Positive scalar curvature and the Euler class

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A B S T R A C T

We prove the following generalization of the classical Lichnerowicz vanishing theorem: if F is an oriented flat vector bundle over a closed spin manifold M such that TM carries a metric of positive scalar curvature, then $\langle \widehat{A}(TM) e(F), [M] \rangle = 0$, where $e(F)$ is the Euler class of F.

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1. Introduction

The classical Lichnerowicz vanishing theorem [1] states that if a closed spin manifold M admits a Riemannian metric of positive scalar curvature, then its Hirzebruch $\widehat{A}$-genus (cf. [2, §1.6.3]) vanishes: $\widehat{A}(M) = 0$.

Let F be an oriented real flat vector bundle over a closed manifold M. Let $\widehat{A}(TM)$ denote the Hirzebruch $\widehat{A}$-class of TM (cf. [2, p. 13]) and $e(F)$ denote the Euler class of F (cf. [2, §3.4]). The purpose of this paper is to prove the following generalization of the Lichnerowicz vanishing theorem.

Theorem 1.1. *If TM is spin and carries a metric of positive scalar curvature, then $\langle \widehat{A}(TM) e(F), [M] \rangle = 0$.*

Remark 1.2. Recall that Milnor [3] constructs on any oriented closed surface Σ_g of genus $g > 1$ a rank two oriented flat vector bundle F_g such that $\langle e(F_g), [\Sigma_g] \rangle \neq 0$. Let M be any closed spin manifold such that TM carries a metric of positive scalar curvature, then $T(M \times \Sigma_g)$ also carries a metric of positive scalar curvature, and one gets by Theorem 1.1 that

$$0 = \langle \widehat{A}\left(T\left(M \times \Sigma_g\right)\right) e\left(\pi^* F_g\right), \left[M \times \Sigma_g\right] \rangle = \widehat{A}(M) \langle e\left(F_g\right), \left[\Sigma_g\right] \rangle, \tag{1.1}$$

where $\pi : M \times \Sigma_g \to \Sigma_g$ denotes the natural projection, which implies $\widehat{A}(M) = 0$. Thus Theorem 1.1 indeed recovers the Lichnerowicz theorem.

The following consequence of Theorem 1.1, which generalizes the well-known fact that $T\Sigma_g$ $(g > 1)$ does not carry a metric of positive scalar curvature, is of independent interest.

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Corollary 1.3. *Let F be an oriented flat vector bundle over a closed spin manifold M such that* $\mathrm{rk}(F) = \dim M$. *If* $\langle e(F), [M] \rangle \neq 0$, *then TM does not carry a metric of positive scalar curvature.*

Inspired by [4], we call a closed manifold M a Smillie manifold if TM carries a flat connection, while the Euler characteristic $\chi(M) \neq 0$. By Corollary 1.3, one sees that a closed spin Smillie manifold does not admit a Riemannian metric of positive scalar curvature.

On the other hand, Theorem 1.1 can also be reformulated as follows.

Theorem 1.4. *If a closed spin manifold M admits a Riemannian metric of positive scalar curvature, then for any representation* $\rho : \pi_1(M) \to GL(q, \mathbb{R})^+$, $q \in \mathbb{N}$, *one has*

$$\left\langle \widehat{A}(TM) f^* \left(e \left(E\pi \times_\rho \mathbb{R}^q \right) \right), [M] \right\rangle = 0, \tag{1.2}$$

where $E\pi \times_\rho \mathbb{R}^q$, *with* $\pi = \pi_1(M)$, *is the oriented real flat vector bundle over* $B\pi$ *associated to the representation* ρ, *and* $f : M \to B\pi$ *is the classifying map.*

Thus our result provides a new evidence to support the famous Gromov–Lawson conjecture on the vanishing of higher $\widehat{A}$-genera [5].

The main difficulty in proving Theorem 1.1 is that the flat connection on F need not preserve any metric on F. Similar difficulties have appeared in the foliation situations studied in [6] and [7], where one uses the Connes fibration introduced in [6] to overcome such difficulties.

Our proof of Theorem 1.1 is index theoretic. On one hand, it is inspired by [7] and makes use of the Connes fibration, as well as the constructions of deformed sub-Dirac operators. On the other hand, a significant technical difference with respect to what in [7] is that while in [7], one identifies the characteristic number in question with the indices of certain nonexplicit pseudodifferential operators on a closed manifold, here we will work intrinsically on a sub-manifold with boundary of the Connes fibration and apply directly the analytic localization techniques developed in [8,9] and [10].

The rest of this paper is organized as follows. In Section 2, we provide an index theoretic interpretation of $\langle \widehat{A}(TM)e(F), [M] \rangle$, and prove a simple vanishing result for it. In Section 3, we construct the sub-Dirac operators as well as the key deformations of them on the Connes fibration associated to the flat vector bundle F, and reduce Eq. (1.1) to an estimating result concerning the Atiyah–Patodi–Singer elliptic boundary valued problem. In Section 4, we prove the above estimating result.

2. Flat vector bundles and the twisted index

In this section, we provide an index theoretic interpretation of the characteristic number under consideration, and prove a simple vanishing result.

This section is organized as follows. In Section 2.1, we construct certain vector bundles associated to the given flat vector bundle. In Section 2.2, we construct a twisted Dirac operator of which the index is equal to the given characteristic number. A simple vanishing result is also established.

2.1. Flat vector bundles and the signature splitting

Let (F, ∇^F) be an oriented real flat vector bundle over a closed manifold M. Let g^F be a Euclidean metric on F. As in [11, (4.1)], set

$$\omega \left(F, g^F \right) = \left(g^F \right)^{-1} \nabla^F g^F. \tag{2.1}$$

Then F carries a canonical Euclidean connection (see [11, (4.3)])

$$\nabla^{F,e} = \nabla^F + \frac{1}{2} \omega \left(F, g^F \right). \tag{2.2}$$

By [11, Proposition 4.3], its curvature is given by

$$\left(\nabla^{F,e} \right)^2 = -\frac{1}{4} \left(\omega \left(F, g^F \right) \right)^2, \tag{2.3}$$

which is usually nonzero.

Let $\Lambda(F^*)$ be the (complex) exterior algebra bundle of F. Then $\Lambda(F^*)$ carries a Hermitian metric canonically induced from g^F and a Hermitian connection $\nabla^{\Lambda(F^*),u}$ induced from $\nabla^{F,e}$.

For any $f \in F$, let $f^* \in F^*$ be the metric dual of f with respect to g^F. Let $c(f), \widehat{c}(f)$ be the Clifford actions on $\Lambda(F^*)$ defined by

$$c(f) = f^* \wedge -i_f, \quad \widehat{c}(f) = f^* \wedge +i_f, \tag{2.4}$$

where $f^* \wedge$ and i_f are the exterior and interior multiplications of f^* and f.

Let $\{f_\mu\}_{\mu=1}^{\text{rk}(F)}$ be an orthonormal basis of (F, g^F), and $\{f^\mu\}_{\mu=1}^{\text{rk}(F)}$ be its dual basis. Then by [12, (1.26)], the curvature $(\nabla^{\Lambda(F^*),u})^2$ of $\nabla^{\Lambda(F^*),u}$ can be computed as follows,

$$\left(\nabla^{\Lambda(F^*),u}\right)^2 = \sum_{\mu,\nu=1}^{\text{rk}(F)} \left\langle \left(\nabla^{F,e}\right)^2 f_\mu, f_\nu \right\rangle f^\nu \wedge i_{f_\mu}. \tag{2.5}$$

From (2.3)–(2.5), we deduce that

$$\left(\nabla^{\Lambda(F^*),u}\right)^2 = -\frac{1}{16} \sum_{\mu,\nu=1}^{\text{rk}(F)} \left\langle f_\mu, \omega\left(F, g^F\right)^2 f_\nu \right\rangle \left(\widehat{c}\left(f_\mu\right) \widehat{c}\left(f_\nu\right) - c\left(f_\mu\right) c\left(f_\nu\right)\right). \tag{2.6}$$

On the other hand, when $\text{rk}(F)$ is even, set

$$\tau\left(F, g^F\right) = \left(\frac{1}{\sqrt{-1}}\right)^{\frac{\text{rk}(F)}{2}} c\left(f_1\right) \dots c\left(f_{\text{rk}(F)}\right). \tag{2.7}$$

Then $(\tau(F, g^F))^2 = \text{Id}\big|_{\Lambda(F^*)}$, and $\tau(F, g^F)$ induces a $\mathbb{Z}_2$-graded splitting

$$\Lambda\left(F^*\right) = \Lambda_+\left(F^*\right) \oplus \Lambda_-\left(F^*\right), \tag{2.8}$$

where

$$\Lambda_\pm\left(F^*\right) = \left\{\alpha \in \Lambda\left(F^*\right) \mid \tau\left(F, g^F\right)\alpha = \pm\alpha\right\}. \tag{2.9}$$

Moreover, $c(f)$ exchanges $\Lambda_\pm(F^*)$ for any $f \in F$.

