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A lower bound for the first eigenvalue in the Laplacian operator on compact Riemannian manifolds

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ABSTRACT

This paper gives a simple proof of the main result of Ling [J. Ling, Lower bounds of the eigenvalues of compact manifolds with positive Ricci curvature, Ann. Global Anal. Geom. 31 (2007) 385–408] in an in-depth study of the sharp lower bound for the first eigenvalue in the Laplacian operator on compact Riemannian manifolds with nonnegative Ricci curvature. Although we use Ling's methods on the whole, to some extent we deal with the singularity of test functions and greatly simplify many of the calculations involved. This may provide a new way for estimating eigenvalues.

1. Introduction

Suppose (M, g) is an n-dimensional Riemannian manifold with Ricci curvature satisfying

$$\mathrm{Ric}(M) \geqslant (n-1)K \tag{1.1}$$

for some nonnegative constant K.

Unlike upper bound estimates, lower bound estimates for an eigenvalue are difficult to obtain. Studies on the lower bound of the first positive eigenvalue in the Laplacian operator on compact Riemannian manifolds have a long history with many studies. Results reported by Li [1,2], Li and Yau [3,4], Zhong and Yang [5], Yang [6], Ling [7–10], Ling and Lu [11], Shi and Zhang [12], Qian et al. [13], Andrews and Ni [14], and Andrews and Clutterbuck [15] are all well known. Here we outline just some of the important work carried out.

We recall the following lower bound estimate of the first eigenvalue, first reported by Lichnerowicz [16] and then Obata [17], for the case in which M is a compact manifold without a boundary. Under assumption (1.1), Escobar proved that if a compact manifold has a weakly convex boundary, the first nonzero Neumann eigenvalue of M has lower bound (1.2) as well [18].

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Theorem 1.1 (*[6]*). *Assume that* $\mathrm{Ric}(M) \geqslant (n-1)K > 0$. *Let* λ_1 *be the first positive eigenvalue on* M *(with either a Dirichlet or Neumann boundary condition if* $\partial M \neq \emptyset$*). If* $\partial M \neq \emptyset$*, we also assume that* ∂M *is of nonnegative mean curvature* $\mathrm{tr}S \geqslant 0$ *if* λ_1 *is the first Dirichlet eigenvalue and* ∂M *is of nonnegative definite second fundamental form* $S \geqslant 0$ *if* λ_1 *is the first Neumann eigenvalue. Then*

$$\lambda_1 \geqslant nK. \tag{1.2}$$

This estimate provides no information when the constant K vanishes. For such a case, Li and Yau [3] and Zhong and Yang [5] provided another lower bound.

An interesting problem is identification of a unified lower bound for the first nonzero eigenvalue λ_1 in terms of the lower bound $(n-1)K$ of the Ricci curvature and the diameter d, the inscribed radius r and other geometric quantities, which do not vanish as K vanishes, of the manifold with positive Ricci curvature.

Later on, the maximum principle method, which is rather different from the one above, was first used by Li in proving eigenvalue estimates for compact manifolds [1]. Since then the method has been refined and used by many authors [3,5,6] to obtain sharper eigenvalue estimates.

Using an improved maximum principle method, Li and Yau derived the following elegant result for $K = 0$ [3].

Theorem 1.2 (*[3]*). *Let* M *be a compact Riemannian manifold, and let* $\partial M = \phi$ *and* $\mathrm{Ric}(M) \geqslant 0$. *Then* $\lambda_1 \geqslant \frac{\pi^2}{4d^2}$*, where* d *is the diameter of* M.

The above result was improved by Li [2] to $\lambda_1 \geqslant \frac{\pi^2}{2d^2}$ for $K = 0$. Li also conjectured that the first positive eigenvalue should satisfy

$$\lambda_1 \geqslant \frac{\pi^2}{d^2} + (n-1)K. \tag{1.3}$$

This conjecture motivated many related studies. We recall some of the main results in the following.

First, we recall the well-known Bonnet–Myers theorem.

Theorem 1.3. *Suppose that* M *is an* n*-dimensional complete Riemannian manifold with Ricci curvature bounded below by* $(n-1)K(K > 0)$. *Then* M *is compact and its diameter* $d(M)$ *satisfies the estimate*

$$d(M) \leqslant \frac{\pi}{\sqrt{K}}. \tag{1.4}$$

Combining (1.3) with (1.4), we can deduce (1.2). Thus, (1.3) is usually regarded as the sharp lower bound on λ_1 in terms of the diameter for manifolds with a Ricci curvature satisfying (1.1). Obviously, an optimal estimate of the lower bound for the first eigenvalue would be perfect. It seems that any further progress requires a refined gradient estimate relevant to the first eigenfunction.

By sharpening the method of Li and Yau and giving a more delicate estimate, Zhong and Yang obtained the sharp estimate $\lambda_1 \geqslant \frac{\pi^2}{d^2}$ for $K = 0$ [5].

Theorem 1.4 (*[5]*). *Let* M *be a compact Riemannian manifold without a boundary and with nonnegative Ricci curvature and let* d *be the diameter of* M. *Then*

$$\lambda_1 \geqslant \frac{\pi^2}{d^2}. \tag{1.5}$$

Note that attempts to prove the so-called Li conjecture should unify the estimates of Yang and Zhong with that of Lichnerowicz. There have been several efforts to prove (1.3), particularly to improve inequalities of the form

$$\lambda_1 \geqslant \frac{\pi^2}{d^2} + C(n-1)K \tag{1.6}$$

for some constant C [6–15,19,20]. We recall some of these results in brief.

Using the methods noted above, but constructing a more complicated test function, Yang made some progress for Li's conjecture [6], as shown by the following results.

Theorem 1.5 (*[6]*). *Let* M^n *be a closed Riemannian manifold with* $\mathrm{Ric}(M^n) \geqslant (n-1)K \geqslant 0$ *and diameter* d. *Then the first positive eigenvalue* λ_1 *on* M *satisfies the lower bound*

$$\lambda_1 \geqslant \frac{\pi^2}{d^2} + \frac{(n-1)K}{4}.$$

Theorem 1.6 (*[6]*). *Let* M^n *be a compact manifold with nonempty boundary and with* $\mathrm{Ric}(M^n) \geqslant (n-1)K \geqslant 0$.

