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Invariant connections and ∇-Einstein structures on isotropy irreducible spaces

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ABSTRACT

This paper is devoted to a systematic study and classification of invariant affine or metric connections on certain classes of naturally reductive spaces. For any non-symmetric, effective, strongly isotropy irreducible[2] homogeneous Riemannian manifold $(M = G/K, g)$, we compute the dimensions of the spaces of G-invariant affine and metric connections. For such manifolds we also describe the space of invariant metric connections with skew-torsion. For the compact Lie group U_n we classify all bi-invariant metric connections, by introducing a new family of bi-invariant connections whose torsion is of vectorial type. Next we present applications related with the notion of ∇-Einstein manifolds with skew-torsion. In particular, we classify all such invariant structures on any non-symmetric strongly isotropy irreducible homogeneous space.

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0. Introduction

Motivation. Given ahomogeneous space $M = G/K$ with a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, a G-invariant affine connection ∇ is nothing but a connection in the frame bundle $F(M) = G \times_K \mathrm{GL}(\mathfrak{m})$ of M which is also G-invariant. The first studies of invariant connections were performed by Nomizu [36], Wang [44] and Kostant [30] during the fifties. After that, homogeneous connections on principal bundles have attracted the interest of both mathematicians and physicists, with several different perspectives; for example Cartan connections and parabolic geometries [16], Lie triple systems and Yamaguti–Lie algebras [23,12], Yang–Mills and gauge theories [27,31], etc. From another point of view, invariant connections are crucial in the holonomy theory of naturally reductive spaces and Dirac operators, mainly due to the special properties of the *canonical connection* (or the characteristic connection in terms of special structures, see [29,38,37,20,1,4,6]).

According to [44], given a G-homogeneous principal bundle $P \to G/K$ with structure group U, there is a bijective correspondence between G-invariant connections on P and certain linear maps $\Lambda : \mathfrak{g} \to \mathfrak{u}$, where $\mathfrak{g}, \mathfrak{u}$ are the Lie algebras of G and U, respectively. Wang's correspondence was successfully used by Laquer [32,33] during the nineties to describe the set of

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2 We will use the abbreviation SII for *strongly isotropy irreducible*.

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Table 1
Invariant connections on compact irreducible symmetric spaces due to [32,33].

Type I	$M = G/K$	Invariant connections $\mathcal{A}ff_G(F(M))$
AI	$\mathrm{SU}_n/\mathrm{SO}_n$ $(n \geq 3)$	1-dimensional family
AII	$\mathrm{SU}_{2n}/\mathrm{Sp}_n$ $(n \geq 3)$	1-dimensional family
EIV	$\mathrm{E}_6/\mathrm{F}_4$	1-dimensional family
	All the other cases	Canonical connection ≡ Levi-Civita connection
Type II	$M = (G \times G)/\Delta G$	Bi-invariant connections $\mathcal{A}ff_{G\times G}(F(M))$
	SU_n $(n \geq 3)$	2-dimensional family
	All the other simple Lie groups	1-dimensional family (inducing the flat ± 1-connections)

invariant affine connections, denoted by $\mathcal{A}ff_G(F(G/K))$, on compact irreducible Riemannian symmetric spaces $M = G/K$. For most cases, Laquer proved that $\mathcal{A}ff_G(F(G/K))$ consists of the canonical connection (simple Lie groups admit a line of canonical connections), except for a few cases where new 1-parameter families arise, see Table 1. By contrast, much less is known about invariant connections on *non-symmetric* homogeneous spaces, even in the isotropy irreducible case. For example, the first author in [17], considered invariant connections on manifolds G/K diffeomorphic to a symmetric space, which however do not induce a symmetric pair (G, K), e.g. $\mathrm{G}_2/\mathrm{SU}_3 \cong \mathrm{S}^6$ and $\mathrm{Spin}_7/\mathrm{G}_2 \cong \mathrm{S}^7$. There, it was shown that the space of G_2-invariant affine or metric connections on the sphere $\mathrm{S}^6 = \mathrm{G}_2/\mathrm{SU}(3)$ is 2-dimensional, while the space of Spin_7-invariant affine or metric connections on the 7-sphere $\mathrm{S}^7 = \mathrm{Spin}_7/\mathrm{G}_2$ is 1-dimensional. This is a remarkable result, since the only SO_7- (resp. SO_8)-invariant affine (or metric) connection on the symmetric space $\mathrm{S}^6 = \mathrm{SO}_7/\mathrm{SO}_6$ (resp. $\mathrm{S}^7 = \mathrm{SO}_8/\mathrm{SO}_7$) is the canonical connection.

Motivated by this simple result, in this article we classify invariant affine connections on *(compact) non-symmetric strongly isotropy irreducible homogeneous Riemannian manifolds*. A connected effective homogeneous space G/K is called isotropy irreducible if K acts irreducibly on $T_o(G/K)$ via the isotropy representation. If the identity component K_0 of K also acts irreducibly on $T_o(G/K)$, then G/K is called *strongly isotropy irreducible*. Obviously, any strongly isotropy irreducible space is also isotropy irreducible but the converse is false, see [13]. Non-symmetric strongly isotropy irreducible homogeneous spaces were originally classified by Manturov (see for example [13,45]) and were later studied by Wolf [46] and others. Any SII space admits a unique invariant Einstein metric, the so-called Killing metric and in the non-compact case such a manifold is a symmetric space of non-compact type. In fact, SII spaces share many properties with symmetric spaces and indeed, any irreducible (as Riemannian manifold) symmetric space is strongly isotropy irreducible. A conceptual relationship between symmetric spaces and SII spaces was explained in [45]. More recently, isotropy irreducible homogeneous spaces endowed with their canonical connection ∇^c was shown to have a special relationship with geometric structures with torsion (see [20,19]).

Outline and classification results. After recalling preliminaries in Section 1 about (invariant) metric connections and their torsion types, in Section 2 we fix a reductive homogeneous space $(M = G/K, \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m})$ and introduce the notion of generalized derivations of a tensor $F : \otimes^p\mathfrak{m} \to \mathfrak{m}$. When F is $\mathrm{Ad}(K)$-invariant and $\mu \in \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$ is a K-intertwining map, we prove that μ induces a generalized derivation of F if and only if F is ∇^μ-parallel, where ∇^μ is the invariant connection on M associated to μ (Theorem 2.5). Moreover, we conclude that for an invariant tensor field F the operation induced by a generalized derivation $\mu \in \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$, coincides with the covariant derivative $\nabla^\mu F$ (Corollary 2.6). Then we consider derivations on $\mathfrak{m}$ and provide necessary and sufficient conditions for their existence (Theorem 2.8), generalizing results from [17].

Next, in Section 3 we present a series of new results related to invariant connections and their torsion type, on compact, effective, naturally reductive Riemannian manifolds. In particular, we examine both the symmetric and non-symmetric case and we develop some theory available for the classification of all G-invariant metric connections, with respect to a naturally reductive metric (see Lemma 3.9, Lemma 3.12, Theorem 3.13). In fact, in this way we correct some wrong conclusions given in [6,17]. For example, for the compact Lie group U_n $(n \geq 3)$ endowed with a bi-invariant metric we present a class of bi-invariant metric connections whose torsion is not a 3-form, but of vectorial type (Theorem 3.15, Proposition 3.20).

In Section 4 we focus on (compact) non-symmetric, strongly isotropy irreducible homogeneous Riemannian manifolds $(M = G/K, g = -B|_{\mathfrak{m}})$ with aim the classification of all G-invariant affine or metric connections. We always work with an effective G-action, and based on our previous results on effective naturally reductive spaces we first prove that a G-invariant metric connection on $(M = G/K, g = -B|_{\mathfrak{m}})$ cannot admit a component of vectorial type (Proposition 4.1). Then, in the spin case we describe an application about the formal self-adjointness of Dirac operators associated to invariant metric connections on such types of homogeneous spaces (Corollary 4.4).

Notice now that any (effective) non-symmetric SII space $M = G/K$ admits a family of invariant metric connections induced by the $\mathrm{Ad}(K)$-invariant bilinear map $\eta^\alpha : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ with $\eta^\alpha(X, Y) := \frac{1-\alpha}{2}[X, Y]_{\mathfrak{m}}$. In full details, this family, which we call the *Lie bracket family*, has the form

$$\nabla^\alpha_X Y = \nabla^c_X Y + \eta^\alpha(X, Y) = \nabla^g_X Y + \frac{\alpha}{2} T^c(X, Y),$$

where ∇^c denotes the canonical connection associated to $\mathfrak{m}$ and ∇^g the Levi-Civita connection of the Killing metric.[2] Hence, its torsion is an invariant 3-form on M, given by $T^\alpha = \alpha \cdot T^c$, where T^c is the torsion of ∇^c (see [1,17]). However, we will show that in general the family ∇^α does not exhaust all G-invariant metric connections, even with skew-torsion. In particular, for the classification of invariant connections on $M = G/K$ one needs to decompose the modules $\Lambda^2(\mathfrak{m})$ and $\mathrm{Sym}^2(\mathfrak{m})$ into irreducible submodules. For such a procedure we mainly use the LiE program,[3] but also provide examples of how such spaces can be treated only by pure representation theory arguments, without the aid of a computer (see Section 4.5). As a result, for any effective non-symmetric (compact) SII homogeneous Riemannian manifold $(M = G/K, g = -B|_{\mathfrak{m}})$ we state the dimension of the space $\mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$ (see Theorem 4.7, Tables 4, 5). In addition to this, for any such homogeneous space we present the space of G-invariant torsion-free connections and classify the dimension of the space of G-invariant metric connections. Moreover, we state the multiplicity of the (real) trivial representation inside the space $\Lambda^3(\mathfrak{m})$ of 3-forms. This last step yields finally the presentation of the subclass of G-invariant metric connections with skew-torsion. Note that all these desired multiplicities were also obtained in [19], up to some errors/omissions, see Remark 4.5 and Table 4 for corrections. We summarize our results as follows:

Theorem A.1. *Let $(M = G/K, g = -B|_{\mathfrak{m}})$ be an effective non-symmetric SII space. Then:*
(i) The family $\{\nabla^\alpha : \alpha \in \mathbb{R}\}$ exhausts all G-invariant affine or metric connections on $M = G/K$, if and only if $G = \mathrm{Sp}_n$, or $M = G/K$ is one of the manifolds

$$\mathrm{SO}_{14}/\mathrm{Sp}_3, \quad \mathrm{SO}_{4n}/\mathrm{Sp}_n \times \mathrm{Sp}_1\ (n \geq 2), \quad \mathrm{SO}_7/\mathrm{G}_2, \quad \mathrm{SO}_{16}/\mathrm{Spin}_9, \quad \mathrm{G}_2/\mathrm{SO}_3,$$
$$\mathrm{F}_4/(\mathrm{G}_2 \times \mathrm{SU}_2), \quad \mathrm{E}_7/(\mathrm{G}_2 \times \mathrm{Sp}_3), \quad \mathrm{E}_7/(\mathrm{F}_4 \times \mathrm{SU}_2), \quad \mathrm{E}_8/(\mathrm{F}_4 \times \mathrm{G}_2).$$

