation of the canonical spectral triple of the Euclidean plane, the spectral distance d on the Moyal plane does not appear to be a deformation of the Euclidean distance on $\mathbb{R}^2$. It has rather a quantum mechanics interpretation. Indeed the pure states $\mathcal{P}(\mathcal{A})$ of $\mathcal{A}$ correspond to Wigner transition eigenfunctions of the quantum harmonic oscillator, and we show in this paper (Proposition 3.6) that the eigenstates of the quantum harmonic oscillator form a one-dimensional lattice of the Moyal plane, with the distance between two subsequent energy levels E_{m-1}, E_m proportional to $m^{-\frac{1}{2}}$. We also point out some states at infinite distance from one another, meaning that the topology induced by d on $\mathcal{S}(\mathcal{A})$ is *not* the weak* topology. Therefore the spectral triple for the Moyal plane introduced in [17] is not a *spectral metric space* [18,19]. This leads us to study truncations of the Moyal spectral triple, which turn out to be *compact quantum metric spaces* in the sense of Rieffel [20,21].

The paper is organized as follows. In Section 2 we recall some basics on the Moyal product, the associated spectral triple, and the pure states of the Moyal algebra, and we establish several technical results. In Section 3, we explicitly compute the spectral distance between the eigenstates of the quantum harmonic oscillator. We also derive some bounds for the distance between any pure states and show that the induced topology is not the weak* topology. In Section 4 we introduce a spectral triple on $M_n(\mathbb{C})$ obtained by truncating the Moyal spectral triple. We explicitly compute the associated spectral distance for $n = 2$, and for $n \geq 2$ we prove that the truncation gives a compact quantum metric space (Proposition 4.2). This is an interesting result since the "natural" spectral triple on $M_n(\mathbb{C})$ studied in [12] did not induce a quantum metric space. In Section 5 we draw conclusions and illustrate open problems.

2. Moyal non-compact spin geometries

2.1. The Moyal product, matrix basis and relevant algebras of tempered distribution on $\mathbb{R}^2$

The main properties of the Moyal machinery can be found e.g. in [22,23] to which we refer the reader for more details. Various related algebras have appeared in the literature, some of which will be recalled below. An extension of the Connes real spectral triple to the non-compact case, to which we will refer heavily throughout this paper, has been carried out in [17]; the corresponding action functionals and spectral actions have also been considered in [24]. Constructions of various derivation based differential calculi on Moyal algebras have been carried out in [25–27] together with applications to the construction of Yang–Mills–Higgs models on noncommutative Moyal spaces. In this work, we will only consider the two-dimensional case.

Let $\mathcal{S} = \mathcal{S}(\mathbb{R}^2)$ be the space of complex Schwartz functions on $\mathbb{R}^2$ and $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^2)$ its topological dual. Let $\theta > 0$ be a fixed positive real parameter.

Proposition 2.1 ([28,29]). *The associative Moyal $\star$ product is defined for all a, b in $\mathcal{S}$ by $\star : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$:*

$$(a \star b)(x) = \frac{1}{(\pi\theta)^2} \int \mathrm{d}^2y\, \mathrm{d}^2z\, a(x+y) b(x+z) \mathrm{e}^{-\mathrm{i}\, 2y\, \Theta^{-1} z}, \tag{2.1}$$

where $y\, \Theta^{-1} z \equiv y^\mu \Theta^{-1}_{\mu\nu} z^\nu$, $\Theta_{\mu\nu} = \theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

The complex conjugation is an involution for the $\star$ product; the integral is a faithful trace,

$$\int \mathrm{d}^2x (a \star b)(x) = \int \mathrm{d}^2x\, (b \star a)(x) = \int \mathrm{d}^2x\, a(x) b(x);$$

and the Leibniz rule is satisfied: $\partial_\mu (a \star b) = \partial_\mu a \star b + a \star \partial_\mu b$, $\forall a, b \in \mathcal{S}$.

Throughout the paper, we use the notation

$$\mathcal{A} := (\mathcal{S}, \star) \tag{2.2}$$

for the non-unital involutive algebra of Schwartz functions equipped with the Moyal product.

Our analysis will use the matrix basis whose relevant properties are summarized below.

Proposition 2.2 ([22]). *The matrix basis $\{f_{mn}\}_{m,n\in\mathbb{N}} \subset \mathcal{S} \subset L^2(\mathbb{R}^2)$ is the family of Wigner transition eigenfunctions of the one-dimensional harmonic oscillator,*

$$f_{mn} = \frac{1}{(\theta^{m+n} m! n!)^{1/2}}\, \bar{z}^{\star m} \star f_{00} \star z^{\star n}$$

where $f_{00} = 2e^{-2H/\theta}$, $H = \frac{1}{2}(x_1^2 + x_2^2)$, $X^{\star n} = \underbrace{X \star X \star \cdots \star X}_{n \text{ times}}$ *and*

$$\bar{z} = \frac{1}{\sqrt{2}}(x_1 - \mathrm{i}x_2), \qquad z = \frac{1}{\sqrt{2}}(x_1 + \mathrm{i}x_2).$$

(i) *Writing as* $\langle ., . \rangle$ *the inner product*[1] *on* $L^2(\mathbb{R}^2)$, *one has the relations*

$$f_{mn} \star f_{pq} = \delta_{np} f_{mq}, \qquad f_{mn}^* = f_{nm}, \qquad \langle f_{mn}, f_{kl} \rangle = (2\pi\theta)\delta_{mk}\delta_{nl}.$$

(ii) $\mathcal{A}$ *is a Fréchet pre-*C^**-algebra, with semi-norms*[2] $\rho_{\alpha\beta}(a) = \sup_{x \in \mathbb{R}^2} |x^\beta D^\alpha a|$, *isomorphic to the Fréchet algebra of rapid decay matrices* $(a_{mn})_{m,n \in \mathbb{N}}$ *with semi-norms*

$$\rho_k^2(a) := \sum_{m,n \in \mathbb{N}} \theta^{2k} \left(m + \frac{1}{2}\right)^k \left(n + \frac{1}{2}\right)^k |a_{mn}|^2. \tag{2.3}$$

The isomorphism is given by $(a_{mn}) \to \sum_{m,n} a_{mn} f_{mn} \in \mathcal{S}$, *with inverse*

$$a \in \mathcal{S} \to a_{mn} = \frac{1}{2\pi\theta}\langle f_{mn}, a \rangle = \frac{1}{2\pi\theta} \int \mathrm{d}^2x\, a(x) f_{mn}^*(x) = \frac{1}{2\pi\theta} \int \mathrm{d}^2x\, a(x) f_{nm}(x).$$

The matrix basis diagonalizes the Hamiltonian of the harmonic oscillator,

$$H \star f_{mn} = \theta(2m + 1) f_{mn}, \qquad f_{mn} \star H = \theta(2n + 1) f_{mn},$$

and is also called the *twisted Hermite basis* (an explicit decomposition in the Hermite functions h_m is provided by the Wigner operator $W(f_{mn}) = h_m \otimes h_n$; see [22][3] and references therein). In particular the f_{mm}, $m \in \mathbb{N}$, are the eigenstates of the mth energy level of the harmonic oscillator, and in Proposition 3.6 we compute the spectral distance between any two of them. In a different context, the authors of [30,31] have used H as (the square of) a distance operator, yielding a notion of quantized distance in the Moyal plane which is distinct from the spectral distance. Finding the link between these two approaches will be the subject of a future work [32]. For subsequent computations, let us write out the derivatives of the f_{mn}'s, that are obtained by easy calculation.

Proposition 2.3. *Define* $\partial = \frac{1}{\sqrt{2}}(\partial_1 - \mathrm{i}\partial_2)$, $\bar{\partial} = \frac{1}{\sqrt{2}}(\partial_1 + \mathrm{i}\partial_2)$. *For any* $m, n \in \mathbb{N}$,

$$\partial f_{mn} = \sqrt{\frac{n}{\theta}} f_{m,n-1} - \sqrt{\frac{m+1}{\theta}} f_{m+1,n}; \qquad \bar{\partial} f_{mn} = \sqrt{\frac{m}{\theta}} f_{m-1,n} - \sqrt{\frac{n+1}{\theta}} f_{m,n+1}.$$

The $\star$ product (2.1) is extended to spaces larger than $\mathcal{S}$, obtained by completing the Schwartz algebra with respect to the norm

$$\|a\|_{s,t}^2 = \sum_{m,n} \theta^{s+t} \left(m + \frac{1}{2}\right)^s \left(n + \frac{1}{2}\right)^t |a_{mn}|^2$$

with $s, t \in \mathbb{R}$. Calling $\mathcal{G}_{s,t}$ the completion, one has that for any $a = \sum_{m,n} a_{mn} f_{mn} \in \mathcal{G}_{s,t}$, $b = \sum_{m,n} b_{mn} f_{mn} \in \mathcal{G}_{q,r}$ with $t + q \geq 0$, the sequence $c_{mn} = \sum_p a_{mp} b_{pn}$ converges in $\mathcal{G}_{s,r}$, thus defining a map $\star : \mathcal{G}_{s,t} \times \mathcal{G}_{q,r} \to \mathcal{G}_{s,r}$. In particular $\mathcal{G}_{0,0} = L_2(\mathbb{R}^2)$ and one has the dense inclusions $\mathcal{S} \subset \mathcal{G}_{s,t} \subset \mathcal{S}'$ for any $s, t \in \mathbb{R}$. Using continuity and duality of $\star$ (for more details, see e.g. [22,23]) one can define the star product on various subspaces of $\mathcal{S}'$. The following algebras in particular are relevant for the study of Moyal spaces.

Proposition 2.4 (*[17]*). *Let* $\mathcal{D}_{L^2}$ *denote the space of square integrable smooth functions on* $\mathbb{R}^2$ *having all their derivatives in* $L^2(\mathbb{R}^2)$, $\mathcal{B}$ *the space of smooth functions on* $\mathbb{R}^2$ *that are bounded together with all their derivatives and*

$$\mathbf{A}_\theta = \{a \in \mathcal{S}' / a \star b \in L^2(\mathbb{R}^2),\ \forall b \in L^2(\mathbb{R}^2)\}.$$

(i) $\mathbf{A}_\theta$ *is a unital* $C*$*-algebra with operator norm*

$$\|a\|_\theta = \|L(a)\|_{op} = \sup_{0 \neq b \in L^2(\mathbb{R}^2)} \left\{ \frac{\|a \star b\|_2}{\|b\|_2} \right\} \quad \forall a \in \mathbf{A}_\theta, \tag{2.4}$$

[1] Note the change of convention with respect to [17] in which $\langle ., . \rangle$ denotes the inner product on $L^2(\mathbb{R}^2)$ divided by $\pi\theta$.

[2] With standard multi-index notation $x^\alpha D^\beta := x_1^{\beta_1} x_2^{\beta_2} \partial_1^{\alpha_1} \partial_2^{\alpha_2}$.

[3] The authors there use the convention $\theta = 2$.

where $L(.)$ is the left multiplication operator and $\|.\|_2$ denotes the $L^2(\mathbb{R}^2)$ norm. Moreover $\mathbf{A}_\theta$ is isomorphic to the algebra of bounded operators on $L^2(\mathbb{R})$.

