



Biharmonic holomorphic maps into Kähler manifolds

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ABSTRACT

We study biharmonic holomorphic maps from an almost Hermitian manifold into a Kähler manifold. First, by a simple observation of the curvature term in the biharmonic equation, we establish non-existence results of biharmonic holomorphic maps into Kähler manifolds with non-positive holomorphic bisectional curvature, which extend the similar results of biharmonic maps between Riemannian manifolds. Second, by applying the second variation formula of biharmonic maps, we prove a non-existence result of stable biharmonic holomorphic maps into complex projective space.

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1. Introduction

The theory of harmonic maps plays an important role in geometry. Let (M, g) and (N, h) be two Riemannian manifolds, for smooth maps $\varphi : M \rightarrow N$, the *energy functional* is defined by

$$E(\varphi) = \frac{1}{2} \int_M |d\varphi|^2 dv_g,$$

where dv_g denotes the volume element of g . Harmonic maps are critical points of E . On the other hand, the Euler–Lagrange equation of E is $\tau(\varphi) = \text{trace} \nabla d\varphi = 0$, where $\tau(\varphi)$ is called the *tension field* of φ . A map $\varphi : M \rightarrow N$ is called a *harmonic map* if $\tau(\varphi) = 0$.

In 1983, Eells and Lemaire [1] proposed the problem to consider the biharmonic maps which are critical maps of the *bienergy functional*

$$E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 dv_g.$$

It is a generalization of harmonic maps. In 1986, Jiang [2] derived the first and the second variation formulas of the bienergy functional, and he also proved that when M is compact and N has non-positive sectional curvature, every biharmonic map $\varphi : M \rightarrow N$ is harmonic. This is not true when M is non-compact. During the past decades, many efforts have been devoted to find conditions to ensure that biharmonic maps are harmonic (see [3–8]).

Now we talk about holomorphic maps. It is well known that holomorphic maps between compact Kähler manifolds are harmonic. When the Kähler structure is removed, holomorphic maps will not be harmonic in general. Lichnerowicz [9]

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investigated conditions on almost complex structures (a cosymplectic domain and (1,2)-symplectic target) ensuring that holomorphic maps are harmonic. Following Lichnerowicz's approach, Benyounes, Loubeau and Slobodeanu [10] determined conditions for holomorphic maps between almost Hermitian manifolds to be biharmonic.

In this paper, we consider maps from almost Hermitian manifolds into Kähler manifolds which are both biharmonic and holomorphic, and investigate conditions for them to be harmonic.

In Section 3, we extend the non-existence results of biharmonic maps between Riemannian manifolds by Jiang [2] and Luo [4,5].

Recall that Jiang proved the following:

Theorem 1.1 ([2]). *Let $\varphi : (M^m, g) \rightarrow (N^n, h)$ be a biharmonic map from a compact Riemannian manifold M into a Riemannian manifold N with non-positive sectional curvature, then φ is harmonic.*

We extend this theorem to biharmonic holomorphic maps and prove the following:

Theorem 1.2. *Let $\varphi : (M^m, J^M, g) \rightarrow (N^n, J^N, h)$ be a biharmonic holomorphic map from a compact almost Hermitian manifold M into a Kähler manifold N with non-positive holomorphic bisectional curvature, then φ is harmonic.*

When M is non-compact, Luo investigated conditions for biharmonic maps to be harmonic. Before stating Luo's theorems, we first give a definition.

Definition 1.3 ([4]). Let (N, h) be a Riemannian manifold. We say a point $x_0 \in N$ is a *hyperbolic point* if the sectional curvature of any tangent plane at x_0 is negative.

Luo proved the following:

Theorem 1.4 ([4]). *Let $\varphi : (M^m, g) \rightarrow (N^n, h)$ be a biharmonic map from a complete Riemannian manifold M into a Riemannian manifold N with non-positive sectional curvature. If $\int_M |\tau(\varphi)|^p dv_g < \infty$, where $2 \leq p < \infty$ is a real constant, and N has at least one hyperbolic point. Then φ is harmonic.*

Theorem 1.5 ([5]). *Let $\varphi : (M^m, g) \rightarrow (N^n, h)$ be a biharmonic map from a complete Riemannian manifold M into a Riemannian manifold N with non-positive sectional curvature. Let p, q be constants satisfying $1 \leq q \leq \infty$, $1 < p < \infty$. If one of the following condition holds:*

(i) $|d\varphi|$ is bounded in $L^q(M)$ and $\int_M |\tau(\varphi)|^p dv_g < \infty$;

(ii) $\text{Vol}(M, g) = \infty$ and $\int_M |\tau(\varphi)|^p dv_g < \infty$.

Then φ is harmonic.

We extend these theorems to biharmonic holomorphic maps and prove the following:

Theorem 1.6. *Let $\varphi : (M^m, J^M, g) \rightarrow (N^n, J^N, h)$ be a biharmonic holomorphic map from a complete almost Hermitian manifold M into a Kähler manifold N with non-positive holomorphic bisectional curvature. Let p, q be constants satisfying $1 \leq q \leq \infty$, $1 < p < \infty$. If any of the following three conditions holds:*

(i) $|d\varphi|$ is bounded in $L^q(M)$ and $\int_M |\tau(\varphi)|^p dv_g < \infty$;

(ii) $\text{Vol}(M, g) = \infty$ and $\int_M |\tau(\varphi)|^p dv_g < \infty$;

(iii) N has at least one hyperbolic point and $\int_M |\tau(\varphi)|^p dv_g < \infty$.

Then φ is harmonic.

In Section 4, we apply the results in Section 3 to biharmonic holomorphic submersions.

