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Conformal invariants of twisted Dirac operators and positive scalar curvature

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ABSTRACT

For a closed, spin, odd dimensional Riemannian manifold (Y, g), we define the rho invariant $\rho_{spin}(Y, \mathcal{E}, H, [g])$ for the twisted Dirac operator $\eth_H^{\mathcal{E}}$ on Y, acting on sections of a flat Hermitian vector bundle $\mathcal{E}$ over Y, where $H = \sum i^{j+1} H_{2j+1}$ is an odd-degree closed differential form on Y and H_{2j+1} is a real-valued differential form of degree $2j + 1$. We prove that it only depends on the conformal class $[g]$ of the metric g. In the special case when H is a closed 3-form, we use a Lichnerowicz–Weitzenböck formula for the square of the twisted Dirac operator, which in this case has no first order terms, to show that $\rho_{spin}(Y, \mathcal{E}, H, [g]) = \rho_{spin}(Y, \mathcal{E}, [g])$ for all $|H|$ small enough, whenever g is conformally equivalent to a Riemannian metric of positive scalar curvature. When H is a top-degree form on an oriented three dimensional manifold, we also compute $\rho_{spin}(Y, \mathcal{E}, H)$.

0. Introduction

In an earlier paper [1], we extended some of the results of Atiyah, Patodi and Singer [2–4] on the signature operator on an oriented, compact manifold with boundary, to the twisted case. Atiyah, Patodi and Singer also studied the *Dirac operator* $\eth^{\mathcal{E}}$ on an odd dimensional, closed, spin manifold, which is self-adjoint and elliptic, having a spectrum in the real numbers. For this (and other elliptic self-adjoint operators), they introduced the eta invariant which measures the spectral asymmetry of the operator and is a spectral invariant. Coupling with flat bundles, they introduced the closely related rho invariant, which has the striking property that it is independent of the choice of a Riemannian metric needed in its definition, when it is reduced modulo $\mathbb{Z}$. In this paper we generalize the construction of Atiyah–Patodi–Singer to the twisted Dirac operator $\eth_H^{\mathcal{E}}$ with a closed, odd-degree differential form as flux and with coefficients in a flat vector bundle.

More precisely, let X be a $2m$-dimensional compact, spin Riemannian manifold without boundary, $\mathcal{E}$ a flat Hermitian vector bundle over X and H a closed, odd degree differential form on Y. Consider the twisted Dirac operator $\eth_H^{\mathcal{E}} = c \circ \nabla^{\mathcal{E},H} = \eth^{\mathcal{E}} + c(H)$ where c denotes Clifford multiplication, $\nabla^{\mathcal{E},H}$ denotes the flat superconnection $\nabla_X^{\mathcal{E}} + c(H)$, and $\nabla^{\mathcal{E}}$ is the canonical

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flat Hermitian connection on $\mathcal{E}$. Then $\eth_H^{\mathcal{E}}$ anticommutes with the usual grading involution and it is self-adjoint if and only if

$$H = \sum i^{j+1} H_{2j+1} \tag{1}$$

where H_{2j+1} are real-valued differential forms of degree $2j+1$. It is only in this case that one gets a generalization of the usual Dirac operator on X, in contrast to the case of the twisted de Rham complex; cf. [5–10].

When the compact spin manifold X has non-empty boundary and assuming that the Riemannian metric is of product type near the boundary and that H satisfies the absolute boundary condition, we explicitly identify the twisted Dirac operator near the boundary to be $\eth_{\tilde{H}}^{\tilde{\mathcal{E}}} = \sigma\left(\frac{\partial}{\partial r} + \eth_H^{\mathcal{E}}\right)$, where r is the coordinate in the normal direction, σ is a bundle isomorphism and finally, the self-adjoint elliptic operator given on $\Omega^{2h}(\partial X, \mathcal{E})$ by $\eth_H^{\mathcal{E}}$ is the Dirac operator on the boundary. $\eth_H^{\mathcal{E}}$ is an elliptic self-adjoint operator, and by [2], the non-local boundary condition given by $P^+(s|_{\partial X}) = 0$, where P^+ denotes the orthogonal projection onto the eigenspaces with positive eigenvalues, makes the pair $(\eth_H^{\mathcal{E}}; P^+)$ into an elliptic boundary value problem. Applying the Atiyah–Patodi–Singer index theorem, and computing the local contribution when H is closed, we get

$$\mathrm{Index}(\eth_{\tilde{H}}^{\tilde{\mathcal{E}}}; P^+) = \mathrm{Rank}(\mathcal{E}) \int_Y \alpha_0(H) - \frac{1}{2}\left(\dim(\ker(\eth_H^{\mathcal{E}})) + \eta(\eth_H^{\mathcal{E}})\right),$$

where $\eta(\eth_H^{\mathcal{E}})$ denotes the eta invariant of the self-adjoint operator $\eth_H^{\mathcal{E}}$ and $\alpha_0(H)$ is a differential form which does not depend on the flat Hermitian bundle $\mathcal{E}$. This is the main tool used to prove our results about the twisted Dirac rho invariant, defined below.

