

Accepted Manuscript

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Paul Baird

PII: S0393-0440(13)00152-6

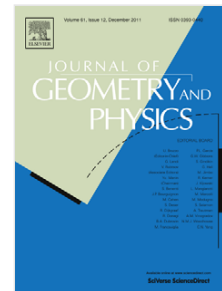
DOI: <http://dx.doi.org/10.1016/j.geomphys.2013.08.005>

Reference: GEOPHY 2240

To appear in: *Journal of Geometry and Physics*

Received date: 18 May 2012

Accepted date: 11 August 2013



Please cite this article as: P. Baird, Emergence of geometry in a combinatorial universe, *Journal of Geometry and Physics* (2013), <http://dx.doi.org/10.1016/j.geomphys.2013.08.005>

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EMERGENCE OF GEOMETRY IN A COMBINATORIAL UNIVERSE

PAUL BAIRD

ABSTRACT. Our objective is to construct an elementary universe populated entirely by graphs from which geometry and dynamics emerge. The universe is based on a binary relation between objects: are they connected by an edge or not? It is only this relation that matters; the nature of the objects being irrelevant. Our perspective is that, out of the graphs so formed, further implicit structure is present, defined by a geometric spectrum and corresponding fields. This implicit structure comes into play when graphs correlate. Thus we propose ways in which graphs can interact and so dynamics, that is *change*, appears. With a suitable definition of time, this change can be ordered to give a universe endowed with local geometry and time.

1. INTRODUCTION

The setting for this article is that of graph theory, whereby a binary relation exists between vertices, which we represent by the existence or otherwise of a connecting edge. Quantum computing provides an illustration of how certain quantum processes can be translated into binary data [11]. The idea that combinatorial structure could be at the basis of the description of matter, was put forward in 1971 by R. Penrose [12].

Apart from its combinatorial structure, a graph carries further implicit information. We begin by discussing how geometry can be represented in terms of a complex-valued field defined on a graph. A graph together with such a field φ now satisfies a local lifting property which we discuss in Section 3. Thus we may lift a given vertex and its neighbours to what we refer to as an *invariant configured star* in a Euclidean space. This is a rigid framework with one internal vertex and n external vertices, with the property that it satisfies equation (2) below at

2000 *Mathematics Subject Classification.* 05C10, 52C99, 52B11, 39A14.

Key words and phrases. Discrete geometry, Theorem of Axonometry, regular polytope, finite graph, quadratic difference equation, orthogonal projection, body-bar framework, emergent phenomena.

The author is grateful for support provided by the Australian Research Council and to the Mathematical Sciences Institute at the Australian National University for support and hospitality.

the central vertex independently of any dilation or scaling of the framework. In essence, the presence of the field φ enables us to attach a “best-fit” polytope to the graph at the vertex in question. Apart from special cases, this lift is unique up to a two-fold ambiguity; but independently of this ambiguity, edge-length and a star axis are well-defined, which leads to notions of distance and curvature.

Our perspective is that the graph carries with it an ensemble of such fields, which are encoded in what we call the *geometric spectrum* (Section 2). No geometry is present as such: the graph is not in any geometric state. Geometry emerges upon *correlation* between graphs. Here, correlation between graphs means simply that a multiply connected graph changes following some specified rules. By analogy with quantum mechanics, in order to correlate, two (or more) connected components must fall into a particular geometric state. The connected components we refer to as *particles*, so that correlation simulates particle interaction. From a sufficiently complex combinatorial structure, geometry should now emerge as a statistically dominant state.

2. THE GEOMETRIC SPECTRUM

The Theorem of Axonometry of Gauss [8] states that the end points of three line segments emanating from the origin in 3-dimensional Euclidean space \mathbb{R}^3 correspond to adjacent vertices of a cube if and only if their images z_1, z_2, z_3 under orthogonal projection to the complex plane satisfy:

$$z_1^2 + z_2^2 + z_3^2 = 0,$$

and conversely, any three complex numbers satisfying this equation, arise in this way, from orthogonal projection of adjacent vertices of one of two possible cubes. One way to rewrite this is as follows.

Consider a cube placed in \mathbb{R}^3 and let $\varphi : \mathbb{R}^3 \rightarrow \mathbb{C}$ be any orthogonal projection onto the complex plane, for example $\varphi(x_1, x_2, x_3) = x_1 + ix_2$. Let V be the collection of points in \mathbb{R}^3 corresponding to the vertex set of the cube. Then φ satisfies the equation

$$\sum_{y \sim x} (\varphi(y) - \varphi(x))^2 = 0$$

for each $x \in V$, where $y \sim x$ means that y is adjacent to x along an edge of the cube.

The converse is not quite true. Thus if φ satisfies the above equation, it may not arise as the orthogonal projection of the vertices of a cube. Rather, we only have local lifting of a vertex and its neighbours.

The above construction can be generalized to more general polytopes [7]. For example, if z_1, z_2, \dots, z_{N+1} are the orthogonal projections to \mathbb{C} of the vertices of a regular simplex in \mathbb{R}^N , then

$$(z_1 + \dots + z_{N+1})^2 = (N+1)(z_1^2 + \dots + z_{N+1}^2).$$

Thus if V is the vertex set of a regular simplex in \mathbb{R}^N and φ is the function that associates to each element of V its value in the complex plane after orthogonal projection, then φ satisfies the equation:

$$(1) \quad \frac{1}{N+1} \left(\sum_{y \sim x} (\varphi(y) - \varphi(x)) \right)^2 = \sum_{y \sim x} (\varphi(y) - \varphi(x))^2.$$

The same applies to all the regular polytopes, where the factor $1/(N+1)$ is replaced by some other constant [7, 1].

From a physical point of view, the above equation has some appealing aspects. Firstly, it is invariant under any Euclidean motion, or any dilation of the simplex. In particular, absolute position and size have no relevance, which is as we would wish in a universe where only *relational* properties have significance. Secondly, the geometry of the simplex is contained in the combinatorial relationship between the vertices of the underlying graph and the coefficient $1/(N+1)$ that occurs on the left-hand side of the equation, as we explain more fully below. Our perspective is to regard the underlying background space \mathbb{R}^N as a secondary object and equation (1) as a primary object, from which geometry emerges. Let us put this into a more formal setting.