In Section 5, inspired by Jiang [2], we apply the second variation formula of biharmonic maps and prove a non-existence result of stable biharmonic holomorphic maps into \mathbb{CP}^n .

Recall that Jiang proved the following:

Theorem 1.7 ([2]). *Let $\varphi : (M^m, g) \rightarrow \mathbb{CP}^n$ be a stable biharmonic map from a compact Riemannian manifold M into \mathbb{CP}^n with constant holomorphic sectional curvature $C > 0$, which satisfies the conservation law, and $|\tau(\varphi)|^2 = \text{constant}$. If the energy density function of φ satisfies $e(\varphi) < |\tau(\varphi)|/6\sqrt{C}$, then φ is harmonic.*

In our case, since φ is holomorphic, the restrictions on $|\tau(\varphi)|$ and $e(\varphi)$ can be removed and we prove the following:

Theorem 1.8. *Let $\varphi : (M^m, J^M, g) \rightarrow \mathbb{CP}^n$ be a stable biharmonic holomorphic map from a compact almost Hermitian manifold M which satisfies the conservation law, then φ is harmonic.*

2. Preliminaries

Let (M^m, g) and (N^n, h) be two Riemannian manifolds with dimensions m and n respectively. We denote by ∇ and ∇^N , the Levi-Civita connections on (M, g) and (N, h) , respectively and by $\bar{\nabla}$ the induced connection on $\varphi^{-1}TN$.

Let us first recall the definition of harmonic maps. For smooth maps $\varphi : M \rightarrow N$, the *energy functional* is defined by

$$E(\varphi) = \frac{1}{2} \int_M |d\varphi|^2 dv_g,$$

where dv_g denotes the volume element of g . The Euler–Lagrange equation of E is

$$\tau(\varphi) = \sum_{i=1}^m \{\bar{\nabla}_{e_i} d\varphi(e_i) - d\varphi(\nabla_{e_i} e_i)\} = 0,$$

where $\{e_i\}_{i=1}^m$ is a local orthonormal frame on M . $\tau(\varphi)$ is called the *tension field* of φ . A map $\varphi : M \rightarrow N$ is called a *harmonic map* if $\tau(\varphi) = 0$.

For smooth maps $\varphi : (M^m, g) \rightarrow (N^n, h = \langle \cdot, \cdot \rangle)$, the *stress–energy tensor* is defined by $S_\varphi = e(\varphi)g - \varphi^*h$, where $e(\varphi) = \frac{1}{2}|d\varphi|^2$ is the energy density function. It is well known that for any $X \in \mathfrak{X}(M)$, it holds that

$$(\operatorname{div} S_\varphi)(X) = -\langle \tau(\varphi), d\varphi(X) \rangle.$$

φ is said to satisfy the *conservation law* if $\operatorname{div} S_\varphi = 0$, i.e., $\langle \tau(\varphi), d\varphi(X) \rangle = 0, \forall X \in \mathfrak{X}(M)$.

In 1983, Eells and Lemaire [1] proposed the problem to consider the k -harmonic maps: critical maps of the functional $E_k(\varphi) = \frac{1}{2} \int_M |(d + d^*)^k \varphi|^2 dv_g$. When $k = 2$, we have the *bienergy functional*

$$E_2(\varphi) = \frac{1}{2} \int_M |d^* d\varphi|^2 dv_g = \frac{1}{2} \int_M |\tau(\varphi)|^2 dv_g,$$

the biharmonic maps are critical points of E_2 on the space of smooth maps between two Riemannian manifolds. In 1986, Jiang [2] derived the first and the second variation formulas of the bienergy functional. The Euler–Lagrange equation of E_2 is

$$\tau_2(\varphi) = \bar{\Delta} \tau(\varphi) - \sum_{i=1}^m R^N(\tau(\varphi), d\varphi(e_i)) d\varphi(e_i) = 0, \quad (1)$$

where $\bar{\Delta} = \bar{\nabla}^* \bar{\nabla} = -\sum_{i=1}^m (\bar{\nabla}_{e_i} \bar{\nabla}_{e_i} - \bar{\nabla}_{\nabla_{e_i} e_i})$ is the Rough Laplacian, and R^N is the Riemannian curvature tensor of (N, h) given by $R^N(X, Y)Z = \nabla_X^N \nabla_Y^N Z - \nabla_Y^N \nabla_X^N Z - \nabla_{[X, Y]}^N Z$ for $X, Y, Z \in \mathfrak{X}(N)$. A map $\varphi : M \rightarrow N$ is called a *biharmonic map* if $\tau_2(\varphi) = 0$.

By Eq. (1), harmonic maps are trivially biharmonic. As for non-trivial biharmonic maps, Jiang [2] proved that the embedding of Clifford torus into the unit sphere is biharmonic but not harmonic. Ou [11] summarized most of the known examples of non-trivial biharmonic maps, and he also constructed some new examples.

An *almost Hermitian manifold* (M^m, J, g) is defined to be a $2m$ dimensional orientable Riemannian manifold M admitting an almost complex structure J which satisfies $J^2 = -\operatorname{Id}$ and $g(X, Y) = g(JX, JY)$ for any $X, Y \in \mathfrak{X}(M)$.

A *Kähler manifold* $(N^n, J, h = \langle \cdot, \cdot \rangle)$ is defined to be an almost Hermitian manifold N satisfying $\nabla^N J = 0$.

Let (M^m, J^M, g) and (N^n, J^N, h) be two almost Hermitian manifolds. A smooth map $\varphi : M \rightarrow N$ is said to be *holomorphic* if $d\varphi \circ J^M = J^N \circ d\varphi$.