(a) *Assume that the boundary* ∂M *is weakly convex, that is, the second fundamental form with respect to the outward normal is nonnegative. Then the first positive Neumann eigenvalue* λ_1 *on* M^n *satisfies the same lower bound in Theorem 1.5.*

(b) *Assume that the mean curvature with respect to the outward normal of the boundary ∂M is nonnegative. Then the first positive Dirichlet eigenvalue λ_1 on M^n satisfies the lower bound estimate*

$$\lambda_1 \geqslant \frac{1}{4}\left[\frac{\pi^2}{r^2} + (n-1)K\right],$$

where r is the inscribed radius of M^n.

To improve previous results via the maximum principle method, we need to construct suitable test functions requiring detailed technical work. Ling provided new estimates that improve the lower bound in part [7,9]. The main results are the following three theorems.

Theorem 1.7 ([7]). *Let (M, g) be an n-dimensional compact Riemannian manifold with a boundary. Suppose that the Ricci curvature of M is bounded below by $(n-1)K$ for some constant $K > 0$,*

$$\mathrm{Ric}(M) \geqslant (n-1)K,$$

and that the mean curvature of the boundary ∂M with respect to the outward normal is nonnegative. Then the first Dirichlet eigenvalue λ_1 of the Laplacian Δ of M has the lower bound

$$\lambda_1 \geqslant \frac{\pi^2}{\tilde{d}^2} + \frac{1}{2}(n-1)K,$$

where $\tilde{d}$ is the diameter of the largest interior ball in M, that is, $d = 2\sup_{x\in M}\{\mathrm{dist}(x, \partial M)\}$.

Theorem 1.8 ([9]). *Let M be an n-dimensional compact Riemannian manifold that has an empty or nonempty boundary whose second fundamental form is nonnegative with respect to the outward normal (i.e., weakly convex). Suppose that its Ricci curvature has a lower bound $(n-1)K$ for some constant $K > 0$, that is,*

$$\mathrm{Ric}(M) \geqslant (n-1)K > 0.$$

Then the first nonzero (closed or Neumann) eigenvalue λ_1 of the Laplacian on M has the lower bound

$$\lambda_1 \geqslant \frac{\pi^2}{d^2} + \frac{3}{8}(n-1)K \quad \text{for } n = 2$$

and

$$\lambda_1 \geqslant \frac{\pi^2}{d^2} + \frac{31}{100}(n-1)K \quad \text{for } n \geqslant 3,$$

where d is the diameter of M.

Theorem 1.9 ([9]). *Let $a \in (0, 1)$ and α be defined by (2.2) and (4.2), respectively. We assume that $\mu\alpha\pi^2/8 \leqslant 1$, that is,*

$$\lambda_1 \geqslant \frac{\pi^2\mu(n-1)K}{8a} \tag{1.7}$$

for a constant $\mu \in (0, 1]$. The other assumptions are as in Theorem 1.8. Then the first nonzero (closed or Neumann) eigenvalue λ_1 of the Laplacian on M has the following lower bound:

$$\lambda_1 \geqslant \frac{\pi^2}{d^2} + \frac{\mu(n-1)K}{2}. \tag{1.8}$$

However, these findings have been updated by more recent results reported by Shi and Zhang [12], Qian et al. [13], and Andrews and Clutterbuck [15]. More precisely, Shi and Zhang [12] soon got the following result.

Theorem 1.10 ([12]). *Let M be a compact n-dimensional Riemannian manifold without a boundary (or with a convex boundary) and let $\mathrm{Ric}(M) \geqslant (n-1)K$. Then its first nonzero (Neumann) eigenvalue $\lambda_1(M)$ satisfies*

$$\lambda_1(M) \geqslant 4s(1-s)\frac{\pi^2}{d^2} + s(n-1)K \quad \text{for all } s \in (0, 1), \tag{1.9}$$

where d is the diameter of M.

Following the argument of Shi and Zhang [12], Qian et al. extended this result to the case in which M is an Alexandrov space [13].

Theorem 1.11 ([13]). *Let M be a compact n $(\geqslant 2)$-dimensional Alexandrov space without a boundary and let $\mathrm{Ric}(M) \geqslant (n-1)K$. Then its first nonzero eigenvalue $\lambda_1(M)$ satisfies (1.9), where d is the diameter of M.*

Qian et al. provided the following remarks [13].

Remark 1.1. (1) If let $s = \frac{1}{2}$, the estimate (1.9) becomes

$$\lambda_1(M) \geqslant \frac{\pi^2}{d^2} + \frac{1}{2}(n-1)K. \tag{1.10}$$

This improves the results of Chen and Wang for both $K > 0$ and $K < 0$ [21,22]. It also improves Ling's results [9].

(2) If $K > 0$, Theorem 1.11 implies that

$$\lambda_1(M) \geqslant \frac{3}{4}\left[\frac{\pi^2}{d^2} + (n-1)K\right].$$

(3) For $n \leqslant 5$ and $K > 0$, by choosing some suitable constant s, Qian et al. obtained the following estimate [13]:

$$\lambda_1(M) \geqslant \frac{\pi^2}{d^2} + \frac{1}{2}(n-1)K + \frac{(n-1)^2k^2d^2}{16\pi^2}.$$

Andrews and Clutterbuck recently showed that the first nonzero Neumann eigenvalue λ_1 satisfies (1.10) when M is a compact n-dimensional Riemannian manifold with a weakly convex boundary [15]. Their contribution is a rather simple proof that uses the long-term behavior of the heat equation, which is probably much easier than previous arguments. In particular, their argument avoids any problems arising from possible asymmetry of the first eigenfunction. Andrews and Ni had already used a similar argument to prove the sharp lower bound for λ_1 on a Bakry–Emery manifold [14]. Andrews and Clutterbuck also showed that the inequality with $C = \frac{1}{2}$ is the best possible constant for this type of estimate, in other words, the Li conjecture is false [15].