The same family exhausts also all SU_{2q}-invariant metric connections on the homogeneous space $\mathrm{SU}_{2q}/\mathrm{SU}_2 \times \mathrm{SU}_q$ $(q \geq 3)$, but not all the SU_{2q}-invariant affine connections.
(ii) The family $\{\nabla^\alpha : \alpha \in \mathbb{C}\}$ exhausts all G-invariant affine or metric connections on $M = G/K$, if and only if $M = G/K$ is one of the manifolds

$$\mathrm{SO}_8/\mathrm{SU}_3, \quad \mathrm{G}_2/\mathrm{SU}_3, \quad \mathrm{F}_4/(\mathrm{SU}_3 \times \mathrm{SU}_3), \quad \mathrm{E}_6/(\mathrm{SU}_3 \times \mathrm{SU}_3 \times \mathrm{SU}_3),$$
$$\mathrm{E}_7/(\mathrm{SU}_3 \times \mathrm{SU}_6), \quad \mathrm{E}_8/\mathrm{SU}_9, \quad \mathrm{E}_8/(\mathrm{E}_6 \times \mathrm{SU}_3).$$

For invariant metric connections different from ∇^α, we prove that

Theorem A.2. *Let $(M = G/K, g = -B|_{\mathfrak{m}})$ be an effective non-symmetric SII space which admits at least one invariant metric connection, different from the Lie bracket family. Then, $M = G/K$ is isometric to a space given in Table 2. In this table we present the dimensions of the spaces $\mathrm{Hom}_K(\mathfrak{m}, \Lambda^2\mathfrak{m})$ and $(\Lambda^3\mathfrak{m})^K$, which respectively parametrize the space of invariant metric connections and the space of invariant metric connections with totally skew-symmetric torsion. In particular:*
(i) Any homogeneous space in Table 2 whose isotropy representation is of real type and which is not isometric to $\mathrm{SO}_{10}/\mathrm{Sp}_2$, admits a 2-dimensional space of G-invariant metric connections with skew-torsion. For $\mathrm{SO}_{10}/\mathrm{Sp}_2$, the unique family of SO_{10}-invariant metric connections with skew-torsion is given by ∇^α $(\alpha \in \mathbb{R})$. However, the space of all SO_{10}-invariant metric connections is 2-dimensional.
(ii) Any homogeneous space in Table 2 whose isotropy representation is of complex type, admits a 6-dimensional space of G-invariant metric connections and a 4-dimensional subspace of G-invariant metric connections with skew-torsion.

For invariant connections induced by some $0 \neq \mu \in \mathrm{Hom}_K(\mathrm{Sym}^2\,\mathfrak{m}, \mathfrak{m})$, we prove the following:

Theorem B. *Let $(M = G/K, g = -B|_{\mathfrak{m}})$ be an effective non-symmetric SII space, which admits at least one invariant affine connection ∇^μ, induced by some $0 \neq \mu \in \mathrm{Hom}_K(\mathrm{Sym}^2\,\mathfrak{m}, \mathfrak{m})$. Then:*
*(i) If the associated isotropy representation is of real type, then $M = G/K$ is isometric to a manifold in Table 3. In this table, **s** states for the dimension of the module $\mathrm{Hom}_K(\mathrm{Sym}^2\,\mathfrak{m}, \mathfrak{m})$, which parametrizes the space of such invariant connections.*
(ii) If the associated isotropy representation is of complex type, then $M = G/K$ is isometric to one of the manifolds $\mathrm{SO}_{n^2-1}/\mathrm{SU}_n$ $(n \geq 4)$ or $\mathrm{E}_6/\mathrm{SU}_3$, where the dimension of the space of such invariant connections is 2 and 4, respectively.
(iii) The G-invariant connection ∇^μ does not preserve the Killing metric $g = -B|_{\mathfrak{m}}$. Thus, ∇^μ is not metric with respect to any G-invariant metric.

Now, a small combination of Theorems A.1, A.2 and B yields the desired dimension of the space of all *G-invariant affine connections* for any non-symmetric (compact) SII space $M = G/K$,

$$\mathcal{N} := \dim_{\mathbb{R}} \mathcal{A}ff_G(F(G/K)) = \dim_{\mathbb{R}} \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m}).$$

We refer to Tables 4 and 5, where the number $\mathcal{N}$ is explicitly indicated. Note that for SII homogeneous spaces $M = G/K$ of the Lie group $G = \mathrm{SU}_n$, we can describe explicitly some of the SU_n-invariant affine connections induced by a symmetric K-intertwining map $0 \neq \mu \in \mathrm{Hom}_K(\mathrm{Sym}^2\,\mathfrak{m}, \mathfrak{m})$ (and in a few cases all such connections, see Corollary 4.8). We also conclude that the space of invariant *torsion-free connections* on a non-symmetric SII space $M = G/K$, denoted by $\mathcal{A}ff_G^0(F(G/K))$,

[2] The parameter α can be a real or complex number, depending on the type of the isotropy representation $\mathfrak{m}$.
[3] http://wwwmathlabo.univ-poitiers.fr/~maavl/LiE/.

Table 2
Non-symmetric SII spaces carrying new G-invariant metric connections and the dimension of the space of global G-invariant 3-forms.

Real type			
$M = G/K$ (families)	$\dim_{\mathbb{R}} M$	Invariant metric connections $\dim_{\mathbb{R}} \mathrm{Hom}_K(\mathfrak{m}, \Lambda^2\mathfrak{m})$	skew-torsion $\dim_{\mathbb{R}}(\Lambda^3\mathfrak{m})^K$
$\mathrm{SU}_{pq}/\mathrm{SU}_p \times \mathrm{SU}_q$ $(p, q \geq 3)$	$(p^2-1)(q^2-1)$	2	2
$\mathrm{SO}_{\frac{n(n-1)}{2}}/\mathrm{SO}_n$ $(n \geq 9)$	$\frac{1}{8}(6n - 5n^2 - 2n^3 + n^4)$	3	2
$\mathrm{SO}_{\frac{(n-1)(n+2)}{2}}/\mathrm{SO}_n$ $(n \geq 7)$	$\frac{1}{8}(8 - 2n - 9n^2 + 2n^3 + n^4)$	3	2
$\mathrm{SO}_{(n-1)(2n+1)}/\mathrm{Sp}_n$ $(n \geq 4)$	$\frac{1}{2}(2 + n - 9n^2 - 4n^3 + 4n^4)$	3	2
$\mathrm{SO}_{n(2n+1)}/\mathrm{Sp}_n$ $(n \geq 3)$	$\frac{1}{2}(-3n - 5n^2 + 4n^3 + 4n^4)$	3	2
(low-dim cases)			
$\mathrm{SO}_{21}/\mathrm{SO}_7$	189	3	2
$\mathrm{SO}_{28}/\mathrm{SO}_8$	350	4	2
$\mathrm{SO}_{14}/\mathrm{SO}_5$	81	3	2
$\mathrm{SO}_{20}/\mathrm{SO}_6$	175	3	2
$\mathrm{SO}_{10}/\mathrm{Sp}_2$	35	2	1
(exceptions)			
$\mathrm{SO}_{14}/\mathrm{G}_2$	70	2	2
$\mathrm{SO}_{26}/\mathrm{F}_4$	273	2	2
$\mathrm{SO}_{42}/\mathrm{Sp}_4$	825	2	2
$\mathrm{SO}_{52}/\mathrm{F}_4$	1274	2	2
$\mathrm{SO}_{70}/\mathrm{SU}_8$	2352	2	2
$\mathrm{SO}_{248}/\mathrm{E}_8$	30 380	2	2
$\mathrm{SO}_{78}/\mathrm{E}_6$	2925	2	2
$\mathrm{SO}_{128}/\mathrm{Spin}_{16}$	8008	2	2
$\mathrm{SO}_{133}/\mathrm{E}_7$	8645	2	2
$\mathrm{E}_7/\mathrm{SU}_3$	125	2	2
Complex type			
$M = G/K$	$\dim_{\mathbb{R}} M$	Invariant metric connections $\dim_{\mathbb{R}} \mathrm{Hom}_K(\mathfrak{m}, \Lambda^2(\mathfrak{m}))$	skew-torsion $\dim_{\mathbb{R}}(\Lambda^3\mathfrak{m})^K$
$\mathrm{SO}_{n^2-1}/\mathrm{SU}_n$ $(n \geq 4)$	$\frac{1}{2}(4 - 5n^2 + n^4)$	6	4
$\mathrm{E}_6/\mathrm{SU}_3$	70	6	4

Table 3
Non-symmetric SII spaces of real type carrying G-invariant affine connections induced by $0 \neq \mu \in \mathrm{Hom}_K(\mathrm{Sym}^2\mathfrak{m}, \mathfrak{m})$.

Real type		
$\mathbf{s} = 1$	$\mathbf{s} = 2$	$\mathbf{s} = 3$
$\mathrm{SU}_{10}/\mathrm{SU}_5$	$\mathrm{SU}_{\frac{n(n-1)}{2}}/\mathrm{SU}_n$ $(n \geq 6)$	$\mathrm{SO}_{28}/\mathrm{SO}_8$
$\mathrm{SO}_{\frac{n(n-1)}{2}}/\mathrm{SO}_n$ $(n \geq 9)$	$\mathrm{SU}_{\frac{n(n+1)}{2}}/\mathrm{SU}_n$ $(n \geq 3)$	$\mathrm{E}_7/\mathrm{SU}_3$
$\mathrm{SO}_{\frac{(n-1)(n+2)}{2}}/\mathrm{SO}_n$ $(n \geq 7)$	$\mathrm{SO}_{20}/\mathrm{SO}_6$	
$\mathrm{SO}_{21}/\mathrm{SO}_7$	$\mathrm{SU}_{27}/\mathrm{E}_6$	
$\mathrm{SO}_{14}/\mathrm{SO}_5$	$\mathrm{SU}_{pq}/\mathrm{SU}_p \times \mathrm{SU}_q$ $(p, q \geq 3)$	
$\mathrm{SO}_{(n-1)(2n+1)}/\mathrm{Sp}_n$ $(n \geq 4)$		
$\mathrm{SO}_{n(2n+1)}/\mathrm{Sp}_n$ $(n \geq 3)$		
$\mathrm{SU}_{2q}/\mathrm{SU}_2 \times \mathrm{SU}_q$ $(q \geq 3)$		
$\mathrm{SO}_{10}/\mathrm{Sp}_2$		
$\mathrm{SU}_{16}/\mathrm{Spin}_{10}$		
$\mathrm{SO}_{70}/\mathrm{SU}_8$		
$\mathrm{E}_6/\mathrm{G}_2$		
$\mathrm{E}_6/(\mathrm{G}_2 \times \mathrm{SU}_3)$		

is parametrized by an affine subspace of $\mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$, which is modelled on $\mathrm{Hom}_K(\mathrm{Sym}^2\mathfrak{m}, \mathfrak{m})$ and contains the Levi-Civita connection, see Lemma 1.4 and Remark 1.6. In particular, for any G-invariant affine connection ∇^μ induced by $\mu \in \mathrm{Hom}_K(\mathrm{Sym}^2\mathfrak{m}, \mathfrak{m})$, the invariant connection $\nabla := \nabla^\mu - \frac{1}{2}T^\mu$ is torsion-free. Thus, the following is a direct consequence of Theorem B.

Corollary of Theorem B. *The classification of non-symmetric SII spaces which admit new invariant torsion-free connections, in addition to the Levi-Civita connection, is given by the manifolds of Theorem B. In particular, for a space in Table 3 we have* $\dim_{\mathbb{R}} \mathcal{A}ff^0_G(F(G/K)) = \mathbf{s}$, *and for the almost complex homogeneous spaces in Theorem B it is* $\dim_{\mathbb{R}} \mathcal{A}ff^0_G(F(G/K)) = 2$ *or* 4, *respectively.*

Classification of ∇-Einstein structures with skew-torsion. After obtaining Theorems A.1, A.2 and B, in the final Section 5 we turn our attention to more geometric problems. We use our classification results of Table 2 to examine ∇-Einstein structures

Table 4
The multiplicities **a**, **s**, $\mathcal{N}$ and ℓ for (non-symmetric) SII homogeneous spaces—Classical families.