Let now Y be a closed, oriented, $(2m-1)$-dimensional Riemannian spin manifold and $H = \sum i^{j+1} H_{2j+1}$ an odd-degree, closed differential form on Y where H_{2j+1} is a real-valued differential form of degree $2j+1$. Denote by $\mathcal{E}$ a Hermitian flat vector bundle over Y with the canonical flat connection $\nabla^{\mathcal{E}}$. Consider the twisted Dirac operator $\eth_H^{\mathcal{E}} = c \circ \nabla^{\mathcal{E},H} = \eth^{\mathcal{E}} + c(H)$. Then $\eth_H^{\mathcal{E}}$ is a self-adjoint elliptic operator and let $\eta(\eth_H^{\mathcal{E}})$ denote its eta invariant. The twisted Dirac rho invariant $\rho_{\mathrm{spin}}(Y, \mathcal{E}, H, [g])$ is defined to be the difference

$$\rho_{\mathrm{spin}}(Y, \mathcal{E}, H, [g]) = \frac{1}{2}\left(\dim(\ker(\eth_H^{\mathcal{E}})) + \eta(\eth_H^{\mathcal{E}})\right) - \mathrm{Rank}(\mathcal{E})\frac{1}{2}\left(\dim(\ker(\eth_H)) + \eta(\eth_H)\right),$$

where $\eth_H$ is the same twisted Dirac operator corresponding to the trivial line bundle. Although the eta invariant $\eta(\eth_H^{\mathcal{E}})$ is a priori only a spectral invariant, we show that the twisted Dirac rho invariant, $\rho_{\mathrm{spin}}(Y, \mathcal{E}, H, [g])$, depends only on the conformal class of the Riemannian metric $[g]$. We compute it for 3-dimensional manifolds with a degree three flux form, Corollary 3.2. This is done via the important method of spectral flow as in [4,11]. In Section 4, we analyse the special case when H is a closed 3-form, using a Lichnerowicz–Weitzenböck formula for the square of the twisted Dirac operator, which in this case has no first order terms, to show that $\rho_{\mathrm{spin}}(Y, \mathcal{E}, H, [g]) = \rho_{\mathrm{spin}}(Y, \mathcal{E}, [g])$ for all $|H|$ small enough, whenever g is conformally equivalent to a Riemannian metric of positive scalar curvature.

The twisted analogue of analytic torsion was studied in [8,9,12] and is another source of inspiration for this paper. We mention that twisted Dirac operators, known as *cubic Dirac operators*, have been studied in representation theory of Lie groups on homogeneous spaces [13,14]. It also appears in the study of Dirac operators on loop groups and their representation theory [15].

1. Twisted Dirac operator and Lichnerowicz–Weitzenböck formulae

1.1. The twisted analogue of the Dirac operator

Assume that X is an oriented manifold of dimension $n = 2m$ and that H is a given complex differential form of positive degree on X. We denote by $\check{H}$ and $\hat{H}$ the differential forms

$$\check{H} = \sum_{k\geqslant 1} H_k/k \quad \text{and} \quad \hat{H} = \sum_{k\geqslant 1} kH_k, \quad \text{if } H = \sum_{k\geqslant 1} H_k.$$

We assume that X is endowed with a spin structure and denote by $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ the $\mathbb{Z}_2$-graded spin bundle. We fix a unitary-flat Hermitian bundle $\mathcal{E}$, $\nabla^{\mathcal{E}}$ and denote by γ the grading involution obtained on $\mathcal{S} \otimes \mathcal{E}$. We are interested in the twisted Dirac operator $\eth_H^{\mathcal{E}} = c \circ \nabla^{\mathcal{E},\check{H}}$ where V is a vector field

$$\nabla_V^{\mathcal{E},\check{H}} = \nabla_V^{\mathcal{E}} + i_V \check{H}\cdot,$$

and $\nabla^{\mathcal{E}}$ is the canonical flat Hermitian connection on $\mathcal{E}$. So

$$\eth_H^{\mathcal{E}} = \eth^{\mathcal{E}} + c(H),$$

where $c(H)$ is the action of the differential form H by Clifford multiplication on the Clifford module $\mathcal{S} \otimes \mathcal{E}$. Then we have the following proposition.

Proposition 1.1. *In the notation above,*

- $\partial\!\!\!/_H^{\mathcal{E}}\,\gamma = -\gamma\,\partial\!\!\!/_H^{\mathcal{E}} \Longleftrightarrow H \in \Omega^{\mathrm{odd}}(X, \mathbb{C})$.
- *The twisted Dirac operator* $\partial\!\!\!/_H^{\mathcal{E}}$ *is self-adjoint if and only if*

$$H = \sum_{k\geqslant 1}(A_{4k} + iB_{4k-2}) + (A_{4k-1} + iB_{4k-3}),$$

where the differential forms A_j *and* B_j *are real differential forms of degree* j *on* X.

So, in the even dimensional case, it is only in the case $H \in \Omega^{4\bullet+3} + i\Omega^{4\bullet+1}$ that one gets a twisted version of the usual Dirac operator. When the ambient manifold is odd dimensional, then we may also consider even forms. Compare with [5,7–9]. Notice that this condition coincides with (and explains) the one obtained for the twisted signature operator in [1].

Proof. We have $\partial\!\!\!/_H^{\mathcal{E}} = \partial\!\!\!/^{\mathcal{E}} + c(H)$, where $\partial\!\!\!/^{\mathcal{E}}$ is the usual Dirac operator acting on sections of $\mathcal{S}(X) \otimes \mathcal{E}$. It is a classical result that $\partial\!\!\!/^{\mathcal{E}}\,\gamma = -\gamma\,\partial\!\!\!/^{\mathcal{E}}$ and that $\partial\!\!\!/^{\mathcal{E}}$ is self-adjoint. Recall that Clifford multiplication by a differential form of degree j is self-adjoint if and only if j is congruent to 0 or 3 modulo 4 and is skew adjoint otherwise. This proves the second item.

Recall now that γ is given locally using an orthonormal basis by Clifford multiplication with the differential volume form

$$\omega = i^m e_1 \cdots e_{2m}.$$

A straightforward local computation then shows that for any complex differential form α of degree j, we have $c(\alpha) \circ \gamma - (-1)^j \gamma \circ c(\alpha) = 0$. □

1.2. Lichnerowicz–Weitzenböck formulae for odd degree twist

Let $H \in \Omega^{\mathrm{odd}}(X)$ be a closed differential form of odd degree and $c(H)$ be the image of H in the sections of the Clifford algebra bundle Cliff(TX). Then $\nabla^{\mathcal{E}} + H$ is a superconnection on the trivially graded, flat bundle $\mathcal{E}$ over X. Then $(\nabla^{\mathcal{E}} + H)^2 = dH = 0$ is the curvature of the superconnection which is flat.