2.2. Dirac operators and an easy vanishing result

We assume that M is spin and of even dimension, and that $\text{rk}(F)$ is even. Let g^{TM} be a Riemannian metric on M, and ∇^{TM} be the associated Levi-Civita connection. Let $S(TM)$ be the Hermitian bundle of spinors associated to (TM, g^{TM}) with the $\mathbb{Z}_2$-graded splitting

$$S(TM) = S_+(TM) \oplus S_-(TM). \tag{2.10}$$

Then ∇^{TM} induces naturally a Hermitian connection $\nabla^{S(TM)}$ on $S(TM)$ preserving the $\mathbb{Z}_2$-grading.

From the $\mathbb{Z}_2$-graded vector bundles in (2.8) and (2.10), we form the following $\mathbb{Z}_2$-graded tensor product (see [13, p. 11])

$$S(TM)\widehat{\otimes}\Lambda\left(F^*\right) = \left(S(TM)\widehat{\otimes}\Lambda\left(F^*\right)\right)_+ \oplus \left(S(TM)\widehat{\otimes}\Lambda\left(F^*\right)\right)_-, \tag{2.11}$$

which carries the $\mathbb{Z}_2$-graded tensor product connection

$$\nabla^u = \nabla^{S(TM)} \otimes \text{Id}\big|_{\Lambda(F^*)} + \text{Id}_{S(TM)} \otimes \nabla^{\Lambda(F^*),u}. \tag{2.12}$$

For $e \in TM$, we denote by $c(e)$ the Clifford action of e on $S(TM)$. Then it extends to an action $c(e)\otimes\text{Id}\big|_{\Lambda(F^*)}$ on $S(TM)\widehat{\otimes}\Lambda(F^*)$, which we still denote by $c(e)$.

Take an oriented orthonormal basis $\{e_i\}_{i=1}^{\dim M}$ of (TM, g^{TM}). Let

$$D^M = \sum_{i=1}^{\dim M} c\left(e_i\right) \nabla^u_{e_i} : \Gamma\left(M, S(TM)\widehat{\otimes}\Lambda\left(F^*\right)\right) \longrightarrow \Gamma\left(M, S(TM)\widehat{\otimes}\Lambda\left(F^*\right)\right) \tag{2.13}$$

be the corresponding twisted Dirac operator, and denote

$$D^M_\pm = D^M\big|_{\Gamma\left(M,(S(TM)\widehat{\otimes}\Lambda(F^*))_\pm\right)}. \tag{2.14}$$

Since (F, ∇^F) is flat, by the Atiyah–Singer index theorem [14], we get

$$\begin{aligned}
\text{ind}\left(D^M_+\right) &= \left\langle \widehat{A}(TM) \text{ch}\left(\Lambda_+\left(F^*\right) - \Lambda_-\left(F^*\right)\right), [M]\right\rangle \\
&= 2^{\frac{\text{rk}(F)}{2}} \left\langle \widehat{A}(TM) e(F), [M]\right\rangle.
\end{aligned} \tag{2.15}$$

Let $\Delta^u = \sum_{i=1}^{\dim M}(\nabla^u_{e_i}\nabla^u_{e_i} - \nabla^u_{\nabla^{TM}_{e_i}e_i})$ be the Bochner Laplacian. Let k^{TM} denote the scalar curvature of (TM, g^{TM}). We have the following standard Lichnerowicz formula [1],

$$\left(D^M\right)^2 = -\Delta^u + \frac{k^{TM}}{4} + \frac{1}{2} \sum_{i,j=1}^{\dim M} c\left(e_i\right) c\left(e_j\right) \left(\nabla^{\Lambda(F^*),u}\right)^2 \left(e_i, e_j\right). \tag{2.16}$$

From (2.15) and (2.16), one obtains the following easy vanishing result.

Proposition 2.1. *If there holds over M that*

$$\frac{k^{TM}}{4} + \frac{1}{2} \sum_{i,j=1}^{\dim M} c\left(e_i\right) c\left(e_j\right) \left(\nabla^{\Lambda(F^*),u}\right)^2 \left(e_i, e_j\right) > 0, \tag{2.17}$$

then one has $\langle \widehat{A}(TM) e(F), [M] \rangle = 0$.

In the next two sections, we will eliminate the summation term in (2.17).

3. Connes fibration and sub-Dirac operators

In this section, we reduce the proof of Theorem 1.1 to an estimating result of certain Atiyah–Patodi–Singer elliptic boundary valued problems for deformed sub-Dirac operators constructed on the Connes fibration associated to a given flat vector bundle.

This section is organized as follows. In Section 3.1, we present the construction of the Connes fibration associated to a flat vector bundle as well as certain basic properties of the Connes fibration. In Section 3.2, we construct the needed sub-Dirac operator on the Connes fibration. In Section 3.3, we study the induced sub-Dirac operator on the boundary. In Section 3.4, we introduce certain deformations of the sub-Dirac operators and reduce Theorem 1.1 to an estimating result.

3.1. Connes fibration associated with a flat bundle

Let (M, g^{TM}) be a closed Riemannian manifold, and (F, ∇^F) an oriented real flat vector bundle over M. Following [6, §5] (cf. [7, §2.1]), let $\pi : \mathcal{M} \to M$ be the Connes fibration over M such that for any $x \in M$, $\mathcal{M}_x$ is the space of Euclidean metrics on the vector space F_x. Let $\mathcal{E}^{\perp} = T^V \mathcal{M}$ denote the vertical tangent bundle of this fibration. Then it carries a naturally induced metric $g^{\mathcal{E}^{\perp}}$ such that each $\mathcal{M}_x = \pi^{-1}(x), x \in M$, is of nonpositive sectional curvature. In particular, any two points $p_1, p_2 \in \mathcal{M}_x$ can be joined by a unique geodesic in $\mathcal{M}_x$ (cf. [15]). Let $d^{\mathcal{M}_x}(p_1, p_2)$ denote the length of this geodesic .

By using the flat connection ∇^F, one lifts TM to an integrable horizontal subbundle $\mathcal{E} = T^H \mathcal{M}$ of $T\mathcal{M}$ so that we have a canonical splitting $T\mathcal{M} = \mathcal{E} \oplus \mathcal{E}^{\perp}$.

Set $\mathcal{F} = \pi^* F$. Then there is a canonical Euclidean metric $g^{\mathcal{F}}$ on $\mathcal{F}$ defined as follows: by construction, any $p \in \mathcal{M}$ determines a Euclidean metric on $F_{\pi(p)}$, which in turn determines a metric on $\mathcal{F}_p \simeq \pi^* F_{\pi(p)}$.

Lemma 3.1. *(1). The Bott connection on $(\mathcal{E}^{\perp}, g^{\mathcal{E}^{\perp}})$ is leafwise Euclidean.*
(2). There exists a canonical Euclidean connection on $(\mathcal{F}, g^{\mathcal{F}})$ such that for any $X, Y \in \Gamma(\mathcal{M}, \mathcal{E})$, one has

$$\left(\nabla^{\mathcal{F}}\right)^2 (X, Y) = 0. \tag{3.1}$$

Proof. Let $\widehat{F}$ denote the total space of the flat vector bundle $\pi_F : F \to M$. Then TM lifts to an integrable subbundle $T^H \widehat{F}$ of $T\widehat{F}$ such that $T^H \widehat{F}|_M = TM$, and that $(T\widehat{F}/T^H \widehat{F})|_M \simeq F$.