Classical families and their associated low-dimensional cases							
G	$M=G/K$	$\mathfrak{m}^{\mathbb{C}}$	**a**	**s**	$\mathcal{N}$	ℓ	Type
SU_n	(1) $\mathrm{SU}_{\frac{n(n-1)}{2}}/\mathrm{SU}_n$ $(n\geq 6)$	$R(\pi_2+\pi_{n-2})$	1	2	3	1	r
	(1^*_α) $\mathrm{SU}_{10}/\mathrm{SU}_5$	$R(\pi_2+\pi_3)$	1	1	2	1	r
	(2) $\mathrm{SU}_{\frac{n(n+1)}{2}}/\mathrm{SU}_n$ $(n\geq 3)$	$R(2\pi_1+2\pi_{n-1})$	1	2	3	1	r
	(3) $\mathrm{SU}_{pq}/\mathrm{SU}_p\times\mathrm{SU}_q$ $(p,q\geq 3)$	$R(\pi_1+\pi_{p-1})\hat{\otimes}R(\pi_1+\pi_{q-1})$	2	2	4	2	r
	(3^*_α) $\mathrm{SU}_{2q}/\mathrm{SU}_2\times\mathrm{SU}_q$ $(q\geq 3)$	$R(2\pi_1)\hat{\otimes}R(\pi_1+\pi_{q-1})$	1	1	2	1	r
SO_n	(4) $\mathrm{SO}_{n^2-1}/\mathrm{SU}_n$ $(n\geq 4)$	$R(2\pi_1+\pi_{n-2})\oplus R(\pi_2+2\pi_{n-1})$	6	2	8	4	c
	(4_α) $\mathrm{SO}_8/\mathrm{SU}_3$	$R(3\pi_1)\oplus R(3\pi_2)$	2	0	2	2	c
	(5) $\mathrm{SO}_{\frac{n(n-1)}{2}}/\mathrm{SO}_n$ $(n\geq 9)$	$R(\pi_1+\pi_3)$	3	1	4	2	r
	(5_α) $\mathrm{SO}_{21}/\mathrm{SO}_7$	$R(\pi_1+2\pi_3)$	3	1	4	2	r
	(5_β) $\mathrm{SO}_{28}/\mathrm{SO}_8$	$R(\pi_1+\pi_3+\pi_4)$	4	3	7	2	r
	(6) $\mathrm{SO}_{\frac{(n-1)(n+2)}{2}}/\mathrm{SO}_n$ $(n\geq 7)$	$R(2\pi_1+\pi_2)$	3	1	4	2	r
	(6_α) $\mathrm{SO}_{14}/\mathrm{SO}_5$	$R(2\pi_1+2\pi_2)$	3	1	4	2	r
	(6^*_β) $\mathrm{SO}_{20}/\mathrm{SO}_6$	$R(2\pi_1+\pi_2+\pi_3)$	3	2	5	2	r
	(7) $\mathrm{SO}_{(n-1)(2n+1)}/\mathrm{Sp}_n$ $(n\geq 4)$	$R(\pi_1+\pi_3)$	3	1	4	2	r
	(7_α) $\mathrm{SO}_{14}/\mathrm{Sp}_3$	$R(\pi_1+\pi_3)$	1	0	1	1	r
	(8) $\mathrm{SO}_{n(2n+1)}/\mathrm{Sp}_n$ $(n\geq 3)$	$R(2\pi_1+\pi_2)$	3	1	4	2	r
	(8_α) $\mathrm{SO}_{10}/\mathrm{Sp}_2$	$R(2\pi_1+\pi_2)$	2	1	3	1	r
	(9^*) $\mathrm{SO}_{4n}/\mathrm{Sp}_n\times\mathrm{Sp}_1$ $(n\geq 2)$	$R(\pi_2)\hat{\otimes}R(2\pi_1)$	1	0	1	1	r
Sp_n	(10) $\mathrm{Sp}_n/\mathrm{SO}_n\times\mathrm{Sp}_1$ $(n\geq 5)$	$R(2\pi_1)\hat{\otimes}R(2\pi_1)$	1	0	1	1	r
	(10_α) $\mathrm{Sp}_3/\mathrm{SO}_3\times\mathrm{Sp}_1$	$R(4\pi_1)\hat{\otimes}R(2\pi_1)$	1	0	1	1	r
	(10_β) $\mathrm{Sp}_4/\mathrm{SO}_4\times\mathrm{Sp}_1$	$R(2\pi_1+2\pi_2)\hat{\otimes}R(2\pi_1)$	1	0	1	1	r

with skew-torsion. Roughly speaking, such a structure consists of a n-dimensional connected Riemannian manifold (M,g) endowed with a metric connection ∇ which has non-trivial skew-torsion $0\neq T\in\Lambda^3(T^*M)$ and whose Ricci tensor has symmetric part a multiple of the metric tensor, i.e. (see [25,5,3,17,18,22])

$$\mathrm{Ric}^{\nabla}_S=\frac{\mathrm{Scal}^{\nabla}}{n}g.$$

For $T=0$ the whole notion reduces to the original Einstein metrics. In fact, like Einstein metrics on compact Riemannian manifolds, in [3] it was shown that ∇-Einstein structures can be characterized variationally. On the other hand, the classification of ∇-Einstein structures with skew-torsion on a fixed Riemannian manifold (M,g), is initially based on the classification of all metric connections on M whose torsion is a non-trivial 3-form. For example, for odd dimensional spheres $\mathrm{S}^{2n+1}\cong\mathrm{SU}_{n+1}/\mathrm{SU}_n$ endowed with their Sasakian structure, a classification of SU_{n+1}-*invariant* ∇-*Einstein structures with skew-torsion* has been very recently given in [22], and it follows only after the classification of SU_{n+1}-invariant metric connections (with skew-torsion) and their description in terms of tensor fields related to the special structure (see also [3]).

As far as we know, most well-understood examples of ∇-Einstein manifolds appear in the context of non-integrable geometries, where a metric connection with skew-torsion $0\neq T$ is adapted to the geometry under consideration, the so-called *characteristic connection* ∇^c (see [25]). This connection, which in the homogeneous case coincides with the canonical connection, plays a crucial role in the theory of special geometries and nowadays is a traditional approach to describing the associated non-integrable structure in terms of ∇^c (or the very closely related *intrinsic torsion*). Moreover, the articles [25,5,3] provide some nice classes of ∇^c-Einstein structures, e.g. nearly Kähler manifolds in dimension 6, nearly-parallel G_2-manifolds in dimension 7, or 7-dimensional 3-Sasakian manifolds. Notice that these special structures admit (∇^c-parallel) real Killing spinors and hence, in some cases one can describe a deeper relation between the ∇-Einstein condition and a class of spinor fields, known as *Killing spinors with torsion*. These are natural generalizations of the original Killing spinor fields, satisfying the Killing spinor equation with respect to a metric connection with skew-torsion. Their existence is known for several types of special geometries (see [2,11,18]). For example, on 6-dimensional nearly Kähler manifolds, 7-dimensional nearly parallel G_2-manifolds, or even on $\mathrm{S}^3\cong\mathrm{SU}_2$, such spinors are induced by the associated ∇^c-parallel spinors and their description is given in terms of whole 1-parameter families $\{\nabla^s : s\in\mathbb{R}\}$ of metric connections with skew-torsion. Moreover, their existence imposes the following strong geometric constraint: $\mathrm{Ric}^s=\frac{1}{n}\mathrm{Scal}^s g$ for any $s\in\mathbb{R}$ [18] (although in general this is not the case, see [11]). The special value $s=1/4$ corresponds to the characteristic connection (which has parallel torsion T), while the parameter $s=0$ induces the original Einstein metric related with the existent real Killing spinor.

Beside these classes of ∇-Einstein manifolds, the first author in [17] studies homogeneous ∇-Einstein structures for more general manifolds, e.g. on compact isotropy irreducible spaces and a class of normal homogeneous manifolds with two isotropy summands. An important result for us from [17], is that any effective compact isotropy irreducible homogeneous space $M=G/K$ which is not a symmetric space of Type I, is a ∇^α-Einstein manifold for any parameter $\alpha\neq 0$, where ∇^α is the Lie bracket family. As a consequence of the results in Section 4, we conclude that any (effective) non-symmetric SII homogeneous space $(M=G/K, g=-B|_{\mathfrak{m}})$ is a ∇^α-Einstein manifold for any parameter $\alpha\neq 0$. Moreover, our Lemma 3.12

Table 5
The multiplicities **a**, **s**, $\mathcal{N}$ and ℓ for (non-symmetric) SII homogeneous spaces—Exceptions.

Exceptions ("exceptions" in terms of [13, p. 203])							
G	$M=G/K$	$\mathfrak{m}^{\mathbb{C}}$	**a**	**s**	$\mathcal{N}$	ℓ	Type
SU_n	$\mathrm{SU}_{16}/\mathrm{Spin}_{10}$	$R(\pi_4+\pi_5)$	1	1	2	1	r
	$\mathrm{SU}_{27}/\mathrm{E}_6$	$R(\pi_1+\pi_6)$	1	2	3	1	r
SO_n	$\mathrm{SO}_7/\mathrm{G}_2$	$R(\pi_1)$	1	0	1	1	r
	$\mathrm{SO}_{14}/\mathrm{G}_2$	$R(3\pi_1)$	2	0	2	2	r
	$\mathrm{SO}_{16}/\mathrm{Spin}_9$	$R(\pi_3)$	1	0	1	1	r
	$\mathrm{SO}_{26}/\mathrm{F}_4$	$R(\pi_3)$	2	0	2	2	r
	$\mathrm{SO}_{42}/\mathrm{Sp}_4$	$R(2\pi_3)$	2	0	2	2	r
	$\mathrm{SO}_{52}/\mathrm{F}_4$	$R(\pi_2)$	2	0	2	2	r
	$\mathrm{SO}_{70}/\mathrm{SU}_8$	$R(\pi_3+\pi_5)$	2	1	3	2	r
	$\mathrm{SO}_{248}/\mathrm{E}_8$	$R(\pi_7)$	2	0	2	2	r
	$\mathrm{SO}_{78}/\mathrm{E}_6$	$R(\pi_4)$	2	0	2	2	r
	$\mathrm{SO}_{128}/\mathrm{Spin}_{16}$	$R(\pi_6)$	2	0	2	2	r
	$\mathrm{SO}_{133}/\mathrm{E}_7$	$R(\pi_3)$	2	0	2	2	r
Sp_n	$\mathrm{Sp}_2/\mathrm{SU}_2$	$R(6\pi_1)$	1	0	1	1	r
	$\mathrm{Sp}_7/\mathrm{Sp}_3$	$R(2\pi_3)$	1	0	1	1	r
	$\mathrm{Sp}_{10}/\mathrm{SU}_6$	$R(2\pi_3)$	1	0	1	1	r
	$\mathrm{Sp}_{16}/\mathrm{Spin}_{12}$	$R(2\pi_6)$ or $R(2\pi_5)$	1	0	1	1	r
	$\mathrm{Sp}_{28}/\mathrm{E}_7$	$R(2\pi_7)$	1	0	1	1	r
G_2	$\mathrm{G}_2/\mathrm{SU}_3$	$R(\pi_1)\oplus R(\pi_2)$	2	0	2	2	c
	$\mathrm{G}_2/\mathrm{SO}_3$	$R(10\pi_1)$	1	0	1	1	r
F_4	$\mathrm{F}_4/(\mathrm{SU}_3^1\times\mathrm{SU}_3^2)$	$(R(2\pi_1)\hat{\otimes}R(\omega_1))\oplus(R(2\pi_2)\hat{\otimes}R(\omega_2))$	2	0	2	2	c
	$\mathrm{F}_4/(\mathrm{G}_2\times\mathrm{SU}_2)$	$R(\pi_1)\hat{\otimes}R(4\omega_1)$	1	0	1	1	r
E_6	$\mathrm{E}_6/\mathrm{SU}_3$	$R(4\pi_1+\pi_2)\oplus R(\pi_1+4\pi_2)$	6	4	10	4	c
	$\mathrm{E}_6/(\mathrm{SU}_3\times\mathrm{SU}_3\times\mathrm{SU}_3)$	$(R(\pi_1)\hat{\otimes}R(\omega_1)\hat{\otimes}R(\theta_1))\oplus$ $(R(\pi_2)\hat{\otimes}R(\omega_2)\hat{\otimes}R(\theta_2))$	2	0	2	2	c
	$\mathrm{E}_6/\mathrm{G}_2$	$R(\pi_1+\pi_2)$	1	1	2	1	r
	$\mathrm{E}_6/(\mathrm{G}_2\times\mathrm{SU}_3)$	$R(\pi_1)\hat{\otimes}R(\omega_1+\omega_2)$	1	1	2	1	r
E_7	$\mathrm{E}_7/\mathrm{SU}_3$	$R(4\pi_1+4\pi_2)$	2	3	5	2	r
	$\mathrm{E}_7/(\mathrm{SU}_3\times\mathrm{SU}_6)$	$(R(\pi_1)\hat{\otimes}R(\omega_2))\oplus(R(\pi_2)\hat{\otimes}R(\omega_5))$	2	0	2	2	c
	$\mathrm{E}_7/(\mathrm{G}_2\times\mathrm{Sp}_3)$	$R(\pi_1)\hat{\otimes}R(\omega_2)$	1	0	1	1	r
	$\mathrm{E}_7/(\mathrm{F}_4\times\mathrm{SU}_2)$	$R(\pi_4)\hat{\otimes}R(2\omega_1)$	1	0	1	1	r
E_8	$\mathrm{E}_8/\mathrm{SU}_9$	$R(\pi_3)\oplus R(\pi_6)$	2	0	2	2	c
	$\mathrm{E}_8/(\mathrm{F}_4\times\mathrm{G}_2)$	$R(\pi_4)\hat{\otimes}R(\omega_1)$	1	0	1	1	r
	$\mathrm{E}_8/(\mathrm{E}_6\times\mathrm{SU}_3)$	$(R(\pi_1)\hat{\otimes}R(\omega_1))\oplus(R(\pi_6)\hat{\otimes}R(\omega_2))$	2	0	2	2	c