Given a graph $\Gamma = (V, E)$, with vertex set V and edge set E , together with a real-valued function $\gamma : V \rightarrow \mathbb{R}$, consider the equation:

$$(2) \quad \frac{\gamma(x)}{n(x)} \left(\sum_{y \sim x} (\varphi(y) - \varphi(x)) \right)^2 = \sum_{y \sim x} (\varphi(x) - \varphi(y))^2,$$

at each vertex x , where $\varphi : V \rightarrow \mathbb{C}$ is a complex-valued function and $n(x)$ is the degree of Γ at x (the number of vertices adjacent to x). The special case when γ vanishes identically has been considered in a combinatorial formulation of twistor theory, where solutions are referred to as *holomorphic functions* [3]. Note that the equations are invariant by the replacement of φ by $\tilde{\varphi} = \lambda\varphi + \mu$ where λ and μ are complex numbers, as well as with respect to complex conjugation $\varphi \mapsto \bar{\varphi}$.

It is convenient to write $\Delta\varphi(x) = \frac{1}{n(x)} \sum_{y \sim x} (\varphi(y) - \varphi(x))$ (the Laplacian) and $(d\varphi)^2(x) = \frac{1}{n(x)} \sum_{y \sim x} (\varphi(y) - \varphi(x))^2$ (the symmetric square of the derivative), whereby equation (2) has the more economic expression:

$$\gamma(x)\Delta\varphi(x)^2 = (d\varphi)^2(x).$$

All our graphs will be *simple*, that is we do not allow loops or multiple edges, although much of what we discuss can be generalized to non-simple graphs. We are interested in graphs which possess a latent geometry that will arise from an embedding in Euclidean space. Such objects are traditionally called body-bar frameworks, or just frameworks.

A *framework* \mathcal{F} in \mathbb{R}^N is a finite collection of points $\{\vec{x}_1, \dots, \vec{x}_n\}$ connected by edges which are straight line segments. We shall continue to use the notation $\mathcal{F} = (V, E)$ to distinguish the vertices and edges of the underlying graph. The edges are often called *bars* and such body-bar frameworks have been studied in respect of questions about rigidity [5].

Let $\varphi : \mathbb{R}^N \rightarrow \mathbb{C}$ be an orthogonal projection. Then we say that the framework is *invariant* if φ satisfies (2) for some fixed function $\gamma : V \rightarrow \mathbb{R}$, independently of any orthogonal transformation, translation, or dilation of the framework. Specifically, if $A : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is an orthogonal transformation, then we require $\varphi_A = \varphi \circ A|_V : V \rightarrow \mathbb{C}$ also satisfy (2) with corresponding γ_A satisfying $\gamma_A(A\vec{x}) = \gamma(\vec{x})$ for all $\vec{x} \in V$. Note that if (2) is satisfied, this always remains the case if the framework is translated or dilated, since this just corresponds to a transformation $\varphi \mapsto \lambda\varphi + \mu$ (now with λ real).

Before proceeding, we note the following elementary lemma which justifies our bound on the geometric spectrum defined below.

Lemma 1. *Suppose the equation:*

$$\frac{\gamma}{n} \left(\sum_{\ell=1}^n z_{\ell} \right)^2 = \sum_{\ell=1}^n z_{\ell}^2,$$

is satisfied for γ real and for z_{ℓ} real and not all zero. Then $\gamma \geq 1$. In particular, if φ solves (2) at a vertex x of degree n ; if $n \geq 2$ and $\gamma < 1$, then in any normalization which has $\varphi(x) = 0$, provided φ does not vanish on all neighbouring vertices, at least one of $\varphi(y)$ ($y \sim x$) must be complex.

Proof. The Cauchy-Schwarz inequality shows that for any set $\{a_1, \dots, a_n\}$ of complex numbers, one has the inequality

$$(3) \quad n \sum_{\ell=1}^n a_{\ell} \bar{a}_{\ell} \geq \left(\sum_{\ell} a_{\ell} \right) \left(\sum_{\ell} \bar{a}_{\ell} \right)$$

with equality if and only if $a_1 = a_2 = \dots = a_n$. Then for z_{ℓ} real and not all zero satisfying (2), we have:

$$\sum_{\ell} z_{\ell}^2 = \frac{\gamma}{n} \left(\sum_{\ell} z_{\ell} \right)^2 \leq \gamma \sum_{\ell} z_{\ell}^2.$$

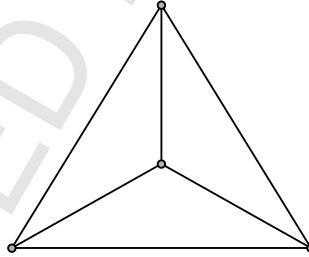
□

For a given graph $\Gamma = (V, E)$, we define the *geometric spectrum* of Γ to be the collection of equivalence classes of functions:

$$\Sigma = \{\gamma : V \rightarrow [-\infty, 1] : \exists \text{ non-const. } \varphi : V \rightarrow \mathbb{C} \text{ satisfying } \gamma(\Delta\varphi)^2 = d\varphi^2\},$$

where two functions are identified when they determine a common solution φ and agree on the complement of the set $\{x \in V : \Delta\varphi(x) = (d\varphi)^2(x) = 0\}$.

Note that we allow γ to take on the values $-\infty$ and 1 ; the case of planar polygons discussed below provides justification for this extension. This ‘spectrum’ may be discrete, or have continuous components. For example, the complete graph¹ on three vertices has geometric spectrum consisting of a single constant function: $\Sigma = \{\gamma \equiv 2/3\}$ corresponding to the invariant planar framework defined by an equilateral triangle. Similarly, the complete graph on four vertices has geometric spectrum consisting of the single constant function: $\Sigma = \{\gamma \equiv 3/4\}$ corresponding to the invariant 3-dimensional framework defined by a regular tetrahedron. In this case we have a special position of the tetrahedron which projects to the planar graph illustrated below. At the central vertex both $\Delta\varphi$ and $(d\varphi)^2$ vanish.



A polygonal configuration in the plane satisfies (2) if and only if its edges have equal length. In this case $\gamma(x) = 2 \cos \theta(x) / (\cos \theta(x) - 1)$, where $\theta(x) \in [-\pi, \pi]$ is the external angle at vertex x . Note that γ can take on the value 1 when $\theta = \pm\pi$, or $-\infty$ when $\theta = 0$. The geometric spectrum now has continuous components, corresponding to continuous deformations of such a polygon. It should be noted that objects with a high degree of symmetry such as the regular polytopes, all correspond to solutions of (2) with γ constant.