Let $\partial\!\!\!/^{\mathcal{E}}$ denote the Dirac operator acting on $\mathcal{E}$-valued spinors on X. If $\{e_1, \ldots, e_n\}$ is a local orthonormal basis of TX, then we have the expression

$$\partial\!\!\!/^{\mathcal{E}} = \sum_{j=1}^{n} c(e_j)\nabla^{\mathcal{E}}_{e_j}.$$

Let R^g denotes the scalar curvature of the Riemannian metric. Then as shown in [16] the spinor Laplacian

$$\Delta_H^{\mathcal{E}} = -\sum_{j=1}^{n}\left(\nabla^{\mathcal{E}}_{e_j} + c(\iota_{e_j}H)\right)^2$$

is a positive operator that does not depend on the local orthonormal basis $\{e_1, \ldots, e_n\}$ of TX. Here ι_{e_j} denotes contraction by the vector e_j.

Then the following is a consequence of Theorem 1.1 in [16].

Theorem 1.2 (*Lichnerowicz–Weitzenböck Formulae [16]*)**.** *Let H be a closed, odd degree differential form on Y. Then the following identities hold:*

$$\left(\partial\!\!\!/_H^{\mathcal{E}}\right)^2 = \Delta_H^{\mathcal{E}} + \frac{R^g}{4} + c(H)^2 + \sum_{j=1}^{n} c(\iota_{e_j}H)^2,$$

where R^g denotes the scalar curvature of the Riemannian spin manifold X and the last 2 terms on the right hand side satisfy

$$c(H)^2 + \sum_{j=1}^{n} c(\iota_{e_j}H)^2 = \sum_{j_1<j_2\cdots<j_k,\, k\geqslant 2} (-1)^{\frac{k(k+1)}{2}}(1-k)c((\iota_{e_{j_1}}\iota_{e_{j_2}}\cdots\iota_{e_{j_k}}H)^2).$$

As a corollary of Theorem 1.2, one has the following special Lichnerowicz–Weitzenböck formula.

Theorem 1.3 (*[16,17]*)**.** *Let H be a closed differential 3-form. Then*

$$\left(\partial\!\!\!/_H^{\mathcal{E}}\right)^2 = \Delta_H^{\mathcal{E}} + \frac{R^g}{4} - 2|H|^2$$

where R^g denotes the scalar curvature of the Riemannian spin manifold X and $|H|$ the length of H.

2. Eta invariants of twisted Dirac operators

In this section, we define rho invariant $\rho_{\rm spin}(X, \mathcal{E}, H)$ and eta invariant $\eta(\not\partial_H^{\mathcal{E}})$ of the twisted Dirac operators. But let us review the boundary Dirac operator first.

Let X be an $2m$ dimensional compact spin manifold with boundary Y, $\mathcal{E}$ a flat Hermitian vector bundle over X and H is a closed, odd degree differential form on X such that

$$H \in \Omega^{4\bullet+3}(X, \mathbb{R}) \oplus i\Omega^{4\bullet+1}(X, \mathbb{R}).$$

Denote by $\mathcal{S}$ the spin bundle and consider as above the involution $\gamma_X = i^m c_X(e_1 \cdots e_{2m}) \otimes 1_{\mathcal{E}}$ on $\mathcal{S} \otimes \mathcal{E}$-sections over X. According to this involution we decompose as usual $\mathcal{S} \otimes \mathcal{E}$ into

$$\mathcal{S} \otimes \mathcal{E} = \mathcal{S}^+ \otimes \mathcal{E} \oplus \mathcal{S}^- \otimes \mathcal{E}.$$

We may and shall assume that the local orthonormal basis $(e_k)_{1\leqslant k\leqslant 2m}$ is decomposed on the boundary into the local orthonormal basis $(e_k)_{1\leqslant k\leqslant 2m-1}$ and the inward unit vector $e_{2m} = \partial/\partial r$ which is orthogonal to the boundary. Clifford multiplication by e_{2m} is denoted below by σ. We shall then also consider the self-adjoint involution $\gamma_Y = i^m c(e_1 \cdots e_{2m-1})$.

We denote by i^*H the restriction of H to the boundary $Y = \partial X$. We shall say that H is boundary-compatible if there exists a collar neighbourhood $p_\epsilon : X_\epsilon \cong (-\epsilon, 0] \times Y \to Y$ such that

$$H|_{X_\epsilon} = p_\epsilon^*(i^*H). \tag{2}$$

As usual the spinor bundle $\mathcal{S}_Y$ on the boundary manifold Y for the induced spin structure from the fixed one on X is naturally identified with $\mathcal{S}^+$ so that the Clifford representations are related by

$$c_Y(U) = -\sigma \cdot c_X(U), \quad \text{for } U \in TY \subset TX.$$

Lemma 2.1. *Under the above assumptions, suppose further that the Riemannian metric on Y and the Hermitian metric on $\mathcal{E}$ are of product type near the boundary and that the closed, odd degree differential form H is boundary-compatible. We identify as usual the restriction of $\mathcal{S}^+$ to the boundary Y with the spin bundle $\mathcal{S}_Y$. Then, in the collar neighbourhood X_ϵ, we explicitly identify the twisted Dirac operator $\not\partial_H^{X,\mathcal{E}}$ with the operator*

$$\sigma\left(\frac{\partial}{\partial r} + \not\partial_{H|_Y}^{Y,\mathcal{E}|_Y}\right),$$

where $\not\partial_{H|_Y}^{Y,\mathcal{E}|_Y}$ is the self-adjoint elliptic twisted Dirac operator on Y defined as before by $\not\partial_{H|_Y}^{Y,\mathcal{E}|_Y} = c \circ \nabla^{\mathcal{E},\check{H}_{\partial Y}}$ acting on $\mathcal{S}(\partial Y) \otimes \mathcal{E}$.