(ii) *$(\mathcal{D}_{L^2}, \star)$ is a Fréchet sub-algebra of $(\mathbf{A}_\theta, \star)$, with Fréchet topology of $L^2(\mathbb{R}^2)$-convergence for all derivatives; $(\mathcal{B}, \star)$ is a Fréchet sub-algebra of $(\mathbf{A}_\theta, \star)$, with topology given by the family of semi-norms $\{p_m\}_{m\in\mathbb{N}}$, $p_m(a) := \max_{|\alpha|<m} \|\partial^\alpha(a)\|_\infty$.*

(iii) *$(\mathcal{D}_{L^2}, \star)$ and $(\mathcal{B}, \star)$ are respectively non-unital and unital pre C^*-algebras with respect to $\|.\|_\theta$. Denoting by an overbar their C^*-completions, one has the inclusions*

$$\mathcal{A} \subset \bar{\mathcal{D}}_{L^2} \subset \bar{\mathcal{B}} \subset \mathbf{A}_\theta. \tag{2.5}$$

The above algebras are related but not identical to the maximal unitization $\mathcal{M} = (\mathcal{M}_L \cap \mathcal{M}_R, \star)$ of $\mathcal{A}$ underlying most of the studies on noncommutative field theories and noncommutative gauge theories on Moyal spaces (see e.g. [33,34,25–27] and references therein), where

$$\mathcal{M}_L := \{a \in \mathcal{S}'/a \star b \in \mathcal{S},\ \forall b \in \mathcal{S}\}, \qquad \mathcal{M}_R := \{a \in \mathcal{S}'/b \star a \in \mathcal{S},\ \forall b \in \mathcal{S}\}.$$

$\mathcal{M}$ is unsuitable here as it cannot be represented by bounded operators on the Hilbert space $L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$ used in the non-compact spectral triple that we now recall.

2.2. The spectral triple for the Moyal plane

A *unital* spectral triple [1] is the datum of a unital involutive algebra A together with a representation π on a Hilbert space $\mathcal{H}$ and a self-adjoint not necessarily bounded operator D (called the Dirac operator) on $\mathcal{H}$, such that $[D, \pi(a)]$ is bounded for any $a \in A$ and the resolvent of D is compact. In the non-compact case, i.e. for non-unital A, one asks instead [35] that the operators $\pi(a)(D - \lambda)^{-1}$ are compact for any $\lambda \notin \mathrm{Sp}D$. A natural candidate spectral triple for the Moyal plane is the isospectral deformation [7] of the canonical spectral triple of the Euclidean plane $\mathbb{R}^2$ built around the algebra of Schwartz functions; namely

$$(\mathcal{A}, \mathcal{H}, D) \tag{2.6}$$

with $\mathcal{A}$ defined in (2.2),

$$\mathcal{H} = L^2(\mathbb{R}^2) \otimes \mathbb{C}^2 \equiv \mathcal{H}_0 \otimes \mathbb{C}^2$$

the Hilbert space of square integrable sections of the trivial spinor bundle $\mathbb{S} = \mathbb{R}^2 \times \mathbb{C}^2$ with inner product

$$\langle \psi, \phi \rangle = \int (\psi_1^* \phi_1 + \psi_2^* \phi_2) \mathrm{d}^2 x \quad \forall\ \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},\ \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in \mathcal{H},$$

and, using the Einstein convention of summing over repeated indices,

$$D = -i\sigma^\mu \partial_\mu = -i\sqrt{2} \begin{pmatrix} 0 & \bar{\partial} \\ \partial & 0 \end{pmatrix} \tag{2.7}$$

where

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

span an irreducible representation of the Clifford algebra $C\ell_2(\mathbb{C}) = M_2(\mathbb{C})$. $\mathcal{A}$ acts faithfully on $\mathcal{H}$ via the representation $\pi(a) = L(a) \otimes \mathbb{I}_2$, namely

$$\pi(a)\psi = (a \star \psi_1, a \star \psi_2) \quad \forall a \in \mathcal{A}, \psi \in \mathcal{H}. \tag{2.8}$$

(2.6) does satisfy the property of a non-compact spectral triple: $\pi(a)$ is bounded by (2.5) and (2.8) together with $\mathbf{A}_\theta \simeq \mathcal{B}(L^2(\mathbb{R}))$. The boundedness of $[D, \pi(a)]$ comes from $[D, \pi(a)] = -\mathrm{i}L(\partial_\mu a) \otimes \sigma^\mu$ (see (3.6)), combined with $L(\partial_\mu a) \in \mathcal{B}(L^2(\mathbb{R}))$. D is the usual Dirac operator on $\mathbb{R}^2$ so it is essentially self-adjoint. Note that $\partial^\dagger = -\bar{\partial}$, on its domain $\mathcal{D}_{L^2} \otimes \mathbb{C}^2$, can be easily seen, noticing that the adjoint of ∂ is $\partial^\dagger = -\bar{\partial}$. The resolvent condition is more involved and we refer the reader to [17] for the details.

In order to establish the equivalence between (unital) commutative spectral triples and (compact, oriented, without boundary) smooth manifolds, one further asks for five extra conditions on $(\mathcal{A}, \mathcal{H}, D)$ [10] (dimension, order 1, regularity, orientability, finiteness), that can be completed by two other conditions (reality, Poincaré duality) in order to recover spin and Riemannian structures from purely algebraic data. This yields the definition of *real* spectral triples [35]. Among these conditions, several easily translate to the non-compact case (dimension, regularity, reality, first-order condition); one still lacks a formulation in the non-compact case (Poincaré duality). For the remaining two (finiteness, orientability), an adaptation to the non-unital case has been proposed in [17], based on a preferred unitization $\mathcal{A}_1 := (\mathcal{B}, \star)$ of $\mathcal{A}$. However, as can be checked in Definition 3.1, none of these conditions enters the definition of the spectral distance. So, regarding the metric aspect of the Moyal plane, it is more natural to work with the spectral triple (2.6) as we do in the following than with the unitization $\mathcal{A}_1$. We come back to this point in Section 4 and in the conclusion.

3. Spectral distance on the Moyal plane

3.1. The distance formula and pure states

Let us begin with technical precision. A state on a complex C^*-algebra A is a positive linear map $A \to \mathbb{C}$ of norm 1, and strictly speaking this notion is reserved for C^*-algebras. However given a pre-C^*-algebra A, a state on its C^*-completion $\bar{A}$ defines by restriction a unique positive linear map of norm 1 from A to $\mathbb{C}$, and conversely by continuity any state of $\bar{A}$ is uniquely determined by its restriction to A. So it is also legitimate to talk about states for pre-C^*-algebras.

The metric aspect of the noncommutative geometry $(\mathcal{A}, \mathcal{H}, D)$ introduced in (2.6) is fully encoded within the spectral distance defined as follows.

Definition 3.1. The spectral distance between two states ω_1 and ω_2 of $\mathcal{A}$ is

$$d(\varphi_1, \varphi_2) = \sup_{a\in\mathcal{A}} \big\{ |\varphi_1(a) - \varphi_2(a)|;\ \|[D, \pi(a)]\|_{\mathrm{op}} \leq 1 \big\} \tag{3.1}$$

where $\|.\|_{\mathrm{op}}$ is the operator norm for the representation of $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$.

One easily checks that (3.1) has all the properties of a distance, except that it may take the value $+\infty$. Recall that in the commutative case, i.e. for $\mathcal{A} = C_0^\infty(M)$ for M a complete Riemannian spin manifold, $\mathcal{H}$ the space of spinors square integrable with respect to the volume form associated with the Riemannian metric, and D the Dirac operator of the Levi-Civita connection, then pure states are evaluations at $x \in M$:

$$\delta_x(f) = f(x) \quad \forall f \in C_0^\infty(M), \tag{3.2}$$

and $d(\delta_x, \delta_y)$ coincides with the geodesic distance on M. Therefore it is appealing, in the noncommutative case, to consider the spectral distance between pure states as a good equivalent of the geodesic distance between points.

However, on the deformed algebra $\mathcal{A}$ the evaluation (3.2) is no longer a state, for $(f^* \star f)(x)$ has no reason to be positive. Having in mind Moyal spaces as a quantized version of Euclidean spaces, one could be tempted to look at the pure states of $\mathcal{A}$ as a deformation of the pure states of $C_0^\infty(\mathbb{R}^2)$. A nice approach for obtaining states for a Rieffel deformation of a C^*-algebra by deformation of states of the undeformed algebra has been developed in [36], but the "purity" of the state is not addressed there. In fact, rather than using the x-representation (2.1), pure states of $\mathcal{A}$ are more easily determined in the twisted Hermite basis $\{f_{mn}\}_{m,n\in\mathbb{N}}$.

Let e_n be the canonical orthonormal basis of $\ell^2(\mathbb{N})$ and η the natural representation of rapid decay matrices, given by row by column multiplication. By Proposition 2.2 the isometry $L^2(\mathbb{R}^2) \to \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ given by

$$U : f_{mn} \mapsto \sqrt{2\pi\theta}\, e_m \otimes e_n$$

is an intertwiner of the left regular representation L and the representation $\eta \otimes 1$. One can see that η is faithful and irreducible,[4] while L is only faithful (it is the direct sum of infinitely many copies of η, in the same way as the left regular representation of $M_n(\mathbb{C})$ with Hilbert–Schmidt inner product is unitary equivalent to the direct sum of n copies of the irreducible representation). Therefore

$$\mathcal{A} \simeq L(\mathcal{A}) \simeq \eta(\mathcal{A}).$$

Since the norm of any operator a on $\ell^2(\mathbb{N})$ coincides with the norm of $a\otimes 1$ on $\ell^2(\mathbb{N})\otimes\ell^2(\mathbb{N})$, the closure of $L(\mathcal{A})$ in $\mathcal{B}(L^2(\mathbb{R}^2))$ is isometrically $*$-isomorphic to the closure of $\eta(\mathcal{A})$ in $\mathcal{B}(\ell^2(\mathbb{N}))$: we name as $\bar{\mathcal{A}}$ this C^*-algebra.

Notice that

$$\bar{\mathcal{A}} \supset \eta(\mathcal{A}) \supset M_\infty(\mathbb{C}) := \bigcup_{k\geq 1} M_k(\mathbb{C})$$

where $M_k(\mathbb{C})$ are identified with $a = ((a_{mn})) \in \mathcal{A}$ such that $a_{mn} = 0$ if $m \geq k$ or $n \geq k$. By [37, II.8.2.2] the closure of $M_\infty(\mathbb{C})$ is the C^*-algebra $\mathbb{K}$ of compact operators on $\ell^2(\mathbb{N})$, proving that $\bar{\mathcal{A}} = \overline{\eta(\mathcal{A})} \supset \mathbb{K}$. On the other hand, η maps any rapid decay matrix a into a Hilbert–Schmidt operator, since the norm $\rho_0(a)$ in (2.3) is exactly the Hilbert–Schmidt norm of $\eta(a)$. Hence $\eta(a)$ is a compact operator for any $a \in \mathcal{A}$ and $\overline{\eta(\mathcal{A})} \subset \mathbb{K}$, proving that $\bar{\mathcal{A}} \simeq \mathbb{K}$.