in combination with Schur's lemma, yield a natural parameterization of the set of G-invariant ∇-Einstein structures with skew-torsion, by the space of invariant metric connections with non-trivial skew-torsion, or equivalently of the vector space of (global) invariant 3-forms. Hence, in this case the space of all homogeneous ∇-Einstein structures with skew-torsion on $(M=G/K, g=-B|_{\mathfrak{m}})$, denoted by $\mathcal{E}^{sk}_G(\mathrm{SO}(G/K,-B|_{\mathfrak{m}}))$, can be viewed as an affine subspace of the space of all G-invariant metric connections. Combining with our classification results on G-invariant metric connections with skew-torsion (see Theorems A.1, A.2, Table 2), we finally deduce that

Theorem C. *Let $(M=G/K, g=-B|_{\mathfrak{m}})$ be an effective non-symmetric SII space and assume that the family ∇^α exhausts all G-invariant metric connections. Then, the associated space of G-invariant ∇-Einstein structures with skew-torsion has dimension either*

$$\dim_{\mathbb{R}}\mathcal{E}^{sk}_G(\mathrm{SO}(G/K,-B|_{\mathfrak{m}}))=1,\quad \text{or}\quad \dim_{\mathbb{R}}\mathcal{E}^{sk}_G(\mathrm{SO}(G/K,-B|_{\mathfrak{m}}))=2,$$

for spaces with isotropy representation of real or complex type, respectively, and the manifold is one of the manifolds of Theorem A.1 or $\mathrm{SO}_{10}/\mathrm{Sp}_2$.

For the new invariant metric connections with skew-torsion, different from the family ∇^α, an explicit description seems difficult (for dimensional reasons, see Table 2). However, we prove that

Theorem D. *Let $(M=G/K, g=-B|_{\mathfrak{m}})$ be an effective non-symmetric SII space of Table 2, whose isotropy representation χ is of real type and assume that M is not isometric to $\mathrm{SO}_{10}/\mathrm{Sp}_2$. Then, the Ricci tensor associated to the 1-parameter family of invariant metric connections with skew-torsion, orthogonal to the Lie bracket family ∇^α, is also symmetric. Moreover,*

$$\dim_{\mathbb{R}}\mathcal{E}^{sk}_G(\mathrm{SO}(G/K,-B|_{\mathfrak{m}}))=2.$$

This result is based on Theorem A.2 (Table 2) and the fact $(\Lambda^2\mathfrak{m})^K=0$ for real representations of real type. This means that the space of skew-symmetric 2-forms $\Lambda^2\mathfrak{m}$ associated to a space $M=G/K$ of Theorem D (or even for the space $\mathrm{SO}_{10}/\mathrm{Sp}_2$), does not

contain the trivial representation, hence there do not exist G-invariant 2-forms. Consequently, the co-differential of the torsion associated to any existent G-invariant affine metric connection on M must vanish and our assertion follows by Schur's lemma in combination with the expression of the Ricci tensor for a metric connection with skew-torsion.

Theorems C and D give the complete classification of all existent G-homogeneous ∇-Einstein structures on any effective, non-symmetric, SII space $M = G/K$, except of the quotients $\mathrm{SO}_{n^2-1}/\mathrm{SU}_n$ ($n \geq 4$) and $\mathrm{E}_6/\mathrm{SU}_3$. These are privileged manifolds with respect to Theorem A.2; the associated space of G-invariant metric connections with skew-torsion is 4-dimensional. Moreover, they both admit an invariant almost complex structure and hence $\Lambda^2(\mathfrak{m})$ contains a copy of the trivial representation $\mathbb{R}$ (Lemma 5.4), i.e. there exist G-invariant (global) 2-forms. However, since we are interested only on the symmetric part of Ric^∇ and the isotropy representation is (strongly) irreducible, again a combination of the results of Theorem A.2 with Schur's lemma, yields that

Theorem E. *Let $(M = G/K, g = -B|_{\mathfrak{m}})$ be one of the manifolds $\mathrm{SO}_{n^2-1}/\mathrm{SU}_n$ $(n \geq 4)$ or $\mathrm{E}_6/\mathrm{SU}_3$. Then, the space of G-homogeneous ∇-Einstein structures with skew-torsion has dimension*

$$\dim_{\mathbb{R}} \mathcal{E}^{sk}_G(\mathrm{SO}(G/K, -B|_{\mathfrak{m}})) = 4.$$

1. Preliminaries

1.1. Metric connections and their types

Consider a connected, oriented Riemannian manifold (M^n, g) and identify the tangent and cotangent bundle $TM \cong T^*M$ via the bundle isomorphism provided by the metric tensor g. Any metric connection $\nabla : \Gamma(TM) \to \Gamma(T^*M \otimes TM) \cong \Gamma(TM \otimes TM)$ on M can be written as $\nabla_X Y = \nabla^g_X Y + A(X, Y)$ for any $X, Y \in \Gamma(TM)$, for some tensor $A \in TM \otimes \Lambda^2(TM)$, where ∇^g is the Levi-Civita connection. Let us denote by $A(X, Y, Z) := g(A(X, Y), Z)$ the induced tensor obtained by contraction with g. The affine connections on M which are compatible with g, form an affine space modelled on the sections of the tensor bundle

$$\mathcal{A} := \{A \in \otimes^3 TM : A(X, Y, Z) + A(X, Z, Y) = 0\} \cong TM \otimes \Lambda^2(TM),$$

which has fibre dimension $n^2(n-1)/2$. It is well-known that $\mathcal{A}$ coincides with the space of torsion tensors

$$\mathcal{T} = \{A \in \otimes^3 TM : A(X, Y, Z) + A(Y, X, Z) = 0\} \cong \Lambda^2(TM) \otimes TM.$$

Moreover, under the action of the structure group SO_n it decomposes into three irreducible representations $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \mathcal{A}_3$, defined by

$$\begin{aligned}
\mathcal{A}_1 &:= \{A \in \mathcal{A} : A(X, Y, Z) = g(X, Y)\varphi(Z) - g(X, Z)\varphi(Y),\ \varphi \in \Gamma(T^*M)\} \cong TM,\\
\mathcal{A}_2 &:= \{A \in \mathcal{A} : \mathfrak{S}^{X,Y,Z} A(X, Y, Z) = 0,\ \Phi(A) = 0\},\\
\mathcal{A}_3 &:= \{A \in \mathcal{A} : A(X, Y, Z) + A(Y, X, Z) = 0\} \cong \Lambda^3 TM.
\end{aligned}$$

Here, the map $\Phi : \mathcal{A} \to T^*M$ is given by $\Phi(A)(Z) := \mathrm{tr} A_Z := \sum_i A(e_i, e_i, Z)$, for a vector field $Z \in \Gamma(TM)$ and a (local) orthonormal frame $\{e_i\}$ of M. The torsion $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$ of ∇ satisfies the relation $T(X, Y) = A(X, Y) - A(Y, X)$ and conversely, A is expressed in terms of T by the condition

$$2A(X, Y, Z) = T(X, Y, Z) - T(Y, Z, X) + T(Z, X, Y), \quad \forall X, Y, Z \in \Gamma(TM). \tag{1.1}$$

We say that ∇ is of *vectorial type* (and the same for its torsion) if $A \in \mathcal{A}_1 \cong TM$, of *Cartan type, or traceless cyclic* if $A \in \mathcal{A}_2$ and finally (totally) *skew-symmetric* (or, of skew-torsion) if $A \in \mathcal{A}_3 \cong \Lambda^3 TM$. Notice that for $n = 2$, $\mathcal{A} \cong \mathbb{R}^2$ is irreducible. For $n \geq 3$, the mixed types occur by taking the direct sums of $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$:

$$\begin{aligned}
\mathcal{A}_1 \oplus \mathcal{A}_2 &= \{A \in \mathcal{A} : \mathfrak{S}^{X,Y,Z} A(X, Y, Z) = 0\},\\
\mathcal{A}_2 \oplus \mathcal{A}_3 &= \{A \in \mathcal{A} : \Phi(A) = 0\},\\
\mathcal{A}_1 \oplus \mathcal{A}_3 &= \{A \in \mathcal{A} : A(X, Y, Z) + A(Y, X, Z) = 2g(X, Y)\varphi(Z) - g(X, Z)\varphi(Y)\\
&\qquad\qquad - g(Y, Z)\varphi(X),\ \varphi \in \Gamma(T^*M)\}.
\end{aligned}$$

Usually, connections of type $\mathcal{A}_1 \oplus \mathcal{A}_2$ are called *cyclic* and connections of type $\mathcal{A}_2 \oplus \mathcal{A}_3$ are known as *traceless* connections.