Note that for manifolds with a small diameter, Theorems 1.5–1.9 are better than the estimate (1.2). Therefore, these results generalize Theorem 1.4. For more information, we refer to excellent surveys in the literature [11,13,15,19] for further results on eigenvalue estimates. With the rapid development of spectral geometry, eigenvalue estimates are increasingly important.

In the present study we only give a simple proof of Theorem 1.9 using the original approach of Zhong and Yang [5], but constructing a suitable test function that is our main contribution. Our argument is based on several early studies [2,3,5,7,9]. One interesting feature of our argument is that it avoids various problems arising from the singularity of $|\nabla u|^2/(1-u^2)$ encountered previously. Although in many ways analogous to the strategy used by Ling [7,9], our approach can readily handle this singularity and reduces the computational complexity to some degree. This may be a new way of estimating eigenvalues.

The remainder of the paper is organized as follows. Section 2 introduces the terminology and notation, which are consistent with Schoen and Yau [23], who defined the corresponding terms in a more general setting. In Section 3, for $\partial M \neq \emptyset$ we establish a lemma that is a version of Yangs Lemma 2.2 [6]. Using this lemma and the maximum principle, we establish a rough estimate of $F(\theta)$, defined in (2.8). A more precise estimate of $F(\theta)$ is provided at the end of Section 4 via the barrier function method. This improved estimate is essential for the proof of Theorem 1.9. Finally, as an application of this estimate, a proof of Theorem 1.9 is presented in Section 5.

2. Notation and preliminaries

Let $\{e_1, e_2, \ldots, e_n\}$ be a local orthonormal frame field on M. We adopt the convention that subscripts i, j, and k, with $1 \leqslant i, j, k \leqslant n$, denote covariant differentiations in the e_i, e_j, and e_k directions, respectively.

The Laplacian operator on M in term of local coordinates associated with the above orthonormal frame is defined by differentiating once more in the direction of e_i and summing over $i = 1, 2, \ldots, n$, that is,

$$\Delta u = \sum_i u_{ii}.$$

We denote by u the normalized eigenfunction with respect to the first eigenvalue $-\lambda_1$ of Δ, that is,

$$\begin{cases} \Delta u = -\lambda_1 u, \\ \max u = 1, \\ \min u = -k, \quad 0 < k \leqslant 1. \end{cases} \tag{2.1}$$

Let

$$\begin{cases} \tilde{u} = \left(u - \dfrac{1-k}{2}\right) \Big/ \dfrac{1+k}{2} \\ a = \dfrac{1-k}{1+k}, \quad 0 \leqslant a < 1. \end{cases} \tag{2.2}$$

Therefore, (2.1) can be rewritten as

$$\begin{cases} \Delta \tilde{u} = -\lambda_1(\tilde{u} + a), \\ \max \tilde{u} = 1, \qquad \min \tilde{u} = -1. \end{cases} \tag{2.3}$$

Throughout this paper,

$$\theta(x) = \arcsin[\tilde{u}(x)], \quad \forall\, x \in M$$

and we define a subset of M as follows:

$$\Sigma_* = \left\{x \in M : \theta(x) = \frac{\pi}{2} \text{ or } \theta(x) = -\frac{\pi}{2}\right\}.$$

Thus,

$$\tilde{u}(x) = \sin[\theta(x)], \quad \forall\, x \in M$$

and

$$-\frac{\pi}{2} \leqslant \theta(x) \leqslant \frac{\pi}{2}, \quad \forall\, x \in M.$$

The above terms apply unless stated otherwise.

By (2.3), a straightforward calculation shows that θ satisfies

$$\cos\theta \cdot \Delta\theta - \sin\theta \cdot |\nabla\theta|^2 = -\lambda_1(\sin\theta + a). \tag{2.4}$$

In particular,

$$\Delta\theta = \frac{\sin\theta}{\cos\theta} \cdot |\nabla\theta|^2 - \frac{\lambda_1(\sin\theta + a)}{\cos\theta} \tag{2.5}$$

whenever $x \in M \setminus \Sigma_*$. From (2.4), it is evident that

$$|\nabla\theta|^2 = \lambda_1(1 - a) \quad \text{as } \theta = -\frac{\pi}{2} \tag{2.6}$$

and

$$|\nabla\theta|^2 = \lambda_1(1 + a) \quad \text{as } \theta = \frac{\pi}{2}. \tag{2.7}$$

We also define the function $F : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \mapsto \mathbb{R}$ as follows:

$$F(\theta_0) = \max_{x\in M,\, \theta(x)=\theta_0} |\nabla\theta(x)|^2, \quad \forall\, \theta_0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \tag{2.8}$$

Obviously F is well defined. In fact, $F(\theta_0)$ is nothing but an extreme value of f with the condition $\theta(x) = \theta_0$. It is very easy to verify that $F(\theta)$ is continuous in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Moreover, by (2.6) and (2.7), if we define

$$F\left(-\frac{\pi}{2}\right) = F\left(-\frac{\pi}{2} + 0\right) = \lambda_1(1 - a)$$

and

$$F\left(\frac{\pi}{2}\right) = F\left(\frac{\pi}{2} - 0\right) = \lambda_1(1 + a),$$

then $F(\theta)$ can be extended to a continuous function on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

3. Rough estimate of $|\nabla\theta|^2$

In a similar way to previous studies [6,7,9,23–25] we obtain the following lemma, which can be viewed as another version of Yangs Lemma 2.2 [6].

Lemma 3.1. *Suppose that $\partial M \neq \emptyset$. Let $G(x)$ be a function defined as*

$$G(x) = \frac{1}{2}\,|\nabla\theta(x)|^2 + g[\theta(x)], \quad \forall\, x \in M,$$

where $g(\theta)$ is a smooth function defined on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Assume that the second fundamental form of ∂M is nonnegative with respect to the outward normal (i.e., weakly convex) and u satisfies the Neumann boundary condition. If $G(x)$ attains its maximum at $x_0 \in \partial M \setminus \Sigma_*$, then $\nabla\theta(x_0) = 0$. Furthermore, $\nabla G(x_0) = 0$.