3. THE LIFTING PROBLEM AND THE EMERGENCE OF GEOMETRY

By a *star* $K_{1,n}$, we mean a graph with one internal vertex connected to n external vertices, with no other connections. Consider a star $K_{1,n}$ embedded

¹A *complete graph* is one for which every vertex is connected to every other vertex by an edge

in \mathbb{R}^N in the following way. Let (y_1, \dots, y_N) be standard coordinates for \mathbb{R}^N ($N \geq 2$); write vectors as columns for the purpose of matrix multiplication. Let $\{\vec{e}_1, \dots, \vec{e}_N\}$ be the canonical basis and write I_N for the $N \times N$ -identity matrix. The internal vertex \vec{x}_0 is located at the origin, while the external vertices $\vec{x}_1, \dots, \vec{x}_n$ are situated at distinct points in the hyperplane $y_N = c$ (constant):

$$(4) \quad \vec{x}_\ell = \begin{pmatrix} \vec{v}_\ell \\ c \end{pmatrix} \quad (\ell = 1, \dots, n),$$

We require further that the $(N-1) \times n$ -matrix $U = (\vec{v}_1 | \vec{v}_2 | \dots | \vec{v}_n)$ with columns the components $v_{\ell j}$ of \vec{v}_ℓ ($j = 1, \dots, N-1$; $\ell = 1, \dots, n$), satisfies:

$$(5) \quad UU^t = \rho I_{N-1}, \quad \sum_{\ell=1}^n \vec{v}_\ell = \vec{0},$$

for some non-zero constant ρ (necessarily positive), where $\vec{0}$ denotes the zero vector in \mathbb{R}^{N-1} and U^t denotes the transpose of U .

Any star, which, up to orthogonal transformation of \mathbb{R}^N is embedded in this way, we will call a *configured star*. We shall also say that the vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$ form a *configuration in \mathbb{R}^{N-1}* and call U the associated *configuration matrix*. Provided the star does not lie in any proper linear subspace, we say that the star is *full*. If further, $\|\vec{x}_\ell\| = r$ (constant) for $\ell = 1, \dots, n$, we refer to the star as *regular of radius r* . An *invariant* of the star is a quantity which is invariant under orthogonal transformation.

Lemma 2. *Consider a configured star in \mathbb{R}^N ($N \geq 2$) with internal vertex the origin connected to n external vertices $\{\vec{x}_1, \dots, \vec{x}_n\}$ ($n \geq N$). Let $W = (\vec{x}_1 | \vec{x}_2 | \dots | \vec{x}_n)$ be the $N \times n$ -matrix whose columns are the components $x_{\ell j}$ of \vec{x}_ℓ ($j = 1, \dots, N$; $\ell = 1, \dots, n$). Then*

$$(6) \quad WW^t = \rho I_N + \sigma \vec{u} \vec{u}^t, \quad \sum_{\ell=1}^n \vec{x}_\ell = \sqrt{n(\sigma + \rho)} \vec{u},$$

where $\vec{u} \in \mathbb{R}^N$ is a unit vector called the *axis* of the star, $\rho > 0$ and $\rho + \sigma > 0$. The quantities n, ρ, σ are all invariants of the star; the vector \vec{u} is normal to the affine plane containing $\vec{x}_1, \dots, \vec{x}_n$.

Conversely, any matrix $W = (\vec{x}_1 | \vec{x}_2 | \dots | \vec{x}_n)$ satisfying (6) determines a configured star with central vertex the origin and external vertices $\vec{x}_1, \dots, \vec{x}_n$.

Proof. Consider a configured star in standard position given by (4) and (5). Set

$$V = \left(\begin{array}{c|c|c|c} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ \hline c & c & \dots & c \end{array} \right)$$

and let $A : \mathbb{R}^N \rightarrow \mathbb{R}^N$ be an orthogonal transformation; set $\vec{x}_n = A \begin{pmatrix} \vec{v}_n \\ c \end{pmatrix}$.

Then $W = (\vec{x}_1 | \vec{x}_2 | \cdots | \vec{x}_n) = AV$ and

$$WW^t = AVV^t A^t = \rho I_N + \sigma(A\vec{e}_N)(A\vec{e}_N)^t,$$

where

$$(7) \quad \sigma = nc^2 - \rho.$$

Furthermore $\sum_{\ell=1}^n \vec{x}_\ell = ncA\vec{e}_N$, which gives the form (6) with $\vec{u} = A\vec{e}_N$. The independence of the quantities n, ρ, σ under the orthogonal transformation A is clear.

Conversely, suppose we are given an $N \times n$ -matrix $W = (\vec{x}_1 | \vec{x}_2 | \cdots | \vec{x}_n)$ satisfying (6). Let A be an orthogonal transformation such that $A\vec{u} = \vec{e}_N$ and let $V = AW$. Write

$$V = \left(\begin{array}{c|c|c|c} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ \hline y_{1N} & y_{2N} & \cdots & y_{nN} \end{array} \right).$$

Then

$$(8) \quad VV^t = \rho I_N + \sigma \vec{e}_N \vec{e}_N^t \quad \text{and} \quad \sum_{\ell} \begin{pmatrix} \vec{v}_\ell \\ y_{\ell N} \end{pmatrix} = \sqrt{n(\sigma + \rho)} \vec{e}_N,$$

so that $\sum_{\ell} \vec{v}_\ell = 0$ and $\sum_{\ell} y_{\ell N} = \sqrt{n(\sigma + \rho)}$. Furthermore, (8) implies that $\sum_{\ell} y_{\ell N}^2 = \rho + \sigma$. In particular

$$n \sum_{\ell} y_{\ell N}^2 = \left(\sum_{\ell} y_{\ell N} \right)^2.$$

But then (3) implies that $y_{1N} = y_{2N} = \cdots = y_{nN} = \sqrt{(\sigma + \rho)/n}$. \square

As a consequence, the function which assigns the values after projection of the vertices of a configured star to the complex plane satisfies (2) at the internal vertex, independently of the position of the star.

Corollary 3. *Let $W = (\vec{x}_1 | \vec{x}_2 | \cdots | \vec{x}_n)$ define a configured star and let $\varphi : \mathbb{R}^N \rightarrow \mathbb{C}$ be orthogonal projection $\varphi(y_1, \dots, y_N) = y_1 + iy_N$. Then if $z_\ell = \varphi(\vec{x}_\ell) = x_{\ell 1} + ix_{\ell 2}$, we have*

$$(9) \quad \frac{\sigma}{n(\sigma + \rho)} \left(\sum_{\ell=1}^n z_\ell \right)^2 = \sum_{\ell=1}^n z_\ell^2,$$

where ρ and σ are given by (6). In particular, $\gamma = \sigma/(\sigma + \rho)$ is real and depends only on the star invariants.