Definition 2.1. Let X, Y be two unit vectors at a point of a Kähler manifold $(N^m, J, \langle \cdot, \cdot \rangle)$. Then the *holomorphic bisectonal curvature* is defined by

$$\operatorname{HBR}(X, Y) = \langle R^N(X, JX)JY, Y \rangle.$$

We say N has non-positive (resp. non-negative) holomorphic bisectonal curvature if

$$\operatorname{HBR}(X, Y) \leq 0 \quad (\text{resp. } \geq 0)$$

for any unit vectors $X, Y \in \mathfrak{X}(N)$.

Remark 2.2. In the above definition, since N is Kähler, by Bianchi identity, we have

$$\begin{aligned} \operatorname{HBR}(X, Y) &= \langle R^N(X, JX)JY, Y \rangle \\ &= \langle R^N(X, JY)JY, X \rangle + \langle R^N(X, Y)Y, X \rangle, \end{aligned}$$

that is, $\operatorname{HBR}(X, Y)$ is determined by the sectional curvature of N , and the holomorphic bisectonal curvature carries less information than the sectional curvature. An example to illustrate the difference between these two curvature conditions is the following. A theorem of Wu says that any simply-connected, complete Kähler manifold with non-positive sectional curvature must be a Stein manifold. In this theorem, the condition of sectional curvature can not be replaced by bisectonal curvature. For more details, see [12], page 182–183.

3. Non-existence of biharmonic holomorphic maps

In this section, we investigate sufficient conditions for a biharmonic holomorphic map to be harmonic. Let $\varphi : (M^m, J^M, g) \rightarrow (N^n, J^N, h = \langle \cdot, \cdot \rangle)$ be a biharmonic holomorphic map from an almost Hermitian manifold M into a Kähler manifold N . Let $\{e_1, \dots, e_m, f_1 = J^M e_1, \dots, f_m = J^M e_m\}$ be a local orthonormal frame on M .

First, we have the following lemma (see also [13], page 178):

Lemma 3.1. *Let $\varphi : (M^m, J^M, g) \rightarrow (N^n, J^N, h)$ be a holomorphic map from an almost Hermitian manifold M into a Kähler manifold N . We have*

$$R^N(\tau(\varphi), d\varphi(e_i))d\varphi(e_i) + R^N(\tau(\varphi), d\varphi(f_i))d\varphi(f_i) = R^N(d\varphi(e_i), J^N d\varphi(e_i))J^N \tau(\varphi). \quad (2)$$

Proof. Since φ is holomorphic and N is Kähler, we have

$$\begin{aligned} & R^N(\tau(\varphi), d\varphi(e_i))d\varphi(e_i) + R^N(\tau(\varphi), d\varphi(f_i))d\varphi(f_i) \\ &= -R^N(\tau(\varphi), d\varphi(e_i))J^N d\varphi(f_i) + R^N(\tau(\varphi), d\varphi(f_i))J^N d\varphi(e_i) \\ &= -J^N R^N(\tau(\varphi), d\varphi(e_i))d\varphi(f_i) + J^N R^N(\tau(\varphi), d\varphi(f_i))d\varphi(e_i) \\ &= J^N R^N(d\varphi(e_i), \tau(\varphi))d\varphi(f_i) + J^N R^N(\tau(\varphi), d\varphi(f_i))d\varphi(e_i) \\ &= -J^N R^N(d\varphi(f_i), d\varphi(e_i))\tau(\varphi) \quad (\text{By Bianchi identity}) \\ &= R^N(d\varphi(e_i), J^N d\varphi(e_i))J^N \tau(\varphi). \quad \square \end{aligned}$$

Lemma 3.2. *Let $\varphi : (M^m, J^M, g) \rightarrow (N^n, J^N, h = \langle \cdot, \cdot \rangle)$ be a biharmonic holomorphic map from an almost Hermitian manifold M into a Kähler manifold N . Then we have*

$$\frac{1}{2} \Delta |\tau(\varphi)|^2 = |\bar{\nabla} \tau(\varphi)|^2 - \sum_{i=1}^m \langle R^N(d\varphi(e_i), J^N d\varphi(e_i))J^N \tau(\varphi), \tau(\varphi) \rangle. \quad (3)$$

Proof. First, from Weitzenböck formula (See [1], page 11) or its corollary (See [1], page 13), we have

$$\frac{1}{2} \Delta |\tau(\varphi)|^2 = |\bar{\nabla} \tau(\varphi)|^2 + \langle -\bar{\nabla}^* \bar{\nabla} \tau(\varphi), \tau(\varphi) \rangle. \quad (4)$$

By the biharmonic equation (1),

$$\begin{aligned} \bar{\nabla}^* \bar{\nabla} \tau(\varphi) &= \bar{\Delta} \tau(\varphi) \\ &= \sum_{i=1}^m \{R^N(\tau(\varphi), d\varphi(e_i))d\varphi(e_i) + R^N(\tau(\varphi), d\varphi(f_i))d\varphi(f_i)\}. \end{aligned}$$

Taking this into (4), and apply Lemma 3.1, then we get (3). \square

The following theorem extends Theorem 1.1 to biharmonic holomorphic maps.