Let us finally recall that a tensor field $A \in \mathcal{A}$ satisfying $\nabla A = 0 = \nabla R$, where R denotes the curvature of the metric connection $\nabla = \nabla^g + A$ is called a *homogeneous structure*. The existence of a metric connection with these properties implies that (M, g) is locally homogeneous and if in addition (M, g) is complete, then it is locally isometric to a homogeneous Riemannian manifold. In particular, a complete, connected and simply-connected Riemannian manifold (M, g) endowed with a metric connection ∇ solving the equations $\nabla A = 0 = \nabla R$ is a homogeneous Riemannian manifold, see [43] for more details and proofs.

1.2. Connections with skew-torsion and ∇-Einstein manifolds

Let (M^n, g) be a connected Riemannian manifold carrying a metric connection ∇ with skew-torsion $0 \neq T \in \Lambda^3(TM)$, i.e.

$$g(\nabla_X Y, Z) = g(\nabla^g_X Y, Z) + \frac{1}{2}T(X, Y, Z).$$

We normalize the length of T such that $\|T\|^2 := (1/6)\sum_{i,j} g(T(e_i, e_j), T(e_i, e_j))$ and we denote by $\delta^\nabla T = -\sum_{i=1}^n e_i \lrcorner \nabla_{e_i} T$ the co-differential of T. It is easy to check that $\delta^g T = \delta^\nabla T$. It is also known that (see for example [28,21,25])

Lemma 1.1. *The Ricci tensor associated to ∇ is given by*

$$\mathrm{Ric}^\nabla(X, Y) \equiv \mathrm{Ric}(X, Y) = \mathrm{Ric}^g(X, Y) - \frac{1}{4}\sum_{i=1}^n g(T(e_i, X), T(e_i, Y)) - \frac{1}{2}(\delta^g T)(X, Y).$$

Thus, in contrast to the Riemannian Ricci tensor Ric^g, the Ricci tensor of ∇ is not symmetric; it decomposes into a symmetric and anti-symmetric part $\mathrm{Ric} = \mathrm{Ric}_S + \mathrm{Ric}_A$, given by

$$\mathrm{Ric}_S(X, Y) := \mathrm{Ric}^g(X, Y) - \frac{1}{4}S(X, Y), \quad \mathrm{Ric}_A(X, Y) := -\frac{1}{2}(\delta^g T)(X, Y),$$

respectively, where S is the symmetric tensor defined by $S(X, Y) = \sum_{i=1}^n g(T(e_i, X), T(e_i, Y))$.

Definition 1.2 ([3])**.** A triple (M, g, T) is called a ∇-*Einstein manifold with non-trivial skew-torsion* $0 \neq T \in \Lambda^3(TM)$, or for short, a ∇-*Einstein manifold*, if the symmetric part Ric_S of the Ricci tensor associated to the metric connection $\nabla = \nabla^g + \frac{1}{2}T$ satisfies the equation

$$\mathrm{Ric}_S = \frac{\mathrm{Scal}}{n}g, \tag{1.2}$$

where $\mathrm{Scal} \equiv \mathrm{Scal}^\nabla$ is the scalar curvature associated to ∇ and $n = \dim_{\mathbb{R}} M$. If $\nabla T = 0$, then (M, g, T) is called a ∇-*Einstein manifold with parallel skew-torsion*.

Notice that in contrast to the Riemannian case, for a ∇-Einstein manifold the scalar curvature $\mathrm{Scal}^\nabla \equiv \mathrm{Scal} = \mathrm{Scal}^g - \frac{3}{2}\|T\|^2$ is not necessarily constant (for details see [3]). For parallel torsion, i.e. $\nabla T = 0$, one has $\delta^\nabla T = 0$ and the Ricci tensor becomes symmetric $\mathrm{Ric} = \mathrm{Ric}_S$. If in addition $\delta\,\mathrm{Ric}^g = 0$, then the scalar curvature is constant, similarly to an Einstein manifold. This is the case for any ∇-Einstein manifold (M, g, ∇, T) with *parallel skew-torsion* [3, Prop. 2.7].

1.3. Invariant connections

Consider a Lie group G acting transitively on a smooth manifold M and let us denote by $\pi : P \to M$ a G-homogeneous principal bundle over M with structure group U. Let K be the isotropy subgroup at the point $o = \pi(p_0) \in M$ with $p_0 \in P$ (this is a closed subgroup $K \subset G$). Then, there is a Lie group homomorphism $\lambda : K \to U$ and hence an action of K on U, given by $ku = \lambda(k)u$. This induces a G-homogeneous principal U-bundle $P_\lambda \to M = G/K$, defined by $P_\lambda := G \times_K U = G \times_\lambda U = G \times U/\sim$, where $(g, u) \sim (gk, \lambda(k^{-1})u)$ for any $g \in G, u \in U, k \in K$. Because the left action of G on P restricts to a left action of K on the fibre P_o of P over a base point $o = eK \in G/K$, for the original bundle P we have $P \cong G \times_K P_o$. But fixing a point $u_0 \in P_o$ we see that the map $U \to P_o, u \mapsto u_0 u$ is a diffeomorphism and hence we identify $P \cong G \times_K P_o = G \times_K U = P_\lambda$, see also [16,32].

For G-homogeneous principal U-bundles $P \cong P_\lambda \to G/K$, it makes sense to speak about G-invariant connections, i.e. connections for which the horizontal subspaces $\mathcal{H}_p$ are also invariant by the left G-action, $(L_g)_* \mathcal{H}_p = \mathcal{H}_{gp}$ for any $g \in G$ and $p \in P$. In other words, a connection in P_λ is G-invariant if and only if the associated connection form $Z \in \Omega^1(P, \mathfrak{u})$ is such that $(\tau'_g)^* Z = Z$, for all $g \in G$, where $\tau'_g : P \to P$ is the (right) U-equivariant bundle map.

Theorem 1.3 ([44])**.** *Let $P \cong P_\lambda \to G/K$ be a G-homogeneous principal U-bundle associated to a homomorphism $\lambda : K \to U$, as above. Then, G-invariant connections on P_λ are in a bijective correspondence with linear mappings $\Lambda : \mathfrak{g} \to \mathfrak{u}$ satisfying the following conditions:*

(a) $\Lambda(X) = \lambda_*(X)$, for all $X \in \mathfrak{k} = T_eK$, where $\lambda_* : \mathfrak{k} \to \mathfrak{u}$ is the differential of λ,

(b) $\Lambda(\mathrm{Ad}(k)X) = \mathrm{Ad}(\lambda(k))\Lambda(X)$, for all $X \in \mathfrak{g} = T_eG, k \in K$.

1.4. Reductive homogeneous spaces

Consider now a reductive homogeneous space $M = G/K$, i.e. we assume that there is an orthogonal decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ of $\mathfrak{g} = T_eG$ with $\mathrm{Ad}(K)\mathfrak{m} \subset \mathfrak{m}$. Then we may identify $\mathfrak{m} = T_oM$ at $o = eK \in M$ and the isotropy representation $\chi : K \to \mathrm{Aut}(\mathfrak{m})$ of K with the restriction of the adjoint representation $\mathrm{Ad}|_K$ on $\mathfrak{m}$. Therefore, there

is a direct sum decomposition $\mathrm{Ad}\big|_K = \mathrm{Ad}_K \oplus \chi$ where Ad_K is the adjoint representation of K (see [10]). As a further consequence, we identify the tangent bundle TM and the frame bundle $F(M)$ of $M = G/K$ with the homogeneous vector bundle $G\times_K \mathfrak{m}$ and the homogeneous principal bundle $G\times_K \mathrm{GL}(\mathfrak{m})$, respectively, the latter with structure group $\mathrm{GL}(\mathfrak{m}) = \mathrm{GL}_n\,\mathbb{R}$ ($n = \dim_{\mathbb{R}} \mathfrak{m} = \dim_{\mathbb{R}} M$).

An invariant affine connection on $M = G/K$ is a principal connection on $F(G/K)$ that is G-invariant. By Theorem 1.3 such an affine connection is described by a $\mathbb{R}$-linear map $\Lambda : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$ which is equivariant under the isotropy representation, i.e. $\Lambda(\mathrm{Ad}(k)X) = \mathrm{Ad}(k)\Lambda(X)\,\mathrm{Ad}(k)^{-1}$ for any $X \in \mathfrak{m}$ and $k \in K$. Let us denote by $\mathrm{Hom}_K(\mathfrak{m}, \mathfrak{gl}(\mathfrak{m}))$ the set of such linear maps. The assignment $\Lambda(X)Y = \eta(X, Y)$ provides an identification of $\mathrm{Hom}_K(\mathfrak{m}, \mathfrak{gl}(\mathfrak{m}))$ (and hence of the space of G-invariant affine connections on $M = G/K$) with the set of all $\mathrm{Ad}(K)$-equivariant bilinear maps $\eta : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$, i.e.

$$\eta(\mathrm{Ad}(k)X, \mathrm{Ad}(k)Y) = \mathrm{Ad}(k)\eta(X, Y), \tag{1.3}$$

for any $X, Y \in \mathfrak{m}$ and $k \in K$. Moreover, since any such map $\eta : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ induces a unique linear map $\tilde{\eta} : \mathfrak{m} \otimes \mathfrak{m} \to \mathfrak{m}$ with $\tilde{\eta}(X \otimes Y) = \eta(X, Y)$, one may further identify (see [32, Thm. 5.1])

$$\mathcal{A}\!\mathit{ff}_G\big(F(G/K)\big) \cong \mathrm{Hom}_K(\mathfrak{m}, \mathfrak{gl}(\mathfrak{m})) \cong \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m}),$$

where in general $\mathcal{A}\!\mathit{ff}_G(P)$ denotes the affine space of G-invariant affine connections on a homogeneous principal bundle $P \to G/K$ over $M = G/K$ and $\mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$ is the space of K-intertwining maps $\mathfrak{m} \otimes \mathfrak{m} \to \mathfrak{m}$. Usually we shall work with K connected and in this case we may identify $\mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m}) = \mathrm{Hom}_{\mathfrak{k}}(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$. Due to the orthogonal splitting $\mathfrak{m} \otimes \mathfrak{m} = \Lambda^2\mathfrak{m} \oplus \mathrm{Sym}^2\,\mathfrak{m}$ we also remark that

$$\mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m}) = \mathrm{Hom}_K(\Lambda^2\mathfrak{m}, \mathfrak{m}) \oplus \mathrm{Hom}_K(\mathrm{Sym}^2, \mathfrak{m}). \tag{1.4}$$

The linear map $\Lambda : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$ is usually called *Nomizu map* or *connection map* (for details see [9,29]) and it satisfies the relation $\Lambda(X) = -(\nabla_X - L_X)_o$, where L_X is the Lie derivative with respect to X. Hence it encodes most of the properties of ∇; for example, the torsion $T \in \Lambda^2(\mathfrak{m}) \otimes \mathfrak{m}$ and curvature $R \in \Lambda^2(\mathfrak{m}) \otimes \mathfrak{k}$ of ∇ are given by:

$$T(X, Y)_o = \Lambda(X)Y - \Lambda(Y)X - [X, Y]_{\mathfrak{m}}, \quad R(X, Y)_o = [\Lambda(X), \Lambda(Y)] - \Lambda([X, Y]_{\mathfrak{m}}) - \mathrm{ad}([X, Y]_{\mathfrak{k}}).$$

Lemma 1.4 (*[29]*). *Let $M = G/K$ be a homogeneous space with a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Let Λ, $\Lambda' \in \mathrm{Hom}_{\mathfrak{k}}(\mathfrak{m}, \mathfrak{gl}(\mathfrak{m}))$ be two connection maps and let ∇, $\nabla' \in \mathcal{A}\!\mathit{ff}_G\big(F(G/K)\big)$ be the associated G-invariant affine connections. Set $\eta := \Lambda - \Lambda'$. Then*
(i) ∇ and ∇' have the same geodesics if and only if $\eta \in \mathrm{Hom}_K(\Lambda^2\mathfrak{m}, \mathfrak{m})$.
(ii) ∇ and ∇' have the same torsion if and only if $\eta \in \mathrm{Hom}_K(\mathrm{Sym}^2\,\mathfrak{m}, \mathfrak{m})$.