Proof. Let $\vec{u} = (u_1, \dots, u_N)$ be the unit normal to the plane of the star. Then for each $j = 1, \dots, N$, we have

$$\sum_{\ell=1}^n x_{\ell j} = \sqrt{n(\sigma + \rho)} u_j.$$

Thus

$$\begin{aligned} \left(\sum_{\ell=1}^n z_{\ell} \right)^2 &= \sum_{k,\ell=1}^n (x_{k1}x_{\ell 1} - x_{k2}x_{\ell 2} + 2ix_{k1}x_{\ell 2}) \\ &= n(\sigma\rho)(u_1^2 - u_2^2 + 2iu_1u_2) = n(\sigma\rho)(u_1 + iu_2)^2, \end{aligned}$$

whereas

$$\begin{aligned} \sum_{\ell=1}^n z_{\ell}^2 &= \sum_{\ell=1}^n (x_{\ell 1}^2 - x_{\ell 2}^2 + 2ix_{\ell 1}x_{\ell 2}) \\ &= (WW^t)_{11} - (WW^t)_{22} + 2i(WW^t)_{12} \\ &= \sigma(u_1 + iu_2)^2. \end{aligned}$$

The formula now follows. \square

Given a solution φ to (2), at each vertex x , our aim is to construct a configured star in some Euclidean space \mathbb{R}^N whose external vertices project to the points $\varphi(y) - \varphi(x)$ ($y \sim x$) of the complex plane. To do this, we establish a converse to Corollary 3. We shall refer to the problem of constructing such a star as *the lifting problem*. At a vertex of degree three with φ holomorphic, this is the Theorem of Axonometry of Gauss [8]. It turns out that provided $\gamma < 1$, the lifting problem can always be solved. When $N = 3$, modulo a special case that we make precise below, the configured star is unique up to a sign ambiguity. Relative edge-length and so relative distance on the graph, can now be defined in terms of virtual configured stars.

Fix a vertex x of degree n and label its neighbours y_1, \dots, y_n . Set $z_{\ell} = \varphi(y_{\ell}) - \varphi(x)$ ($\ell = 1, \dots, n$), which we suppose not all zero. From (2):

$$(10) \quad \frac{\gamma}{n} \left(\sum_{\ell=1}^n z_{\ell} \right)^2 = \sum_{\ell=1}^n z_{\ell}^2 \quad (\gamma \in \mathbb{R}).$$

For a given N with $2 \leq N \leq n$, we wish to construct a configured star $W = (\vec{x}_1|\vec{x}_2|\dots|\vec{x}_n)$ in \mathbb{R}^N with z_{ℓ} the orthogonal projection of \vec{x}_{ℓ} . For convenience,

write $z_\ell = x_{\ell 1} + ix_{\ell 2} = \alpha_\ell + i\beta_\ell$, so that

$$W = \left(\begin{array}{c|c|c|c} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \beta_1 & \beta_2 & \cdots & \beta_n \\ x_{13} & x_{23} & \cdots & x_{n3} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1N} & x_{2N} & \cdots & x_{nN} \end{array} \right).$$

For $N \geq 3$, we are required to solve the system:

$$(11) \quad WW^t = \rho I_N + \sigma \vec{u} \vec{u}^t, \quad \sum_{\ell=1}^n \vec{x}_\ell = \sqrt{n(\sigma + \rho)} \vec{u},$$

for $x_{\ell j}$ ($\ell = 1, \dots, n$; $j = 3, \dots, N$), $\rho > 0$, σ such that $\rho + \sigma > 0$ and $\vec{u} \in \mathbb{R}^N$ unit, with $\gamma = \sigma/(\sigma + \rho)$. This is a matter of linear algebra which we now detail for the case $N = 3$.

Let $\{z_1, \dots, z_n; \gamma\}$ be a non-trivial solution to (10) satisfying $\gamma < 1$. Set

$$(12) \quad \rho = \frac{1}{2} \sum_{\ell} z_{\ell} \bar{z}_{\ell} - \frac{\gamma}{2n} \left(\sum_{\ell} z_{\ell} \right) \left(\sum_{\ell} \bar{z}_{\ell} \right) > 0,$$

and

$$(13) \quad \sigma = \frac{\gamma \rho}{1 - \gamma} \quad (\Rightarrow \sigma + \rho = \rho/(1 - \gamma) > 0).$$

Define

$$(14) \quad u_1 = \frac{1}{\sqrt{n(\sigma + \rho)}} \sum_{\ell=1}^n \alpha_{\ell}, \quad u_2 = \frac{1}{\sqrt{n(\sigma + \rho)}} \sum_{\ell=1}^n \beta_{\ell};$$

and let $u_3 = \sqrt{1 - u_1^2 - u_2^2}$. Set

$$A := \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \beta_1 & \beta_2 & \cdots & \beta_n \\ 1 & 1 & \cdots & 1 \end{pmatrix}, \quad X := \begin{pmatrix} x_{13} \\ x_{23} \\ \vdots \\ x_{n3} \end{pmatrix}.$$

Then (11) is equivalent to solving

$$(15) \quad AX = B := u_3 \begin{pmatrix} \sigma u_1 \\ \sigma u_2 \\ \sqrt{n(\sigma + \rho)} \end{pmatrix}$$

subject to the constraint:

$$(16) \quad X^t X = \rho I_{N-2} + \sigma u_3^2.$$

It is important to note the sign ambiguity: the equations are invariant under the simultaneous replacement of u_3 by $-u_3$ and of X by $-X$. This ambiguity represents two choices for the configured star.

When A has maximal rank 3, the system (15) and (16) has the unique solution $X = A^+B$, where $A^+ = A^t(AA^t)^{-1}$ (together with the sign ambiguity discussed above). If the rows of A are dependent then AA^t is no longer invertible and $u_1^2 + u_2^2 = 1 \Rightarrow u_3 = 0$, so we are required to solve the system $AX = 0$ with the constraint $X^tX = \rho$. There is now a 1-parameter family of solutions. This case occurs if and only if the complex numbers z_ℓ in (10) satisfy

$$n \sum_{\ell=1}^n |z_\ell|^2 + (\gamma - 2) \left| \sum_{\ell=1}^n z_\ell \right|^2 = 0.$$

The solution to the local lifting problem given above enables us to define edge-length and so distance on a graph $\Gamma = (V, E)$ admitting a solution to (2), provided that at each vertex we have $\gamma < 1$.