Theorem 3.3. *Let $\varphi : (M^m, J^M, g) \rightarrow (N^n, J^N, h)$ be a biharmonic holomorphic map from a compact almost Hermitian manifold M into a Kähler manifold N with non-positive holomorphic bisectional curvature, then φ is harmonic.*

Proof. By Lemma 3.2, we have

$$\frac{1}{2} \Delta |\tau(\varphi)|^2 = |\bar{\nabla} \tau(\varphi)|^2 - \sum_{i=1}^m \langle R^N(d\varphi(e_i), J^N d\varphi(e_i))J^N \tau(\varphi), \tau(\varphi) \rangle \geq 0. \quad (5)$$

By Green's theorem, $\int_M \Delta |\tau(\varphi)|^2 = 0$. It follows that $\Delta |\tau(\varphi)|^2 = 0$, so $|\tau(\varphi)|^2$ is a constant. Again by (5), we have

$$\bar{\nabla}_{e_i} \tau(\varphi) = 0, \quad \bar{\nabla}_{f_i} \tau(\varphi) = 0 \quad \forall i = 1, \dots, m.$$

Then, define a global vector field $X_\varphi = \sum_{i=1}^m \{\langle d\varphi(e_i), \tau(\varphi) \rangle e_i + \langle d\varphi(f_i), \tau(\varphi) \rangle f_i\} \in \mathfrak{X}(M)$, and its divergence is

$$\operatorname{div}(X_\varphi) = \langle \tau(\varphi), \tau(\varphi) \rangle + \langle d\varphi(e_i), \bar{\nabla}_{e_i} \tau(\varphi) \rangle + \langle d\varphi(f_i), \bar{\nabla}_{f_i} \tau(\varphi) \rangle = \langle \tau(\varphi), \tau(\varphi) \rangle.$$

Integrating over M , we have

$$0 = \int_M \operatorname{div}(X_\varphi) dv_g = \int_M \langle \tau(\varphi), \tau(\varphi) \rangle dv_g$$

which implies $\tau(\varphi) = 0$. \square

When M is non-compact, Luo investigated conditions for biharmonic maps to be harmonic in [Theorems 1.4](#) and [1.5](#). Now we extend his results to biharmonic holomorphic maps. First, we recall the following lemma proved by Luo [\[5\]](#).

Lemma 3.4 ([\[5\]](#)). *Let $\varphi : (M^m, g) \rightarrow (N^n, h)$ be a biharmonic map from a complete Riemannian manifold M into a Riemannian manifold N with non-positive sectional curvature. If $\int_M |\tau(\varphi)|^p dv_g < \infty$ for some $p > 1$, then $|\tau(\varphi)|$ is constant and $\bar{\nabla}\tau(\varphi) = 0$.*

We extend this lemma to biharmonic holomorphic maps and prove the following.

Lemma 3.5. *Let $\varphi : (M^m, J^M, g) \rightarrow (N^n, J^N, h = \langle \cdot, \cdot \rangle)$ be a biharmonic holomorphic map from a complete almost Hermitian manifold M into a Kähler manifold N with non-positive holomorphic bisectional curvature. If $\int_M |\tau(\varphi)|^p dv_g < \infty$ for some $p > 1$, then $|\tau(\varphi)|$ is constant and $\bar{\nabla}\tau(\varphi) = 0$.*

Proof. The proof is similar to the proof of [Lemma 3.4](#).

For $\varepsilon > 0$, by direct computation, we have

$$\Delta(|\tau(\varphi)|^2 + \varepsilon)^{\frac{1}{2}} = (|\tau(\varphi)|^2 + \varepsilon)^{-\frac{3}{2}} \left(\frac{1}{2}(|\tau(\varphi)|^2 + \varepsilon)\Delta|\tau(\varphi)|^2 - \frac{1}{4}|\nabla|\tau(\varphi)|^2|^2 \right). \quad (6)$$

Since $\nabla|\tau(\varphi)|^2 = 2\langle \bar{\nabla}\tau(\varphi), \tau(\varphi) \rangle$, we have

$$|\nabla|\tau(\varphi)|^2|^2 \leq 4(|\tau(\varphi)|^2 + \varepsilon)|\bar{\nabla}\tau(\varphi)|^2.$$

Therefore, we deduce that

$$\frac{1}{2}(|\tau(\varphi)|^2 + \varepsilon)\Delta|\tau(\varphi)|^2 - \frac{1}{4}|\nabla|\tau(\varphi)|^2|^2 \geq \frac{1}{2}(|\tau(\varphi)|^2 + \varepsilon)(\Delta|\tau(\varphi)|^2 - 2|\bar{\nabla}\tau(\varphi)|^2). \quad (7)$$

By [Lemma 3.2](#), we have

$$\begin{aligned} \frac{1}{2}\Delta|\tau(\varphi)|^2 &= |\bar{\nabla}\tau(\varphi)|^2 - \sum_{i=1}^m \langle R^N(d\varphi(e_i), J^N d\varphi(e_i))J^N \tau(\varphi), \tau(\varphi) \rangle \\ &\geq |\bar{\nabla}\tau(\varphi)|^2, \end{aligned} \quad (8)$$

since N has non-positive holomorphic bisectional curvature. Combining [\(7\)](#) and [\(8\)](#), we have

$$\frac{1}{2}(|\tau(\varphi)|^2 + \varepsilon)\Delta|\tau(\varphi)|^2 - \frac{1}{4}|\nabla|\tau(\varphi)|^2|^2 \geq 0. \quad (9)$$

By [\(6\)](#) and [\(9\)](#), we get

$$\Delta(|\tau(\varphi)|^2 + \varepsilon)^{\frac{1}{2}} \geq 0.$$

Now let $\varepsilon \rightarrow 0$, we have

$$\Delta|\tau(\varphi)| \geq 0.$$

By Yau's generalized maximum principle (see [\[14\]](#)), we see that if $\int_M |\tau(\varphi)|^p dv_g < \infty$ for some $p > 1$, then $|\tau(\varphi)|$ is constant. Moreover, by [\(8\)](#), we see that $\bar{\nabla}\tau(\varphi) = 0$. This completes the proof. \square

The following theorem extends [Theorem 1.5](#).