Consider now a homogeneous Riemannian manifold $(M = G/K, g)$. In this case G can be considered as a closed subgroup of the full isometry group $\mathrm{Iso}(M, g)$, which implies that K and the Lie subgroup $\mathrm{Ad}(K) \subset \mathrm{Ad}(G)$ are compact subgroups. Hence, there is always a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ with respect to some $\mathrm{Ad}(K)$-invariant inner product in the Lie algebra $\mathfrak{g}$. We shall denote by $\langle\ ,\ \rangle$ the $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{m}$ induced by g. We equivariantly identify the K-modules $\mathfrak{so}(\mathfrak{m}, g) \equiv \mathfrak{so}(\mathfrak{m}) = \Lambda^2(\mathfrak{m})$ via the isomorphism $X \wedge Y \mapsto \langle X, \cdot\rangle Y - \langle Y, \cdot\rangle X$, for any $X, Y \in \mathfrak{m}$. Consider the $\mathrm{SO}(\mathfrak{m})$-principal bundle $\mathrm{SO}(G/K) \to G/K$ of $\langle\ ,\ \rangle$-orthonormal frames. This is a homogeneous principal bundle and an invariant metric connection on $M = G/K$ is a principal connection on $\mathrm{SO}(G/K)$ that is G-invariant. It follows that

Lemma 1.5. *A G-invariant affine connection ∇ on $(M = G/K, g)$ preserves the G-invariant Riemannian metric g if and only if the associated Nomizu map satisfies $\Lambda(X) \in \mathfrak{so}(\mathfrak{m}, g)$ for any $X \in \mathfrak{m}$.*

Notice that the existence of an invariant metric means that the isotropy representation of $M = G/K$ is self-dual, $\mathfrak{m} \simeq \mathfrak{m}^*$. Thus we may equivariantly identify

$$\mathfrak{gl}(\mathfrak{m}) \simeq \mathrm{End}(\mathfrak{m}) \simeq \mathfrak{m} \otimes \mathfrak{m}, \quad \mathrm{Hom}_K(\mathfrak{m}, \mathrm{End}(\mathfrak{m})) = (\mathfrak{m}^* \otimes \mathfrak{m}^* \otimes \mathfrak{m})^K \simeq (\otimes^3\mathfrak{m})^K.$$

In the last case, a K-equivariant map Λ on the left hand side is equivalent to a K-invariant tensor on the right hand side: $\mathrm{Hom}_K(\mathfrak{m}, \mathrm{End}(\mathfrak{m})) = (\otimes^3\mathfrak{m})^K$. The latter space has the following obvious K-submodules: $\Lambda^2\mathfrak{m} \otimes \mathfrak{m}$, $\mathrm{Sym}^2\,\mathfrak{m} \otimes \mathfrak{m}$, $\mathfrak{m} \otimes \mathrm{Sym}^2\,\mathfrak{m}$ and $\mathfrak{m} \otimes \Lambda^2\mathfrak{m}$. Of these, the last space corresponds to the $\mathfrak{so}(\mathfrak{m})$-valued Nomizu maps, i.e. the space of homogeneous metric connections which we denote by $\mathcal{M}_G(\mathrm{SO}(G/K))$. In particular, there is an equivariant isomorphism

$$\mathcal{M}_G(\mathrm{SO}(G/K, g)) \cong \mathrm{Hom}_K(\mathfrak{m}, \Lambda^2\mathfrak{m}).$$

Remark 1.6. The other submodules have different interpretations. For example, $\mathrm{Sym}^2\,\mathfrak{m} \otimes \mathfrak{m}$ is the vector space on which the affine space of invariant torsion-free connections $\mathcal{A}\!\mathit{ff}^0_G(F(G/K))$ is modelled, and $\Lambda^2\mathfrak{m} \otimes \mathfrak{m}$ is the vector space on which the affine space of possible invariant torsion tensors is modelled. In fact, since the rearrangement of indices is equivariant (even with respect to the bigger algebra $\mathfrak{gl}(\mathfrak{m})$), one has the following isomorphisms: $\Lambda^2\mathfrak{m} \otimes \mathfrak{m} \simeq \mathfrak{m} \otimes \Lambda^2\mathfrak{m}$ and $\mathrm{Sym}^2\,\mathfrak{m} \otimes \mathfrak{m} \simeq \mathfrak{m} \otimes \mathrm{Sym}^2\,\mathfrak{m}$. Let us now relate this to the question of multiplicities of $\mathfrak{m}$ inside $\otimes^2\mathfrak{m} = \mathrm{End}(\mathfrak{m})$. Suppose we have a copy of $\mathfrak{m}$ inside the invariant decomposition of $\Lambda^2\mathfrak{m}$ (or respectively, in $\mathrm{Sym}^2\,\mathfrak{m}$). This is equivalent to a map $\theta : \mathfrak{m} \to \Lambda^2\mathfrak{m}$ (respectively $\mathfrak{m} \to \mathrm{Sym}^2\,\mathfrak{m}$). We may then raise all indices of θ to produce a K-invariant element of $\otimes^3\mathfrak{m}$. However through our freedom to rearrange indices, we may change to which of our four submodules this tensor belongs. For example, one may interpret the tensor corresponding to the instance of $\mathfrak{m}$ in $\Lambda^2\mathfrak{m}$ either as a metric connection in $\mathfrak{m} \otimes \Lambda^2\mathfrak{m}$, or a potentially non-metric connection in $\Lambda^2\mathfrak{m} \otimes \mathfrak{m}$. These coincide up to a scalar when $\theta \in \Lambda^3\mathfrak{m}$.

On a homogeneous Riemannian manifold $(M = G/K, g)$ the Levi-Civita connection ∇^g is the unique G-invariant metric connection determined by (cf. [13,35])

$$\langle \nabla^g_X Y, Z\rangle = -\frac{1}{2}\big[\langle [X,Y]_{\mathfrak{m}}, Z\rangle + \langle [Y,Z]_{\mathfrak{m}}, X\rangle - \langle [Z,X]_{\mathfrak{m}}, Y\rangle\big], \quad \forall\, X,Y,Z \in \mathfrak{m}. \tag{$\star$}$$

On the other hand, the *canonical connection* on $M = G/K$ is induced by the principal K-bundle $G \to G/K$ and depends on the choice of the reductive complement $\mathfrak{m}$. It is defined by the horizontal distribution $\{\mathcal{H}_g := d\ell_g(\mathfrak{m}) : g \in G\}$, where ℓ_g denotes the left translation on G and its Nomizu map is given by $\Lambda^c : \mathfrak{g} = \mathfrak{k}\oplus\mathfrak{m} \xrightarrow{\mathrm{pr}_{\mathfrak{k}}} \mathfrak{k} \xrightarrow{\chi_*} \mathfrak{so}(\mathfrak{m})$, i.e. $\Lambda^c = \chi_* \circ \mathrm{pr}_{\mathfrak{k}}$. Thus, $\Lambda^c(X) = 0$ for any $X \in \mathfrak{m}$ (cf. [29,9]). Both the torsion $T^c(X,Y) = -[X,Y]_{\mathfrak{m}}$ and the curvature $R^c(X,Y) = -\mathrm{ad}([X,Y]_{\mathfrak{k}})$ of ∇^c are parallel objects, in particular any G-invariant tensor field on $M = G/K$ is ∇^c-parallel (cf. [36,29]). Hence, any homogeneous Riemannian manifold $(M = G/K, g)$ admits a homogeneous structure $A^c \in \mathfrak{m}\otimes\Lambda^2\mathfrak{m} \cong \mathcal{A}$ induced by the canonical connection ∇^c associated to the reductive decomposition $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{m}$. In the following, we shall refer to this homogeneous structure as the *canonical homogeneous structure*, adapted to $\mathfrak{m}$ and G. Using ($\star$) it is easy to see that $A^c := \nabla^c - \nabla^g$ satisfies the relation

$$A^c(X,Y,Z) = \frac{1}{2}T^c(X,Y,Z) - \langle U(X,Y), Z\rangle, \quad \forall\, X,Y,Z \in \mathfrak{m}, \tag{1.5}$$

where $U : \mathfrak{m}\times\mathfrak{m} \to \mathfrak{m}$ is the symmetric bilinear mapping defined by

$$2\langle U(X,Y), Z\rangle = \langle [Z,X]_{\mathfrak{m}}, Y\rangle + \langle X, [Z,Y]_{\mathfrak{m}}\rangle. \tag{1.6}$$

2. Invariant connections and derivations

Given a reductive homogeneous space $M = G/K$ endowed with a G-invariant affine connection ∇, in the following we examine $\mathrm{Ad}(K)$-equivariant derivations on $\mathfrak{m}$ induced by ∇ in terms of Nomizu maps. For the case of a compact Lie group G, this problem has been analysed in [17].

2.1. Derivations and generalized derivations

For the rest of this section let us fix a (connected) homogeneous manifold $M = G/K$ with a reductive decomposition $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{m}$. For simplicity we assume that the transitive G-action is effective. We consider a bilinear mapping $\mu : \mathfrak{m}\otimes\mathfrak{m} \to \mathfrak{m}$ and denote by $\Lambda : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$ the adjoint map, defined by $\Lambda(X)Y = \mu(X,Y)$.

Definition 2.1. The endomorphism $\Lambda(Z) : \mathfrak{m} \to \mathfrak{m}$ $(Z \in \mathfrak{m})$ is called a *derivation* of $\mathfrak{m}$, with respect to the Lie bracket operation $\mathrm{ad}_{\mathfrak{m}} := [\ ,\ ]_{\mathfrak{m}} : \mathfrak{m}\times\mathfrak{m} \to \mathfrak{m}$, $\mathrm{ad}_{\mathfrak{m}}(X,Y) := [X,Y]_{\mathfrak{m}}$, if and only if $\mathfrak{der}^{\mu}(X,Y;Z) = 0$ identically, where for any $X,Y,Z \in \mathfrak{m}$ we set

$$\begin{aligned}\mathfrak{der}^{\mu}(X,Y;Z) &:= \Lambda(Z)[X,Y]_{\mathfrak{m}} - [\Lambda(Z)X,Y]_{\mathfrak{m}} - [X,\Lambda(Z)Y]_{\mathfrak{m}}\\ &= \mu(Z,[X,Y]_{\mathfrak{m}}) - [\mu(Z,X),Y]_{\mathfrak{m}} - [X,\mu(Z,Y)]_{\mathfrak{m}}.\end{aligned}$$

From now on, let us denote by $\mathrm{Der}(\mathrm{ad}_{\mathfrak{m}};\mathfrak{m}) \equiv \mathrm{Der}(\mathfrak{m})$ the vector space of all derivations on $\mathfrak{m}$. We mention that given a bilinear map $\mu : \mathfrak{m}\otimes\mathfrak{m} \to \mathfrak{m}$, the condition $\mu \in \mathrm{Der}(\mathfrak{m})$ is equivalent to say that the associated connection map Λ is valued in $\mathrm{Der}(\mathfrak{m})$, i.e. $\Lambda \in \mathrm{Hom}(\mathfrak{m}, \mathrm{Der}(\mathfrak{m}))$. Restricting on K-intertwining maps $\mu \in \mathrm{Hom}_K(\mathfrak{m}\otimes\mathfrak{m},\mathfrak{m})$ the vector space $\mathrm{Der}(\mathfrak{m})$ becomes a K-module, denoted by $\mathrm{Der}_K(\mathfrak{m})$. In fact, in this case we shall speak about $\mathrm{Ad}(K)$-equivariant derivations on $\mathfrak{m}$. So, let us focus on $\mathrm{Ad}(K)$-equivariant derivations induced by invariant connections on $M = G/K$.