Specifically, given an edge \overline{xy} (now directed) connecting vertex x to vertex y , we can define its *length* $\ell(\overline{xy})$ *relative to the endpoint* x to be the length of the corresponding edge of the lifted configured star at x . However, it may be that $\ell(\overline{xy}) \neq \ell(\overline{yx})$. In this case, we can define the length to be the mean of its lengths relative to x and to y . Note that edge-length so defined is *relative* to the solution φ of (2), which is only defined up to $\varphi \mapsto \lambda\varphi + \mu$ for $\lambda, \mu \in \mathbb{C}$, so that the only meaningful quantities are *relative* lengths between edges.

Curvature can be defined, either at a vertex by analogy with the classical notion for polyhedra in terms of angular deficiency: $\delta = 2\pi - \sum_i \theta_i$ where the θ_i are angles between successive edges. Thus given a vertex x and a lift of x and its neighbours to a configured star in \mathbb{R}^3 , we can define the *vertex-curvature* at x to be the quantity $\delta(x) = 2\pi - \sum_i \theta_i$ where now the θ_i are the angles between the edges of the star. For this to be well-defined, we require an ordering of the edges.

A natural curvature, which we call *edge-curvature* is defined for an edge e joining two vertices x and y at which we have defined a unique choice of configured star in \mathbb{R}^3 . Let \vec{u} and \vec{v} be the axes of the stars at x and y , respectively (see Lemma 2); then define the *edge-curvature* $\theta(e)$ to be the angle between them. Define the *normal curvature* of e to be the quantity $k(e) = \theta/\ell(e)$, where $\ell(e)$ is the length of e as we have defined it above. This notion gives a combinatorial version of normal curvature as it occurs in smooth differential geometry, where at a point of a curve in a submanifold of Euclidean space, it is defined to be the reciprocal of the radius of a circle with identical curvature.

A striking aspect of this approach is the way it corresponds to how we, as sentient beings, observe the world. Information is transmitted to our 2-dimensional retinæ via light rays, which is then processed by our brain to reconstruct a 3-dimensional world. The cube is a particular case: when we see the projection of its 1-skeleton on the flat page of a piece of paper, we perceive rather the 3-dimensional object.

Our point of view henceforth is to suppose that a graph carries around its latent geometry encoded in the geometric spectrum. This is implicit to the graph and depends only on its combinatorial structure. However, at this point no geometry has emerged. For this we require a graph to fall into a particular state which occurs upon *correlation*; we discuss this in the next section.

4. PARTICLES AND STATES

The initial data for our universe is a graph Γ made up of a finite number of connected components $\Gamma_1, \Gamma_2, \dots, \Gamma_K$. In addition to the binary relation between vertices (whether or not they are connected by an edge), there is an additional relation between them: whether or not they belong to the same connected component of Γ . A priori there is no reason to give greater emphasis to the property that two vertices be connected by an edge, rather than that they are not so connected. The representation by drawing an edge just gives a convenient way to visualize the relation.

A *particle* is a connected component Γ_k of Γ . A *state of the particle* is an equivalence class of solutions (φ, γ) ($\gamma \leq 1$) to equation (2) on Γ_k , where two solutions $(\varphi_1, \gamma_1) \sim (\varphi_2, \gamma_2)$ if and only if (i) $\varphi_2 = \lambda\varphi_1 + \mu$ for $\lambda, \mu \in \mathbb{C}$ with $\lambda \neq 0$ and (ii) $\gamma_1(x) = \gamma_2(x)$ at all vertices x where $\Delta\varphi_1(x) \neq 0$. A member of an equivalence class will be called a *representative state* and we shall write $[\varphi]$ for the equivalence class determined by the state φ (for given γ). A state for which γ is constant will be called an *isostate*. We allow point particles, consisting of a single vertex with all elements of \mathbb{C} as representative states.

A particle is not deemed to be in any state, but carries with it, its ensemble of states. By analogy with quantum mechanics, when two particles correlate, each falls into a particular state; that is, states are chosen with a certain probability. For a cyclic graph, isostates occur as critical states of the energy functional $\mathcal{E} = \sum_k (1 + \cos \theta_k)$ where θ_k measures the exterior angle at vertex k , further these states include the regular star polygons [1]. Such empirical evidence suggest isostates may be favoured. After correlation, further states may be present in the combined graph, permitting new correlations with other graphs that were not possible prior to correlation. An *evolution* of the universe is a sequence

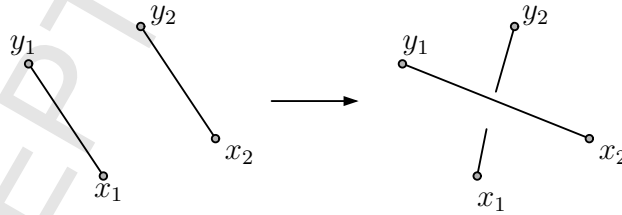
$\Gamma \mapsto \Gamma' \mapsto \Gamma'' \mapsto \dots$ of graphs, whereby a subsequent graph is obtained from the previous one by specific rules to be defined. The order of the sequence is dictated by the rules. There are three *changes* in our universe that we now specify: correlation between particles; internal mutation of a particle; separation of a particle into two or more particles.

(i) *Correlation.* A *correlation* between K particles $\Gamma_\ell = (V_\ell, E_\ell)$ ($\ell = 1, \dots, K$) (with disjoint vertex sets) is a new particle $\Gamma = \Gamma_1 \star \Gamma_2 \star \dots \star \Gamma_K = (V = V_1 \cup V_2 \cup \dots \cup V_K, E)$, i.e. vertices are preserved but the edge set may change. For a correlation to occur, we require Γ_ℓ be in a state $[\varphi_\ell]$ ($\ell = 1, \dots, K$), Γ be in a state $[\varphi]$ such that $\varphi|_{V_\ell} \in [\varphi_\ell]$ ($\ell = 1, \dots, K$). After correlation, the new particle carries its ensemble of states and is not considered to be in any particular state. Correlation between isostates can occur as follows.

Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be two graphs with disjoint vertex sets endowed with isostates with identical γ ; thus we have two functions $\varphi_\ell : V_\ell \rightarrow \mathbb{C}$ ($\ell = 1, 2$) which both satisfy (2) with identical γ constant. Let $x_1 \sim y_1$ in Γ_1 and $x_2 \sim y_2$ in Γ_2 be such that

$$\begin{aligned} \text{either} \quad & \text{(i)} \quad \varphi_1(x_1) \neq \varphi_1(y_1) \text{ and } \varphi_2(x_2) \neq \varphi_2(y_2); \\ \text{or} \quad & \text{(ii)} \quad \varphi_1(x_1) = \varphi_1(y_1) \text{ and } \varphi_2(x_2) = \varphi_2(y_2). \end{aligned}$$

Normalize φ_2 such that $\varphi_2(y_2) = \varphi_1(y_1)$ and $\varphi_2(x_2) = \varphi_1(x_1)$. Consider the new graph formed from the union of the vertices: $V = V_1 \cup V_2$, where we replace the edge $\overline{x_1 y_1}$ by $\overline{x_1 y_2}$ and the edge $\overline{x_2 y_2}$ by $\overline{x_2 y_1}$. All other edges remain fixed.