Following [6, §5] and [7, §2.1], let $\widehat{\pi} : \widehat{\mathcal{F}} \to \widehat{F}$ be the Connes fibration such that for any $x \in \widehat{F}$, $\widehat{\mathcal{F}}_x = \widehat{\pi}^{-1}(x)$ is the space of Euclidean metrics on $T\widehat{F}_x/T^H \widehat{F}_x$. Then one verifies that

$$\mathcal{M} \simeq \widehat{\pi}^{-1}(M). \tag{3.2}$$

By restricting [7, Lemma 1.5] from $\widehat{\mathcal{F}}$ to $\mathcal{M}$, one gets Lemma 3.1. We leave the details to the interested reader. $\square$

Let $g^{\mathcal{E}} = \pi^* g^{TM}$ be the pullback Euclidean metric on $\mathcal{E}$. Let $g^{T\mathcal{M}}$ be the Riemannian metric on $\mathcal{M}$ given by the orthogonal splitting

$$T\mathcal{M} = \mathcal{E} \oplus \mathcal{E}^{\perp}, \quad g^{T\mathcal{M}} = g^{\mathcal{E}} \oplus g^{\mathcal{E}^{\perp}}. \tag{3.3}$$

Let p and $p^{\perp}$ be the orthogonal projections from $T\mathcal{M}$ to $\mathcal{E}$ and $\mathcal{E}^{\perp}$. Let $\nabla^{T\mathcal{M}}$ be the Levi-Civita connection of $g^{T\mathcal{M}}$. Set

$$\nabla^{\mathcal{E}} = p\nabla^{T\mathcal{M}} p, \quad \nabla^{\mathcal{E}^{\perp}} = p^{\perp} \nabla^{T\mathcal{M}} p^{\perp}. \tag{3.4}$$

Then $\nabla^{\mathcal{E}^{\perp}}$ does not depend on $g^{\mathcal{E}}$. Moreover, by Lemma 3.1, one has

$$\left(\nabla^{\mathcal{E}^{\perp}}\right)^2 (X, Y) = 0, \text{ for any } X, Y \in \Gamma(\mathcal{M}, \mathcal{E}). \tag{3.5}$$

Take a Euclidean metric g^F on F, which amounts to taking an embedded section $\jmath : M \hookrightarrow \mathcal{M}$ of M into the Connes fibration $\pi : \mathcal{M} \to M$. Then we have the canonical inclusion $\jmath(M) \subset \mathcal{M}$.

For any $p \in \mathcal{M} \setminus \jmath(M)$, we connect p and $\jmath(\pi(p)) \in \jmath(M)$ by the unique geodesic in $\mathcal{M}_{\pi(p)}$. Let $\sigma(p) \in \mathcal{E}^{\perp}|_p$ denote the unit vector tangent to this geodesic. Set $\rho(p) = d^{\mathcal{M}_{\pi(p)}}(p, \jmath(\pi(p)))$.

By (3.2) and [7, Lemma 2.1], we have the following estimating result.

Lemma 3.2. *There exists $C > 0$, which depends only on the embedding $\jmath : M \hookrightarrow \mathcal{M}$, such that for any $X \in \Gamma(\mathcal{M}, \mathcal{E})$ with $|X| \leqslant 1$, the following pointwise inequality holds on $\mathcal{M}$,*

$$\left| \nabla_X^{\mathcal{E}^\perp} (\rho \sigma) \right| \leqslant C. \tag{3.6}$$

3.2. Sub-Dirac operators on the Connes fibration

Without loss of generality, we can and we will assume that both $\dim M$ and $\mathrm{rk}(F)$ are divisible by 4. Then both $\dim \mathcal{M}$ and $\mathrm{rk}(\mathcal{E}^\perp)$ are even. By passing to a double covering if necessary, we also assume that $\mathcal{E}^\perp$ is oriented.

We assume from now on that M is spin, then $\mathcal{E} = \pi^*(TM)$ is spin and carries a naturally induced spin structure. Let

$$S(\mathcal{E}) = S_+(\mathcal{E}) \oplus S_-(\mathcal{E}) \tag{3.7}$$

be the $\mathbb{Z}_2$-graded Hermitian bundle of spinors associated to $(\mathcal{E}, g^{\mathcal{E}})$. Then $\nabla^{\mathcal{E}}$ induces naturally a Hermitian connection $\nabla^{S(\mathcal{E})}$ on $S(\mathcal{E})$ preserving the $\mathbb{Z}_2$-grading. For $e \in \mathcal{E}$, let $c(e)$ denote the Clifford action of e on $S(\mathcal{E})$, which exchanges $S_\pm(\mathcal{E})$.

Let $\Lambda(\mathcal{E}^{\perp,*})$ be the (complex) exterior algebra bundle of $\mathcal{E}^\perp$ with the $\mathbb{Z}_2$-graded splitting

$$\Lambda\left(\mathcal{E}^{\perp,*}\right) = \Lambda^{\mathrm{even}}\left(\mathcal{E}^{\perp,*}\right) \oplus \Lambda^{\mathrm{odd}}\left(\mathcal{E}^{\perp,*}\right). \tag{3.8}$$

Then $\Lambda(\mathcal{E}^{\perp,*})$ carries a Hermitian metric canonically induced from $g^{\mathcal{E}^\perp}$ and a Hermitian connection $\nabla^{\Lambda(\mathcal{E}^{\perp,*})}$ induced from $\nabla^{\mathcal{E}^\perp}$. For $h \in \mathcal{E}^\perp$, let $c(h)$ and $\widehat{c}(h)$ be the actions of h on $\Lambda(\mathcal{E}^{\perp,*})$ defined as in (2.4).

Let $\Lambda(\mathcal{F}^*)$ be the (complex) exterior algebra bundle of $\mathcal{F}$. The Euclidean connection $\nabla^{\mathcal{F}}$ on $\mathcal{F}$ naturally induces a connection $\nabla^{\Lambda(\mathcal{F}^*)}$ on $\Lambda(\mathcal{F}^*)$, which preserves the metric on $\Lambda(\mathcal{F}^*)$ induced by $g^{\mathcal{F}}$. We denote by $c(\cdot)$ and $\widehat{c}(\cdot)$ the actions of $\mathcal{F}$ on $\Lambda(\mathcal{F}^*)$ defined as in (2.4). Let

$$\Lambda(\mathcal{F}^*) = \Lambda_+(\mathcal{F}^*) \oplus \Lambda_-(\mathcal{F}^*) \tag{3.9}$$

be the $\mathbb{Z}_2$-graded splitting of $\Lambda(\mathcal{F}^*)$ determined as in (2.7)–(2.9).

Using the $\mathbb{Z}_2$-graded vector bundles in (3.7)–(3.9), we form the following $\mathbb{Z}_2$-graded tensor product (see [13, p. 11])

$$\begin{aligned} &S(\mathcal{E}) \widehat{\otimes} \Lambda\left(\mathcal{E}^{\perp,*}\right) \widehat{\otimes} \Lambda\left(\mathcal{F}^*\right) \\ &= \left(S(\mathcal{E}) \widehat{\otimes} \Lambda\left(\mathcal{E}^{\perp,*}\right) \widehat{\otimes} \Lambda\left(\mathcal{F}^*\right)\right)_+ \oplus \left(S(\mathcal{E}) \widehat{\otimes} \Lambda\left(\mathcal{E}^{\perp,*}\right) \widehat{\otimes} \Lambda\left(\mathcal{F}^*\right)\right)_-, \end{aligned} \tag{3.10}$$

which carries the tensor product connection $\nabla^{S(\mathcal{E}) \widehat{\otimes} \Lambda(\mathcal{E}^{\perp,*}) \widehat{\otimes} \Lambda(\mathcal{F}^*)}$ induced by $\nabla^{S(\mathcal{E})}$, $\nabla^{\Lambda(\mathcal{E}^{\perp,*})}$ and $\nabla^{\Lambda(\mathcal{F}^*)}$.