Theorem 3.6. *Let $\varphi : (M^m, J^M, g) \rightarrow (N^n, J^N, h)$ be a biharmonic holomorphic map from a complete almost Hermitian manifold M into a Kähler manifold N with non-positive holomorphic bisectional curvature. Let p, q be constants satisfying $1 \leq q \leq \infty$, $1 < p < \infty$.*

- (i) *If $|d\varphi|$ is bounded in $L^q(M)$ and $\int_M |\tau(\varphi)|^p dv_g < \infty$, then φ is harmonic.*
- (ii) *If $\text{Vol}(M, g) = \infty$ and $\int_M |\tau(\varphi)|^p dv_g < \infty$, then φ is harmonic.*

Proof. The proof is similar to Luo's proof of [Theorem 1.5](#). By [Lemma 3.5](#), we already have that $|\tau(\varphi)| = c$ is constant and $\bar{\nabla}\tau(\varphi) = 0$. Hence if $\text{Vol}(M, g) = \infty$, we must have $c = 0$, and this proves (ii).

To prove (i), if $c \neq 0$, we see that $\text{Vol}(M, g) < \infty$, and we will get a contradiction in the following. Define a 1-form on M by

$$\omega(X) := \langle d\varphi(X), \tau(\varphi) \rangle, \quad (X \in \mathfrak{X}(M)).$$

Then we have

$$\begin{aligned} \int_M |\omega| dv_g &= \int_M \left(\sum_{i=1}^m (|\omega(e_i)|^2 + |\omega(f_i)|^2) \right)^{\frac{1}{2}} dv_g \\ &\leq \int_M |\tau(\varphi)| |d\varphi| dv_g \\ &\leq c \text{Vol}(M)^{1-\frac{1}{q}} \left(\int_M |d\varphi|^q dv_g \right)^{\frac{1}{q}} \\ &< \infty, \end{aligned}$$

for $1 \leq q < \infty$. For $q = \infty$, the above inequality still holds.

Now we consider $-\delta\omega = \sum_{i=1}^m \{(\nabla_{e_i}\omega)(e_i) + (\nabla_{f_i}\omega)(f_i)\}$. By a little calculation, we have $-\delta\omega = |\tau(\varphi)|^2$. Then by Gaffney's theorem (see [5]) and the above inequality, we have

$$0 = \int_M -\delta\omega = \int_M |\tau(\varphi)|^2 dv_g = c^2 \text{Vol}(M),$$

which implies that $c = 0$, a contradiction. We refer the reader to [5], page 196 for more details of the proof. \square

The following theorem extends [Theorem 1.4](#).

Theorem 3.7. *Let $\varphi : (M^m, J^M, g) \rightarrow (N^n, J^N, h)$ be a biharmonic holomorphic map from a complete almost Hermitian manifold M into a Kähler manifold N with non-positive holomorphic bisectional curvature and at least one hyperbolic point. Let p be a constant satisfying $1 < p < \infty$. If $\int_M |\tau(\varphi)|^p dv_g < \infty$, then φ is harmonic.*

Proof. The proof is similar to Luo's proof of [Theorem 1.4](#). By [Lemma 3.5](#), we already have that $|\tau(\varphi)| = c$ is constant and $\bar{\nabla}\tau(\varphi) = 0$. We need to prove that $c = 0$. If not, assume that x_0 is a hyperbolic point on N ; then by smoothness we see that there is a neighborhood $U \subset N$ around x_0 where every point is hyperbolic. Now we discuss two possibilities.

(1) $d\varphi = 0$ everywhere in U . Then $\tau(\varphi) = \text{trace} \nabla d\varphi = 0$ in U , i.e., $c = 0$.

(2) There exists a hyperbolic point $y_0 \in U$ and $q \in M$, $\varphi(q) = y_0$, such that $d\varphi_q$ is non-degenerate. We choose geodesic coordinates $\{x_i, i = 1, \dots, 2m\}$ on M around q such that $J^N(\partial_{x_i}) = \partial_{x_{m+i}}$, for $i = 1, \dots, m$ and $d\varphi|_V(\partial_{x_1}) \neq 0$ (possibly in a smaller neighborhood V of q). Then by [Lemma 3.2](#), we have in V

$$\begin{aligned} 0 &= \frac{1}{2} \Delta |\tau(\varphi)|^2 \\ &= |\bar{\nabla}\tau(\varphi)|^2 - \sum_{i=1}^m \langle R^N(d\varphi(\partial_{x_i}), J^N d\varphi(\partial_{x_i})) J^N \tau(\varphi), \tau(\varphi) \rangle \\ &= - \sum_{i=1}^m \langle R^N(d\varphi(\partial_{x_i}), J^N d\varphi(\partial_{x_i})) J^N \tau(\varphi), \tau(\varphi) \rangle. \end{aligned}$$

If $d\varphi(\partial_{x_1}) \neq \tau(\varphi)$ at some point $q_1 \in V$ and $c \neq 0$, then by [Remark 2.2](#), at q_1 we have

$$\begin{aligned} 0 &= \sum_{i=1}^m \langle R^N(d\varphi(\partial_{x_i}), J^N d\varphi(\partial_{x_i})) J^N \tau(\varphi), \tau(\varphi) \rangle \\ &= \sum_{i=1}^m \langle R^N(d\varphi(\partial_{x_i}), \tau(\varphi)) \tau(\varphi), d\varphi(\partial_{x_i}) \rangle \\ &\quad + \sum_{i=1}^m \langle R^N(d\varphi(\partial_{x_i}), J^N \tau(\varphi)) J^N \tau(\varphi), d\varphi(\partial_{x_i}) \rangle \\ &\leq \langle R^N(d\varphi(\partial_{x_1}), \tau(\varphi)) \tau(\varphi), d\varphi(\partial_{x_1}) \rangle \\ &< 0, \end{aligned}$$

which is a contradiction. Note that the two inequalities above hold because q_1 is a hyperbolic point.