Proposition 2.2. *Let $\nabla \equiv \nabla^{\mu}$ be a G-invariant connection on $M = G/K$ corresponding to $\mu \in \mathrm{Hom}_K(\mathfrak{m}\otimes\mathfrak{m},\mathfrak{m})$. Then, ∇^{μ} induces a $\mathrm{Ad}(K)$-equivariant derivation $\mu \in \mathrm{Der}_K(\mathfrak{m})$, if and only if $\mathrm{ad}_{\mathfrak{m}} := [\ ,\ ]_{\mathfrak{m}}$ is ∇^{μ}-parallel, i.e. $\nabla^{\mu}\,\mathrm{ad}_{\mathfrak{m}} = 0$ (which is equivalent to say that the torsion T^c of the canonical connection ∇^c associated to the reductive complement $\mathfrak{m}$ is ∇^{μ}-parallel, i.e. $\nabla^{\mu}T^c = 0$).*

Proof. The equivalence $\mu \in \mathrm{Der}(\mathrm{ad}_{\mathfrak{m}};\mathfrak{m}) \equiv \mathrm{Der}(\mathfrak{m}) \Leftrightarrow \nabla^{\mu}\,\mathrm{ad}_{\mathfrak{m}} \equiv 0$ is an immediate consequence of the identity

$$\mathfrak{der}^{\mu}(X,Y;Z) = (\nabla^{\mu}_Z\,\mathrm{ad}_{\mathfrak{m}})(X,Y) = -(\nabla^{\mu}_Z T^c)(X,Y), \quad \forall\, X,Y,Z \in \mathfrak{m}. \tag{2.1}$$

The proof of (2.1) relies on the fact that G-invariant tensor fields are ∇^c-parallel, where ∇^c is the canonical connection associated to $\mathfrak{m}$. In particular, since ∇ is a G-invariant connection we write $\nabla^{\mu}_Z = \nabla^c_Z + \Lambda(Z)$, for any $Z \in \mathfrak{m}$, where $\Lambda : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$ is the associated Nomizu map. Then, for any $X,Y,Z \in \mathfrak{m}$ we obtain that

$$\begin{aligned}(\nabla^{\mu}_Z\,\mathrm{ad}_{\mathfrak{m}})(X,Y) &= \nabla^{\mu}_Z\,\mathrm{ad}_{\mathfrak{m}}(X,Y) - \mathrm{ad}_{\mathfrak{m}}(\nabla^{\mu}_Z X,Y) - \mathrm{ad}_{\mathfrak{m}}(X,\nabla^{\mu}_Z Y)\\ &= \big[\nabla^c_Z\,\mathrm{ad}_{\mathfrak{m}}(X,Y) - \mathrm{ad}_{\mathfrak{m}}(\nabla^c_Z X,Y) - \mathrm{ad}_{\mathfrak{m}}(X,\nabla^c_Z Y)\big]\\ &\quad + \big[\Lambda(Z)\,\mathrm{ad}_{\mathfrak{m}}(X,Y) - \mathrm{ad}_{\mathfrak{m}}(\Lambda(Z)X,Y) - \mathrm{ad}_{\mathfrak{m}}(X,\Lambda(Z)Y)\big]\\ &= (\nabla^c_Z\,\mathrm{ad}_{\mathfrak{m}})(X,Y) + \mathfrak{der}^{\mu}(X,Y;Z) = \mathfrak{der}^{\mu}(X,Y;Z),\end{aligned}$$

where the last equality follows since $\nabla^c\,\mathrm{ad}_{\mathfrak{m}} \equiv 0$. Similarly for the second equality in (2.1). □

Example 2.3. The canonical connection ∇^c associated to the reductive complement $\mathfrak{m}$ induces a derivation on $\mathfrak{m}$ (the zero one, corresponding to $0 \in \mathrm{Der}_K(\mathfrak{m})$), since $\nabla^c T^c = 0$, or in other words since T^c is ∇^μ-parallel, where $\mu = 0 \in \mathrm{Hom}_K(\mathfrak{m}\otimes\mathfrak{m}, \mathfrak{m})$.

Let us now generalize the notion of derivations on $\mathfrak{m}$, as follows:

Definition 2.4. Consider a tensor $F : \otimes^p\mathfrak{m} \to \mathfrak{m}$. Then, a bilinear mapping $\mu : \mathfrak{m}\otimes\mathfrak{m} \to \mathfrak{m}$ is said to be a *generalized derivation* of F on $\mathfrak{m}$, if and only if μ satisfies the relation

$$\begin{aligned}\mu(Z, F(X_1,\dots,X_p)) &= F(\mu(Z,X_1),X_2,\dots,X_p)+\cdots+F(X_1,\dots,X_{p-1},\mu(Z,X_p)) \;\Leftrightarrow\\ \Lambda(Z)F(X_1,\dots,X_p) &= F(\Lambda(Z)X_1,X_2,\dots,X_p)+\cdots+F(X_1,\dots,X_{p-1},\Lambda(Z)X_p),\end{aligned}$$

for any $Z, X_1, \dots, X_p \in \mathfrak{m}$, where $\Lambda \in \mathrm{Hom}(\mathfrak{m}, \mathfrak{gl}(\mathfrak{m}))$ is the adjoint map induced by μ.

For a tensor $F : \otimes^p\mathfrak{m} \to \mathfrak{m}$, the definition of a generalized derivation implies that if $\mu_1, \mu_2 : \mathfrak{m}\otimes\mathfrak{m} \to \mathfrak{m}$ are two such bilinear mappings, then the linear combination $a\mu_1 + b\mu_2$ is also a generalized derivation of F on $\mathfrak{m}$. Hence, the set $\mathrm{Der}(F;\mathfrak{m})$ of all generalized derivations of F on $\mathfrak{m}$ is a vector space. Obviously, for $F = \mathrm{ad}_\mathfrak{m}$, a generalized derivation is just a classical derivation on $\mathfrak{m}$. Notice however that F can be much more general than the Lie bracket restriction, e.g. the torsion, or the curvature of a G-invariant connection ∇ on $M = G/K$ induced by some $\mu \in \mathrm{Hom}_K(\mathfrak{m}\otimes\mathfrak{m}, \mathfrak{m})$, or even μ itself. In particular, one may restrict Definition 2.4 on K-intertwining maps $\mu \in \mathrm{Hom}_K(\mathfrak{m}\otimes\mathfrak{m}, \mathfrak{m})$; then, the space $\mathrm{Der}(F;\mathfrak{m})$ becomes a K-module, which we shall denote by $\mathrm{Der}_K(F;\mathfrak{m})$. If moreover we focus on G-invariant tensor fields, then similarly to Proposition 2.2 we conclude that

Theorem 2.5. *Let $(M = G/K, \mathfrak{g} = \mathfrak{k}\oplus\mathfrak{m})$ be a reductive homogeneous space endowed with an $\mathrm{Ad}(K)$-invariant tensor $F : \otimes^p\mathfrak{m} \to \mathfrak{m}$. Consider a K-intertwining map $\mu \in \mathrm{Hom}_K(\mathfrak{m}\otimes\mathfrak{m}, \mathfrak{m})$ and let us denote by ∇^μ the associated G-invariant affine connection. Then, μ is an $\mathrm{Ad}(K)$-equivariant generalized derivation of F if and only if F is ∇^μ-parallel, i.e. $\mu \in \mathrm{Der}_K(F;\mathfrak{m}) \Leftrightarrow \nabla^\mu F \equiv 0$.*

Proof. A direct computation shows that the evaluation of the covariant differentiation ∇F at the point $o = eK \in G/K$ gives rise to the following $\mathrm{Ad}(K)$-invariant tensor on $\mathfrak{m}$:

$$\begin{aligned}(\nabla_Z F)(X_1,\dots,X_p) &= \nabla_Z F(X_1,\dots,X_p) - \sum_{i=1}^p F(X_1,\dots,\nabla_Z X_i,\dots,X_p)\\ &= \nabla^c_Z F(X_1,\dots,X_p) + \Lambda(Z)F(X_1,\dots,X_p) - \sum_{i=1}^p F(X_1,\dots,\nabla^c_Z X_i,\dots,X_p)\\ &\quad - \sum_{i=1}^p F(X_1,\dots,\Lambda(Z)X_i,\dots,X_p)\\ &= \nabla^c_Z F(X_1,\dots,X_p) - \sum_{i=1}^p F(X_1,\dots,\nabla^c_Z X_i,\dots,X_p)\\ &\quad + \Lambda(Z)F(X_1,\dots,X_p) - \sum_{i=1}^p F(X_1,\dots,\Lambda(Z)X_i,\dots,X_p)\\ &= (\nabla^c_Z F)(X_1,\dots,X_p) + (\mathcal{D}^\mu_Z F)(X_1,\dots,X_p),\end{aligned}$$

where we set $(\mathcal{D}^\mu_Z F)(X_1,\dots,X_p) := \Lambda(Z)F(X_1,\dots,X_p) - \sum_{i=1}^p F(X_1,\dots,\Lambda(Z)X_i,\dots,X_p)$. However, F is by assumption G-invariant, hence $\nabla^c F = 0$ and our claim immediately follows. □

Moreover, we see that

Corollary 2.6. *On a reductive homogeneous space $(M = G/K, \mathfrak{g} = \mathfrak{k}\oplus\mathfrak{m})$, given an $\mathrm{Ad}(K)$-invariant tensor $F : \otimes^p\mathfrak{m} \to \mathfrak{m}$ and some K-intertwining map $\mu \in \mathrm{Hom}_K(\mathfrak{m}\otimes\mathfrak{m}, \mathfrak{m})$, the operation*

$$(\mathcal{D}^\mu_Z F)(X_1,\dots,X_p) := \Lambda(Z)F(X_1,\dots,X_p) - \sum_{i=1}^p F(X_1,\dots,\Lambda(Z)X_i,\dots,X_p)$$

coincides with the covariant differentiation of F with respect to the connection $\nabla = \nabla^\mu$ induced on $M = G/K$ by μ, i.e. $(\nabla^\mu_Z F)(X_1,\dots,X_p) = (\mathcal{D}^\mu_Z F)(X_1,\dots,X_p)$ for any $X_1,\dots,X_p, Z \in \mathfrak{m}$.

For a bilinear mapping $\mu : \mathfrak{m}\otimes\mathfrak{m} \to \mathfrak{m}$ let us now introduce the tensor $\mathcal{C}^\mu$, defined by

$$\mathcal{C}^\mu(X,Y;Z) := (\nabla^\mu_Z\mu)(X,Y) - (\nabla^\mu_Z\mu)(Y,X),$$

for any $X, Y, Z \in \mathfrak{m}$. If $\mu \in \mathrm{Hom}_K(\mathfrak{m}\otimes\mathfrak{m}, \mathfrak{m})$, then we get the further identification $\mathcal{C}^\mu(X,Y;Z) := (\mathcal{D}^\mu_Z\mu)(X,Y) - (\mathcal{D}^\mu_Z\mu)(Y,X)$. In terms of $\mathcal{C}^\mu$ it is easy to check that

Proposition 2.7. *Let $\nabla = \nabla^\mu$ be a G-invariant affine connection on a reductive homogeneous space ($M = G/K$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$), corresponding to some $\mu \in \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$. Then, $\mu \in \mathrm{Der}_K(\mathfrak{m})$, if and only if*

$$(\nabla_Z T)(X,Y) \equiv (\mathcal{D}^\mu_Z T)(X,Y) = \mathcal{C}^\mu(X,Y;Z), \quad \forall\, X,Y,Z \in \mathfrak{m},$$

where $T = T^\mu$ is the torsion associated to ∇^μ.