It is well known [38, Corollary 10.4.4] that all pure states of $\mathbb{K}$ are vector states of the (unique) irreducible representation on $\ell^2(\mathbb{N})$. Hence:

Proposition 3.2. *Any unit vector $\psi = \sqrt{2\pi\theta}\sum_m \psi_m e_m \in \ell^2(\mathbb{N})$, i.e. with normalization $\sum_m |\psi_m|^2 = \frac{1}{2\pi\theta}$, defines a pure state ω_ψ of $\mathcal{A}$,*

$$\omega_\psi(a) := 2\pi\theta \sum_{m,n\in\mathbb{N}} \psi_m^* \psi_n a_{mn}. \tag{3.3}$$

Moreover any pure state of $\mathcal{A}$ comes from such a unit vector.

[4] It is irreducible since its restriction to $M_\infty(\mathbb{C})$ is irreducible.

Notice that ψ can be obtained from

$$v_n := U^*(\psi \otimes e_n) = \sum_{m\geq 0} \psi_m f_{mn} \tag{3.4}$$

for any $n \in \mathbb{N}$, and we have $\omega_\psi(a) \equiv \langle v_n, L(a)v_n\rangle$.

3.2. The unit ball

In order to compute the distance (3.1) on the Moyal plane, one needs to conveniently characterize the unit ball

$$\mathcal{B}_D := \left\{a \in \mathcal{A},\ \|[D, \pi(a)]\|_{\text{op}} \leq 1\right\}. \tag{3.5}$$

The first step is to determine the relation between the coefficients of a in the matrix basis and those of $[D, \pi(a)]$. Noticing that $[\partial_\mu, L(a)] = L(\partial_\mu a)$ by Leibniz rule one obtains

$$[D, \pi(a)] = -i\sqrt{2}\begin{pmatrix} 0 & L(\bar\partial a) \\ L(\partial a) & 0 \end{pmatrix}. \tag{3.6}$$

Thus

$$\begin{aligned} \|[D, \pi(a)]\|^2_{\text{op}} &= \left\|[D, \pi(a)]^*[D, \pi(a)]\right\|_{\text{op}} = 2\left\|\begin{pmatrix} L(\partial a)^*L(\partial a) & 0 \\ 0 & L(\bar\partial a)^*L(\bar\partial a)\end{pmatrix}\right\|_{\text{op}} \\ &= 2\max\left(\left\|L(\partial a)^*L(\partial a)\right\|_{\text{op}}, \left\|L(\bar\partial a)L(\bar\partial a)^*\right\|_{\text{op}}\right) \\ &= 2\max\left(\|\partial a\|^2_\theta, \left\|\bar\partial a\right\|^2_\theta\right) \end{aligned} \tag{3.7}$$

and the aim is now to find the relation between the coefficients of a and those of ∂a, $\bar\partial a$.

Proposition 3.3. *For any $a \in \mathcal{A}$, $a = \sum_{m,n} a_{mn} f_{mn}$, we define*

$$\partial a := \sum_{m,n} \alpha_{mn} f_{mn}, \qquad \bar\partial a := \sum_{m,n} \beta_{mn} f_{mn}.$$

(i) *The coefficients of ∂a, $\bar\partial a$ as functions of the coefficients of a are given by*

$$\alpha_{m+1,n} = \sqrt{\frac{n+1}{\theta}}a_{m+1,n+1} - \sqrt{\frac{m+1}{\theta}}a_{m,n}, \qquad \alpha_{0,n} = \sqrt{\frac{n+1}{\theta}}a_{0,n+1}, \tag{3.8}$$

$$\beta_{m,n+1} = \sqrt{\frac{m+1}{\theta}}a_{m+1,n+1} - \sqrt{\frac{n+1}{\theta}}a_{m,n}, \qquad \beta_{m,0} = \sqrt{\frac{m+1}{\theta}}a_{m+1,0}, \tag{3.9}$$

for all $m, n \in \mathbb{N}$.

(ii) *One has the inversion formula*

$$a_{p,q} = \delta_{p,q}a_{0,0} + \sqrt{\theta}\sum_{k=0}^{\min(p,q)} \frac{\alpha_{p-k,q-k-1} + \beta_{p-k-1,q-k}}{\sqrt{p-k}+\sqrt{q-k}}, \quad \forall\ p, q \in \mathbb{N},\ p+q>0$$

where by convention terms with negative indices do not appear in the sum.

Proof. (i) is proved by standard calculation using Proposition 2.3. To prove (ii) one combines (3.8) and (3.9) to obtain

$$a_{p+1,q+1} = a_{p,q} + \sqrt{\theta}\,\frac{\alpha_{p+1,q} + \beta_{p,q+1}}{\sqrt{p+1}+\sqrt{q+1}}.$$

This yields by induction

$$a_{p,q} = \begin{cases} a_{0,q-p} + \sqrt{\theta}\displaystyle\sum_{k=0}^{p-1} \frac{\alpha_{p-k,q-k-1} + \beta_{p-k-1,q-k}}{\sqrt{p-k}+\sqrt{q-k}} & \text{if } 0 \leq p < q, \\ a_{0,0} + \sqrt{\theta}\displaystyle\sum_{k=0}^{p-1} \frac{\alpha_{p-k,p-k-1} + \beta_{p-k-1,p-k}}{2\sqrt{p-k}} & \text{if } 0 \leq p = q, \\ a_{p-q,0} + \sqrt{\theta}\displaystyle\sum_{k=0}^{q-1} \frac{\alpha_{p-k,q-k-1} + \beta_{p-k-1,q-k}}{\sqrt{p-k}+\sqrt{q-k}} & \text{if } 0 \leq q < p, \end{cases}$$

and (ii) follows from a further use of the second relations for $\alpha_{0,n}$ and $\beta_{0,n}$. □

The second step carried out in order to characterize $\mathcal{B}_D$ would be computing $\|[D, a]\|$ as an explicit function $f(\alpha_{mn})$, then using Proposition 3.3 to transfer the constraints $f(\alpha_{mn}) \leq 1$ to the coefficients a_{mn}. However we are dealing with infinite dimensional matrices and one may well not have such a function f. Here we simply exhibit some necessary constraints on the a_{mn}'s such that $a \in \mathcal{B}_D$. Quite remarkably these constraints are sufficient for explicitly computing, in the next section, the distance for interesting classes of pure states.

Lemma 3.4. *Let $a \in \mathcal{B}_D$. Then:*

(i) *For any unit vector $\varphi = \sum_{m,n} \varphi_{mn} f_{mn} \in \mathcal{H}_0$ one has*

$$\sum_p |\alpha_{mp}\|\varphi_{pn}| \leq \frac{1}{2\sqrt{\pi\theta}} \quad \text{and} \quad \sum_p |\beta_{mp}\|\varphi_{pn}| \leq \frac{1}{2\sqrt{\pi\theta}}, \quad \forall m, n \in \mathbb{N}; \tag{3.10}$$

(ii) $|\alpha_{mn}| \leq \frac{1}{\sqrt{2}}$ *and* $|\beta_{mn}| \leq \frac{1}{\sqrt{2}}$, $\forall m, n \in \mathbb{N}$.

Proof. Let φ be a unit vector in $\mathcal{H}_0$, that is to say $\|\varphi\|_2^2 = 2\pi\theta \sum_{mn} |\varphi_{mn}|^2 = 1$. Using the matrix basis, a standard calculation yields

$$\|\partial a \star \varphi\|_2^2 = 2\pi\theta \sum_{m,n} \left| \sum_p \alpha_{mp}\varphi_{pn} \right|^2 .$$

By (3.7), $a \in \mathcal{B}_D$ implies $\|\partial a\|_\theta \leq \frac{1}{\sqrt{2}}$; hence, owing to the definition of $\|\partial a\|_\theta$,

$$\sum_{m,n} \left| \sum_p \alpha_{mp}\varphi_{pn} \right|^2 \leq \frac{1}{4\pi\theta}.$$

This implies

$$\left| \sum_p \alpha_{mp}\varphi_{pn} \right| \leq \frac{1}{2\sqrt{\pi\theta}} \quad \forall m, n \in \mathbb{N} \tag{3.11}$$

together with a similar relation stemming from $\|\bar{\partial} a\|_\theta$ with the α_{mn}'s replaced by β_{mn}. Now (3.11) holds true for any unit $\varphi \in \mathcal{H}_0$ and in particular, given a value $m \in \mathbb{N}$, for $\tilde{\varphi}^m$ defined by $\tilde{\varphi}^m_{pn} := \mathrm{e}^{-\mathrm{iArg}(\alpha_{mp})}|\varphi_{pn}|$. This implies

$$\sum_p |\alpha_{mp}||\varphi_{pn}| \leq \frac{1}{2\sqrt{\pi\theta}}, \quad \forall n \in \mathbb{N} \tag{3.12}$$

and the first part of (3.10) by repeating the procedure for other values of m. The second part is obtained by similar considerations applied to the β_{mn}'s. (ii) is obtained by considering in (3.12) the unit vector $\varphi = \frac{1}{\sqrt{2\pi\theta}} f_{mn}$. □

Notice that by the triangle inequality, (3.12) implies (and thus is equivalent to) (3.11).

Remark 3.5. When the algebra $\mathcal{A}$ is unital, the supremum in the distance formula can be equivalently searched for on the positive unit sphere [12]:

$$\mathcal{S}_D^+ := \left\{ a \in \mathcal{A}^+, \|[D, \pi(a)]\|_{\mathrm{op}} = 1 \right\}$$

where $\mathcal{A}^+$ denotes the positive elements of $\mathcal{A}$. When the algebra is not unital, the supremum can be searched for on the self-adjoint elements of the unit ball:

$$\mathcal{B}_D^{\mathrm{sa}} := \left\{ a = a^* \in \mathcal{A}, \|[D, \pi(a)]\|_{\mathrm{op}} \leq 1 \right\}.$$

In this paper we are interested in the spectral distance on the non-unital algebra $\mathcal{A}$, so we assume that a is always self-adjoint, which implies $(\partial a)^* = (\bar{\partial} a)$, that is to say $\beta_{mn} = \bar{\alpha}_{nm}$.

3.3. The distance on the diagonal

The set of diagonal elements of $\mathcal{A}$ (i.e. the maximal abelian sub-algebra) is the set of rapid decay sequences $\mathcal{s}(\mathbb{N})$, whose pure states ω_m, $m \in \mathbb{N}$ are defined in (3.3) with $\psi = e_m$ (i.e. from (3.4): $U^*(\psi \otimes e_q) = \frac{1}{\sqrt{2\pi\theta}} f_{mq}$), namely

$$\omega_m(a) = a_{mm}.$$

Since on $\mathcal{s}(\mathbb{N})$ the operator norm and the max norm coincide, $\|a\|_{\mathrm{op}} \equiv \max_{p,q} |a_{pq}|$, one can expect the distance between any two ω_m, ω_n to be easily computable.