Now define $\varphi : V_1 \cup V_2 \rightarrow \mathbb{C}$ by $\varphi(x) = \varphi_1(x)$ for $x \in V_1$ and $\varphi(x) = \varphi_2(x)$ for $x \in V_2$. Note that the degree of each vertex has been preserved. Then clearly φ satisfies (2) with constant γ identical to those of the isostates before correlation. An example of the correlation of two isostates corresponding to the invariant framework given by a regular tetrahedron is illustrated below. The resulting isostate after correlation can be realised as a non-embedded invariant framework consisting of two tetrahedra identified along a common edge (which as a framework is considered as two distinct edges). In this example $\gamma = 3/4$.

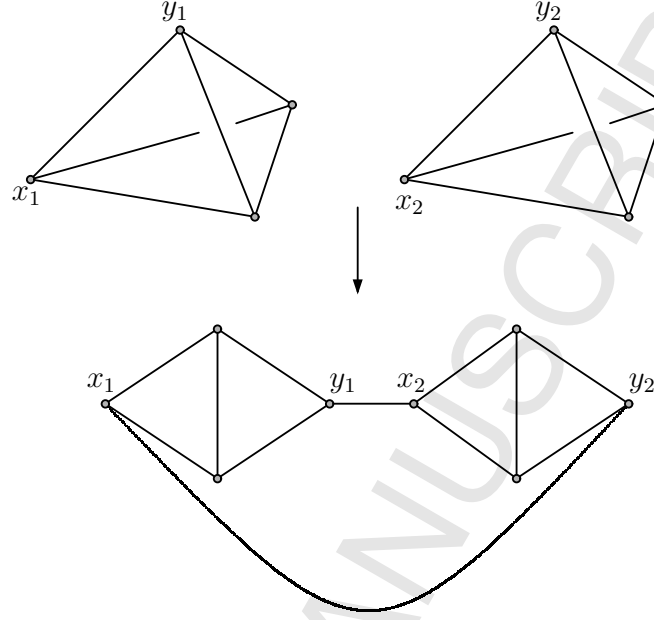


Fig. 1. An example of correlation: the two particles at the top correlate to form a single particle; all are in isostates arising from invariant frameworks with $\gamma = 3/4$.

The analogous example consisting of the correlation of two triangles rather than two tetrahedra corresponds to the hexagon in the isostate with $\gamma = 2/3$ realised as two circuits of a triangle. This highlights a similarity with the procedure from Riemann surface theory, whereby we cut and glue copies of a surface endowed with a multi-valued holomorphic function to obtain a new surface endowed with a (single-valued) holomorphic function. Clearly in the above discussion, we can replace the graphs Γ_1 and Γ_2 with K graphs, all in isostates with identical γ , and proceed to replace edges $\overline{x_1 y_1}$ with $\overline{x_1 y_2}$, $\overline{x_2 y_2}$ with $\overline{x_2 y_3}$, \dots , $\overline{x_K y_K}$ with $\overline{x_K y_1}$.

The correlation between different particles in isostates with identical γ , leads to particles in isostates whose underlying graph has arbitrarily high order and complexity (the latter word not used in any precise sense here).

Separation. A *separation* of a particle is a dissociation of $\Gamma = (V, E)$ into K particles $\Gamma_\ell = (V_\ell, E_\ell)$ ($\ell = 1, \dots, K$), with Γ a correlation of the Γ_ℓ . For separation to occur, we require Γ be in a state $[\varphi]$, Γ_ℓ be in a state $[\varphi_\ell]$ with $\varphi|_{V_\ell} \in [\varphi_\ell]$. An example of separation is given as follows. Two concentric triangles connected by edges as indicated, falls into the state corresponding to the spectral value $\gamma = 1$. The edges joining vertices on which the field has a common value are removed to give two disjoint triangles.

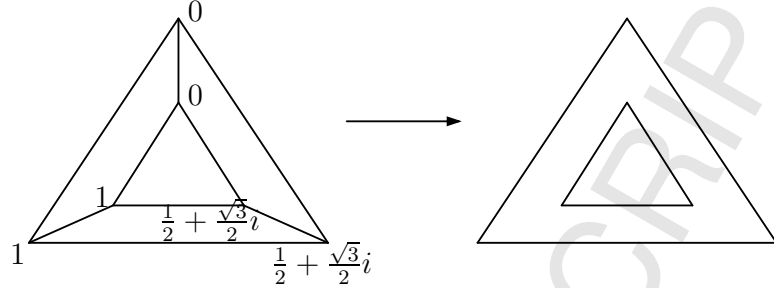


Fig.2. An example of separation

Mutation. A *mutation* of a particle $\Gamma = (V, E) \rightarrow \Sigma = (W, F)$, where Σ is a new particle with $V = W$. For mutation to occur, we require Γ be in a state $[\varphi]$, Σ be in a state $[\psi]$ with $\varphi \in [\psi]$. Collapsing is a particular example of mutation provided it does not disconnect the particle, whereby we remove edges that connect vertices on which φ takes on the same value. If collapsing disconnects the particle, it falls into the category of separation. Note that collapsing doesn't in general preserve isostates, since the degree of a vertex may change. An example of mutation is illustrated in the following figure.

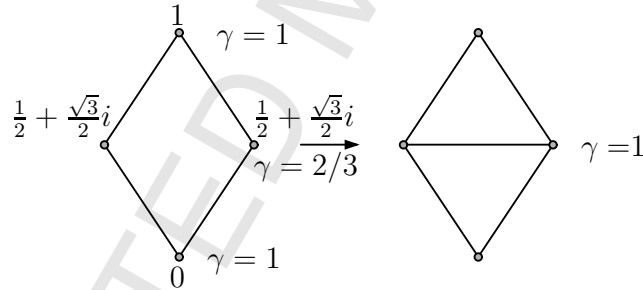


Fig. 3. The particle on the left mutates into an isostate by the addition of an edge.

Another illustration is given by the final particle of Figure 1, which, after mutation, produces the 1-skeleton of a cube.