The action of $\mathcal{E}$ on $S(\mathcal{E})$ and that of $\mathcal{E}^\perp$ on $\Lambda(\mathcal{E}^{\perp,*})$ as well as that of $\mathcal{F}$ on $\Lambda(\mathcal{F}^*)$ extend to actions on $S(\mathcal{E}) \widehat{\otimes} \Lambda(\mathcal{E}^{\perp,*}) \widehat{\otimes} \Lambda(\mathcal{F}^*)$ in an obvious way. We still use the same notation to denote these extended actions.

Let $\{h_i\}_{i=1}^{\dim M}$ (resp. $\{h_j\}_{j=\dim M+1}^{\dim \mathcal{M}}$) be an oriented orthonormal basis of $(\mathcal{E}, g^{\mathcal{E}})$ (resp. $(\mathcal{E}^\perp, g^{\mathcal{E}^\perp})$). Let $\mathcal{S}$ be the $\mathrm{End}(T\mathcal{M})$-valued one-form on $\mathcal{M}$ defined by

$$\mathcal{S}(\cdot) = \nabla_{\cdot}^{T\mathcal{M}} - \left(\nabla_{\cdot}^{\mathcal{E}} \oplus \nabla_{\cdot}^{\mathcal{E}^\perp}\right). \tag{3.11}$$

Following [16] and [7, (1.60)], we define a Hermitian connection

$$\widehat{\nabla}_{\cdot} = \nabla_{\cdot}^{S(\mathcal{E}) \widehat{\otimes} \Lambda(\mathcal{E}^{\perp,*}) \widehat{\otimes} \Lambda(\mathcal{F}^*)} + \frac{1}{4} \sum_{i,j=1}^{\dim \mathcal{M}} \left\langle \mathcal{S}(\cdot) h_i, h_j \right\rangle c(h_i) c(h_j) \tag{3.12}$$

on $S(\mathcal{E}) \widehat{\otimes} \Lambda(\mathcal{E}^{\perp,*}) \widehat{\otimes} \Lambda(\mathcal{F}^*)$.

As in [16] and [7, (1.61)], let

$$\begin{aligned} D^{\mathcal{M}} = \sum_{i=1}^{\dim \mathcal{M}} c(h_i) \widehat{\nabla}_{h_i} : \ &\Gamma\left(\mathcal{M}, S(\mathcal{E}) \widehat{\otimes} \Lambda\left(\mathcal{E}^{\perp,*}\right) \widehat{\otimes} \Lambda\left(\mathcal{F}^*\right)\right) \\ &\longrightarrow \Gamma\left(\mathcal{M}, S(\mathcal{E}) \widehat{\otimes} \Lambda\left(\mathcal{E}^{\perp,*}\right) \widehat{\otimes} \Lambda\left(\mathcal{F}^*\right)\right) \end{aligned} \tag{3.13}$$

be the sub-Dirac operator with respect to the spinor bundle $S(\mathcal{E})$. Then $D^{\mathcal{M}}$ is a formally self-adjoint first order elliptic differential operator, which exchanges the $\mathbb{Z}_2$-grading in (3.10). Moreover, as indicated in [7, Remark 1.8], $D^{\mathcal{M}}$ can locally be viewed as a twisted Dirac operator.

Let $\Delta^{\mathcal{M}} = \sum_{i=1}^{\dim \mathcal{M}}(\widehat{\nabla}_{h_i}\widehat{\nabla}_{h_i} - \widehat{\nabla}_{\nabla_{h_i}^{T\mathcal{M}} h_i})$ be the Bochner Laplacian, and $k^{T\mathcal{M}}$ be the scalar curvature of $(T\mathcal{M}, g^{T\mathcal{M}})$. We have the following Lichnerowicz formula,

$$\left(D^{\mathcal{M}}\right)^2 = -\Delta^{\mathcal{M}} + \frac{k^{T\mathcal{M}}}{4}$$

$$+ \frac{1}{8} \sum_{k,\,l=\dim M+1}^{\dim \mathcal{M}} \sum_{i,\,j=1}^{\dim \mathcal{M}} \left\langle h_k, \left(\nabla^{\mathcal{E}^\perp}\right)^2 (h_i, h_j) h_l \right\rangle \cdot c\,(h_i)\, c\,(h_j)\, \widehat{c}\,(h_k)\, \widehat{c}\,(h_l)$$

$$+ \frac{1}{8} \sum_{\mu,\,\nu=1}^{\mathrm{rk}(\mathcal{F})} \sum_{i,\,j=1}^{\dim \mathcal{M}} \left\langle f_\mu, \left(\nabla^{\mathcal{F}}\right)^2 (h_i, h_j) f_\nu \right\rangle \cdot c\,(h_i)\, c\,(h_j)\, \left(\widehat{c}\,(f_\mu)\, \widehat{c}\,(f_\nu) - c\,(f_\mu)\, c(f_\nu)\right), \tag{3.14}$$

where $\{f_\mu\}_{\mu=1}^{\mathrm{rk}\,\mathcal{F}}$ is an orthonormal basis of $(\mathcal{F}, g^{\mathcal{F}})$.

3.3. Induced sub-Dirac operators on the boundary

For any $R > 0$, denote $\mathcal{M}_R = \{p \in \mathcal{M} \mid \rho(p) \leqslant R\}$. Then $\mathcal{M}_R$ is a compact smooth manifold with boundary $\partial \mathcal{M}_R$. We follow the convention as in [17].

Let $\epsilon_R > 0$ be a sufficiently small positive number. We use the inward geodesic flow to identify a neighborhood of $\partial \mathcal{M}_R$ with the collar $\partial \mathcal{M}_R \times [0, \epsilon_R)$. Let $e_{\dim \mathcal{M}}$ be the inward unit normal vector field to $T\partial \mathcal{M}_R$ so that $e_1, \ldots, e_{\dim \mathcal{M}}$ is an oriented orthonormal basis of $T\mathcal{M}|_{\partial \mathcal{M}_R}$. Then using parallel transport with respect to $\nabla^{T\mathcal{M}}$ along the unit speed geodesics perpendicular to $\partial \mathcal{M}_R$, $e_1, \ldots, e_{\dim \mathcal{M}}$ forms an oriented orthonormal basis of $T\mathcal{M}$ over $\partial \mathcal{M}_R \times [0, \epsilon_R)$.

For $1 \leqslant i, j \leqslant \dim \mathcal{M} - 1$, let $\pi_{ij} = \langle \nabla^{T\mathcal{M}}_{e_i} e_j, e_{\dim \mathcal{M}} \rangle|_{\partial \mathcal{M}_R}$ be the second fundamental form of the isometric embedding $i_{\partial \mathcal{M}_R} : \partial \mathcal{M}_R \hookrightarrow \mathcal{M}_R$. Let

$$D^{\partial \mathcal{M}_R} : \Gamma \left(\partial \mathcal{M}_R, \left(S(\mathcal{E}) \widehat{\otimes} \Lambda \left(\mathcal{E}^{\perp,*}\right) \widehat{\otimes} \Lambda \left(\mathcal{F}^*\right)\right)\big|_{\partial \mathcal{M}_R} \right)$$
$$\longrightarrow \Gamma \left(\partial \mathcal{M}_R, \left(S(\mathcal{E}) \widehat{\otimes} \Lambda \left(\mathcal{E}^{\perp,*}\right) \widehat{\otimes} \Lambda \left(\mathcal{F}^*\right)\right)\big|_{\partial \mathcal{M}_R} \right)$$

be the differential operator on $\partial \mathcal{M}_R$ defined by

$$D^{\partial \mathcal{M}_R} = - \sum_{i=1}^{\dim \mathcal{M}-1} c\,(e_{\dim \mathcal{M}})\, c\,(e_i)\, \widehat{\nabla}_{e_i} + \frac{1}{2} \sum_{i=1}^{\dim \mathcal{M}-1} \pi_{ii}\,. \tag{3.15}$$

By [17, Lemmas 2.1 and 2.2], $D^{\partial \mathcal{M}_R}$ is a formally self-adjoint first order elliptic differential operator intrinsically defined on $\partial \mathcal{M}_R$. Also, it preserves the $\mathbb{Z}_2$-grading of $(S(\mathcal{E}) \widehat{\otimes} \Lambda(\mathcal{E}^{\perp,*}) \widehat{\otimes} \Lambda(\mathcal{F}^*))|_{\partial \mathcal{M}_R}$ induced by (3.10).