Proof. Choose an orthonormal basis $(e_1, \ldots, e_{2m})$ as above near a point on the boundary Y such that $e_{2m} = \frac{\partial}{\partial r}$, σ anti commutes with the grading involution γ_X and is itself a well defined involution of $\mathcal{S}|_{X_\epsilon}$. Given a spinor $\phi \in \Gamma(Y, \mathcal{S}_Y \otimes \mathcal{E}_Y)$ over the boundary manifold Y, the pull-back section $\pi_\epsilon^*\phi$ is identified with a section of $\mathcal{S}^+ \otimes \mathcal{E} \cong p_\epsilon^*(\mathcal{S}_Y \otimes \mathcal{E}_Y)$. Therefore, for any smooth function f on $(-\epsilon, 0]$, we compute

$$\begin{aligned}
\sigma \cdot \not\partial_H^{X,\mathcal{E}}(f\pi_\epsilon^*\phi) &= f\sum_{i=1}^{2m-1} \sigma c_X(e_i)\pi_\epsilon^*(\nabla_{e_i}^Y\phi) + \frac{\partial f}{\partial r}\sigma^2\pi_\epsilon^*\phi + f\sigma(H\cdot\pi_\epsilon^*\phi)\\
&= -f\sum_{i=1}^{2m-1} c_Y(e_i)\pi_\epsilon^*(\nabla_{e_i}^Y\phi) - \frac{\partial f}{\partial r}\pi_\epsilon^*\phi + f\sigma c_X(H)(\pi_\epsilon^*\phi)\\
&= -f\pi_\epsilon^*\left(\sum_{i=1}^{2m-1} c_Y(e_i)\nabla_{e_i}^Y\phi\right) - \frac{\partial f}{\partial r}\pi_\epsilon^*\phi - f\pi_\epsilon^*(c_Y(H)(\phi))\\
&= -f\pi_\epsilon^*(\not\partial^{Y,\mathcal{E}|_Y,H|_Y}\phi) - \frac{\partial f}{\partial r}\pi_\epsilon^*\phi.
\end{aligned}$$

Composing again by σ and using $\sigma^2 = -1$, we conclude. □

Let us briefly recall the definition of the eta invariant. Given a self-adjoint elliptic differential operator A of order d on a closed oriented manifold Y of dimension $2m-1$, the *eta-function* of A was defined in [3] as

$$\eta(s, A) := \mathrm{Tr}'(A|A|^{-s-1}),$$

where Tr' stands for the trace restricted to the subspace orthogonal to $\ker(A)$. By [2–4], $\eta(s, A)$ is holomorphic when $\Re(s) > 2m - 1/d$ and can be extended meromorphically to the entire complex plane with possible simple poles only. It is related to the heat kernel by a Mellin transform

$$\eta(s, A) = \frac{1}{\Gamma\left(\frac{s+1}{2}\right)}\int_0^\infty t^{\frac{s-1}{2}}\,\mathrm{Tr}(Ae^{-t\widetilde{A}^2})\,dt.$$

The eta function is then known to be holomorphic at $s = 0$; more precisely by [4] one has the following theorem.

Theorem 2.2 ([4]). *Let A be a first order self-adjoint elliptic operator on the odd dimensional closed manifold X. Then the eta function $\eta(s, A)$ of A has a meromorphic continuation to the complex plane with no pole at $s = 0$. The eta invariant $\eta(A)$ of A is then defined as $\eta(0, A)$.*

The *eta-invariant* of A is thus defined as

$$\eta(A) = \eta(0, A).$$

Consider now our twisted Dirac operator $\not{\partial}_H^{\mathcal{E}}$ defined in the previous subsection for general odd dimensional spin manifold Y and flat bundle $\mathcal{E} \to Y$, and which is a self-adjoint elliptic differential operator and let $\eta(\not{\partial}_H^{Y,\mathcal{E}})$ then denote its well defined eta invariant.

Following [3], we set

$$\xi_{\text{spin}}(Y, \mathcal{E}, H) := \frac{1}{2}\left(\dim(\ker(\not{\partial}_H^{\mathcal{E}})) + \eta(\not{\partial}_H^{\mathcal{E}})\right). \tag{3}$$

Definition 2.3. The *twisted Dirac rho invariant* $\rho_{\text{spin}}(Y, \mathcal{E}, H)$ is defined to be

$$\rho_{\text{spin}}(Y, \mathcal{E}, H) := \xi_{\text{spin}}(Y, \mathcal{E}, H) - \text{rank}(\mathcal{E})\, \xi_{\text{spin}}(Y, H),$$

where $\xi_{\text{spin}}(Y, H)$ is the invariant ξ corresponding to the case where the flat Hermitian bundle $\mathcal{E}$ is the trivial line bundle.

As for the untwisted case, we denote this reduction of $\rho_{\text{spin}}(Y, \mathcal{E}, H)$ mod $\mathbb{Z}$ by $\bar{\rho}_{\text{spin}}(Y, \mathcal{E}, H)$. Then, the reduced twisted rho invariant $\bar{\rho}_{\text{spin}}(Y, \mathcal{E}, H)$ is independent of the choice of the Riemannian metric on X and the Hermitian metric on $\mathcal{E}$. It is also a cobordism invariant of the triple $(Y, \mathcal{E}, H)$. In the case of positive scalar curvature, we shall though be able to work with the real invariant $\rho_{\text{spin}}(Y, \mathcal{E}, H)$. These results will be established in Section 2.2.

2.1. The twisted Dirac index for manifolds with boundary

The goal of this section is to review the Atiyah–Patodi–Singer index theorem for the twisted Dirac operator $\not{\partial}_H^{X,\mathcal{E}}$ with non-local boundary conditions. Here and as before, $\mathcal{E}$ is a unitary flat Hermitian bundle on X and H is a closed, odd degree differential form on X which is boundary compatible and belongs to $\Omega^{4\bullet+3}(X, \mathbb{R}) \oplus i\Omega^{4\bullet+1}(X, \mathbb{R})$. The notation $H|_Y$ stands for the restriction i^*H of H to the boundary manifold $\partial X = Y$. Proposition 2.4 can be deduced from [2].