Proof. Choose a local orthonormal frame $\{e_1, e_2, \ldots, e_n\}$ around x_0 such that e_1 is the unit normal ∂M pointing outwards to M. We also denote by $\frac{\partial}{\partial x_1}$ the restriction on ∂M of the directional derivative corresponding to e_1.

Clearly, the maximality of $G(x_0)$ implies that

$$G_i(x_0) = 0 \quad \text{for } 2 \leqslant i \leqslant n \tag{3.1}$$

and

$$0 \leqslant \frac{\partial G}{\partial x_1}(x_0) = \sum_{i=1}^{n} \theta_i(x_0) \cdot \theta_{i1}(x_0) + g'[\theta(x_0)] \cdot \theta_1(x_0). \tag{3.2}$$

In addition, since u satisfies the Neumann boundary condition,

$$\begin{aligned} \theta_1 &= \frac{1}{\sqrt{1-\tilde{u}^2}} \cdot \tilde{u}_1 = \frac{1}{\sqrt{1-\tilde{u}^2}} \cdot \frac{\partial \tilde{u}}{\partial x_1} \\ &= \frac{1}{\sqrt{\left(\frac{1+k}{2}\right)^2 - \left(u - \frac{1-k}{2}\right)^2}} \cdot \frac{\partial u}{\partial x_1} = 0 \quad \text{on } \partial M. \end{aligned}$$

Therefore,

$$\theta_1(x_0) = 0. \tag{3.3}$$

Substituting (3.3) into (3.2), we obtain

$$0 \leqslant \frac{\partial G}{\partial x_1}(x_0) = \sum_{i=2}^{n} \theta_i(x_0) \cdot \theta_{i1}(x_0). \tag{3.4}$$

Note that $\theta_1(x_0) = 0$ and recall the definition of the second fundamental form with respect to the outward normal. We can then derive that, for $2 \leqslant i \leqslant n$,

$$\begin{aligned} \theta_{i1} &= e_i e_1 \theta - (\nabla_{e_i} e_1)\theta = e_i(\theta_1) - (\nabla_{e_i} e_1, e_j)\theta_j \\ &= -(\nabla_{e_i} e_1, e_j)\theta_j = -\sum_{j=2}^{n} h_{ij}\theta_j \quad \text{at } x_0, \end{aligned}$$

that is, for $2 \leqslant i \leqslant n$,

$$\theta_{i1} = -\sum_{j=2}^{n} h_{ij}\theta_j \quad \text{at } x_0, \tag{3.5}$$

where $(h_{ij})_{2 \leqslant i,j \leqslant n}$ is the second fundamental form of ∂M relative to e_1. Substituting (3.5) into (3.4), we obtain

$$0 \leqslant \frac{\partial G}{\partial x_1}(x_0) = -\sum_{i,j=2}^{n} \theta_i(x_0) h_{ij}(x_0) \theta_j(x_0) \leqslant 0 \tag{3.6}$$

since $(h_{ij})_{2 \leqslant i,j \leqslant n}$ is nonnegative (i.e., ∂M is weakly convex). Hence, $\theta_i(x_0) = 0$ for $2 \leqslant i \leqslant n$. By (3.3), we have $\nabla\theta(x_0) = 0$. Finally, $\nabla G(x_0) = 0$ follows from (3.1) and (3.6). This completes the proof. $\square$

As previously pointed out [5], the estimate of the upper bound of $|\nabla\theta|^2$ plays an important role in the estimate of the lower bound for λ_1. In the following we establish a rough estimate for $|\nabla\theta|^2$.

Lemma 3.2 ([5]). *Assume that* $\mathrm{Ric}(M) \geqslant 0$. *The other assumption is as in Theorem 1.8. In any case the following estimate is valid:*

$$|\nabla\theta(x)|^2 \leqslant \lambda_1(1+a), \quad \forall\, x \in M. \tag{3.7}$$

Moreover,

$$F(\theta) \leqslant \lambda_1(1+a). \tag{3.8}$$

Proof. Suppose that $|\nabla\theta|^2$ attains its local maximum at x_0. Clearly, (2.6) and (2.7) imply that (3.7) holds in the case $x_0 \in \Sigma_*$. Without loss of generality, we can assume that $x_0 \in M \setminus \Sigma_*$ in the rest of the proof, and thus $\theta_0 = \theta(x_0) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. When $\partial M \neq \emptyset$, we know from Lemma 3.1 that $\nabla G(x_0) = 0$ if $x_0 \in \partial M \setminus \Sigma_*$. According to the maximum principle, the maximality of $G(x_0)$ implies that at x_0,

$$\nabla(|\nabla\theta|^2) = 0 \quad \text{and} \quad \Delta(|\nabla\theta|^2) \leqslant 0, \tag{3.9}$$

regardless of $x_0 \in \partial M \setminus \Sigma_*$ or $x_0 \in M \setminus (\partial M \cup \Sigma_*)$. Applying the Bochner formula to θ, we have

$$\frac{1}{2}\,\Delta(|\nabla\theta|^2) = \left|\nabla^2\theta\right|^2 + \nabla\theta\cdot\nabla(\Delta\theta) + \mathrm{Ric}(\nabla\theta,\nabla\theta), \tag{3.10}$$

where $\mathrm{Ric}(\nabla\theta,\nabla\theta)$ is the Ricci curvature along $\nabla\theta$. Substituting (2.5) into (3.10), we have

$$\begin{aligned}\frac{1}{2}\,\Delta(|\nabla\theta|^2) &= \left|\nabla^2\theta\right|^2 + \nabla\theta\cdot\nabla\left[\frac{\sin\theta}{\cos\theta}\cdot|\nabla\theta|^2 - \frac{\lambda_1(\sin\theta + a)}{\cos\theta}\right] + \mathrm{Ric}(\nabla\theta,\nabla\theta)\\ &= \left|\nabla^2\theta\right|^2 + \nabla\theta\cdot\nabla\left(\frac{\sin\theta}{\cos\theta}\right)\cdot|\nabla\theta|^2 + \nabla\theta\cdot\frac{\sin\theta}{\cos\theta}\cdot\nabla(|\nabla\theta|^2)\\ &\quad - \lambda_1\cdot\nabla\theta\cdot\left[\nabla\left(\frac{\sin\theta}{\cos\theta}\right) + a\cdot\nabla\left(\frac{1}{\cos\theta}\right)\right] + \mathrm{Ric}(\nabla\theta,\nabla\theta).\end{aligned} \tag{3.11}$$