3.4. Deformations of sub-Dirac operators and their indices

Let $\psi : [0, 1] \to [0, 1]$ be a smooth function such that $\psi(t) = 0$ for $0 \leqslant t \leqslant \frac{2}{3}$, while $\psi(t) = 1$ for $\frac{3}{4} \leqslant t \leqslant 1$. For any $0 < \varepsilon \leqslant 1$ and $R > 0$, let

$$g_{\varepsilon,R}^{T\mathcal{M}} = \left(1 - \psi\left(\frac{\rho}{R}\right)\right) g_\varepsilon^{T\mathcal{M}} + \psi\left(\frac{\rho}{R}\right) g^{T\mathcal{M}} \tag{3.16}$$

be the Riemannian metric on $\mathcal{M}_R$, where $g_\varepsilon^{T\mathcal{M}}$ is the Riemannian metric on $\mathcal{M}$ defined by the orthogonal splitting (cf. [7, (1.11)])

$$T\mathcal{M} = \mathcal{E} \oplus \mathcal{E}^\perp, \quad g_\varepsilon^{T\mathcal{M}} = \varepsilon^2 g^{\mathcal{E}} \oplus g^{\mathcal{E}^\perp}. \tag{3.17}$$

Then

$$g_{\varepsilon,R}^{T\mathcal{M}} = g_\varepsilon^{T\mathcal{M}} \text{ over } \mathcal{M}_{\frac{2}{3}R}, \text{ while } g_{\varepsilon,R}^{T\mathcal{M}} = g^{T\mathcal{M}} \text{ over } \mathcal{M}_R \setminus \mathcal{M}_{\frac{3}{4}R}\,. \tag{3.18}$$

In what follows, we will use the subscripts (or superscripts) "ε, R" to decorate the geometric objects with respect to $g_{\varepsilon,R}^{T\mathcal{M}}$. Let $k_{\varepsilon,R}$ be the scalar curvature of $g_{\varepsilon,R}^{T\mathcal{M}}$. Then by Lemma 3.1 and (3.18) (cf. [7, Proposition 1.4]), one has over $\mathcal{M}_{\frac{2}{3}R}$ that

$$k_{\varepsilon,R} = \frac{\pi^* k^{TM}}{\varepsilon^2} + O_R(1), \tag{3.19}$$

where k^{TM} is the scalar curvature of (TM, g^{TM}), and by $O_R(\cdot)$ we mean that the estimating constant might depend on R.

As in Section 3.2, we construct the sub-Dirac operator

$$D_\varepsilon^{\mathcal{M}_R} : \Gamma \left(\mathcal{M}_R, S_{\varepsilon,R}(\mathcal{E}) \widehat{\otimes} \Lambda \left(\mathcal{E}^{\perp,*}\right) \widehat{\otimes} \Lambda \left(\mathcal{F}^*\right) \right)$$
$$\longrightarrow \Gamma \left(\mathcal{M}_R, S_{\varepsilon,R}(\mathcal{E}) \widehat{\otimes} \Lambda \left(\mathcal{E}^{\perp,*}\right) \widehat{\otimes} \Lambda \left(\mathcal{F}^*\right) \right)$$

on $\mathcal{M}_R$ associated with $g_{\varepsilon,R}^{T\mathcal{M}}$, which is given by

$$D_\varepsilon^{\mathcal{M}_R} = \sum_{j=1}^{\dim M} c_{\varepsilon,R}(h_j) \widehat{\nabla}_{h_j}^{\varepsilon,R} + \sum_{j=\dim M+1}^{\dim \mathcal{M}} c(h_j) \widehat{\nabla}_{h_j}^{\varepsilon,R}, \tag{3.20}$$

where $\{h_j\}_{j=1}^{\dim M}$ is an orthonormal basis of $(\mathcal{E}, (\varepsilon^2(1 - \psi(\frac{\rho}{R})) + \psi(\frac{\rho}{R})) g^{\mathcal{E}})$, and $\widehat{\nabla}^{\varepsilon,R}$ denotes the connection on $S_{\varepsilon,R}(\mathcal{E})\widehat{\otimes} \Lambda(\mathcal{E}^{\perp,*})\widehat{\otimes}\Lambda(\mathcal{F}^*)$ associated with $g_{\varepsilon,R}^{T\mathcal{M}}$ as in (3.12). In particular, in view of [7, Remark 1.8], one deduces that for any $V \in \Gamma(\mathcal{M}_R, \mathcal{E}^\perp)$,

$$\left[\widehat{\nabla}^{\varepsilon,R}, \widehat{c}(V)\right] = \widehat{c}\left(\nabla^{\mathcal{E}^\perp}V\right). \tag{3.21}$$

Inspired by [7, (2.21) and Remark 2.6], we introduce the following key deformation of the sub-Dirac operator $D_\varepsilon^{\mathcal{M}_R}$,

$$D_{\varepsilon,R}^{\mathcal{M}_R} = D_\varepsilon^{\mathcal{M}_R} + \frac{\widehat{c}(\rho\sigma)}{\varepsilon R} : \Gamma\left(\mathcal{M}_R, S_{\varepsilon,R}(\mathcal{E})\widehat{\otimes}\Lambda\left(\mathcal{E}^{\perp,*}\right)\widehat{\otimes}\Lambda\left(\mathcal{F}^*\right)\right)$$
$$\longrightarrow \Gamma\left(\mathcal{M}_R, S_{\varepsilon,R}(\mathcal{E})\widehat{\otimes}\Lambda\left(\mathcal{E}^{\perp,*}\right)\widehat{\otimes}\Lambda\left(\mathcal{F}^*\right)\right). \tag{3.22}$$

Recall that $D^{\partial\mathcal{M}_R}$ is defined in (3.15). Set

$$D_\varepsilon^{\partial\mathcal{M}_R} = D^{\partial\mathcal{M}_R} - \frac{1}{\varepsilon}c\left(e_{\dim\mathcal{M}}\right)\widehat{c}(\sigma) : \Gamma\left(\partial\mathcal{M}_R, \left(S(\mathcal{E})\widehat{\otimes}\Lambda\left(\mathcal{E}^{\perp,*}\right)\widehat{\otimes}\Lambda\left(\mathcal{F}^*\right)\right)\big|_{\partial\mathcal{M}_R}\right)$$
$$\longrightarrow \Gamma\left(\partial\mathcal{M}_R, \left(S(\mathcal{E})\widehat{\otimes}\Lambda\left(\mathcal{E}^{\perp,*}\right)\widehat{\otimes}\Lambda\left(\mathcal{F}^*\right)\right)\big|_{\partial\mathcal{M}_R}\right). \tag{3.23}$$

Then $D_\varepsilon^{\partial\mathcal{M}_R}$ is the induced boundary operator of $D_{\varepsilon,R}^{\mathcal{M}_R}$. Write

$$D_{\varepsilon,R,\pm}^{\mathcal{M}_R} = D_{\varepsilon,R}^{\mathcal{M}_R}\Big|_{\Gamma(\mathcal{M}_R, (S_{\varepsilon,R}(\mathcal{E})\widehat{\otimes}\Lambda(\mathcal{E}^{\perp,*})\widehat{\otimes}\Lambda(\mathcal{F}^*))_\pm)},$$

$$D_{\varepsilon,\pm}^{\partial\mathcal{M}_R} = D_\varepsilon^{\partial\mathcal{M}_R}\Big|_{\Gamma(\partial\mathcal{M}_R, (S(\mathcal{E})\widehat{\otimes}\Lambda(\mathcal{E}^{\perp,*})\widehat{\otimes}\Lambda(\mathcal{F}^*))_\pm|_{\partial\mathcal{M}_R})}.$$