Proposition 3.6. *The spectral distance between ω_m, ω_n, $n < m$, is*

$$d(\omega_m, \omega_n) = \sqrt{\frac{\theta}{2}} \sum_{k=n+1}^{m} \frac{1}{\sqrt{k}}.$$

Proof. By Proposition 3.3,

$$\alpha_{n+1,n} = \sqrt{\frac{n+1}{\theta}}(a_{n+1,n+1} - a_{nn}) = \sqrt{\frac{n+1}{\theta}}(\omega_{n+1,n+1}(a) - \omega_{nn}(a)), \quad \forall n \in \mathbb{N}$$

so, for any a in the unit ball, (ii) of Lemma 3.4 yields

$$|\omega_{n+1}(a) - \omega_n(a)| \leq \sqrt{\frac{\theta}{2}} \frac{1}{\sqrt{n+1}}, \quad \forall n \in \mathbb{N}. \tag{3.13}$$

Hence $d(\omega_{n+1}, \omega_n) \leq \sqrt{\frac{\theta}{2}} \frac{1}{\sqrt{n+1}}$ and by a repeated use of the triangular inequality,

$$d(\omega_m, \omega_n) \leq \sqrt{\frac{\theta}{2}} \sum_{k=n+1}^{m} \frac{1}{\sqrt{k}}, \quad \forall m, n \in \mathbb{N},\ n < m. \tag{3.14}$$

This upper bound is attained by any element in the unit ball which saturates (3.13) between m and n, that is to say such that

$$a_{p+1,p+1} - a_{pp} = \sqrt{\frac{\theta}{2}} \frac{1}{\sqrt{p+1}} \quad \text{for } n \leq p \leq m-1. \tag{3.15}$$

For instance the element $a^{(m)}$ (diagonal in the matrix basis) with components

$$a^{(m)}_{pq} = -\delta_{pq} \sqrt{\frac{\theta}{2}} \sum_{k=p}^{m} \frac{1}{\sqrt{k+1}} \tag{3.16}$$

(where an empty sum means zero) obviously satisfies (3.15). Moreover $\left\{a^{(m)}_{pq}\right\}_{p,q\in\mathbb{N}}$ is a rapid decay sequence since only finitely many elements are non-zero; hence $a^{(m)} \in \mathcal{A}$. By (i) of Proposition 3.3, one checks that all the α_{pq}'s vanish except $\alpha_{p+1,p} = \frac{1}{\sqrt{2}}$ for $0 \leq p \leq m$. Therefore for any $\psi \in \mathcal{H}_0$,

$$\begin{aligned} \left\| \partial a^{(m)} \star \psi \right\|_2^2 &= 2\pi\theta \sum_{p,q} \left| \sum_g \alpha_{pg} \psi_{gq} \right|^2 = 2\pi\theta \sum_{p,q} \left| \alpha_{p+1,p} \psi_{pq} \right|^2 \\ &\leq \pi\theta \sum_{p,q} |\psi_{pq}|^2 = \frac{1}{2} \|\psi\|^2, \end{aligned}$$

so $\left\| \partial\, a^{(m)} \right\|_\theta \leq \frac{1}{\sqrt{2}}$, showing by (3.7) that $a^{(m)}$ is in the unit ball. □

Notice that the distance between nearest points ω_{k-1} and ω_k is $\sqrt{\frac{\theta}{2k}}$, and the distance between ω_p and ω_q $(p < q)$ is the sum over the path joining the two points

$$d(\omega_p, \omega_q) = \sum_{k=p+1}^{q} d(\omega_{k-1}, \omega_k).$$

In other words, for any $p < n < q$, ω_n is a middle point between ω_p and ω_q, i.e. the triangle inequality becomes an equality $d(\omega_p, \omega_q) = d(\omega_p, \omega_n) + d(\omega_n, \omega_q)$.

Note also that the element $a^{(m)} \in \mathcal{S}(\mathbb{N})$ that attains the supremum (defined in (3.16)) has a geometrical interpretation, in analogy with the commutative case.

Proposition 3.7. *In the weak topology,*

$$\lim_{m\to\infty} i[D, \pi(a^{(m)})] = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix}$$

where $S = Z^*|Z|^{-1}$, $S^* = |Z|^{-1}Z$ with $Z = L(z)$ the representation of z on $\mathcal{H}_0$.

Proof. First notice that in the matrix basis

$$Zf_{m+1,n} = \sqrt{\theta(m+1)}f_{mn}, \qquad Z^*f_{mn} = \sqrt{\theta(m+1)}f_{m+1,n},$$

for all $m, n \geq 0$, and $Zf_{0,n} = 0$. Hence $|Z| = (ZZ^*)^{\frac{1}{2}}$ acts as

$$|Z|f_{mn} = \sqrt{\theta(m+1)}f_{mn}$$

and is thus invertible, so the bounded operators S, S^* are well defined. Their action is $Sf_{m,n} = f_{m+1,n}$ and $S^*f_{m+1,n} = f_{m,n}$.

From (2.4) one has

$$(\partial\, a^{(l)})f_{mn} = \sum_p \alpha_{p+1,p} f_{p+1,p} \star f_{mn} = \alpha_{m+1,m} f_{m+1,n} = \begin{cases} \dfrac{1}{\sqrt{2}} S f_{mn} & \text{for } m \leq l, \\ 0 & \text{for } m > l, \end{cases}$$

and hence the result follows by (3.6). □

In the commutative limit, i.e. for $\theta = 0$, $S = |z|^{-1}z^*$ is (almost everywhere) equal to $2\partial|z| = \sqrt{2}[\partial, \sqrt{2}|z|]$. The function $z \mapsto \sqrt{2}|z|$ is precisely the function that attains the supremum in the computation of the spectral distance on the positive real half-line, i.e. $d(x, y)$ for $0 \leq x \leq y$. In this sense $\mathcal{P}(\mathcal{J}(\mathbb{N}))$ is a kind of discretization of the positive real half-line within the Moyal plane.

3.4. The lower bound and states at infinite distance

In the previous section we computed the distance between pure states associated with vectors (3.4) with only one non-zero component. Studying vectors with more components is not easy, due to the lack of an explicit formula for $\|[D, a]\|$ (instead, Lemma 3.4 only gives necessary conditions satisfied by the elements of the unit ball). Nevertheless we show in this section how to provide a lower bound for the distance between any two pure states, allowing us to exhibit some states that are at infinite distance from one another.

We first notice that the difference between any two vector states of Proposition 3.2 is naturally related to the universal differential $d_U a = a \otimes \mathbb{I} - \mathbb{I} \otimes a$, $a \in \mathcal{A}$.

Lemma 3.8. *Let $\omega_{\psi'}$ and ω_ψ be two pure states. For any $a \in \mathcal{A}$, one has*

$$\omega_{\psi'}(a) - \omega_\psi(a) = (\omega_{\psi'} \otimes \omega_\psi) d_U a = (2\pi\theta)^2 \sum_{m,n,p,q} (d_U a)_{mn,pq} \psi'^*_m \psi^*_p \psi'_n \psi_q$$

where $(d_U a)_{mn,pq} := (a_{mn}\delta_{pq} - a_{pq}\delta_{mn})$.

Proof. One has

$$\begin{aligned} \omega_{\psi'}(a) - \omega_\psi(a) &= 2\pi\theta \sum_{m,n} a_{mn}(\psi'^*_m \psi'_n - \psi^*_m \psi_n) \\ &= (2\pi\theta)^2 \sum_{m,n,p,q} (a_{mn}\delta_{pq} - a_{pq}\delta_{mn}) \psi'^*_m \psi^*_p \psi'_n \psi_q \end{aligned}$$

where the second equality stems from $\|\psi'\|_2 = \|\psi\|_2 = 1$. □

Lemma 3.9. *For any unit vector $\psi, \psi' \in \mathcal{H}_0$,*

$$d(\omega_\psi, \omega_{\psi'}) \geq (2\pi\theta)^2 \sqrt{\frac{\theta}{2}} \left| \sum_{q=1}^{\infty} \sum_{p=0}^{q-1} \sum_{k=p+1}^{q} \frac{1}{\sqrt{k}} (|\psi_p \psi'_q|^2 - |\psi_q \psi'_p|^2) \right| \tag{3.17}$$

whenever the r.h.s. is absolutely convergent or diverges to infinity.

Proof. Take $a^{(m)} \in \mathcal{B}_D$ defined in (3.16). Since $a^{(m)}$ saturates (3.14), one has

$$\begin{aligned} \omega_\psi(a^{(m)}) - \omega_{\psi'}(a^{(m)}) &= (2\pi\theta)^2 \sum_{p,q} (a^{(m)}_{pp} - a^{(m)}_{qq}) |\psi_p \psi'_q|^2 \\ &= (2\pi\theta)^2 \sum_{p<q} (a^{(m)}_{pp} - a^{(m)}_{qq})(|\psi_p \psi'_q|^2 - |\psi_q \psi'_p|^2) \\ &= (2\pi\theta)^2 \sqrt{\frac{\theta}{2}} \sum_{p<k\leq \min(m+1,q)} \frac{1}{\sqrt{k}} (|\psi_p \psi'_q|^2 - |\psi_q \psi'_p|^2). \end{aligned}$$

The result then follows from $d(\omega_\psi, \omega_{\psi'}) \geq \lim_{m\to+\infty} \omega_\psi(a^{(m)}) - \omega_{\psi'}(a^{(m)})$, whenever the limit exists (finite or not). □

Proposition 3.10. *Consider the two unit vectors $\psi, \psi' \in \mathcal{H}_0$ with components*

$$\psi_q = \frac{1}{\sqrt{2\pi\theta}}\delta_{q0}, \qquad \psi'_q = \frac{1}{\sqrt{2\pi\theta}\sqrt{\zeta(s)q^s}} \quad \text{for } q \neq 0,\ \psi'_0 = 0, \tag{3.18}$$

where $s > 1$ and $\zeta(s)$ is the Riemann zeta function. If $s \leq 3/2$, then $d(\omega_\psi, \omega_{\psi'}) = \infty$.

Proof. From (3.17) we get

$$d(\omega_\psi, \omega_{\psi'}) \geq \sqrt{\frac{\theta}{2}}\frac{1}{\zeta(s)}\sum_{1\leq k\leq q}\frac{1}{q^s\sqrt{k}} \geq \sqrt{\frac{\theta}{2}}\frac{1}{\zeta(s)}\sum_{q=1}^{\infty} q^{\frac{1}{2}-s} \tag{3.19}$$

where we used $\sum_{k=1}^{q}\frac{1}{\sqrt{k}} \geq \sqrt{q}$. The r.h.s term of (3.19) diverges for $s \leq 3/2$. □

Proposition 3.10 shows that the diameter $d_{\mathcal{A}} = \sup\{d(x, y); x, y \in \mathcal{P}(\mathcal{A})\}$ of the metric space $(\mathcal{P}(\mathcal{A}), d)$ is infinite, as the diameter of the Euclidean plane. However the Euclidean distance between any two points in the plane is always finite, although one may choose the points so as to make it arbitrarily large. The situation is different in the Moyal plane: the diameter is infinite and the distance can also be infinite. In this sense, the non-locality of the Moyal product makes the distance larger.