A direct calculation leads to

$$\nabla\left(\frac{\sin\theta}{\cos\theta}\right) = \frac{\nabla(\sin\theta)\cdot\cos\theta - \sin\theta\cdot\nabla(\cos\theta)}{\cos^2\theta} = \frac{1}{\cos^2\theta}\cdot\nabla\theta \tag{3.12}$$

and

$$\nabla\left(\frac{1}{\cos\theta}\right) = \frac{-1}{\cos^2\theta}\cdot(-\sin\theta)\cdot\nabla\theta = \frac{\sin\theta}{\cos^2\theta}\cdot\nabla\theta. \tag{3.13}$$

Substituting (3.12) and (3.13) into (3.11), we obtain

$$\frac{1}{2}\,\Delta(|\nabla\theta|^2) = \left|\nabla^2\theta\right|^2 + \frac{|\nabla\theta|^4}{\cos^2\theta} + \nabla\theta\cdot\frac{\sin\theta}{\cos\theta}\cdot\nabla(|\nabla\theta|^2) - \frac{\lambda_1(1 + a\sin\theta)}{\cos^2\theta}\cdot|\nabla\theta|^2 + \mathrm{Ric}(\nabla\theta,\nabla\theta). \tag{3.14}$$

Under the assumption that $\mathrm{Ric}(M) \geqslant 0$, by virtue of (3.9) and noting that $\left|\nabla^2\theta\right|^2 \geqslant 0$, we deduce from (3.14) that at x_0,

$$0 \geqslant \frac{|\nabla\theta|^4}{\cos^2\theta} - \frac{\lambda_1(1 + a\sin\theta)}{\cos^2\theta}\cdot|\nabla\theta|^2.$$

Dividing by $|\nabla\theta|^2$ and multiplying by $\cos^2\theta$ successively, it follows that at x_0,

$$0 \geqslant |\nabla\theta|^2 - \lambda_1(1 + a\sin\theta).$$

Hence, we have

$$|\nabla\theta(x_0)|^2 \leqslant \lambda_1(1 + a\sin\theta_0) \leqslant \lambda_1(1 + a),$$

which implies the desired conclusion. □

4. Estimate of $F(\theta)$

In the following, we assume $0 < a < 1$. We want to obtain a more precise estimate of $F(\theta)$ than in Lemma 3.2. For this purpose, we introduce the function $\phi(\theta) : M \mapsto \mathbb{R}$ such that

$$F(\theta) = \lambda_1[1 + a\phi(\theta)]. \tag{4.1}$$

By Lemma 3.2, it is easy to see that $\phi(\theta) \leqslant 1$. Conversely,

$$1 + a\phi \geqslant \frac{|\nabla\theta|^2}{\lambda_1} \geqslant 0.$$

From now on we denote

$$\alpha = \frac{(n-1)K}{\lambda_1 a}. \tag{4.2}$$

It follows from (1.2) that

$$0 < \alpha \leqslant \frac{n-1}{na} < \frac{1}{a}.$$

We also need the following lemma to accurately estimate $\phi(\theta)$.

Lemma 4.1. *Assume that* $\mathrm{Ric}(M) \geqslant (n-1)K$ *and the other conditions in Theorem 1.8 hold. If* $h : \left[-\frac{\pi}{2}, -\frac{\pi}{2}\right] \mapsto \mathbb{R}$ *is a function that satisfies the properties*

(1) *h is nondecreasing, that is, $h'(\theta) \geqslant 0$ for all $\theta \in \left[-\frac{\pi}{2}, -\frac{\pi}{2}\right]$;*
(2) *$h(\theta) \geqslant \phi(\theta)$; and*
(3) *There exists some $\theta_0 \in (-\frac{\pi}{2}, -\frac{\pi}{2})$, such that $h(\theta_0) = \phi(\theta_0) \geqslant -1$, then the following estimate holds:*

$$\phi(\theta_0) \leqslant \sin\theta_0 - \sin\theta_0 \cdot \cos\theta_0 \cdot h'(\theta_0) + \frac{\cos^2\theta_0}{2} \cdot h''(\theta_0) - \alpha\cos^2\theta_0. \tag{4.3}$$

Proof. Let

$$f(x) = \frac{1}{2}\left\{|\nabla\theta(x)|^2 - \lambda_1\left[1 + ah\left(\theta(x)\right)\right]\right\}.$$

Obviously, $f(x) \leqslant 0$ for all $x \in M$. By (2.8), we know that there exists some $x_0 \in M \setminus \Sigma_*$ such that $\theta(x_0) = \theta_0$ and $F(\theta_0) = |\nabla\theta(x_0)|^2$. Thus, f achieves its maximum 0 at x_0. More precisely,

$$|\nabla\theta(x_0)|^2 = \lambda_1[1 + a\phi(\theta_0)] = \lambda_1[1 + ah(\theta_0)]. \tag{4.4}$$

By the same argument as in the proof of Lemma 3.2, we easily obtain that at x_0,

$$\nabla f = 0 \quad \text{and} \quad \Delta f \leqslant 0, \tag{4.5}$$

regardless of $x_0 \in \partial M \setminus \Sigma_*$ or $x_0 \in M \setminus (\partial M \cup \Sigma_*)$. Direct computation shows that

$$f_j = \sum_i \theta_i \cdot \theta_{ij} - \frac{\lambda_1 a}{2} h'(\theta) \cdot \theta_j,$$

namely,

$$\nabla f = \frac{1}{2}\left[\nabla(|\nabla\theta|^2) - \lambda_1 a h'(\theta) \cdot \nabla\theta\right] = \nabla\theta \cdot \nabla^2\theta - \frac{\lambda_1 a}{2} h'(\theta) \cdot \nabla\theta.$$