$\not{\partial}_H^{\tilde{\mathcal{E}}}$ is an elliptic self-adjoint operator, and by [2], the non-local boundary condition given by $P^+(s|_{\partial X}) = 0$, where P^+ denotes the orthogonal projection onto the eigenspaces with positive eigenvalues, yields an elliptic boundary value problem $(\not{\partial}_H^{\tilde{\mathcal{E}}}; P^+)$. Recall that $\dim(X) = 2m$. By the Atiyah–Patodi–Singer index theorem [2,4] and its extension in [18], we have the following proposition.

Proposition 2.4. *In the notation above,*

$$\text{Index}(\not{\partial}_H^{X,\mathcal{E}}; P^+) = \text{Rank}(\mathcal{E}) \int_X \alpha_0(H) - \xi_{\text{spin}}(Y, \mathcal{E}|_Y, H|_Y)$$

where $\xi_{\text{spin}}(Y, \mathcal{E}|_Y, H|_Y)$ is as defined in Eq. (3) and $\alpha_0(H)$ is the local contribution given by the APS theorem.

Remark 2.5. The precise form of $\alpha_0(x)$ is unknown for general H. However in the case when $H = 0$, the local index theorem cf. [19] establishes that the Atiyah–Singer $\widehat{A}$-polynomial applied to the curvature of the Levi-Civita connection, wedged by the Chern character of the flat bundle $\mathcal{E}$, is equal to $\alpha_0(x)$ times the rank of $\mathcal{E}$. In the case when degree of H is equal to 3, it follows from a result of Bismut [16] that the Atiyah–Singer $\widehat{A}$-polynomial applied to the curvature of a Riemannian connection defined in terms of the Levi-Civita connection together with a torsion tensor determined by H, wedged by the Chern character of the flat bundle $\mathcal{E}$, is equal to $\alpha_0(x)$ times the rank of $\mathcal{E}$. The proof that in general, one also gets $\alpha_0(x)$ times the rank of $\mathcal{E}$ is exactly as argued in the Appendix to [1].

2.2. Conformal invariance of the twisted spin eta and rho invariant

Here we prove that the twisted spin rho invariant $\rho_{\text{spin}}(X, \mathcal{E}, H, [g])$ depends only on the conformal class $[g]$ of the Riemannian metric g on X and the Hermitian metric on $\mathcal{E}$ needed in its definition. The proof relies on the index theorem for twisted Dirac operator for spin manifolds with boundary, established in Proposition 2.4. We also state and prove the basic functorial properties of the twisted spin rho invariant.

Conformal variation of the Riemannian metrics. We assume that X is a compact spin manifold of dimension $(2m - 1)$. Let g be a Riemannian metric on X and $g_{\mathcal{E}}$, a Hermitian metric on $\mathcal{E}$. Suppose that g is deformed smoothly and conformally along

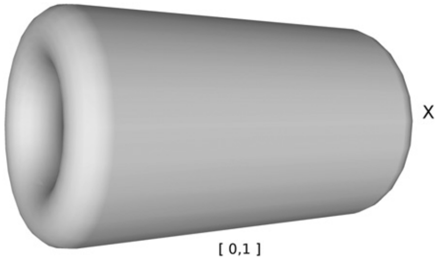

Fig. 1. Manifold with boundary.

a one-parameter family $t \in \mathbb{R}$. The Dirac operator $\not{\partial}_g^{X,\mathcal{E}}$ acting on the bundle of spinors over (X, g) is conformally covariant according to [20], where $\hat{g} = e^f g$, for f a smooth function on X,

$$\not{\partial}_{\hat{g}}^{X,\mathcal{E}} = e^{mf} \circ \not{\partial}_g^{X,\mathcal{E}} \circ e^{-(m-1)f}. \tag{4}$$

It follows that the twisted Dirac operator, $\not{\partial}_H^{X,\mathcal{E}}$, is also conformally covariant with the same weights. Then we have the following theorem.

Theorem 2.6 (*Conformal Invariance of the Spin Rho Invariant*)**.** *Let Y be a compact, spin manifold of dimension $2m-1$, $\mathcal{E}$ a flat Hermitian vector bundle over Y, and $H = \sum i^{j+1} H_{2j+1}$ an odd-degree closed differential form on Y and H_{2j+1} is a real-valued differential form homogeneous of degree $2j+1$. Then the spin rho invariant $\rho_{\mathrm{spin}}(Y, \mathcal{E}, H, [g])$ of the twisted Dirac operator depends only on the conformal class of the Riemannian metric on Y.*

Proof. Consider the manifold with boundary $X = Y \times [0, 1]$, where the boundary $\partial X = Y \times \{0\} - Y \times \{1\}$. Choose a smooth function $a(t),\ t \in [0, 1]$ such that $a(t) \equiv 0$ near $t = 0$ and $a(t) \equiv 1$ near $t = 1$. Consider the metric $h = e^{2a(t)f}(g + dt^2)$ on X, which is also of product type near the boundary, and let $\pi : X \to Y$ denote projection onto the first factor (see Fig. 1).