So far, the only known cases where the spectral distance between two states ω, ω' is infinite have been due to algebraic properties, namely the existence of a non-trivial element a_0 such that

$$[D, \pi(a_0)] = 0, \qquad \omega(a_0) - \omega'(a_0) \neq 0. \tag{3.20}$$

Considering na_0, $n \to +\infty$, one had that $d(\omega, \omega') = +\infty$. Here only constant functions commute with D and the infinity of the distance has analytical origin, owing to the infinite dimension of the algebra.

3.5. The spectral metric space

In [20,21,39] Rieffel introduced the notion of compact quantum metric spaces (that we recall; cf. Definition 4.1) which has been adapted to the non-compact case (i.e. for non-unital algebras) by Latrémolière [18]. This leads to the recent definition of the *spectral metric space*, namely quoting [19]: "A spectral metric space is a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ with additional properties which guarantee that the Connes metric[5] induces the weak* topology on the state space of $\mathcal{A}$".

When the unit ball (3.5) is bounded, these properties have been established in [18], leading to the definition of *bounded* spectral metric space in [19]. When the unit ball is not bounded, the question is still open. In [19, Proposition 5] an example of spectral triples based on $\mathbb{Z}$ is introduced that fails to be a spectral metric space as soon as the unit ball is not norm bounded. A similar situation occurs for the Moyal plane: the unit ball is not norm bounded since $\left\|a^{(m)}\right\|_{\text{op}} \to \infty$ (see (3.16)) and we show below that the topology induced by the spectral distance is not the weak* one.

Lemma 3.11. *Let $\mathcal{X}$ be a topological space whose topology is induced by a pseudo-distance d.*[6] *Then, if $d(\varphi_0, \varphi_1) = \infty$ the points $\varphi_0, \varphi_1 \in \mathcal{X}$ are in distinct connected components.*

Proof. Let us fix $\varphi_0 \in \mathcal{X}$. Define

$$S_0 \doteq \{\varphi \in \mathcal{X}, d(\varphi_0, \varphi) < \infty\},$$

and $B_r(\varphi) \doteq \left\{\varphi' \in \mathcal{X}, d(\varphi, \varphi') < r\right\}$, the open ball or radius $r > 0$ centered at φ. For any $\varphi \in S_0$ and $r > 0$, one has $B_r(\varphi) \subset S_0$. Indeed if $\varphi' \in B_r(\varphi)$, by the triangle inequality,

$$d(\varphi_0, \varphi') \leq d(\varphi_0, \varphi) + d(\varphi, \varphi') < \infty,$$

so $\varphi' \in S_0$. Hence S_0 is open. Similarly, the inequality

$$d(\varphi_0, \varphi) \geq d(\varphi_0, \varphi') - d(\varphi, \varphi') = \infty,$$

for all $\varphi' \notin S_0$ and $\varphi \in B_r(\varphi')$, shows that the complement of S_0 is open too. Hence S_0 is closed. Any clopen set is a union of connected components, and this proves that $\varphi_0 \in S_0$ and $\varphi_1 \notin S_0$ cannot be in the same connected component. □

5 That here we call spectral distance.

6 Following [40], a pseudo-distance is a function that satisfies all the properties of a distance, except that it may be infinite.

Given two states $\varphi_0, \varphi_1 \in \mathcal{S}(\mathcal{A})$, consider the map

$$[0,1] \ni t \mapsto \varphi_t = (1-t)\varphi_0 + t\varphi_1 \in \mathcal{S}(\mathcal{A}). \tag{3.21}$$

The spectral distance satisfies [11, Eq. (1.9)]

$$d_D(\varphi_t, \varphi_{t'}) = |t - t'| d_D(\varphi_0, \varphi_1), \quad \forall\, 0 \leq t,\ t' \leq 1.$$

This implies that the map in (3.21) is continuous for the topology induced by the spectral distance if $d_D(\varphi_0, \varphi_1) < \infty$. Therefore:

Lemma 3.12. *If $d_D(\varphi_0, \varphi_1) < \infty$, then $\varphi_0, \varphi_1 \in \mathcal{S}(\mathcal{A})$ are in the same connected components for the topology induced by the spectral distance. Moreover, connected components are path connected.*

Lemma 3.13. *$\mathcal{S}(\mathcal{A})$ is path connected for the weak* topology.*

Proof. Recall that a *density matrix* P (on $\ell^2(\mathbb{N})$) is a positive trace-class operator on $\ell^2(\mathbb{N})$ with trace 1. A *normal* state $\omega_P \in \mathcal{S}(\mathcal{A})$ is a state that can be written as

$$\omega_P(a) = \mathrm{Tr}(Pa).$$

Since $\bar{\mathcal{A}} \simeq \mathbb{K}$, it follows from Proposition 2.6.13 of [41] that any state of the Moyal algebra is normal, that is $\mathcal{S}(\mathcal{A})$ is identified with the set of density matrices. By [41, Proposition 2.6.15], the weak* topology on $\mathcal{S}(\mathcal{A})$ is equivalent to the uniform topology induced by the trace norm,

$$\|T\|_{\mathrm{Tr}} \doteq \mathrm{Tr}\,|T| \quad \text{for all trace classes } T. \tag{3.22}$$

Similarly to (3.21), for any density matrices P_0, P_1 we define a map

$$[0,1] \ni t \mapsto P_t = (1-t)P_0 + tP_1.$$

For all $0 \leq t, t' \leq 1$ we have $P_t - P_{t'} = (t - t')(P_1 - P_0)$, so

$$\|P_t - P_{t'}\|_{\mathrm{Tr}} = |t - t'|\, \|P_0 - P_1\|_{\mathrm{Tr}}\,. \tag{3.23}$$

Therefore, $t \mapsto P_t$ is continuous in the weak* topology (for all $\epsilon > 0$, called $\delta_\epsilon = \epsilon / \|P_0 - P_1\|_{\mathrm{Tr}}$, from (3.23) we get $|t - t'| < \delta_\epsilon \Rightarrow \|P_t - P_{t'}\|_{\mathrm{Tr}} < \epsilon$) and so $\mathcal{S}(\mathcal{A})$ is path connected, as claimed. □

Proposition 3.14. *The topology induced by the spectral distance is not the weak* topology; thus the spectral triple introduced in* [17] *for the Moyal plane is not a spectral metric space.*

Proof. Suppose that two topologies T_1 and T_2 are equivalent: then, if S is a connected component of T_1, it must also be a connected component of T_2. Let T_1 be the topology induced on $\mathcal{S}(\mathcal{A})$ by the spectral distance and T_2 the weak* topology. By Lemma 3.11 the two pure states in Proposition 3.10 are in different connected components of T_1. On the other hand, from Lemma 3.13 it follows that any two pure states are in the same connected component for the weak* topology ($\mathcal{S}(\mathcal{A})$ is path connected). This concludes the proof. □

4. The truncated Moyal space as compact quantum metric spaces

4.1. Quantum metric spaces

In [21] Rieffel introduces the notion of quantum metric space motivated on the one hand by the Connes distance formula (3.1), on the other hand by the observation that on a compact Hausdorff metric space, (X, ρ) with Lipschitz semi-norm $l(f) = \sup_{x \neq y} |f(x) - f(y)|/\rho(x, y)$ on $A = C(X)$ (the infinite value is permitted), one can define on $\mathcal{S}(A)$ a distance

$$\rho_l(\omega_1, \omega_2) := \sup\{\omega_1(f) - \omega_2(f);\, l(f) \leq 1\} \tag{4.1}$$

whose topology coincides with the weak* topology,

$$\lim_{n\to\infty} \rho_l(\omega_n - \omega) = 0 \quad \text{iff} \quad \lim_{n\to\infty} \omega_n(f) - \omega(f) = 0 \quad \forall f \in A,$$

and whose restriction to $\mathcal{P}(A)$ gives back ρ. When moreover X is a Riemannian spin manifold with Dirac operator D, then $l(f) = \|[D, f]\|$ and (4.1) is nothing but the spectral distance. Stated differently, for a compact commutative spectral triple the topology defined by the spectral distance on the state space coincides with the weak* topology. However there is a priori no reason for this to still hold true for arbitrary spectral triples. This motivates the following definition of *compact quantum metric space*.

Definition 4.1 (*[21]*)**.** A *compact quantum metric space* is an order unit space A equipped with a semi-norm l such that $l(1) = 0$ and the distance

$$d(\omega_1, \omega_2) := \sup\{\omega_1(a) - \omega_2(a);\, l(a) \leq 1\} \tag{4.2}$$

induces the weak* topology on the state space of A.

Note that for technical flexibility, Rieffel [20] uses order 1 spaces rather than algebras. The precise definition can be found in [42,43]. For our purpose we just need to know that any real linear space of self-adjoint operators on a Hilbert space containing the identity operator $\mathbb{I}$ is an order 1 space, and any order 1 space can be realized in this way. Moreover the notion of state naturally extends to order 1 spaces. Therefore, since the supremum in (3.1) can be searched for on self-adjoint elements, it make sense to view a unital spectral triple whose spectral distance induces on $\mathcal{S}(A)$ the weak* topology as a compact quantum metric space, with semi-norm $l(.) = \|[D, .]\|$.

A necessary condition [21] for a semi-norm l on A to define a quantum metric space is

$$l(a) = 0 \Longleftrightarrow a \in \mathbb{R}\mathbb{I}. \tag{4.3}$$

Indeed when (4.3) does not hold, it is not difficult to find states at infinite distance from one another (see (3.20)), so that the metric and weak* topologies cannot coincide since $\mathcal{S}(A)$ – for any order 1 space A – is compact for the weak* topology. However this condition is not sufficient: consider the order 1 space $\mathcal{A}_1^{sa}$ of the Moyal algebra introduced in Section 2.2 with semi-norm $\|[D, .]\|$ and associated distance d_1. The unit vectors ψ, ψ' in (3.18) still define pure states ω_ψ, $\omega_{\psi'}$ of $\mathcal{A}_1$ and, since $\mathcal{A} \subset \mathcal{A}_1$, one has $d \leq d_1$. In particular by Proposition 3.10, $d_1(\omega_\psi, \omega_{\psi'}) = +\infty$, although (4.3) holds true. For this reason the Moyal plane equipped with the distance d_1 is *not* a quantum metric space.

However for spectral triples whose algebra has finite dimension, (4.3) guarantees that the spectral distance induces the weak* topology. Let us recall that any finite dimensional C^*-algebra $\mathcal{A}$ is a direct sum of matrix algebras, $\mathcal{A} = \sum_{n\in I} M_n(\mathbb{C})$, and $\mathcal{P}(\mathcal{A}) = \cup_{n\in I}\, \mathcal{P}(\mathcal{A}_n)$ where $\mathcal{P}(\mathcal{A}_n)$ is the set of vector states

$$a \in M_n(\mathbb{C}) \mapsto \omega_\Psi(a) = \langle \Psi, a\Psi\rangle = \mathrm{Tr}(s_\Psi a) \tag{4.4}$$

where Ψ is a unit vector in $\mathbb{C}^n$, $s_\Psi = |\Psi\rangle\langle\Psi| \in M_n(\mathbb{C})$ is the associated projection and the omission of the symbol π means that one is considering the fundamental representation. $\mathcal{S}(\mathcal{A})$ is in 1-to-1 correspondence with convex sums of rank 1 projections on each $M_n(\mathbb{C})$, i.e.

$$\mathcal{S}(\mathcal{A}) = \left\{s \in \mathcal{A}^+, \mathrm{Tr}\, s = 1\right\} : \mathcal{A} \ni a \mapsto \mathrm{Tr}(sa)$$

where $\mathcal{A}^+$ denotes the set of positive elements of $\mathcal{A}$. In the language of physics, these are the density matrices and one recovers that a state is pure iff the associated density matrix is a projection $M_n(\mathbb{C})$.