Since $\nabla f = 0$ at x_0,

$$\nabla(|\nabla\theta|^2) = 2\nabla\theta \cdot \nabla^2\theta = \lambda_1 a h'(\theta_0) \cdot \nabla\theta \quad \text{at } x_0. \tag{4.6}$$

By directly calculating and applying (2.5), we obtain

$$\begin{aligned}
\frac{1}{2}\Delta\left[\lambda_1(1 + ah)\right] &= \frac{1}{2}\sum_j \left[\lambda_1(1 + ah)\right]_{jj} = \frac{\lambda_1 a}{2}\sum_j \left(h' \cdot \theta_j\right)_j \\
&= \frac{\lambda_1 a}{2}\sum_j (h'' \cdot \theta_j^2 + h' \cdot \theta_{jj}) = \frac{\lambda_1 a}{2}(h'' \cdot |\nabla\theta|^2 + h' \cdot \Delta\theta) \\
&= \frac{\lambda_1 a}{2}\left\{h'' \cdot |\nabla\theta|^2 + h' \cdot \left[\frac{\sin\theta}{\cos\theta} \cdot |\nabla\theta|^2 - \frac{\lambda_1(\sin\theta + a)}{\cos\theta}\right]\right\}.
\end{aligned} \tag{4.7}$$

Combining (3.14) with (4.7), we obtain

$$\begin{aligned}
\Delta f &= \left|\nabla^2\theta\right|^2 + \frac{|\nabla\theta|^4}{\cos^2\theta} + \nabla\theta \cdot \frac{\sin\theta}{\cos\theta} \cdot \nabla(|\nabla\theta|^2) \\
&\quad - \frac{\lambda_1(1 + a\sin\theta)}{\cos^2\theta} \cdot |\nabla\theta|^2 + \mathrm{Ric}(\nabla\theta, \nabla\theta) \\
&\quad - \frac{\lambda_1 a}{2}\left\{h'' \cdot |\nabla\theta|^2 + h' \cdot \left[\frac{\sin\theta}{\cos\theta} \cdot |\nabla\theta|^2 - \frac{\lambda_1(\sin\theta + a)}{\cos\theta}\right]\right\}.
\end{aligned}$$

Recall that $\mathrm{Ric}(\nabla\theta, \nabla\theta) \geqslant (n-1)K\,|\nabla\theta|^2$, so we can obtain

$$\begin{aligned}
\Delta f &= \left|\nabla^2\theta\right|^2 + \frac{|\nabla\theta|^4}{\cos^2\theta} + \nabla\theta \cdot \frac{\sin\theta}{\cos\theta} \cdot \nabla(|\nabla\theta|^2) \\
&\quad - \frac{\lambda_1(1 + a\sin\theta)}{\cos^2\theta} \cdot |\nabla\theta|^2 + (n-1)K\,|\nabla\theta|^2 \\
&\quad - \frac{\lambda_1 a}{2}\left\{h'' \cdot |\nabla\theta|^2 + h' \cdot \left[\frac{\sin\theta}{\cos\theta} \cdot |\nabla\theta|^2 - \frac{\lambda_1(\sin\theta + a)}{\cos\theta}\right]\right\}.
\end{aligned} \tag{4.8}$$

Substituting (4.6) into (4.8), it is easy to deduce that

$$\begin{aligned}\Delta f &= \left|\nabla^2\theta\right|^2 + \frac{|\nabla\theta|^4}{\cos^2\theta} + \lambda_1 a h' \cdot \frac{\sin\theta}{\cos\theta}\cdot|\nabla\theta|^2 \\ &\quad - \frac{\lambda_1(1+a\sin\theta)}{\cos^2\theta}\cdot|\nabla\theta|^2 + (n-1)K\,|\nabla\theta|^2 \\ &\quad - \frac{\lambda_1 a}{2}\left\{h''\cdot|\nabla\theta|^2 + h'\cdot\left[\frac{\sin\theta}{\cos\theta}\cdot|\nabla\theta|^2 - \frac{\lambda_1(\sin\theta+a)}{\cos\theta}\right]\right\}. \qquad (4.9)\end{aligned}$$

By virtue of (4.4) and (4.5), we derive from (4.9) that at x_0,

$$\begin{aligned}0 &\geqslant \left|\nabla^2\theta\right|^2 + \frac{\lambda_1^2(1+ah)^2}{\cos^2\theta} + \lambda_1^2 a h'(1+ah)\frac{\sin\theta}{\cos\theta} \\ &\quad - \lambda_1^2(1+ah)\frac{1+a\sin\theta}{\cos^2\theta} + \lambda_1(1+ah)(n-1)K \\ &\quad - \frac{\lambda_1 a}{2}\left\{h''\cdot\lambda_1(1+ah) + h'\cdot\left[\frac{\sin\theta}{\cos\theta}\cdot\lambda_1(1+ah) - \frac{\lambda_1(\sin\theta+a)}{\cos\theta}\right]\right\} \\ &= \left|\nabla^2\theta\right|^2 + \frac{\lambda_1^2(1+ah)^2}{\cos^2\theta} - \lambda_1^2(1+ah)\frac{1+a\sin\theta}{\cos^2\theta} + \lambda_1(1+ah)(n-1)K \\ &\quad + \frac{\lambda_1^2 a}{2}\left\{-h''(1+ah) + h'\cdot\left[\frac{\sin\theta}{\cos\theta}(1+ah) + \frac{(\sin\theta+a)}{\cos\theta}\right]\right\}. \qquad (4.10)\end{aligned}$$

Obviously the first term above can be dropped since it is nonnegative. Dividing by $\lambda_1^2 a$, multiplying by $\cos^2\theta$ and rearranging the terms successively, we obtain

$$\begin{aligned}0 &\geqslant \frac{(1+ah)^2}{a} - \frac{(1+ah)(1+a\sin\theta)}{a} - \frac{h''(1+ah)\cos^2\theta}{2} \\ &\quad + \frac{h'\cos\theta}{2}[(1+ah)\sin\theta + (\sin\theta+a)] + (1+ah)\alpha\cos^2\theta \\ &= (1+ah)(h-\sin\theta) - \frac{h''(1+ah)\cos^2\theta}{2} \\ &\quad + \frac{h'\cos\theta}{2}[(1+ah)\sin\theta + (\sin\theta+a)] + (1+ah)\alpha\cos^2\theta. \qquad (4.11)\end{aligned}$$

Since $h(\theta_0) = \phi(\theta_0) \geqslant -1$ and $\phi(\theta_0) = \phi(\theta(x_0)) \leqslant 1$, then $|h(\theta_0)| \leqslant 1$.