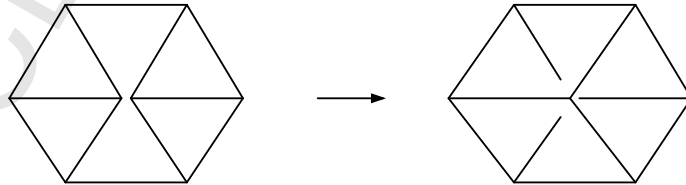


Fig. 4. Mutation occurs when two edges "flip" to connect the middle outer vertices to different central vertices.

The resulting change of state can be realised as a 3-dimensional invariant framework embedded in Euclidean space.

Example 4. A particular evolution of a simple universe is as follows. First take the universe Γ consisting of two copies of the left-hand particle of Figure 3. These then mutate to form Γ' consisting of two copies of the right-hand particle of Figure 3. A correlation then occurs to form the final particle of Figure 1; denote this by Γ'' . Finally a mutation takes place as in Figure 4 to form Γ''' . We view this evolution as irreversible in the sense that if we begin with the 1-skeleton of the cube, given that all isostates with $\gamma = 0$ are equally probable, the probability that it fall into a state φ which has identical values on two diagonally opposite vertices to enable the reciprocal mutation of Figure 4 to occur, would be zero.

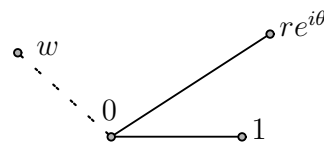
A desirable objective would be to produce a complex universe from a simple initial state. One way to accomplish this is to suppose the existence of virtual point particles that are susceptible to correlate with existing particles. We now explore this possibility in more detail.

Consider a graph $\Gamma = (V, E)$ together with a function $\varphi : V \rightarrow \mathbb{C}$ not necessarily a solution to (2). For ease of representation, suppose that each vertex $x \in V$ be placed at its corresponding position $\varphi(x)$ in the complex plane. We fix our attention on a particular vertex, x_0 say, which by translation, we suppose placed at the origin. Suppose x_0 has k neighbours placed at z_1, \dots, z_k . We now wish to add a new vertex placed at w and to join it to x_0 in such a way as to satisfy (2) at x_0 . Specifically, we wish to consider the locus of points w which can be placed in this way. Since the mapping

$$(17) \quad w \mapsto \frac{z_1^2 + \dots + z_k^2 + w^2}{(z_1 + \dots + z_k + w)^2},$$

is in general holomorphic in w , we expect a 1-parameter family of values of w for which the right-hand side is real.

Example 5. Consider the graph on three vertices as indicated below. It may be that the extremal vertices placed at $re^{i\theta}$ and 1 are joined to other vertices, but for the moment we are just interested in satisfying (2) at the origin.



If we set $w = u + iv$, then it is a routine computation to show that the identity $\gamma(1 + re^{i\theta} + w)^2 = 1 + r^2e^{2i\theta} + w^2$ has γ real if and only if the following algebraic

equation of degree three in u and v is satisfied:

$$(18) \quad \begin{aligned} & (ur \sin \theta - v(1 + r \cos \theta))(u^2 + v^2) + r \sin \theta(u^2 - v^2) - 2uvr \cos \theta \\ & + r \sin \theta(1 - r^2 - 2r \cos \theta)u + (1 + r \cos \theta + r^3 \cos \theta + r^2 \cos 2\theta)v \\ & + r(1 - r^2) \sin \theta = 0. \end{aligned}$$

There are two cases when the solution set can be explicitly written down:

(i) $\theta = \pi/2$. Equation (18) now becomes:

$$(19) \quad (ru - v)(u^2 + v^2) + r(u^2 - v^2) + r(1 - r^2)u + (1 - r^2)v + r(1 - r^2) = 0.$$

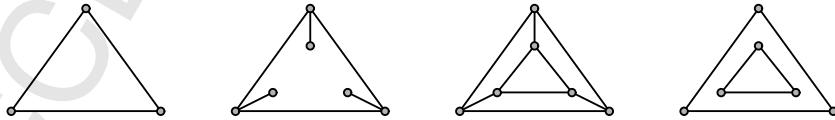
For each $r \neq 0, 1$, this is a smooth curve except at the singular point $(u, v) = (-1, -r)$. When $r = 1$ (so the original graph is holomorphic at the origin), this gives the algebraic set:

$$(u - v)(u^2 + v^2 + u + v) = 0,$$

consisting of the union of the line $u = v$ and the circle $(u + \frac{1}{2})^2 + (v + \frac{1}{2})^2 = \frac{1}{2}$.

A variant on the above procedure is to consider two particles Γ and Σ and to add a new vertex x which correlates with both particles, joining it to x_0 in Γ and y_0 in Σ , say. In order to correlate, suppose Γ falls into state $[\varphi]$ and Σ falls into state $[\psi]$. Now we require that (2) be satisfied at the new vertex x . This is the case if and only if $|\varphi(x) - \varphi(x_0)| = |\psi(y) - \psi(y_0)|$ for representative states. This can be further generalized by creating a new vertex x and attempting to join several vertices to x . Such correlations can lead to discrete phenomena when we combine the various constraints. That is, the various loci determined by the real solutions to (17) will in general intersect in a discrete set of points.

The possibility that point particles may attach themselves to existing (more complex) particles, can lead to duplication and eventually a complex universe. The following sequence of correlation, mutation and separation gives an example of duplication.



We begin with a triangle on the left-hand side. Three isolated vertices then attach themselves in a symmetric way. This must be done so as to preserve the property that equation (2) remain satisfied. Symmetry is preferred since this leads to an isostate. In fact, if the vertices of the left-hand triangle have representative field values $0, 1, \frac{1}{2} + \frac{\sqrt{3}}{2}i$, then the point particle connected to 0 should have representative field value $\frac{3}{2} + \frac{\sqrt{3}}{2}i$, with the other point particles similarly assigned

values to give an isostate with $\gamma = 1$ (so the figure is misleading if we view the field as the position function, but avoids crossing edges). A mutation now occurs whereby the new vertices are joined by edges in the way shown. This produces an non-isostate particle with $\gamma = 7/9$ at the new vertices (with γ still equal to 1 at the original vertices), which then falls into an isostate given by the example on the left-hand side of Figure 2, with spectral value $\gamma = 1$, once more. Finally, separation occurs as in Figure 2.

Time. Time is an ordering on a sequence of universes: (Γ, Γ', \dots) . The ordering must be compatible with the rules for change. Thus $\Gamma^{(j+1)}$ must derive from $\Gamma^{(j)}$ by correlation, separation, mutation, or correlation with virtual point particles. There are two ways to decide such an ordering:

(i) The rules for change: these may determine an irreversible process, such as that given in Example 4. The order $(\Gamma, \Gamma', \Gamma'', \Gamma''')$ is determined by the irreversibility of $\Gamma'' \rightarrow \Gamma'''$.