Proposition 4.2. *A spectral triple $(\mathcal{A}, \mathcal{H}, D)$, with $\mathcal{A}$ a finite dimensional algebra, is a compact quantum metric space iff*

$$[D, \pi(a)] = 0 \Longleftrightarrow a = \lambda\mathbb{I} \quad \textit{for } \lambda \in \mathbb{C}. \tag{4.5}$$

Proof. (i) Assume (4.5) does not hold. Consider $b_0 \neq \lambda\mathbb{I}$ such that $[D, \pi(b_0)] = 0$. For $a_0 = \frac{1}{2}(b_0 + b_0^*) \in \mathcal{B}_D^{sa}$, $\pi(a_0)$ has at least two normalized eigenvectors $\psi, \psi' \in \mathcal{H}$ with distinct eigenvalues α, α'. By (3.20) the corresponding states are at infinite distance since $\omega_\psi(a_0) = \alpha \neq \alpha' = \omega_{\psi'}(a_0)$. Hence the spectral triple is not a compact quantum metric space.

(ii) Assume that (4.5) holds. Two distances d_1 and d_2 induce the same topology on $\mathcal{S}(\mathcal{A})$ if they are strongly equivalent, that is to say if there exist two constants $C, C' > 0$ such that

$$C\, d_1(r, s) \leq d_2(r, s) \leq C'\, d_1(r, s) \quad \forall r, s \in \mathcal{S}(\mathcal{A}). \tag{4.6}$$

In particular two distances defined via (4.2) by semi-norms l_1, l_2 satisfying

$$K' l_1(a) \leq l_2(a) \leq K\, l_1(a) \quad \forall a \in \mathcal{A} \tag{4.7}$$

for some constant K, K' are equivalent with $C' = K'^{-1}$, $C = K^{-1}$. By (3.7), Remark 3.5, and noting that adding a multiple of $\mathbb{I}$ to a does not change $\mathrm{Tr}(ra) - \mathrm{Tr}(sa)$ nor $[D, \pi(a)]$, one has

$$d(r, s) = \sup_{a\in\mathcal{V}\cap\mathcal{B}_D} \left\{\mathrm{Tr}(ra) - \mathrm{Tr}(sa)\right\}$$

where $\mathcal{V}$ denotes the set of self-adjoint $a \in \mathcal{A}$ such that $\mathrm{Tr}(a) = 0$. Moreover $\|[D, \pi(.)]\|$ on the vector space $\mathcal{V}$ is a norm since

$$\|[D, \pi(a)]\|_{\mathrm{op}} = 0 \Longrightarrow a = \lambda\mathbb{I}$$

by (4.5) and $\lambda = n^{-1}\mathrm{Tr}(a) = 0$ with n the dimension of $\mathcal{A}$. All norms on a finite dimensional vector space being equivalent, the first part of the proposition follows from (4.6) and (4.7) as soon as one finds a norm on $\mathcal{V}$ whose associated distance induces the weak* topology.

The L^2 norm $\|a\|_{L_2} := \sqrt{\langle a, a\rangle}$ is such a norm, where $\langle a, b\rangle = \mathrm{Tr}(a^* b)$. Indeed, defining

$$d_2(r, s) \doteq \sup_{a\in\mathcal{V}} \big\{\mathrm{Tr}(ra) - \mathrm{Tr}(sa),\ \|a\|_{L_2} \leq 1\big\},$$

one has that

$$d_2(r, s) = \|r - s\|_{L_2} \tag{4.8}$$

since, by Cauchy–Schwarz and for any $a \in \mathcal{V}$ such that $\|a\|_{L_2} \leq 1$,

$$\mathrm{Tr}(ra) - \mathrm{Tr}(sa) = \langle r - s, a\rangle \leq \|a\|_{L_2}\|r - s\|_{L_2} \leq \|r - s\|_{L_2}$$

and the equality is attained by the unit-norm element $\|r - s\|_{L_2}^{-1}(r - s) \in \mathcal{V}$. (4.8) obviously induces the weak* topology: a sequence of density matrices s_n with components $(s_n)_{ij}$ tends to s, i.e. $\lim_{n\to+\infty} \|s_n - s\|_{L_2} = 0$, iff $|(s_n)_{ij} - s_{ij}| \to 0$ for all i, j. □

Notice that Proposition 4.2 can also be derived from Theorem 4.5 of [39], but the direct proof given here may be of interest.

4.2. Truncation of the Moyal spectral triple

Viewing $\mathcal{A}$ as the inductive limit

$$\mathcal{A} = \overrightarrow{\lim}\, \mathcal{A}_n, \quad \mathcal{A}_n := M_n(\mathbb{C})$$

with morphism the natural embedding of $M_n(\mathbb{C})$ into $M_{n+1}(\mathbb{C})$ and closure with respect the Fréchet norm (2.3), we show below how the restriction of the spectral triple of the Moyal plane to a finite rank n yields a compact quantum metric space. The truncation of the spectral triple (2.6) is the following: the algebra is $\mathcal{A}_n$, acting on $\mathcal{H}_n = M_n(\mathbb{C}) \otimes \mathbb{C}^2$ as $L(a) \otimes \mathbb{I}_2$ where $L(a)$ is the left regular representation of $M_n(\mathbb{C})$ on itself, with inner product normalized as $\langle a, b\rangle_\theta = 2\pi\theta\,\mathrm{Tr}(a^\dagger b)$. The Dirac operator (2.7) is now defined in term of the two derivations

$$\partial a = -[X_-, a], \qquad \bar\partial a = [X_+, a], \tag{4.9}$$

with

$$X_- := \frac{1}{\sqrt{\theta}} \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & \sqrt{2} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \sqrt{n-1} & 0 \end{pmatrix}, \qquad X_+ := X_-^t.$$

For $n = \infty$ one recovers the Dirac operator of the Moyal plane.

With the notation of Proposition 3.2, $\mathcal{P}(\mathcal{A}_n)$ is the set of vector states

$$\omega_\psi(a) = \langle \psi, a\psi\rangle_\theta = \langle s_\psi, a\rangle_\theta$$

with

$$\psi = \begin{pmatrix} \psi_1 & 0 & \dots & 0 \\ \psi_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \psi_n & 0 & 0 & 0 \end{pmatrix}, \qquad s_\psi = \psi\psi^*$$

where the ψ_i's are complex numbers such that $\langle \psi, \psi\rangle_\theta = 2\pi\theta \sum_{j=1}^n |\psi_j|^2 = 1$. Two matrices $\psi, \psi' \in M_n(\mathbb{C})$ define the same pure state iff $\psi\psi^* = \psi'\psi'^*$, e.g., – with the notation of (3.4) – $\psi = f_{mp}$, $\psi' = f_{mq}$. Writing $\Psi \in \mathbb{C}^2$ for the unit vector with component $\sqrt{2\pi\theta}\,\psi_i$, one retrieves the usual form (4.4) with $\langle ., .\rangle$ the inner product on $\mathbb{C}^n$ and $s_\Psi = 2\pi\theta s_\psi$ the projection on Ψ. Two unit vectors equal up to a phase define the same state; hence $\mathcal{P}(\mathcal{A}_n) = S^{2n-1}/S^1 = \mathbb{CP}^{n-1}$. Note that the ambiguity in the choice of ψ is $U(n)$, while the ambiguity in the choice of Ψ is $U(1)$.

Proposition 4.3. *The truncated Moyal plane $(\mathcal{A}_n, \mathcal{H}_n, D)$ is a compact quantum metric space. With the distance d_2 introduced in (4.8), $\mathcal{S}(\mathcal{A}_n)$ has radius smaller than $2\sqrt{1 - \frac{1}{n}}$.*

Proof. The first statement follows from Proposition 4.2, noting that nothing but multiples of the identity commute with D. For any state $s \in \mathcal{S}(\mathcal{A})$,

$$d_2\left(\frac{1}{n}\mathbb{I}, s\right)^2 = \mathrm{Tr}\left(\left(\frac{1}{n}\mathbb{I} - s\right)^2\right) = \mathrm{Tr}\, s^2 - \frac{1}{n} \leq 1 - \frac{1}{n} \tag{4.10}$$

where we noticed that for any positive matrix s of trace 1, $\mathrm{Tr}\, s^2 \leq \mathrm{Tr}\, s$. Hence

$$d_2(r, s) \leq d_2\left(r, \frac{1}{n}\mathbb{I}\right) + d_2\left(\frac{1}{n}\mathbb{I}, s\right) \leq 2\sqrt{1 - \frac{1}{n}}$$

for all $r, s \in \mathcal{S}(\mathcal{A}_n)$. □

The upper bound in (4.10) is attained by projections,[7] so $\mathcal{P}(\mathcal{A}_n)$ is the sphere of radius $\sqrt{1 - \frac{1}{n}}$ centered on $\frac{1}{n}\mathbb{I}$, and $\mathcal{S}(\mathcal{A}_n)$ is the corresponding ball. Alternatively, one may extend the distance d_2 to $\mathcal{A}_n^+$, yielding a characterization of the state space in terms of the unit sphere S^1 and the unit ball B^1 of the metric space $(\mathcal{A}_n^+, d_2)$. With calculations similar to the ones in Proposition 4.3, one gets that $\mathcal{P}(\mathcal{A}_n)$ is the intersection of S^1 with the set of matrices of trace 1; and $\mathcal{S}(\mathcal{A}_n)$ is the intersection of B^1 with the set of matrices of trace 1.

With some more calculation, one finds $Cd_2 \leq d \leq C'd_2$ with

$$C \leq \frac{1}{2}\sqrt{\frac{\theta}{2n}}, \qquad C' \geq (2n^3 + n^2)\sqrt{\frac{\theta}{2}}. \tag{4.11}$$

Proposition 4.3 is particularly interesting in comparison with other spectral distances on $M_n(\mathbb{C})$: in [12] one takes as a Dirac operator $D(a) := Ma + aM$ with $M = M^* \in M_n(\mathbb{C})$. This choice is physically relevant since the coefficients of M can be interpreted as masses, but the spectral triple does not yield a compact quantum metric space: condition (4.3) is not satisfied – the eigenprojections of M commute with D – and $\mathcal{P}(\mathcal{A}_n)$ decomposes into sub-tori, all at infinite distance from one another. In [44] one gets a compact metric space using the operator of matrix transposition as the Dirac operator (in this case the spectral distance coincides with the operator norm); but there is no room in the Dirac operator to have mass parameters ($D(a) = {}^t(a)$ has only eigenvalues ± 1). With Proposition 4.3 one combines n arbitrary masses (the proposition still holds with arbitrary non-zero values instead of $1, \ldots, \sqrt{n-1}$ in X_-) with a metric giving the weak* topology on $\mathcal{S}(\mathcal{A})$.