For any $\lambda \in \mathrm{Sp}\{D_\varepsilon^{\partial\mathcal{M}_R}\}$, the spectrum of $D_\varepsilon^{\partial\mathcal{M}_R}$, let E_λ be the eigenspace corresponding to λ. For any $b \in \mathbb{R}$, denote by $P_{\geqslant b,\varepsilon,R}$ and $P_{>b,\varepsilon,R}$ the orthogonal projections from the L^2-completion of

$$\Gamma\left(\partial\mathcal{M}_R, \left(S(\mathcal{E})\widehat{\otimes}\Lambda\left(\mathcal{E}^{\perp,*}\right)\widehat{\otimes}\Lambda\left(\mathcal{F}^*\right)\right)\big|_{\partial\mathcal{M}_R}\right)$$

onto $\oplus_{\lambda\geqslant b}E_\lambda$ and $\oplus_{\lambda>b}E_\lambda$, respectively. Let $P_{\geqslant b,\varepsilon,R,\pm}$ and $P_{>b,\varepsilon,R,\pm}$ be the restrictions of $P_{\geqslant b,\varepsilon,R}$ and $P_{>b,\varepsilon,R}$ on the L^2-completions of

$$\Gamma\left(\partial\mathcal{M}_R, \left(S(\mathcal{E})\widehat{\otimes}\Lambda\left(\mathcal{E}^{\perp,*}\right)\widehat{\otimes}\Lambda\left(\mathcal{F}^*\right)\right)_\pm\Big|_{\partial\mathcal{M}_R}\right).$$

One verifies (cf. [18] and [17]) that the Atiyah–Patodi–Singer boundary valued problems $(D_{\varepsilon,R,+}^{\mathcal{M}_R}, P_{\geqslant 0,\varepsilon,R,+})$ and $(D_{\varepsilon,R,-}^{\mathcal{M}_R}, P_{>0,\varepsilon,R,-})$ are elliptic, and that $(D_{\varepsilon,R,+}^{\mathcal{M}_R}, P_{\geqslant 0,\varepsilon,R,+})$ is the adjoint of $(D_{\varepsilon,R,-}^{\mathcal{M}_R}, P_{>0,\varepsilon,R,-})$. Set

$$\mathrm{ind}\left(D_{\varepsilon,R,+}^{\mathcal{M}_R}, P_{\geqslant 0,\varepsilon,R,+}\right) = \dim\ker\left(D_{\varepsilon,R,+}^{\mathcal{M}_R}, P_{\geqslant 0,\varepsilon,R,+}\right)$$
$$- \dim\ker\left(D_{\varepsilon,R,-}^{\mathcal{M}_R}, P_{>0,\varepsilon,R,-}\right). \tag{3.24}$$

Let $\|\cdot\|_{\partial\mathcal{M}_R}$ denote the L^2-norm with respect to the volume element $dv_{\partial\mathcal{M}_R}$ of $(\partial\mathcal{M}_R, g^{T\mathcal{M}}|_{\partial\mathcal{M}_R})$. Let e_i, $1 \leqslant i \leqslant \dim\mathcal{M} - 1$, be an orthonormal basis of $(\partial\mathcal{M}, g^{T\mathcal{M}}|_{\partial\mathcal{M}})$.

Proposition 3.3. *For any (fixed) $R > 0$, there exists $\varepsilon_0 > 0$ (which may depend on R) such that for any $0 < \varepsilon \leqslant \varepsilon_0$, one has*

$$\left\|D_\varepsilon^{\partial\mathcal{M}_R}s\right\|_{\partial\mathcal{M}_R}^2 \geqslant \frac{1}{2}\sum_{i=1}^{\dim\mathcal{M}}\left\|\widehat{\nabla}_{e_i}s\right\|_{\partial\mathcal{M}_R}^2 + \frac{1}{4\varepsilon^2}\|s\|_{\partial\mathcal{M}_R}^2, \tag{3.25}$$

for any $s \in \Gamma\left(\partial\mathcal{M}_R, \left(S(\mathcal{E})\widehat{\otimes}\Lambda\left(\mathcal{E}^{\perp,*}\right)\widehat{\otimes}\Lambda\left(\mathcal{F}^*\right)\right)\big|_{\partial\mathcal{M}_R}\right).$

Proof. From (3.23), one deduces that

$$\left(D_\varepsilon^{\partial\mathcal{M}_R}\right)^2 = \left(D^{\partial\mathcal{M}_R}\right)^2 - \frac{1}{\varepsilon}\left[D^{\partial\mathcal{M}_R}, c\left(e_{\dim\mathcal{M}}\right)\widehat{c}(\sigma)\right] + \frac{1}{\varepsilon^2}. \tag{3.26}$$

It is easy to see that $[D^{\partial\mathcal{M}_R}, c(e_{\dim\mathcal{M}})\widehat{c}(\sigma)]$ is of zeroth order. On the other hand, by the Lichnerowicz formula one deduces that

$$\left\langle\left(D^{\partial\mathcal{M}_R}\right)^2s, s\right\rangle_{\partial\mathcal{M}_R} \geqslant \frac{1}{2}\sum_{i=1}^{\dim\mathcal{M}}\left\|\widehat{\nabla}_{e_i}s\right\|_{\partial\mathcal{M}_R}^2 + O_R(1)\|s\|_{\partial\mathcal{M}_R}^2. \tag{3.27}$$

From (3.26) and (3.27), one gets (3.25). $\square$

Recall that D^M is introduced in (2.13) and (2.14).

Since $D_\varepsilon^{\mathcal{M}_R} = D^{\mathcal{M}_R}$ near $\partial \mathcal{M}_R$, by a simple homotopy in the interior of $\mathcal{M}_R$ and a simplified version of the analytic Riemann–Roch property proved in [9, Theorem 2.4], one obtains from Proposition 3.3 the following proposition.

Proposition 3.4. *For any $R > 0$, there exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon \leqslant \varepsilon_0$, then*

$$\mathrm{ind}\left(D_{\varepsilon,R,+}^{\mathcal{M}_R}, P_{\geqslant 0,\varepsilon,R,+}\right) = \mathrm{ind}\left(D_+^M\right). \tag{3.28}$$

Let $dv_{\varepsilon,R}$ denote the volume element of $(\mathcal{M}_R, g_{\varepsilon,R}^{T\mathcal{M}})$, and $\|\cdot\|_{\varepsilon,R}$ denote the corresponding L^2-norm. From (2.15) and Propositions 3.3 and 3.4, one sees that in order to prove Theorem 1.1, one need only to prove the following result.

Theorem 3.5. *Under the assumptions of Theorem 1.1, there exists $R_0 > 0$ such that for any $R \geqslant R_0$, there exist $c_1 > 0$ and $\varepsilon_1 > 0$ such that for any $0 < \varepsilon \leqslant \varepsilon_1$ and smooth section $s \in \Gamma(\mathcal{M}_R, S_{\varepsilon,R}(\mathcal{E})\widehat{\otimes}\Lambda(\mathcal{E}^{\perp,*})\widehat{\otimes}\Lambda(\mathcal{F}^*))$ verifying $P_{\geqslant 0,\varepsilon,R}(s|_{\partial\mathcal{M}_R}) = 0$, one has*

$$\left\|D_{\varepsilon,R}^{\mathcal{M}_R}s\right\|_{\varepsilon,R}^2 \geqslant c_1 \left(\sum_{i=1}^{\dim\mathcal{M}} \left\|\widehat{\nabla}_{h_i}^{\varepsilon,R}s\right\|_{\varepsilon,R}^2 + \frac{1}{\varepsilon^2}\|s\|_{\varepsilon,R}^2\right). \tag{3.29}$$

4. Proof of Theorem 3.5

This section is organized as follows. In Section 4.1, we establish an estimating result near $\partial\mathcal{M}_R$. In Section 4.2, we establish two interior estimating results. In Section 4.3, we complete the proof of Theorem 3.5.