Now if $d\varphi(\partial_{x_1}) = \tau(\varphi)$ and $d\varphi(\partial_{x_i}) = 0$ or $\tau(\varphi)$ everywhere in V , for $i = 2, \dots, 2m$, then we see that at q

$$\tau(\varphi) = \sum_{i=1}^{2m} \bar{\nabla}_{\partial_{x_i}} d\varphi(\partial_{x_i}) = 0,$$

since $\bar{\nabla}\tau(\varphi) = 0$.

If $d\varphi(\partial_{x_1}) = \tau(\varphi)$ everywhere in V , and there exists a point $q_2 \in V$ and $2 \leq j \leq 2m$ such that $d\varphi(\partial_{x_j}) \neq 0$ or $\tau(\varphi)$, then we must have $c = 0$. For if $c \neq 0$, we have at q_2

$$\begin{aligned} 0 &= \sum_{i=1}^m \langle R^N(d\varphi(\partial_{x_i}), J^N d\varphi(\partial_{x_i})) J^N \tau(\varphi), \tau(\varphi) \rangle \\ &= \sum_{i=1}^m \langle R^N(d\varphi(\partial_{x_i}), \tau(\varphi)) \tau(\varphi), d\varphi(\partial_{x_i}) \rangle \\ &\quad + \sum_{i=1}^m \langle R^N(d\varphi(\partial_{x_i}), J^N \tau(\varphi)) J^N \tau(\varphi), d\varphi(\partial_{x_i}) \rangle \\ &\leq \langle R^N(d\varphi(\partial_{x_j}), \tau(\varphi)) \tau(\varphi), d\varphi(\partial_{x_j}) \rangle \\ &< 0, \end{aligned}$$

which is a contradiction.

This completes the proof of the theorem. \square

Remark 3.8. (1) Note that in [Theorem 1.4](#), Luo's assumption is $2 \leq p < \infty$, while in the above theorem, our assumption is $1 < p < \infty$.

(2) It should be pointed out that if we replace Lemma 3.1 in [4] by [Lemma 3.4](#), then [Theorem 1.4](#) can be improved to the following result:

Let $\varphi : (M^m, g) \rightarrow (N^n, h)$ be a biharmonic map from a complete Riemannian manifold M into a Riemannian manifold N with non-positive sectional curvature. If $\int_M |\tau(\varphi)|^p dv_g < \infty$, where $1 < p < \infty$ is a real constant, and N has at least one hyperbolic point. Then φ is harmonic.

4. Biharmonic holomorphic submersions

In this section, we give applications of [Theorems 3.6](#) and [3.7](#) to biharmonic holomorphic submersions.

We first recall some definitions (see [15]).

Let $\varphi : (M, g) \rightarrow (N, h)$ be a smooth map between Riemannian manifolds, and let $x \in M$. Then φ is called *horizontally weakly conformal* at x if either

- (i) $d\varphi_x = 0$, or
- (ii) $d\varphi_x$ maps the horizontal space $\mathcal{H}_x = \{\text{Ker}(d\varphi_x)\}^\perp$ conformally onto $T_{\varphi(x)}N$, such that

$$h(d\varphi_x(X), d\varphi_x(Y)) = \lambda g(X, Y), \quad (X, Y \in \mathcal{H}_x).$$

A map φ is called *horizontally weakly conformal* on M if it is horizontally weakly conformal at every point of M . If further, φ has no critical points, then we call it a *horizontally conformal submersion*. Note that if $\varphi : (M, g) \rightarrow (N, h)$ is a horizontally weakly conformal map and $\dim M < \dim N$, then φ is constant.

If for every harmonic function $f : V \rightarrow \mathbb{R}$ defined on an open subset V of N with $\varphi^{-1}(V)$ non-empty, the composition $f \circ \varphi$ is harmonic on $\varphi^{-1}(V)$, then φ is called a *harmonic morphism*. Harmonic morphisms are characterized as follows:

Theorem 4.1 ([16, 17]). *A smooth map $\varphi : (M, g) \rightarrow (N, h)$ between Riemannian manifolds is a harmonic morphism if and only if φ is both harmonic and horizontally weakly conformal.*

Let $\varphi : (M, g) \rightarrow (N, h)$ be a submersion, then each tangent space $T_x M$ can be decomposed as follows:

$$T_x M = \mathcal{V}_x \oplus \mathcal{H}_x, \tag{10}$$

where $\mathcal{V}_x = \text{Ker}(d\varphi_x)$ is the vertical space and \mathcal{H}_x is the horizontal space. If there exists a positive C^∞ function λ on M such that, for each $x \in M$,

$$h(d\varphi_x(X), d\varphi_x(Y)) = \lambda^2(x)g(X, Y), \quad (X, Y \in \mathcal{H}_x),$$

then λ is called the *dilation*.