4.3. A spectral distance on the 2-sphere

To illustrate our results, we close this paper by explicitly computing the distance associated with the truncated spectral triple for $n = 2$. The algebra $\mathcal{A}_2 = M_2(\mathbb{C})$ acts on $M_2(\mathbb{C}) \otimes \mathbb{C}^2$ as block diagonal matrices and D is given by (4.9) with

$$\partial a = -\frac{1}{\sqrt{\theta}}\left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, a\right], \qquad \bar{\partial} a = (\partial a)^* \quad \forall a = a^* \in \mathcal{A}_2.$$

The pure state space $\mathcal{P}(\mathcal{A}_2) = \mathbb{CP}^1$ is identified with the sphere S^2 via the map

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \longleftrightarrow \begin{cases} x_\Psi := 2\,\mathrm{Re}\bar{\Psi}_1\Psi_2 \\ y_\Psi := 2\,\mathrm{Im}\bar{\Psi}_1\Psi_2 \\ z_\Psi := |\Psi_1|^2 - |\Psi_1|^2. \end{cases}$$

The evaluation on $a \in \mathcal{A}$ with components a_{ij} reads

$$\omega_\Psi(a) = \langle \Psi, a\Psi \rangle = \frac{1 + z_\Psi}{2}a_{00} + \frac{1 - z_\Psi}{2}a_{11} + r\,\Re\left(\mathrm{e}^{\mathrm{i}\Xi}a_{01}\right)$$

with $r\mathrm{e}^{\mathrm{i}\Xi} := x_\Psi + iy_\Psi$. A non-pure state ω_ϕ is given by a probability distribution ϕ on S^2,

$$\omega_\phi(a) = \int_{S^2} \phi(\xi)\omega_\xi(a)\mathrm{d}\xi = \frac{1 + \tilde{z}_\phi}{2}a_{00} + \frac{1 - \tilde{z}_\phi}{2}a_{11} + \tilde{r}_\phi\,\Re\left(\mathrm{e}^{\mathrm{i}\tilde{\Xi}_\phi}a_{01}\right) \tag{4.12}$$

where $\mathrm{d}\xi$ is the $SU(2)$ invariant measure on S^2 normalized to 1 and $\tilde{\mathbf{x}}_\phi = (\tilde{x}_\phi, \tilde{y}_\phi, \tilde{z}_\phi)$ denotes the mean point of ϕ, with

$$\tilde{x}_\phi := \int_{S^2} \phi(\xi)x_\xi\mathrm{d}\xi$$

[7] Let s be such that $\mathrm{Tr}\, s^2 = 1 = \mathrm{Tr}\, s$. Writing $0 \leq \alpha_i \leq 1$ for the eigenvalues of s, $\mathrm{Tr}\, s^2 = \mathrm{Tr}\, s$ yields $\sum_{i=1}^N \alpha_i - \alpha_i^2 = 0$. Each term of the sum being positive, the sum is zero iff $\alpha_i = \alpha_i^2$ for any i, i.e. $\alpha_i = 0$ or 1. Since the trace is 1, this means that all α_i's vanish except one. Hence s is a projection.

and similar notation for $\tilde{y}_\phi, \tilde{z}_\phi$. The point $\tilde{\mathbf{x}}_\phi$ is in the unit ball B^2. When $\omega_\phi = \omega_\psi$ is pure, then $\phi = \delta_{\xi-\psi}$ and one retrieves $\tilde{\mathbf{x}}_\phi = \mathbf{x}_\psi := (x_\psi, y_\psi, z_\psi) \in S^2$. The correspondences $\mathcal{P}(\mathcal{A}_2) \leftrightarrow S^2$ and $\mathcal{S}(\mathcal{A}_2) \leftrightarrow B^2$ are 1-to-1.

Noting that the weak* topology on $\mathcal{S}(\mathcal{A}_n)$ coincides with the Euclidean topology on $\mathcal{B}^2$,

$$\lim_{n\to+\infty} \omega_{\phi^n} = \omega_\phi \iff \lim_{n\to+\infty} \tilde{\mathbf{x}}_{\phi^n} = \tilde{\mathbf{x}}_\phi,$$

Proposition 4.3 indicates that $\mathcal{S}(\mathcal{A}_2)$ with the topology induced by the spectral distance is homeomorphic to the Euclidean closed ball B^2. $\mathcal{P}(\mathcal{A}_2)$ is homeomorphic to the Euclidean sphere S^2.

Although they induce the same topologies, d_2, the Euclidean distance, and the spectral distance are not equal. Writing as

$$s_\phi = \begin{pmatrix} \dfrac{z_\phi + 1}{2} & \dfrac{x_\phi - \mathrm{i}y_\phi}{2} \\ \dfrac{x_\phi + \mathrm{i}y_\phi}{2} & \dfrac{1 - z_\phi}{2} \end{pmatrix}$$

the density matrix associated with ω_Φ, an easy calculation shows that

$$d_2(s_\phi, s_{\phi'}) = \frac{1}{\sqrt{2}} d_{Ec}(\tilde{\mathbf{x}}_\phi, \tilde{\mathbf{x}}_{\phi'}) \tag{4.13}$$

where

$$d_{Ec}(\tilde{\mathbf{x}}_\phi, \tilde{\mathbf{x}}_{\phi'}) = \sqrt{|\tilde{x}_\phi - \tilde{x}_{\phi'}|^2 + |\tilde{y}_\phi - \tilde{y}_{\phi'}|^2 + |\tilde{z}_\phi - \tilde{z}_{\phi'}|^2}$$

denotes the Euclidean distance on B^2. Note that the radius $\frac{2}{\sqrt{2}}$ of $\mathcal{S}(\mathcal{A}_n)$ is exactly the upper bound in (4.10).

Proposition 4.4. *The spectral distance between any two states of $\mathcal{S}(\mathcal{A}_2)$ is*

$$d(\omega_\phi, \omega_{\phi'}) = \sqrt{\frac{\theta}{2}} \times \begin{cases} d_{eq}(\tilde{\mathbf{x}}_\phi, \tilde{\mathbf{x}}_{\phi'}) & \text{if } |\tilde{z}_\phi - \tilde{z}_{\phi'}| \le d_{eq}(\tilde{\mathbf{x}}_\phi, \tilde{\mathbf{x}}_{\phi'}), \\ \dfrac{d_{Ec}(\tilde{\mathbf{x}}_\phi, \tilde{\mathbf{x}}_{\phi'})^2}{2|\tilde{z}_\phi - \tilde{z}_{\phi'}|} & \text{if } |\tilde{z}_\phi - \tilde{z}_{\phi'}| \ge d_{eq}(\tilde{\mathbf{x}}_\phi, \tilde{\mathbf{x}}_{\phi'}), \end{cases}$$

where

$$d_{eq}(\tilde{\mathbf{x}}_\phi, \tilde{\mathbf{x}}_{\phi'}) = \sqrt{|\tilde{x}_\phi - \tilde{x}_{\phi'}|^2 + |\tilde{y}_\phi - \tilde{y}_{\phi'}|^2}$$

is the Euclidean distance between the projections of the points on the equatorial plane $z = 0$.

Proof. With the notation of (4.12) and by (3.7),

$$d(\omega_\phi, \omega_{\phi'}) = \sup_{a\in\mathcal{V}_2} \left\{ \omega_\phi(a) - \omega_{\phi' \phi'}(a),\ \|\partial a\|_{\mathrm{op}} = 2^{-\frac{1}{2}} \right\}. \tag{4.14}$$

Any $a \in \mathcal{V}_2$ is of the form

$$a = \begin{pmatrix} B & A \\ \bar{A} & -B \end{pmatrix}$$

with $A \in \mathbb{C}$, $B \in \mathbb{R}$, and has the derivative

$$\partial a = \frac{1}{\sqrt{\theta}} \begin{pmatrix} A & 0 \\ -2B & -A \end{pmatrix}$$

whose norm can be explicitly computed as $\|\partial a\|_{\mathrm{op}} = \theta^{-\frac{1}{2}} \left(\sqrt{|A|^2 + |B|^2} + |B| \right)$. Writing $|B| + \mathrm{i}|A| = r\mathrm{e}^{\mathrm{i}\lambda}$, one thus gets

$$\|\partial a\|_{\mathrm{op}} = 2^{-\frac{1}{2}} \iff r = \sqrt{\frac{\theta}{2}} \frac{1}{1 + \cos\lambda}. \tag{4.15}$$

Noting by (4.12) that

$$\omega_\phi(a) - \omega_{\phi' \phi'}(a) = BZ + \Re(AX) \quad \text{with } Z := \tilde{z}_\phi - \tilde{z}_{\phi'},\ X := \tilde{r}' \mathrm{e}^{\mathrm{i}\tilde{\varTheta}'} - \tilde{r}\mathrm{e}^{\mathrm{i}\tilde{\varTheta}},$$

(4.15) together with (4.14) yields

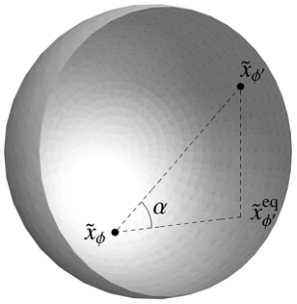


Fig. 1. The vertical plane containing $\tilde{\mathbf{x}}_{\phi}, \tilde{\mathbf{x}}_{\phi'}$.

$$
\begin{aligned}
d(\omega_{\phi}, \omega_{\phi'\phi'}) &\leq \sup_{|A|,|B|\in\mathbb{R}^+} \left\{ |B||Z| + |A||X|,\ \sqrt{|A|^2+|B|^2} + |B| = \sqrt{\frac{\theta}{2}} \right\}, \\
&\leq \sup_{0\leq\lambda\leq\frac{\pi}{2}} \sqrt{\frac{\theta}{2}} \frac{|Z|\cos\lambda + |X|\sin\lambda}{1+\cos\lambda} \\
&= \sqrt{\frac{\theta}{2}} \times \begin{cases} |X| & \text{when } |Z| \leq |X| \\ \dfrac{|X|^2+|Z|^2}{2|Z|} & \text{when } |Z| \geq |X|. \end{cases}
\end{aligned}
$$

These upper bounds are attained by

$$B = 0, \qquad A = \sqrt{\frac{\theta}{2}} \mathrm{e}^{-\mathrm{i}\theta_X}, \quad \theta_X := \mathrm{Arg}\, X$$

if $|Z| \leq |X|$ (which corresponds to $\lambda = \frac{\pi}{2}$), and

$$B = (\mathrm{sign} Z) \frac{|Z|^2 - |X|^2}{2|Z|^2}, \qquad A = \frac{|X|}{|Z|}$$

when $|Z| \geq |X|$ (which corresponds to $\lambda = \arccos(\frac{|Z|^2-|X|^2}{|Z|^2+|X|^2})$). The final result is obtained by noting that $|X| = d_{eq}(\tilde{\mathbf{x}}_{\phi}, \tilde{\mathbf{x}}_{\phi'})$. □