4.1. The estimate near the boundary

In this subsection, we prove the following estimating result near $\partial\mathcal{M}_R$.

Proposition 4.1. *For any $R > 0$, there exist $c_2 > 0$ and $\varepsilon_2 > 0$ such that for any $0 < \varepsilon \leqslant \varepsilon_2$ and any $s \in \Gamma(\mathcal{M}_R, S_{\varepsilon,R}(\mathcal{E})\widehat{\otimes}\Lambda(\mathcal{E}^{\perp,*})\widehat{\otimes}\Lambda(\mathcal{F}^*))$ verifying $\mathrm{Supp}(s) \subseteq \mathcal{M}_R \setminus \mathcal{M}_{\frac{3}{4}R}$ and $P_{\geqslant 0,\varepsilon,R}(s|_{\partial\mathcal{M}_R}) = 0$, one has*

$$\left\|D_{\varepsilon,R}^{\mathcal{M}_R}s\right\|_{\varepsilon,R}^2 \geqslant c_2 \sum_{i=1}^{\dim\mathcal{M}} \left\|\widehat{\nabla}_{h_i}s\right\|_{\varepsilon,R}^2 + \frac{1}{2\,\varepsilon^2}\|s\|_{\varepsilon,R}^2. \tag{4.1}$$

Proof. Since $\psi(\frac{\rho}{R}) = 1$ on $\mathcal{M}_R \setminus \mathcal{M}_{\frac{3}{4}R}$, by (3.20) and (3.22), one has on $\mathcal{M}_R \setminus \mathcal{M}_{\frac{3}{4}R}$ that

$$D_{\varepsilon,R}^{\mathcal{M}_R} = D^{\mathcal{M}_R} + \frac{\widehat{c}(\rho\sigma)}{\varepsilon R}. \tag{4.2}$$

We now proceed as in the proof of [10, Proposition 2.4]. By Green's formula (cf. [17, (2.28)]), one deduces that

$$\left\|D_{\varepsilon,R}^{\mathcal{M}_R}s\right\|_{\varepsilon,R}^2 = \int_{\mathcal{M}_R} \left\langle s, \left(D_{\varepsilon,R}^{\mathcal{M}_R}\right)^2 s\right\rangle dv_{\varepsilon,R}$$
$$+ \int_{\partial\mathcal{M}_R} \left\langle s, c\,(e_{\dim\mathcal{M}})\,D_{\varepsilon,R}^{\mathcal{M}_R}s\right\rangle dv_{\partial\mathcal{M}_R} \tag{4.3}$$

for any $s \in \Gamma(\mathcal{M}_R, S_{\varepsilon,R}(\mathcal{E})\widehat{\otimes}\Lambda(\mathcal{E}^{\perp,*})\widehat{\otimes}\Lambda(\mathcal{F}^*))$ with $\mathrm{Supp}(s) \subseteq \mathcal{M}_R \setminus \mathcal{M}_{\frac{3}{4}R}$.

By (4.2), one has

$$\left(D_{\varepsilon,R}^{\mathcal{M}_R}\right)^2 = \left(D^{\mathcal{M}_R}\right)^2 + \frac{1}{\varepsilon R}\left[D^{\mathcal{M}_R}, \widehat{c}(\rho\sigma)\right] + \frac{\rho^2}{\varepsilon^2 R^2}. \tag{4.4}$$

Following [17, (2.26) and (2.27)], one verifies that on $\partial\mathcal{M}_R$,

$$c(e_{\dim\mathcal{M}})D_{\varepsilon,R}^{\mathcal{M}_R} = -\widehat{\nabla}_{e_{\dim\mathcal{M}}} - D_\varepsilon^{\partial\mathcal{M}_R} + \frac{1}{2}\sum_{i=1}^{\dim\mathcal{M}-1}\pi_{ii}. \tag{4.5}$$

By using the Lichnerowicz formula for $(D^{\mathcal{M}_R})^2$ and proceeding as in [10, (2.10) and (2.11)], one gets for section s with $\mathrm{Supp}(s) \subseteq \mathcal{M}_R \setminus \mathcal{M}_{\frac{3}{4}R}$ that

$$\int_{\mathcal{M}_R} \left\langle s, \left(D^{\mathcal{M}_R}\right)^2 s\right\rangle dv_{\varepsilon,R} - \int_{\partial\mathcal{M}_R} \left\langle s, \widehat{\nabla}_{e_{\dim\mathcal{M}}}\right\rangle dv_{\partial\mathcal{M}_R}$$
$$= \sum_{i=1}^{\dim\mathcal{M}} \left\|\widehat{\nabla}_{h_i}s\right\|^2 + O_R(1)\|s\|^2. \tag{4.6}$$

By Proposition 3.3 and proceeding as in [10, (2.21)], one sees that when $\varepsilon > 0$ is small enough, for any smooth section s verifying $P_{\geq 0, \varepsilon, R}(s|_{\partial \mathcal{M}_R}) = 0$, one has that

$$\int_{\partial \mathcal{M}_R} \langle s, -D_\varepsilon^{\partial \mathcal{M}_R} s \rangle \, dv_{\partial \mathcal{M}_R} + \frac{1}{2} \int_{\partial \mathcal{M}_R} \left\langle s, \sum_{i=1}^{\dim \mathcal{M}-1} \pi_{ii} \, s \right\rangle dv_{\partial \mathcal{M}_R} \geq 0. \tag{4.7}$$

Since $[D^{\mathcal{M}_R}, \widehat{c}(\rho\sigma)]$ is of zeroth order, and $\frac{\rho}{R} \geq \frac{3}{4}$ on $\mathcal{M}_R \setminus \mathcal{M}_{\frac{3}{4}R}$, from (4.3)–(4.7), one gets (4.1). □

4.2. The interior estimates

From now on, we assume that there exists $\delta > 0$ such that

$$k^{TM} \geq \delta \text{ over } M. \tag{4.8}$$

We prove two interior estimating results. The first one is as follows.

Proposition 4.2. *There exists $R_1 > 0$ such that for any $R \geq R_1$, there exist $c_3 > 0$ and $\varepsilon_3 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_3$, one has*

$$\left\| D_{\varepsilon, R}^{\mathcal{M}_R} s \right\|_{\varepsilon, R}^2 \geq c_3 \sum_{i=1}^{\dim \mathcal{M}} \left\| \widehat{\nabla}_{h_i}^{\varepsilon, R} s \right\|_{\varepsilon, R}^2 + \frac{\delta}{9 \, \varepsilon^2} \|s\|_{\varepsilon, R}^2, \tag{4.9}$$

for any $s \in \Gamma(\mathcal{M}_R, S_{\varepsilon, R}(\mathcal{E}) \widehat{\otimes} \Lambda(\mathcal{E}^{\perp, *}) \widehat{\otimes} \Lambda(\mathcal{F}^*))$ supported in $\mathcal{M}_{\frac{2}{3}R}$.