(ii) A statistical parameter. The *thermal time hypothesis* has been developed by Connes and Rovelli [6]. This is based on the Tomita flow associated to a von Neumann algebra. In quantum field theory, the appropriate von Neumann algebra is the closure of the algebra of observables. Then, given a state of a system over this algebra there is always a flow by which the state evolves and we may call this the “flow of time”. In our context, we don’t have an obvious von Neumann algebra that we can exploit. However, there are various parameters that we may consider.

Graph entropy is a well-know concept based on a probability distribution associated to the vertices [9]. Entropy is of course intimately related to the second law of thermodynamics. Intrinsic curvature is also a natural parameter that occurs in smooth Riemannian geometry and curvature flow provides a way by which a manifold may evolve into one of uniform structure.

Curvature can provide a measure of local concentrations of structure. In general, the various notions of curvature we have described depend on the field φ satisfying (2). However, given a particle in a particular state, we could envisage processes whereby the graph could evolve to uniformize the curvature – say the vertex curvature or the edge curvature.

For the purpose of illustration, a simple curvature which depends only on the combinatorial structure is given, for a graph $\Gamma = (V, E)$ with degree function $n : V \rightarrow \mathbb{N}$, by the function $n(x) - 2$. That this could be considered as a measure of curvature appears to have first been suggested by H. Urakawa [13]. In [2], P. Baird and M. Tiba provide an algorithm for sliding edges in order to minimize

the quantity $\sum_{x \in V} (n(x) - 2)^2$ subject to $\sum_{x \in V} n(x)$ remaining constant. This procedure preserves connectedness and may be viewed as a discrete analogue of the scalar curvature flow in Riemannian geometry. This leads to the parameter

$$t(\Gamma) := \frac{\sqrt{\{\sum_{x \in V} (n(x) - 2)^2\}}}{2|E|},$$

as a simple measure of time. In Example 4, the passage from $\Gamma \rightarrow \Gamma'$ increases t , whereas the passage from $\Gamma' \rightarrow \Gamma''$ decreases t , but we don't preclude local increases in t . Indeed, time should be a statistically *dominant* parameter that appears at a macroscopic level.

5. THE SCHRÖDINGER EQUATION AND QUANTUM GRAPHS

To conclude, we note some tentative connections to well-established physical models. In the case of a graph of degree three, it is shown by P. Baird and M. Wehbe that equation (2) with γ vanishing identically provides a discrete analogue of the equation for a shear-free ray congruence on space-time [3]. In the smooth setting such congruences can be used to generate solutions to the zero rest-mass field equations.

Let $\Gamma = (V, E)$ be a finite graph. For $x \in V$ define the *tangent space to Γ at x* to be the set of oriented edges with base point x : $T_x \Gamma = \{\vec{xy} : y \sim x\}$. Define the *tangent bundle to Γ* to be the union: $T\Gamma = \cup_{x \in V} T_x \Gamma$. Then a 1-form on Γ is a map $\omega : T\Gamma \rightarrow \mathbb{C}$ such that $\omega(\vec{xy}) = -\omega(\vec{yx})$. To a function $\varphi : V \rightarrow \mathbb{C}$, we can naturally associate a 1-form, the *derivate* $d\varphi$, by $d\varphi(\vec{xy}) = \varphi(y) - \varphi(x)$.

For two 1-forms ω, η , define their *pointwise symmetric product at $x \in V$* by

$$\langle \omega, \eta \rangle_x = \sum_{y \sim x} \omega(\vec{xy}) \eta(\vec{xy}),$$

and their *(global) symmetric product* by

$$(\omega, \eta) = \sum_{e \in E} \omega(e) \eta(e) = \frac{1}{2} \sum_{x \in V} \sum_{y \sim x} \omega(\vec{xy}) \eta(\vec{xy}).$$

Note that in the first sum the 1-forms act on unoriented edges so that only their product is well-defined; the factor of one half occurs in the second sum, since there, unoriented edges are counted twice.

For functions $\varphi, \psi : V \rightarrow \mathbb{C}$, define their *(global) symmetric product* by

$$(\varphi, \psi) = \sum_{x \in V} n(x) \varphi(x) \psi(x),$$

where $n(x)$ is the degree of vertex x .

The above definitions are the complex symmetric analogues of standard L^2 products that arise in functional analytic theory on a graph; in the latter situation they are replaced by Hermitian products rather than symmetric products [4].

It is now a simple matter to show that if $\varphi : V \rightarrow \mathbb{C}$ is a solution to equation (2) with γ constant, then

$$(\gamma\Delta(\Delta\varphi) + 2n\Delta\varphi + \frac{1}{n}\langle dn, d\varphi \rangle, \varphi) = 0.$$

This suggests a heuristic argument as to why we might consider a pair (Γ, φ) consisting of a connected graph endowed with a solution φ to (2) with γ constant as a particle with mass inversely proportional to $|\gamma|$; in the case when $\gamma = 0$, we can view the pair as representing a massless particle.

Firstly, we do not admit any fixed background with respect to which we can define parameters of equations: the particle creates its own background, so we view an equation of the form $(\mathcal{P}(\varphi), \varphi)$ as appropriate, where \mathcal{P} is some (discrete) differential operator. In the case when n is constant, we now note the relation between the operator $\mathcal{P}(\varphi) = \gamma\Delta(\Delta\varphi) + 2n\Delta\varphi$ and the operator on the left-hand side of the time-independent Schrödinger equation on a fixed smooth background:

$$\left(-\frac{\hbar^2}{2m}\vec{\nabla}^2 + V(x)\right)\psi = E\psi(x),$$

when ψ is identified with $\Delta\varphi$. The case of mass zero (when $\gamma \equiv 0$) is justified in some detail in [3].

A metric graph is a graph each of whose edges are endowed with a length. A quantum graph is a metric graph together with a solution to the 1-dimensional Schrödinger equation along each edge with a compatibility condition at each vertex, see [10]. A particle in a particular state as we have defined it (as a connected graph endowed with a solution φ to (2)), then has a striking resemblance to such an object, except we have dispensed with what we would view as the artificial imposition of a metric structure.

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LABORATOIRE DE MATHÉMATIQUES DE BRETAGNE ATLANTIQUE UMR 6205,
 UNIVERSITÉ DE BRETAGNE OCCIDENTALE 6 AV. VICTOR LE GORGEU – CS 93837,
 29238 BREST CEDEX, FRANCE
E-mail address: Paul.Baird@univ-brest.fr