Applying the index theorem for the twisted Dirac operator, Proposition 2.4, we get

$$\mathrm{Index}\left(\not{\partial}_{\pi^*(H)}^{X,\pi^*(\mathcal{E})}, P_{\mathcal{E}}^+\right) = \mathrm{Rank}(\mathcal{E}) \int_X \alpha_0^H + \xi_{\mathrm{spin}}(\not{\partial}_H^{\mathcal{E}}, g) - \xi_{\mathrm{spin}}(\not{\partial}_H^{\mathcal{E}}, \hat{g}). \tag{5}$$

On the other hand, applying the same theorem to the trivial bundle $\underline{\mathrm{Rank}(\mathcal{E})}$ of rank equal to $\mathrm{Rank}(\mathcal{E})$, we get

$$\mathrm{Index}\left(\not{\partial}_{\pi^*(H)}^{X,\underline{\mathrm{Rank}(\mathcal{E})}}, P^+\right) = \mathrm{Rank}(\mathcal{E}) \int_X \alpha_0^H + \mathrm{Rank}(\mathcal{E}) \left[\xi_{\mathrm{spin}}(\not{\partial}_H, g) - \xi_{\mathrm{spin}}(\not{\partial}_H, \hat{g})\right]. \tag{6}$$

Subtracting the equalities in (5) and (6) above, we get

$$\rho_{\mathrm{spin}}(Y, \mathcal{E}, H, g) - \rho_{\mathrm{spin}}(Y, \mathcal{E}, H, \hat{g}) = \mathrm{Index}\left(\not{\partial}_{\pi^*(H)}^{X,\pi^*(\mathcal{E})}, P_{\mathcal{E}}^+\right) - \mathrm{Index}\left(\not{\partial}_{\pi^*(H)}^{X,\underline{\mathrm{Rank}(\mathcal{E})}}, P^+\right). \tag{7}$$

Each of the index terms on the right hand side of (7) has been shown in [2] to be equal to the L^2-index of $\hat{X}$, which is X together with infinitely long metric cylinders glued onto it at ∂X, plus a correction term (the dimension of the space of, limiting values of right handed spinors). The L^2-index is a conformal invariant by (4), and similarly, the correction term is also a conformal invariant. It follows by (7) that

$$\rho_{\mathrm{spin}}(Y, \mathcal{E}, H, g) = \rho_{\mathrm{spin}}(Y, \mathcal{E}, H, \hat{g}). \quad \square$$

3. Spectral flow and calculations of the twisted Dirac eta invariant

3.1. Spectral flow and the twisted Dirac eta invariant

We employ the method of spectral flow for a path of self-adjoint elliptic operators, which is generically the net number of eigenvalues that cross zero, which was first defined by [4], to get another formula for the difference $\eta(\not{\partial}_H^{\mathcal{E}}) - \eta(\not{\partial}^{\mathcal{E}})$. Extensions of spectral flow to families using entire cyclic cohomology have recently appeared in [21].

Let (Y, g) be an odd-dimensional Riemannian spin manifold. Define $S^{-H} \in \Omega^1(Y, \mathcal{SO}(TY))$, which is a degree one form with values in the skew-symmetric endomorphisms of TY, as follows. For $\alpha, \beta, \gamma \in TY$, set

$$g(S^{-H}(\alpha)\beta, \gamma) = -2H(\alpha, \beta, \gamma).$$

Let ∇^L denote the Levi-Civita connection of (Y, g) and $\nabla^{-H} = \nabla^L + S^{-H}$ be the Riemannian connection whose curvature is denoted by Ω_Y^{-H}.

Proposition 3.1. *Let Y be as before a $(2\ell-1)$-dimensional closed, spin, Riemannian manifold and $\mathcal{E}$ a flat vector bundle associated with an orthogonal or unitary representation of $\pi_1(Y)$. Let $H \in \Omega^3(Y,\mathbb{R})$ be a closed 3-form such that the Dirac operators $\partial\!\!\!/^{\mathcal{E}}$ and $\partial\!\!\!/^{\mathcal{E}}_H$ are both invertible, then*

$$\mathrm{sf}(\partial\!\!\!/^{\mathcal{E}}, \partial\!\!\!/^{\mathcal{E}}_H) = \frac{\mathrm{Rank}(\mathcal{E})}{(-2\pi i)^{\ell+1}} \int_Y H \wedge \widehat{A}(\Omega_Y^{-H}) + \frac{1}{2}(\eta(\partial\!\!\!/^{\mathcal{E}}_H) - \eta(\partial\!\!\!/^{\mathcal{E}})),$$

where $\widehat{A}(\cdot)$ denotes the A-hat genus polynomial, Ω_Y^{-H} denotes the curvature of the Riemannian connection ∇^{-H} and $\mathrm{sf}(\partial\!\!\!/^{\mathcal{E}}, \partial\!\!\!/^{\mathcal{E}}_H)$ denotes the spectral flow of the smooth path of self-adjoint elliptic operators $\{\partial\!\!\!/^{\mathcal{E}}_{uH}, : u \in [0,1]\}$.

Proof. By Theorem 2.6 [11],

$$\mathrm{sf}(\widetilde{\partial\!\!\!/}^{\mathcal{E}}, \widetilde{\partial\!\!\!/}^{\mathcal{E}}_H) = \left(\frac{\epsilon}{\pi}\right)^{1/2} \int_0^1 \mathrm{Tr}\left(c(H) \circ e^{-\epsilon {\partial\!\!\!/^{\mathcal{E}}_{uH}}^2}\right) du + \frac{1}{2}(\eta(\partial\!\!\!/^{\mathcal{E}}_H) - \eta(\partial\!\!\!/^{\mathcal{E}}))$$

where $\widetilde{\partial\!\!\!/}^{\mathcal{E}}_H$ denotes the operator $\partial\!\!\!/^{\mathcal{E}}_H + P_{\mathcal{E},H}$ where $P_{\mathcal{E},H}$ denotes the orthogonal projection onto the nullspace of $\partial\!\!\!/^{\mathcal{E}}_H$. Following the idea of the proof of Theorem 2.8 in [11] the technique of [19], and the Local Index Theorem 1.7 of Bismut [16] (here the fact that H is a closed 3-form on Y is used), we may make the following replacements:

$$\epsilon^{1/2} c(H) \quad \text{by } H \wedge$$
$$e^{-(\epsilon^{1/2}\partial\!\!\!/^{\mathcal{E}}_{uH})^2} \quad \text{by } \widehat{A}(\Omega_Y^{-H}) \wedge e^{(d+uH)^2}$$
$$\mathrm{Tr}(\cdot) \quad \text{by } \frac{-i\pi^{1/2}}{(-2\pi i)^{\ell+1}} \int_Y \mathrm{tr}(\cdot).$$

This then enables us to replace $\left(\frac{\epsilon}{\pi}\right)^{1/2} \int_0^1 \mathrm{Tr}\left(c(H) \circ e^{-\epsilon {\partial\!\!\!/^{\mathcal{E}}_{uH}}^2}\right) du$ by

$$\frac{\mathrm{Rank}(\mathcal{E})}{(-2\pi i)^{\ell+1}} \int_Y H \wedge \widehat{A}(\Omega_Y^{-H}),$$

and the proposition follows. $\square$

As a special case of Proposition 3.1, one obtains the following calculation.