From $|h| = |h(\theta)| \leqslant 1$ at x_0 and $0 < a < 1$, it follows that at x_0,

$$a \geqslant ah\sin\theta \quad \text{and} \quad 1 + ah > 0.$$

Thus, at x_0,

$$\sin\theta + a \geqslant \sin\theta + ah\sin\theta = (1+ah)\sin\theta. \qquad (4.12)$$

Hence, under the assumption that $h'(\theta) \geqslant 0$, using (4.12), we proceed by tackling with (4.11) at x_0 as follows:

$$0 \geqslant (1+ah)(h-\sin\theta) + h'(1+ah)\cos\theta\sin\theta - \frac{h''(1+ah)\cos^2\theta}{2} + (1+ah)\alpha\cos^2\theta.$$

Dividing by $1+ah$, at x_0 we have

$$0 \geqslant (h-\sin\theta) + h'\cos\theta\sin\theta - \frac{h''\cos^2\theta}{2} + \alpha\cos^2\theta, \qquad (4.13)$$

from which (4.3) follows easily. This completes the proof. □

The remainder of the paper follows the literature [5,7,9,23]. For completeness, we sketch a brief proof of Theorem 1.9 below that uses published methods [5,7,9]. Interested readers can consult the relevant references for more details.

Lemma 4.2 (*[7,9]*). *Let*

$$\xi(\theta) = \frac{\cos^2\theta + 2\theta\sin\theta\cos\theta + \theta^2 - \frac{\pi^2}{4}}{\cos^2\theta} \quad \text{in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \qquad (4.14)$$

and $\xi\left(\pm\frac{\pi}{2}\right)=0$. *Then the function* ξ *satisfies*

$$L\xi \equiv \frac{\cos^2\theta}{2}\cdot\xi'' - \cos\theta\sin\theta\cdot\xi' - \xi = 2\cos^2\theta \quad in \left(-\frac{\pi}{2},\frac{\pi}{2}\right). \tag{4.15}$$

Moreover, ξ *also has the following properties:*

$$\xi(-\theta)=\xi(\theta), \quad \forall\,\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right);$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\xi(\theta)\mathrm{d}\theta = 2\int_0^{\frac{\pi}{2}}\xi(\theta)\mathrm{d}\theta = -\pi.$$

Lemma 4.3 ([5]). *We define the function* η *as*

$$\begin{cases}\eta(\theta)=\dfrac{\frac{4}{\pi}(\theta+\cos\theta\sin\theta)-2\sin\theta}{\cos^2\theta}, & \theta\in\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right),\\ \eta\left(-\dfrac{\pi}{2}\right)=-1, \qquad \eta\left(\dfrac{\pi}{2}\right)=1. & \end{cases} \tag{4.16}$$

Then $\eta\in C^0\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\cap C^2\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ *satisfies* $\eta'(\theta)\geqslant 0$ *and*

$$L\eta \equiv \frac{\cos^2\theta}{2}\cdot\eta''(\theta) - \cos\theta\sin\theta\cdot\eta'(\theta) - \eta(\theta) = -\sin\theta \quad in \left(-\frac{\pi}{2},\frac{\pi}{2}\right).$$

Moreover, η *also has the following properties:*

$$|\eta(\theta)|\leqslant 1;$$

$$\eta(-\theta)=-\eta(\theta), \quad \forall\,\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right].$$

Lemma 4.4 ([9]). *The function* $r(\theta)=\xi'(\theta)/\eta'(\theta)$ *is increasing on* $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, *that is,* $r'(\theta)>0$, *and* $|r(\theta)|\leqslant\frac{\pi^2}{4}$ *holds on* $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

Corollary 4.1. *Let*

$$\psi(\theta)=\frac{\mu\alpha}{2}\cdot\xi(\theta)+\eta(\theta), \tag{4.17}$$

where $\mu\in(0,1]$ *is a constant. Then* ψ *satisfies*

$$L\psi \equiv \frac{\cos^2\theta}{2}\cdot\psi''(\theta) - \cos\theta\sin\theta\cdot\psi'(\theta) - \psi(\theta) = \mu\alpha\cos^2\theta - \sin\theta \quad in \left(-\frac{\pi}{2},\frac{\pi}{2}\right). \tag{4.18}$$

Moreover, assume that $\frac{\mu\alpha\pi^2}{8}\leqslant 1$, *that is,* $\lambda_1\geqslant\frac{\pi^2\mu(n-1)K}{8a}$. *Then*

$$\psi'(\theta)\geqslant 0, \quad \forall\,\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right] \tag{4.19}$$

and

$$-1=\psi\left(-\frac{\pi}{2}\right)\leqslant\psi(\theta)\leqslant\psi\left(\frac{\pi}{2}\right)=1, \quad \forall\,\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]. \tag{4.20}$$

Proof. Eq. (4.18) can be directly verified. In addition, by Lemmas 4.3–4.4, we easily obtain

$$\psi'(\theta)=\frac{\mu\alpha}{2}\cdot\xi'(\theta)+\eta'(\theta)=\eta'(\theta)\left(\frac{\mu\alpha}{2}\cdot\frac{\xi'(\theta)}{\eta'(\theta)}+1\right)\geqslant\eta'(\theta)\left(1-\frac{\mu\alpha\pi^2}{8}\right)\geqslant 0.$$

Thus, $\psi(\theta)$ is an increasing function on $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and (4.20) follows easily from this result. This completes the proof. □

Using Lemma 4.1, Corollary 4.1 and the reduction to absurdity, we can easily prove the following conclusion. For the reader's convenience, we provide a proof below taken from the literature [5,7,9,25].