This result has some analogy with the spectral distance computed on $\mathcal{P}(M_2(\mathbb{C}))$ in [12], namely the rotation around the z-axis is an isometry and the distance on a parallel ($z_{\Psi} = z_{\Psi'}$) is proportional to the Euclidean distance within the disk. But remarkably, while the distance on a meridian was infinite in [12], it remains finite here. This allows an easy "classical geometry" interpretation of Proposition 4.4. Assuming for convenience that $\tilde{z}_{\phi} \geq \tilde{z}_{\phi'}$, let us call $\tilde{\mathbf{x}}^{eq}_{\phi'}$ the projection of $\tilde{\mathbf{x}}_{\phi'}$ in the $\tilde{z}_{\phi'}$ plane and as α the angle $(\tilde{\mathbf{x}}_{\phi}\tilde{\mathbf{x}}^{eq}_{\phi'}, \tilde{\mathbf{x}}_{\phi}\tilde{\mathbf{x}}_{\phi'})$ (see Fig. 1). Then

$$d(\omega_{\phi}, \omega_{\phi'}) = \sqrt{\frac{\theta}{2}} \times \begin{cases} \cos\alpha\, d_{Ec}(\tilde{\mathbf{x}}_{\phi}, \tilde{\mathbf{x}}_{\phi'}) & \text{when } \alpha \leq \dfrac{\pi}{4}, \\ \dfrac{1}{2\sin\alpha}\, d_{Ec}(\tilde{\mathbf{x}}_{\phi}, \tilde{\mathbf{x}}_{\phi'}) & \text{when } \alpha \geq \dfrac{\pi}{4}. \end{cases}$$

Note that by (4.13) this result is in agreement with (4.11). Also, two states with the same $\tilde{x}$, $\tilde{y}$ coordinates are at distance $\sqrt{\frac{\theta}{2}}\frac{|\tilde{z}_{\phi}-\tilde{z}_{\phi'}|}{2}$. In particular the distance between the two poles – identified as ω_0, ω_1 – is $\sqrt{\frac{\theta}{2}}$, in agreement with Proposition 3.6.

5. Conclusion

Describing the Moyal plane in terms of a spectral triple based on the algebra of Schwartz functions equipped with Moyal $\star$ product (2.1), we have computed the spectral distance between the eigenfunctions of the quantum harmonic oscillator. We have shown that the distance is not finite on the whole of the pure state space of $\mathcal{A}$, and that it can be made finite by truncating the Moyal spectral triple.

All the results in this paper are expressed in the matrix basis. It would be interesting to study the interpretation in the x-space. Can one see the pure states ω_n as deformations of some states of $C_0(\mathbb{R}^2)$? A relevant tool for that could be the coherent states of the harmonic oscillator.

The limit $\theta \to 0$ also deserves more attention. The pure state space for $\theta \neq 0$ (infinitely many pure states yielding the same representation modulo unitaries) is very different from the pure state space at $\theta = 0$ (infinitely many pure states, the

points of $\mathbb{R}^2$, yielding non-equivalent representations). This question should be addressed by using the continuous field of C^*-algebras, $\theta \mapsto \mathcal{A}_\theta = \overline{(\mathcal{S}, \star_\theta)}$, where $\star_\theta$ is the Moyal product for the given value of the parameter.

One should also question the unitization problem. What is gained, from the metric point of view, by passing to the preferred unitization $\mathcal{A}_1$? In the commutative case, the algebra $C_b(X)$ of bounded continuous functions on a locally compact space X is the maximal unitization of $C_0(X)$, and $\mathcal{P}(C_b(X))$ is the Stone–Čech compactification of $\mathcal{P}(C_0(X)) \simeq X$. This is no longer true in the noncommutative case. In particular $\mathcal{A}_1$ is not the maximal unitization of $\mathcal{A}$ (which is the Moyal multiplier algebra $\mathcal{M}$). The link between $\mathcal{P}(\mathcal{A})$ and $\mathcal{P}(\mathcal{A}_1)$ deserves further study.

Finally, in the commutative case $\mathcal{A} = C_0^\infty(M)$, for M a complete locally compact Riemannian manifold, the spectral distance between non-pure states coincides with the Wasserstein distance of order 1 between probability distributions on M [11]. Is there any possibility of associating a Wasserstein distance with the spectral distance on S^2 computed here?

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References

[1] A. Connes, Noncommutative Geometry, Academic Press, 1994.
[2] B. Iochum, T. Schücker, C. Stephan, On a classification of irreducible almost commutative geometries, J. Math. Phys. 45 (2004) 5003–5041.
[3] J. Madore, Fuzzy space–time, Can. J. Phys./Rev. Can. Phys. 75 (6) (1997) 385–399.
[4] M.A. Rieffel, Deformation quantization for actions of $\mathbb{R}^d$, Mem. Amer. Math. Soc. 506 (1993).
[5] V.G. Drinfeld, Quantum groups, in: Proc. of the ICM, vol. 1, Berkeley, 1986, pp. 798–820.
[6] M. Jimbo, A q-difference analogue of $U(\mathfrak{g})$ and the Yang–Baxter equation, Lett. Math. Phys. 10 (1985) 63–69.
[7] A. Connes, G. Landi, Noncommutative manifolds: the instanton algebra and isospectral deformations, Comm. Math. Phys. 221 (2001) 141–159.
[8] A. Connes, M. Dubois-Violette, Noncommutative finite-dimensional manifolds I. Spherical manifolds and related examples, Comm. Math. Phys. 230 (2002) 539–579.
[9] A. Connes, J. Lott, The metric aspect of noncommutative geometry, in: New Symmetry Principles in Quantum Field Theory, Cargèse 1991, in: NATO Ad. Sci. Inst. Ser. B Phys., vol. 295, Plenum, New York, 1992, pp. 53–93.
[10] A. Connes, On the spectral characterization of manifolds, arXiv:0810.2088 [math.OA].
[11] F. D'Andrea, P. Martinetti, A view on optimal transport from noncommutative geometry, SIGMA 6 (2010) 057.
[12] B. Iochum, T. Krajewski, P. Martinetti, Distances in finite spaces from noncommutative geometry, J. Geom. Phys. 37 (1–2) (2001) 100–125.
[13] G. Bimonte, F. Lizzi, G. Sparano, Distances on a lattice from non-commutative geometry, Phys. Lett. B341 (1994) 139–146.
[14] P. Martinetti, Carnot–Carathéodory metric from gauge fluctuation in noncommutative geometry, Comm. Math. Phys. 265 (2006) 585–616.
[15] P. Martinetti, R. Wulkenhaar, Discrete Kaluza Klein from scalar fluctuations in non-commutative geometry, J. Math. Phys. 43 (2002) 182–204.
[16] P. Martinetti, Spectral distance on the circle, J. Funct. Anal. 255 (2008) 1575–1612.
[17] V. Gayral, J.M. Gracia-Bondía, B. Iochum, T. Schücker, J.C. Várilly, Moyal planes are spectral triples, Comm. Math. Phys. 246 (2004) 569–623.
[18] F. Latrémolière, Bounded-Lipschitz distances on the state space of a C^*-algebra, Taiwanese J. Math. 11 (2) (2007) 447–469.
[19] J.V. Bellissard, M. Marcolli, K. Reihani, Dynamical systems on spectral metric spaces, 2010, arXiv:1008.4617 [math.OA].
[20] M.A. Rieffel, Metric on state spaces, Doc. Math. 4 (1999) 559–600.
[21] M.A. Rieffel, Compact quantum metric spaces, in: Operator Algebras, Quantization, and Noncommutative Geometry, in: Contemp. Math., vol. 365, AMS, Providence, RI, 2004, pp. 315–330.
[22] J.M. Gracia-Bondía, J.C. Várilly, Algebras of distributions suitable for phase–space quantum mechanics. I, J. Math. Phys. 29 (4) (1988) 869–879.
[23] J.M. Gracia-Bondía, J.C. Várilly, Algebras of distributions suitable for phase–space quantum mechanics. II, J. Math. Phys. 29 (4) (1988) 880–887.
[24] V. Gayral, B. Iochum, The spectral action for Moyal planes, J. Math. Phys. 46 (2005) 043503.
[25] E. Cagnache, T. Masson, J.C. Wallet, Noncommutative Yang–Mills–Higgs actions from derivation based differential calculus, J. Noncommut. Geom. 5 (2011) 39–67.
[26] J.C. Wallet, Derivations of the Moyal algebra and noncommutative gauge theories, SIGMA 5 (2009) 013.
[27] A. de Goursac, J.C. Wallet, R. Wulkenhaar, Noncommutative induced gauge theory, Eur. Phys. J. C Part. Fields 51 (2007) 977–987.
[28] J.E. Moyal, Quantum mechanics as a statistical theory, Proc. Cambridge Philos. Soc. 45 (1949) 99–124.
[29] H.J. Groenewold, On the principles of elementary quantum mechanics, Physica 12 (1946) 405–460.
[30] D. Bahns, S. Doplicher, K. Fredenhagen, G. Piacitelli, Quantum geometry on quantum spacetime: distance, area and volume operators, arXiv:1005.2130 [hep-th].
[31] G. Amelino-Camelia, G. Gubitosi, F. Mercati, Discreteness of area in noncommutative space, Phys. Lett. B676 (2009) 180–183.
[32] P. Martinetti, F. Mercati, L. Tomasini, Minimal length in quantum space and integration of the line element in noncommutative geometry, preprint, arXiv:1106.0261.
[33] H. Grosse, R. Wulkenhaar, Renormalisation of φ^4-theory on noncommutative $\mathbb{R}^4$ in the matrix base, Comm. Math. Phys. 256 (2005) 305–374.
[34] V. Rivasseau, F. Vignes-Tourneret, R. Wulkenhaar, Renormalization of noncommutative phi 4-theory by multi-scale analysis, Comm. Math. Phys. 262 (2006) 565–594.
[35] A. Connes, Noncommutative geometry and reality, J. Math. Phys. 36 (1995) 6194–6231.
[36] D. Kaschek, N. Neumaier, S. Waldmann, Complete positivity of Rieffel's deformation quantization by actions of $\mathbb{R}^d$, J. Noncommut. Geom. 3 (2009) 361–375.
[37] B. Blackadar, Operator algebras. Theory of C^*-algebras and von Neumann algebras, Encyclopaedia Math. Sci. 122 (2006).
[38] R.V. Kadison, J.R. Ringrose, Fundamentals of the Theory of Operators Algebras II, Academic Press, 1986.
[39] M.A. Rieffel, Gromov–Hausdorff distance for quantum metric spaces. Matrix algebras converge to the sphere for quantum Gromov–Hausdorff distance, Mem. Amer. Math. Soc. 168 (796) (2004) 1–65.
[40] C. Villani, Optimal Transport: Old and New, Springer, 2009.
[41] O. Bratteli, D.W. Robinson, Operator Algebras and Quantum Statistical Mechanics: vol. 1, Springer-Verlag, 1996.
[42] E.M. Alfsen, Compact Convex Sets and Boundary Integrals, Springer-Verlag, Berlin, New York, 1971.
[43] R.V. Kadison, A representation theory for commutative topological algebra, Mem. Amer. Math. Soc. 7 (1951).
[44] E. Christensen, C. Ivan, Spectral triples for AF C^*-algebras and metrics on the Cantor set, J. Oper. Theory 56:1 (2006) 17–46.