Corollary 3.2. *Let Y be a compact spin Riemannian manifold of dimension 3 and $\mathcal{E}$ a flat vector bundle associated with an orthogonal or unitary representation of $\pi_1(Y)$. Let H be a closed 3-form on Y. Consider the smooth path of self-adjoint elliptic operators $\{\partial\!\!\!/^{\mathcal{E}}_{uH}, : u \in [0,1]\}$ and assume that the Dirac operators $\partial\!\!\!/^{\mathcal{E}}$ and $\partial\!\!\!/^{\mathcal{E}}_H$ are both invertible. Then*

$$\eta(\partial\!\!\!/^{\mathcal{E}}_H) - \eta(\partial\!\!\!/^{\mathcal{E}}) = 2\,\mathrm{sf}(\partial\!\!\!/^{\mathcal{E}}, \partial\!\!\!/^{\mathcal{E}}_H) + \frac{h}{2\pi^2}$$

where $h = [H] \in \mathbb{R} \cong H^3(Y,\mathbb{R})$.

4. Positive scalar curvature

Here we give applications of our results to closed spin manifolds that admit a Riemannian metric of positive scalar curvature. Foundational works on metrics of positive scalar curvature are due to for instance [22–25]. Viewing the eta and rho invariants of the Dirac operator as an obstruction to the existence of Riemannian metrics of positive scalar curvature on compact spin manifolds was done in [26]; obstructions arising from covering spaces using the von Neumann trace in [27,28] and on foliations in [28,29], are some amongst many papers on the subject. As a corollary to Theorem 1.3 one has

Corollary 4.1. *In the notation of Proposition 3.1, if the scalar curvature R^g of the odd-dimensional, compact Riemannian spin manifold Y is positive, then there exists $u_0 > 0$ such that for all $u \in [0,u_0]$, the twisted Dirac operator $\partial\!\!\!/^{\mathcal{E}}_{uH}$ has trivial nullspace, where H is a closed degree 3 form on Y. In particular, the spectral flow $\mathrm{sf}(\partial\!\!\!/^{\mathcal{E}}, \partial\!\!\!/^{\mathcal{E}}_{u_0H})$ of the family of twisted Dirac operators $\{\partial\!\!\!/^{\mathcal{E}}_{uH} \mid u \in [0,u_0]\}$ is trivial and we deduce that*

$$\eta(\partial\!\!\!/^{\mathcal{E}}_{uH}) - \eta(\partial\!\!\!/^{\mathcal{E}}) = -u\frac{\mathrm{Rank}(\mathcal{E})}{(-4\pi i)^{\ell+1}} \int_Y H \wedge \widehat{A}(\Omega_Y^{-uH})$$

Therefore for all $u \in [0,u_0]$, one has

$$\rho_{\mathrm{spin}}(Y,\mathcal{E},uH,[g]) = \rho_{\mathrm{spin}}(Y,\mathcal{E},[g]).$$

Corollary 4.1 above establishes conformal obstructions to the existence of metrics of positive scalar curvature. By Proposition 2.4 and Theorem 1.3, we have the following corollary.

Corollary 4.2. *Let X be an even dimensional Riemannian spin manifold with spin boundary $\partial X = Y$. Let $\widetilde{H}$ be a closed degree 3 form on X which restricts to the closed degree 3-form H on the boundary Y. Let $\Omega_X^{-u\widetilde{H}}$ denote the curvature of the Riemannian connection on X with torsion tensor determined by $\widetilde{H}$ and suppose that the scalar curvature $R^g > 0$. Suppose also that the flat bundle $\widetilde{\mathcal{E}}$ on X restricts to the flat bundle $\mathcal{E}$ on Y. Then there exists $u_0 > 0$ such that for all $u \in [0, u_0]$,*

$$\mathrm{Rank}(\mathcal{E}) \int_X \widehat{A}(\Omega_X^{-u\widetilde{H}}) = \eta(\eth^{\mathcal{E}}_{uH}),$$

so that

$$\eta(\eth^{\mathcal{E}}_{uH}) - \eta(\eth^{\mathcal{E}}) = \mathrm{Rank}(\mathcal{E}) \int_X \left[\widehat{A}(\Omega_X^{-u\widetilde{H}}) - \widehat{A}(\Omega_X)\right].$$

Therefore for all $u \in [0, u_0]$, one has

$$\rho_{\mathrm{spin}}(Y, \mathcal{E}, uH, [g]) = \rho_{\mathrm{spin}}(Y, \mathcal{E}, [g]).$$

The proof of the following uses the Gromov–Lawson–Rosenberg conjecture [22,23] and Corollary 4.1.