When $\varphi : (M^m, g) \rightarrow (N^n, h)$ ($m > n \geq 2$) is a horizontally conformal submersion, the tension field $\tau(\varphi)$ is given by

$$\tau(\varphi) = \frac{n-2}{2} \lambda^2 d\varphi \left(\text{grad}_{\mathcal{H}} \left(\frac{1}{\lambda^2} \right) \right) - (m-n) d\varphi(\hat{\mathbf{H}}), \tag{11}$$

where $\text{grad}_{\mathcal{H}} \left(\frac{1}{\lambda^2} \right)$ is the \mathcal{H} -component of the decomposition according to (10) of $\text{grad} \left(\frac{1}{\lambda^2} \right)$, and $\hat{\mathbf{H}}$ is the trace of the second fundamental form of each fiber which is given by $\hat{\mathbf{H}} = \frac{1}{m-n} \sum_{i=1}^m \mathcal{H}(\nabla_{e_i} e_i)$, where a local orthonormal frame field $\{e_i\}_{i=1}^m$

on M is taken in such a way that $\{e_{ix}|i = 1, \dots, n\}$ belong to \mathcal{H}_x and $\{e_{jx}|j = n+1, \dots, m\}$ belong to \mathcal{V}_x , where x is in a neighborhood in M .

Nakauchi et al. [7], Maeta [6] and Luo [4,5] applied their non-existence results of biharmonic maps to biharmonic submersions. Here we extend these results to biharmonic holomorphic submersion.

Proposition 4.2. *Let $\varphi : (M^m, J^M, g) \rightarrow (N^n, J^N, h)$ ($m > n \geq 2$) be a biharmonic holomorphic horizontally conformal submersion from a complete almost Hermitian manifold M into a Kähler manifold N with non-positive holomorphic bisectional curvature and let p be a real constant satisfying $1 < p < \infty$.*

If

$$\int_M \lambda^p \left| (n-1)\lambda^2 \text{grad}_{\mathcal{H}} \left(\frac{1}{\lambda^2} \right) - 2(m-n)\hat{\mathbf{H}} \right|_g^p dv_g < \infty,$$

and if any of the following three conditions holds:

- (i) λ is bounded in L^q ($1 \leq q \leq \infty$);
 - (ii) $\text{Vol}(M, g) = \infty$;
 - (iii) N has at least one hyperbolic point.
- Then, φ is a harmonic morphism.*

Proof. By (11),

$$\int_M |\tau(\varphi)|^p dv_g = \int_M \lambda^p \left| (n-1)\lambda^2 \text{grad}_{\mathcal{H}} \left(\frac{1}{\lambda^2} \right) - 2(m-n)\hat{\mathbf{H}} \right|_g^p dv_g < \infty,$$

since $|d\varphi(x)|^2 = 2n\lambda^2$, by Theorems 3.6 and 3.7, φ is a harmonic map. Furthermore, since φ is also a horizontally conformal submersion, by Theorem 4.1, φ is a harmonic morphism. \square

5. Stable biharmonic holomorphic maps

In this section, we apply the second variation formula of bienergy functional to prove a non-existence theorem for stable biharmonic holomorphic maps.

First, we recall the second variation formula derived by Jiang [2]. Let $\varphi : (M^m, g) \rightarrow (N^n, h = \langle \cdot, \cdot \rangle)$ be a biharmonic map from a compact Riemannian manifold M into an arbitrary Riemannian manifold N , and $\{\varphi_t\}$ an arbitrary smooth variation of φ with

$$\varphi_0 = \varphi, \quad \frac{\partial \varphi_t}{\partial t} \Big|_{t=0} = V, \quad t \in (-\varepsilon, \varepsilon),$$

for any vector field $V \in \Gamma(\varphi^{-1}TN)$. Then the second variation formula of $E_2(\varphi_t)$ is given by

$$\frac{d^2}{dt^2} \Big|_{t=0} E_2(\varphi_t) = \int_M \{ |\mathcal{J}(V)|^2 - \langle R_2(V), V \rangle \} dv_g, \quad (12)$$

where \mathcal{J} is the Jacobi operator defined by

$$\mathcal{J}(V) = \bar{\Delta}V - \sum_{i=1}^m R^N(V, d\varphi(e_i))d\varphi(e_i),$$

and

$$\begin{aligned} R_2(V) &= R^N(\tau(\varphi), V)\tau(\varphi) \\ &+ 2 \sum_{i=1}^m R^N(d\varphi(e_i), V)\bar{\nabla}_{e_i}\tau(\varphi) + 2 \sum_{i=1}^m R^N(d\varphi(e_i), \tau(\varphi))\bar{\nabla}_{e_i}V \\ &+ \sum_{i=1}^m (\nabla_{d\varphi(e_i)}^N R^N)(d\varphi(e_i), \tau(\varphi))V + \sum_{i=1}^m (\nabla_{\tau(\varphi)}^N R^N)(d\varphi(e_i), V)d\varphi(e_i). \end{aligned}$$

Recall that $\bar{\Delta} = \bar{\nabla}^* \bar{\nabla} = -\sum_{i=1}^m (\bar{\nabla}_{e_k} \bar{\nabla}_{e_k} - \bar{\nabla}_{\nabla_{e_k} e_k})$ is the Rough Laplacian.

Definition 5.1 ([2]). A biharmonic map $\varphi : M \rightarrow N$ from a compact Riemannian manifold M into any Riemannian manifold N is called *stable* if (12) is non-negative for every vector field V .

Now let $N = \mathbb{CP}^n(4k)$ be the complex projective space with constant holomorphic sectional curvature $4k(k > 0)$. The curvature operator of N is given by

$$R^N(X, Y)Z = k\{\langle Y, Z \rangle X - \langle X, Z \rangle Y + \langle JY, Z \rangle JX - \langle JX, Z \rangle JY + 2\langle X, JY \rangle JZ\},$$

for $X, Y, Z \in \mathfrak{X}(N)$, where $\langle \cdot, \cdot \rangle$ is the Riemannian metric on N , and J is the almost complex structure on N .