Lemma 4.5. *Assume that* $\phi(\theta)$ *and* $\psi(\theta)$ *are defined by* (4.1) *and* (4.17), *respectively. Then*

$$\phi(\theta)\leqslant\psi(\theta). \tag{4.21}$$

Proof. Assume that (4.21) is not true. Since $\phi\left(\pm\frac{\pi}{2}\right) = \pm 1 = \psi\left(\pm\frac{\pi}{2}\right)$, then there exists some $\theta_0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that

$$\sigma = \phi(\theta_0) - \psi(\theta_0) = \max_{-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}} \{\phi(\theta) - \psi(\theta)\} > 0. \tag{4.22}$$

Let $\widetilde{h}(\theta) = \psi(\theta) + \sigma$. Obviously, $\widetilde{h}'(\theta) = \psi'(\theta) \geqslant 0$,

$$\widetilde{h}(\theta) = \psi(\theta) + \sigma \geqslant \phi(\theta)$$

and

$$\widetilde{h}(\theta_0) = \phi(\theta_0) = \psi(\theta_0) + \sigma \geqslant -1 + \sigma > -1.$$

Replacing $h(\theta)$ in Lemma 4.1 by $\widetilde{h}(\theta)$, by Lemma 4.1 and Corollary 4.1 we obtain

$$\begin{aligned}
\phi(\theta_0) &\leqslant \sin\theta_0 - \sin\theta_0 \cdot \cos\theta_0 \cdot \widetilde{h}'(\theta_0) + \frac{\cos^2\theta_0}{2} \cdot \widetilde{h}''(\theta_0) - \alpha\cos^2\theta_0 \\
&= \sin\theta_0 - \sin\theta_0 \cdot \cos\theta_0 \cdot \psi'(\theta_0) + \frac{\cos^2\theta_0}{2} \cdot \psi''(\theta_0) - \alpha\cos^2\theta_0 \\
&\leqslant \sin\theta_0 - \sin\theta_0 \cdot \cos\theta_0 \cdot \psi'(\theta_0) + \frac{\cos^2\theta_0}{2} \cdot \psi''(\theta_0) - \mu\alpha\cos^2\theta_0 \\
&= \psi(\theta_0).
\end{aligned}$$

However, this contradicts (4.22), which completes the proof. □

Corollary 4.2. *The following estimate holds:*

$$F(\theta) \leqslant \lambda_1[1 + a\psi(\theta)], \tag{4.23}$$

where $F(\theta)$ and $\psi(\theta)$ are defined by (2.8) and (4.17), respectively.

Our argument above establishes the inequality (4.23), which is an improved estimate of the upper bound for $F(\theta)$ as required.

5. Proof of Theorem 1.9

Following previous arguments [7,9], we now use the estimate of $F(\theta)$ to prove Theorem 1.9 as follows.

Proof. (4.23) implies that

$$\sqrt{\lambda_1} \geqslant \sqrt{\frac{|F(\theta)|}{1 + a\psi(\theta)}} \geqslant \frac{|\nabla\theta|}{\sqrt{1 + a\psi(\theta)}}, \tag{5.1}$$

where $\psi(\theta)$ is defined by (4.16).

Take $x_1, x_2 \in M$ such that $\theta(x_1) = -\frac{\pi}{2}$ and $\theta(x_2) = \frac{\pi}{2}$. Let d' denote the length of the shortest curve γ that connects x_1 with x_2 on M. Let d be the diameter of M. Clearly, $d' \leqslant d$. Using (4.3) and integrating (5.1) along the curve, we derive γ such that

$$\begin{aligned}
\sqrt{\lambda_1}d &\geqslant \sqrt{\lambda_1}d' = \int_\gamma \sqrt{\lambda_1}\mathrm{d}s \geqslant \int_\gamma \frac{1}{\sqrt{1 + a\psi(\theta)}}|\nabla\theta|\,\mathrm{d}s \\
&\geqslant \int_\gamma \frac{1}{\sqrt{1 + a\psi(\theta)}}\mathrm{d}\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\sqrt{1 + a\psi(\theta)}}\mathrm{d}\theta \\
&\geqslant \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}\theta\right)^{\frac{3}{2}} \Bigg/ \left\{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [1 + a\psi(\theta)]\mathrm{d}\theta\right\}^{\frac{1}{2}} \\
&= \pi^{\frac{3}{2}} \Bigg/ \left\{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [1 + a\psi(\theta)]\mathrm{d}\theta\right\}^{\frac{1}{2}}.
\end{aligned}$$

Thus, we have

$$\lambda_1 \geqslant \frac{\pi^3}{d^2} \Bigg/ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [1 + a\psi(\theta)]\mathrm{d}\theta.$$

However,

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}[1+a\psi(\theta)]\mathrm{d}\theta &= \pi+\frac{\mu a\alpha}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\xi(\theta)\mathrm{d}\theta+a\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\eta(\theta)\mathrm{d}\theta\\ &= \pi-\frac{\mu a\alpha}{2}\pi.\end{aligned}$$

Hence, we easily obtain

$$\lambda_1 \geqslant \frac{1}{1-\frac{\mu a\alpha}{2}}\cdot\frac{\pi^2}{d^2},$$

or, equivalently,

$$\lambda_1\left(1-\frac{\mu a\alpha}{2}\right)\geqslant\frac{\pi^2}{d^2}.$$

Therefore, we obtain

$$\lambda_1 \geqslant \frac{\pi^2}{d^2}+\lambda_1\frac{\mu a\alpha}{2}=\frac{\pi^2}{d^2}+\frac{\mu(n-1)K}{2}.$$

This is the required estimate, which completes the proof. □

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