Corollary 4.3. *Let Y be a closed odd dimensional Riemannian spin manifold with positive scalar curvature. Suppose that the Gromov–Lawson–Rosenberg conjecture [22,23] holds for the fundamental group Γ of Y and that H is a closed 3-form on Y such that $[H] = f^*(c)$ where $c \in H^3(B\Gamma)$ and $f : Y \to B\Gamma$ is a continuous map. Then there exists $u_0 > 0$ such that for all $u \in [0, u_0]$,*

$$\eta(\eth^{\mathcal{E}}_{uH}) = \eta(\eth^{\mathcal{E}}).$$

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References

[1] M.T. Benameur, V. Mathai, Index type invariants for twisted signature complexes and homotopy invariance, arXiv:1202.0272.
[2] M.F. Atiyah, V.K. Patodi, I.M. Singer, Spectral asymmetry and Riemannian geometry, I, Math. Proc. Cambridge Philos. Soc. 77 (1975) 43–69. MR0397797, Zbl 0297.58008.
[3] M.F. Atiyah, V.K. Patodi, I.M. Singer, Spectral asymmetry and Riemannian geometry, II, Math. Proc. Cambridge Philos. Soc. 78 (3) (1975) 405–432. MR0397798, Zbl 0314.58016.
[4] M.F. Atiyah, V.K. Patodi, I.M. Singer, Spectral asymmetry and Riemannian geometry, III, Math. Proc. Cambridge Philos. Soc. 79 (1) (1976) 71–99. MR0397799, Zbl 0325.58015.
[5] P. Bouwknegt, A. Carey, V. Mathai, M. Murray, D. Stevenson, Twisted K-theory and K-theory of bundle gerbes, Comm. Math. Phys. 228 (2002) 17–49. MR1911247, Zbl 1036.19005. arXiv:hep-th/0106194.
[6] V. Mathai, D. Stevenson, Chern character in twisted K-theory: equivariant and holomorphic cases, Comm. Math. Phys. 236 (1) (2003) 161–186. MR1977885. arXiv:hep-th/0201010.
[7] M.F. Atiyah, G. Segal, in: P.A. Griffith (Ed.), Twisted K-Theory and Cohomology, in: Nankai Tracts Math., vol. 11, World Sci. Publ., Hackensack, NJ, 2006, pp. 5–43. Inspired by S.S. Chern, MR2307274, Zbl 1138.19003. arXiv:math.AT/0510674.
[8] V. Mathai, S. Wu, Analytic torsion for twisted de Rham complexes, J. Differential Geom. 88 (2) (2011) 297–332. MR2838268. arXiv:math/0810.4204.
[9] V. Mathai, S. Wu, Twisted analytic torsion, Sci. China Math. 53 (3) (2010) 555–563. MR2608312, Zbl 1202.58019. arXiv:math/0912.2184.
[10] R. Rohm, E. Witten, The antisymmetric tensor field in superstring theory, Ann. Phys. 170 (1986) 454–489. MR0851628.
[11] E. Getzler, The odd Chern character in cyclic homology and spectral flow, Topology 32 (3) (1993) 489–507. MR1231957.
[12] V. Mathai, S. Wu, Analytic torsion of $\mathbb{Z}_2$-graded elliptic complexes, Contemp. Math. 546 (2011) 199–212. MR2815136. arXiv:math/1001.3212.
[13] B. Kostant, A cubic Dirac operator and the emergence of Euler number multiplets of representations for equal rank subgroups, Duke Math. J. 100 (3) (1999) 447–501. MR1719734.
[14] S. Slebarski, Dirac operators on a compact Lie group, Bull. London Math. Soc. 17 (6) (1985) 579–583. MR0813743.
[15] G. Landweber, Dirac operators on loop spaces, Harvard University, Ph.D. Thesis, 1999.
[16] J.-M. Bismut, A local index theorem for non-Kähler manifolds, Math. Ann. 284 (4) (1989) 681–699. MR1006380.
[17] I. Agricola, T. Friedrich, On the holonomy of connections with skew-symmetric torsion, Math. Ann. 328 (4) (2004) 711–748. MR2047649.
[18] P.B. Gilkey, Invariance Theory, The Heat Equation, and the Atiyah–Singer Index Theorem, in: Math. Lecture Series, vol. 11, Publish or Perish, Inc., Wilmington, DE, 1984. MR0783634, Zbl 0565.58035;
Studies Adv. Math., second ed., CRC Press, Boca Raton, FL, 1995. MR1396308, Zbl 0856.58001.
[19] N. Berline, E. Getzler, M. Vergne, Heat Kernels and Dirac Operators, in: Grund. Math. Wissen., vol. 298, Springer-Verlag, Berlin, 1992. MR1215720, Zbl 0744.58001.
[20] B.H. Lawson, M.-L. Michelsohn, Spin Geometry, in: Princeton Mathematical Series, vol. 38, Princeton University Press, Princeton, NJ, 1989. MR1031992.
[21] M.T. Benameur, A. Carey, Higher spectral flow and an entire bivariant JLO cocycle, J. K-Theory 11 (2013) 183–232.
[22] M. Gromov, B. Lawson Jr., The classification of simply connected manifolds of positive scalar curvature, Ann. of Math. (2) 111 (3) (1980) 423–434. MR0577131;
M. Gromov, B. Lawson Jr., Spin and scalar curvature in the presence of a fundamental group, I, Ann. of Math. (2) 111 (2) (1980) 209–230. MR0569070.
[23] J. Rosenberg, C^*-algebras, positive scalar curvature, and the Novikov conjecture, Publ. Math. Inst. Hautes Études Sci. 58 (1983) 197–212. MR0720934.

[24] R. Schoen, S.T. Yau, Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with nonnegative scalar curvature, Ann. of Math. (2) 110 (1) (1979) 127–142. MR0541332.
[25] N. Hitchin, Harmonic spinors, Adv. Math. 14 (1974) 1–55. MR0358873.
[26] V. Mathai, Nonnegative scalar curvature, Ann. Global Anal. Geom. 10 (2) (1992) 103–123. MR1175914.
[27] V. Mathai, Von Neumann algebra invariants of Dirac operators, J. Funct. Anal. 152 (1) (1998) 1–21. MR1600078.
[28] M.T. Benameur, J.L. Heitsch, Index theory and non-commutative geometry, II. Dirac operators and index bundles, J. K-Theory 1 (2) (2008) 305–356. MR2434189.
[29] M.T. Benameur, P. Piazza, Index, eta and rho invariants on foliated bundles, Astérisque 327 (2009) 201–287. MR2642361.