Proof. Recall that one has $\psi(\frac{\rho}{R}) = 1$ on $\mathcal{M}_{\frac{2}{3}R}$. From (3.20), one has on $\mathcal{M}_{\frac{2}{3}R}$ that

$$\left(D_{\varepsilon, R}^{\mathcal{M}_R} \right)^2 = \left(D_\varepsilon^{\mathcal{M}_R} \right)^2 + \frac{1}{\varepsilon R} \left[D_\varepsilon^{\mathcal{M}_R}, \widehat{c}(\rho\sigma) \right] + \frac{\rho^2}{\varepsilon^2 R^2}. \tag{4.10}$$

In view of (3.20), (3.21) and (4.10), we compute

$$\left[D_\varepsilon^{\mathcal{M}_R}, \widehat{c}(\rho\sigma) \right] = \sum_{i=1}^{\dim M} c_{\varepsilon, R}(h_i) \widehat{c} \left(\nabla_{h_i}^{\mathcal{E}^\perp}(\rho\sigma) \right) + \sum_{j=\dim M+1}^{\dim \mathcal{M}} c(h_j) \widehat{c} \left(\nabla_{h_j}^{\mathcal{E}^\perp}(\rho\sigma) \right). \tag{4.11}$$

From Lemma 3.2 and (4.11), one finds

$$\left[D_\varepsilon^{\mathcal{M}_R}, \widehat{c}(\rho\sigma) \right] = \frac{O(1)}{\varepsilon} + O_R(1). \tag{4.12}$$

From Lemma 3.1, (3.14), (3.19) and (4.8), one finds (compare with [7, (1.71) and (2.28)])

$$\left(D_\varepsilon^{\mathcal{M}_R} \right)^2 \geq -\Delta_{\varepsilon, R}^{\mathcal{M}_R} + \frac{\delta}{4 \, \varepsilon^2} + \frac{O_R(1)}{\varepsilon}, \tag{4.13}$$

where $-\Delta_{\varepsilon, R}^{\mathcal{M}_R} \geq 0$ is the corresponding Bochner Laplacian.

From (4.10), (4.12) and (4.13), one completes the proof of Proposition 4.2 easily. □

The second interior estimating result can be stated as follows.

Proposition 4.3. *There exists $R_2 > 0$ such that for any $R \geq R_2$, there exist $c_4 > 0$ and $\varepsilon_4 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_4$, one has*

$$\left\| D_{\varepsilon, R}^{\mathcal{M}_R} s \right\|_{\varepsilon, R}^2 \geq c_4 \sum_{i=1}^{\dim \mathcal{M}} \left\| \widehat{\nabla}_{h_i}^{\varepsilon, R} s \right\|_{\varepsilon, R}^2 + \frac{1}{11 \, \varepsilon^2} \|s\|_{\varepsilon, R}^2, \tag{4.14}$$

for any $s \in \Gamma(\mathcal{M}_R, S_{\varepsilon, R}(\mathcal{E}) \widehat{\otimes} \Lambda(\mathcal{E}^{\perp, *}) \widehat{\otimes} \Lambda(\mathcal{F}^*))$ supported in $\mathcal{M}_{\frac{7}{8}R} \setminus \mathcal{M}_{\frac{1}{3}R}$.

Proof. On $\mathcal{M}_{\frac{7}{8}R} \setminus \mathcal{M}_{\frac{1}{3}R}$, one has

$$\frac{\rho}{R} \geq \frac{1}{3}. \tag{4.15}$$

Also, one finds that (4.12) still holds on $\mathcal{M}_{\frac{7}{8}R} \setminus \mathcal{M}_{\frac{1}{3}R}$.

From (4.10), (4.12) and (4.15), one finds that there exists $R_3 > 0$ such that for any $R \geqslant R_3$, when $\varepsilon > 0$ is small enough, one has

$$\left\| D_{\varepsilon,R}^{\mathcal{M}_R} s \right\|_{\varepsilon,R}^2 \geqslant \left\| D_\varepsilon^{\mathcal{M}_R} s \right\|_{\varepsilon,R}^2 + \frac{1}{10\,\varepsilon^2} \left\| s \right\|_{\varepsilon,R}^2, \tag{4.16}$$

for any $s \in \Gamma(\mathcal{M}_R, S_{\varepsilon,R}(\mathcal{E}) \widehat{\otimes} \Lambda(\mathcal{E}^{\perp,*}) \widehat{\otimes} \Lambda(\mathcal{F}^*))$ supported in $\mathcal{M}_{\frac{7}{8}R} \setminus \mathcal{M}_{\frac{1}{3}R}$.

Formula (4.14) follows easily from (4.16). $\quad\square$

4.3. Proof of Theorem 3.5

Let $\beta_1 : [0, 1] \to [0, 1]$ be a smooth function such that $\beta_1(t) = 0$ for $0 \leqslant t \leqslant \frac{3}{4} + \frac{1}{100}$, while $\beta_1(t) = 1$ for $\frac{3}{4} + \frac{1}{50} \leqslant t \leqslant 1$. Let $\beta_2 : [0, 1] \to [0, 1]$ be a smooth function such that $\beta_2(t) = 1$ for $0 \leqslant t \leqslant \frac{1}{2} + \frac{1}{100}$, while $\beta_2(t) = 0$ for $\frac{2}{3} - \frac{1}{50} \leqslant t \leqslant 1$.

Inspired by [8, pp. 115–116], let α_1, α_2 and α_3 be the smooth functions on $\mathcal{M}_R$ defined by

$$\alpha_1 = \frac{\beta_1\left(\frac{\rho}{R}\right)}{\sqrt{\beta_1\left(\frac{\rho}{R}\right)^2 + \beta_2\left(\frac{\rho}{R}\right)^2 + \left(1 - \beta_1\left(\frac{\rho}{R}\right) - \beta_2\left(\frac{\rho}{R}\right)\right)^2}}, \tag{4.17}$$

$$\alpha_2 = \frac{\beta_2\left(\frac{\rho}{R}\right)}{\sqrt{\beta_1\left(\frac{\rho}{R}\right)^2 + \beta_2\left(\frac{\rho}{R}\right)^2 + \left(1 - \beta_1\left(\frac{\rho}{R}\right) - \beta_2\left(\frac{\rho}{R}\right)\right)^2}} \tag{4.18}$$

and

$$\alpha_3 = \frac{1 - \beta_1\left(\frac{\rho}{R}\right) - \beta_2\left(\frac{\rho}{R}\right)}{\sqrt{\beta_1\left(\frac{\rho}{R}\right)^2 + \beta_2\left(\frac{\rho}{R}\right)^2 + \left(1 - \beta_1\left(\frac{\rho}{R}\right) - \beta_2\left(\frac{\rho}{R}\right)\right)^2}}. \tag{4.19}$$

Then $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$ on $\mathcal{M}_R$. Thus, for any $s \in \Gamma(\mathcal{M}_R, S_{\varepsilon,R}(\mathcal{E}) \widehat{\otimes} \Lambda(\mathcal{E}^{\perp,*}) \widehat{\otimes} \Lambda(\mathcal{F}^*))$, one has

$$\begin{aligned}
\left\| D_{\varepsilon,R}^{\mathcal{M}_R} s \right\|_{\varepsilon,R}^2 &= \sum_{i=1}^{3} \left\| \alpha_i D_{\varepsilon,R}^{\mathcal{M}_R} s \right\|_{\varepsilon,R}^2 \\
&\geqslant \frac{1}{2} \sum_{i=1}^{3} \left\| D_{\varepsilon,R}^{\mathcal{M}_R} (\alpha_i s) \right\|_{\varepsilon,R}^2 - \sum_{i=1}^{3} \left\| c_{\varepsilon,R} (d\alpha_i) s \right\|_{\varepsilon,R}^2,
\end{aligned} \tag{4.20}$$

where we identify each $d\alpha_i$ ($i = 1, 2, 3$) with the gradient of α_i.

From Lemma 3.2 and (4.17)–(4.19), one finds (compare with [7, (2.47)])

$$\sum_{i=1}^{3} \left| c_{\varepsilon,R} (d\alpha_i) \right|_{\varepsilon,R} = \frac{O(1)}{\varepsilon R} + O_R(1). \tag{4.21}$$

Clearly, $\mathrm{Supp}(\alpha_1 s) \subseteq \mathcal{M}_R \setminus \mathcal{M}_{\frac{3}{4}R}$, $\mathrm{Supp}(\alpha_2 s) \subseteq \mathcal{M}_{\frac{2}{3}R}$ and $\mathrm{Supp}(\alpha_3 s) \subseteq \mathcal{M}_{\frac{7}{8}R} \setminus \mathcal{M}_{\frac{1}{3}R}$.

From Propositions 4.1–4.3 and formulas (4.20) and (4.21), one completes the proof of Theorem 3.5 easily (compare with [8, pp. 115–117]).

The proof of Theorem 1.1 is thus also completed.

Acknowledgments

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