Recall that a smooth map $\varphi : (M^m, g) \rightarrow (N^n, h = \langle \cdot, \cdot \rangle)$ is said to satisfy the conservation law if $\langle \tau(\varphi), d\varphi(X) \rangle = 0, \forall X \in \mathfrak{X}(M)$ (see Section 2).

The following theorem extends Theorem 1.7 to biharmonic holomorphic maps.

Theorem 5.2. *Let $\varphi : (M^m, J^M, g) \rightarrow \mathbb{CP}^n$ be a stable biharmonic holomorphic map from a compact almost Hermitian manifold M which satisfies the conservation law, then φ is harmonic.*

Proof. Let $\{e_1, \dots, e_m, f_1 = J^M e_1, \dots, f_m = J^M e_m\}$ be a local orthonormal frame on M . Note that $\nabla^N R^N = 0$. If we take $V = \tau(\varphi)$, since φ is biharmonic, we have $\mathcal{J}(\tau(\varphi)) = \tau_2(\varphi) = 0$ by Eq. (1). Then (12) becomes

$$\begin{aligned} \frac{d^2}{dt^2} \Big|_{t=0} E_2(\varphi_t) &= -4 \int_M \sum_{i=1}^m \{ \langle R^N(d\varphi(e_i), \tau(\varphi)) \bar{\nabla}_{e_i} \tau(\varphi), \tau(\varphi) \rangle \\ &\quad + \langle R^N(d\varphi(f_i), \tau(\varphi)) \bar{\nabla}_{f_i} \tau(\varphi), \tau(\varphi) \rangle \} dv_g \\ &= -4k \int_M \sum_{i=1}^m \{ \langle \tau(\varphi), \bar{\nabla}_{e_i} \tau(\varphi) \rangle d\varphi(e_i) \\ &\quad - \langle d\varphi(e_i), \bar{\nabla}_{e_i} \tau(\varphi) \rangle \tau(\varphi) + \langle J\tau(\varphi), \bar{\nabla}_{e_i} \tau(\varphi) \rangle Jd\varphi(e_i) \\ &\quad - \langle Jd\varphi(e_i), \bar{\nabla}_{e_i} \tau(\varphi) \rangle J\tau(\varphi) \\ &\quad + 2\langle d\varphi(e_i), J\tau(\varphi) \rangle J\bar{\nabla}_{e_i} \tau(\varphi), \tau(\varphi) \rangle dv_g \\ &\quad - 4k \int_M \sum_{i=1}^m \{ \langle \tau(\varphi), \bar{\nabla}_{f_i} \tau(\varphi) \rangle d\varphi(f_i) \\ &\quad - \langle d\varphi(f_i), \bar{\nabla}_{f_i} \tau(\varphi) \rangle \tau(\varphi) + \langle J\tau(\varphi), \bar{\nabla}_{f_i} \tau(\varphi) \rangle Jd\varphi(f_i) \\ &\quad - \langle Jd\varphi(f_i), \bar{\nabla}_{f_i} \tau(\varphi) \rangle J\tau(\varphi) \\ &\quad + 2\langle d\varphi(f_i), J\tau(\varphi) \rangle J\bar{\nabla}_{f_i} \tau(\varphi), \tau(\varphi) \rangle dv_g \}. \end{aligned} \quad (13)$$

Since φ satisfies the conservation law, i.e., $\langle \tau(\varphi), d\varphi(X) \rangle = 0, \forall X \in \mathfrak{X}(M)$, we have

$$\langle d\varphi(e_k), \tau(\varphi) \rangle = 0, \quad \langle d\varphi(f_k), \tau(\varphi) \rangle = 0,$$

and

$$\begin{aligned} &\sum_{k=1}^m \{ \langle d\varphi(e_k), \bar{\nabla}_{e_k} \tau(\varphi) \rangle + \langle d\varphi(f_k), \bar{\nabla}_{f_k} \tau(\varphi) \rangle \} \\ &= \sum_{k=1}^m \{ -\langle \bar{\nabla}_{e_k} d\varphi(e_k), \tau(\varphi) \rangle + e_k \langle d\varphi(e_k), \tau(\varphi) \rangle \\ &\quad - \langle \bar{\nabla}_{f_k} d\varphi(f_k), \tau(\varphi) \rangle + f_k \langle d\varphi(f_k), \tau(\varphi) \rangle \} \\ &= -|\tau(\varphi)|^2 - \sum_{k=1}^m \langle d\varphi(\nabla_{e_k} e_k) + d\varphi(\nabla_{f_k} f_k), \tau(\varphi) \rangle \\ &= -|\tau(\varphi)|^2. \end{aligned} \quad (14)$$

Since φ is holomorphic, we have

$$\langle d\varphi(f_k), J\tau(\varphi) \rangle = -\langle Jd\varphi(f_k), \tau(\varphi) \rangle = \langle d\varphi(e_k), \tau(\varphi) \rangle = 0, \quad (15)$$

and

$$\langle d\varphi(e_k), J\tau(\varphi) \rangle = -\langle Jd\varphi(e_k), \tau(\varphi) \rangle = -\langle d\varphi(f_k), \tau(\varphi) \rangle = 0. \quad (16)$$

By (14)–(16) and $\langle J\tau(\varphi), \tau(\varphi) \rangle = 0$, (13) becomes

$$\left. \frac{d^2}{dt^2} \right|_{t=0} E_2(\varphi_t) = -4k \int_M |\tau(\varphi)|^4 dv_g \leq 0. \quad (17)$$

Since φ is stable, (17) implies that $|\tau(\varphi)| = 0$. \square

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