Consequently, for some $\mu \in \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$ the condition $\mu \in \mathrm{Der}_K(\mathfrak{m})$ can also be read in terms of the $\mathrm{Ad}(K)$-invariant tensor $\mathcal{C}^\mu$, which geometrically, represents the difference

$$(\nabla_Z T)(X,Y) - (\nabla_Z T^c)(X,Y) \equiv (\mathcal{D}^\mu_Z T)(X,Y) - (\mathcal{D}^\mu_Z T^c)(X,Y),$$

for any $X, Y, Z \in \mathfrak{m}$. In particular, a combination of Proposition 2.7 and identity (1.4), yields that

Theorem 2.8. *Let $M = G/K$ an effective homogeneous space with a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Then the following hold:*
(1) A G-invariant affine connection $\nabla = \nabla^\mu$ on $M = G/K$ corresponding to $\mu \in \mathrm{Hom}_K(\Lambda^2\mathfrak{m}, \mathfrak{m})$, induces an $\mathrm{Ad}(K)$-equivariant derivation $\mu \in \mathrm{Der}_K(\mathfrak{m})$, if and only if

$$(\nabla^\mu_Z T)(X,Y) \equiv (\mathcal{D}^\mu_Z T)(X,Y) = 2\mathfrak{S}_{X,Y,Z}\mu(X,\mu(Y,Z)), \tag{2.2}$$

for any $X, Y, Z \in \mathfrak{m}$. This is equivalent to say that

$$(\nabla^\mu_Z T)(X,Y) \equiv (\mathcal{D}^\mu_Z T)(X,Y) = 2\big\{R(Z,X)Y + \Lambda(Y)(\Lambda(Z)X - [Z,X]_\mathfrak{m}) + \mathrm{ad}([Z,X]_\mathfrak{k})Y\big\}, \tag{2.3}$$

where R is the curvature tensor associated to ∇.
(2) A G-invariant affine connection $\nabla = \nabla^\mu$ on $M = G/K$ corresponding to $\mu \in \mathrm{Hom}_K(\mathrm{Sym}^2\mathfrak{m}, \mathfrak{m})$, induces an $\mathrm{Ad}(K)$-equivariant derivation $\mu \in \mathrm{Der}_K(\mathfrak{m})$ on $\mathfrak{m}$ if and only if the torsion T^μ associated to ∇^μ is ∇^μ-parallel.
(3) Let $\mu \in \mathrm{Hom}_K(\Lambda^2\mathfrak{m}, \mathfrak{m})$. Then μ is an $\mathrm{Ad}(K)$-equivariant generalized derivation of itself, i.e. $\mu \in \mathrm{Der}_K(\mu;\mathfrak{m})$ if and only if $\mathcal{C}^\mu = 0$ identically.

Proof. For a skew-symmetric mapping $\mu \in \mathrm{Hom}_K(\Lambda^2\mathfrak{m}, \mathfrak{m})$ a simple computation gives that

$$\mathcal{C}^\mu(X,Y;Z) = 2\mathfrak{S}_{X,Y,Z}\mu(X,\mu(Y,Z)).$$

Hence, (2.2) is an immediate consequence of Proposition 2.7. For the second relation (2.3), using the definition of the curvature tensor R and (2.7), for some $\mu \in \mathrm{Hom}_K(\Lambda^2\mathfrak{m}, \mathfrak{m})$ we get that

$$\begin{aligned}(\nabla_Z T)(X,Y) = (\mathcal{D}^\mu_Z T)(X,Y) = {} & 2R(Z,X)Y + 2\Lambda(Y)(\Lambda(Z)X - [Z,X]_\mathfrak{m}) \\ & + 2\,\mathrm{ad}([Z,X]_\mathfrak{k})Y - \mathfrak{der}^\mu(X,Y;Z),\end{aligned} \tag{2.4}$$

for any $X, Y, Z \in \mathfrak{m}$ and our claim immediately follows.

For the second statement and for a symmetric map $\mu \in \mathrm{Hom}_K(\mathrm{Sym}^2\mathfrak{m}, \mathfrak{m})$ it is easy to see that $\mathcal{C}^\mu = 0$. Therefore, our assertion is a direct consequence of Proposition 2.7.

Let us finally prove (3). By definition, it is $\mu \in \mathrm{Der}_K(\mu;\mathfrak{m})$ or equivalent $\Lambda \in \mathrm{Hom}_K(\mathfrak{m}, \mathrm{Der}_K(\mu;\mathfrak{m}))$, if and only if

$$\mu(Z,\mu(X,Y)) = \mu(\mu(Z,X),Y) + \mu(X,\mu(Z,Y))$$

for any $X, Y, Z \in \mathfrak{m}$, which is equivalent to say that $\mathfrak{S}_{X,Y,Z}\mu(X,\mu(Y,Z)) = 0$ identically. But since $\mathcal{C}^\mu(X,Y;Z) = 2\mathfrak{S}_{X,Y,Z}\mu(X,\mu(Y,Z))$, we conclude. □

Remark 2.9. For a compact connected Lie group $G \cong (G \times G)/\Delta G$ endowed with a bi-invariant affine connection ∇ corresponding to a skew-symmetric mapping $\mu \in \mathrm{Hom}_G(\Lambda^2\mathfrak{g}, \mathfrak{g})$, formula (2.3) has been described in [17, Prop. 2.4]. In particular, in this case relation (2.4) reduces to

$$(\nabla_Z T)(X,Y) = 2R(Z,X)Y + 2\Lambda(Y)(\Lambda(Z)X - [Z,X]_\mathfrak{m}) - \mathfrak{der}_\mathfrak{g}(X,Y;Z),$$

for any $X, Y, Z \in \mathfrak{g} = T_eG$, see also [17, Prop. 2.4].

Example 2.10. Let $M = G/K$ an effective homogeneous space with a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. We consider the restricted Lie bracket $\mathrm{ad}_\mathfrak{m} := [-,-]_\mathfrak{m} : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ and denote the associated Nomizu map just by $\Lambda_\mathfrak{m}$. Obviously, $\mathrm{ad}_\mathfrak{m}$ *induces a derivation on $\mathfrak{m}$ if and only if* $\mathrm{Jac}_\mathfrak{m} \equiv 0$, where $\mathrm{Jac}_\mathfrak{m} : \mathfrak{m} \times \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is the trilinear map defined by

$$\mathrm{Jac}_\mathfrak{m}(X,Y,Z) := \mathfrak{S}_{X,Y,Z}[X,[Y,Z]_\mathfrak{m}]_\mathfrak{m} = [X,[Y,Z]_\mathfrak{m}]_\mathfrak{m} + [Y,[Z,X]_\mathfrak{m}]_\mathfrak{m} + [Z,[X,Y]_\mathfrak{m}]_\mathfrak{m},$$

for any $X, Y, Z \in \mathfrak{m}$. The same conclusion follows from Theorem 2.8. Indeed, let us denote by $\nabla^\mathfrak{m}$ the G-invariant connection associated to $\mathrm{ad}_\mathfrak{m}$ and by $T^\mathfrak{m}$ and $R^\mathfrak{m}$ its torsion and curvature, respectively. It is $T^\mathfrak{m}(X,Y) = [X,Y]_\mathfrak{m}$ and

$$(\nabla^\mathfrak{m}_Z T^\mathfrak{m})(X,Y) = (\mathcal{D}^{\alpha_\mathfrak{m}}_Z T^\mathfrak{m})(X,Y) = \mathrm{Jac}_\mathfrak{m}(X,Y,Z),$$

for any $X, Y, Z \in \mathfrak{m}$. Moreover, $R^{\mathfrak{m}}(Z,X)Y = \mathrm{Jac}_{\mathfrak{m}}(X,Y,Z) - [[Z,X]_{\mathfrak{k}}, Y]$ and since $\Lambda_{\mathfrak{m}}(Z)X = [Z,X]_{\mathfrak{m}}$, an application of Theorem 2.8, (1), shows that $\Lambda_{\mathfrak{m}} \in \mathrm{Hom}_K(\mathfrak{m}, \mathrm{Der}(\mathfrak{m}))$ if and only if $\mathrm{Jac}_{\mathfrak{m}}(X,Y,Z) = 0$ for any $X, Y, Z \in \mathfrak{m}$. In fact, for $\mu = \mathrm{ad}_{\mathfrak{m}}$ it is $\mathcal{C}^{\mathrm{ad}_{\mathfrak{m}}}(X,Y;Z) = 2\,\mathrm{Jac}_{\mathfrak{m}}(X,Y,Z)$, hence the same results follows by relation (2.2). Finally, for the same assertion one can even apply Theorem 2.8, (3) for $\mu = \mathrm{ad}_{\mathfrak{m}}$.

Note that if $M = G/K$ is an effective symmetric space, then $\mathrm{Jac}_{\mathfrak{m}}$ is identically zero and $\mathrm{ad}_{\mathfrak{m}}$ is a derivation trivially. For example, any compact connected Lie group $M = G$ with a bi-invariant metric can be viewed as a symmetric space of the form $(G \times G)/\Delta G$. The Cartan decomposition is given by $\mathfrak{g} \oplus \mathfrak{g} = \Delta\mathfrak{g} \oplus \mathfrak{m}$, where $\Delta\mathfrak{g} := \{(X,X) \in \mathfrak{g} \oplus \mathfrak{g} : X \in \mathfrak{g}\} \cong \mathfrak{g}$ and $\mathfrak{m} := \{(X,-X) \in \mathfrak{g} \oplus \mathfrak{g} : X \in \mathfrak{g}\} \cong \mathfrak{g}$, respectively. Obviously, in this case the condition $\mathrm{Jac}_{\mathfrak{m}} \equiv 0$ is the Jacobi identity which leads to the well-known result that the adjoint representation $\Lambda_{\mathfrak{g}} = \mathrm{ad}_{\mathfrak{g}}$ is a derivation of $\mathfrak{g}$. In the following section we examine the condition $\mathrm{Jac}_{\mathfrak{m}} \equiv 0$ also on non-symmetric, compact, effective and simply-connected naturally reductive manifolds, see Corollary 3.6.

3. Invariant connections on naturally reductive manifolds

Next we describe a series of new results related to invariant connections (and their torsion type) on effective naturally reductive spaces. Note that all these results can be applied on an effective, non-symmetric (compact) SII homogeneous Riemannian manifold.

3.1. Naturally reductive spaces

A Riemannian manifold (M,g) is called *naturally reductive* if there exists a closed subgroup G of the isometry group $\mathrm{Iso}(M,g)$ which acts transitively on M with isotropy group K and which induces a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ such that the torsion of the canonical connection ∇^c associated to $\mathfrak{m}$, is a 3-form $T^c \in \Lambda^3(\mathfrak{m})$. This is equivalent to say that $U \equiv 0$ identically, where $U : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is the bilinear map defined by (1.6). Thus, an alternative way to define naturally reductive manifolds is as follows:

Definition 3.1. A naturally reductive manifold is a homogeneous Riemannian manifold $(M = G/K, g)$ with a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ such that canonical homogeneous structure $A^c \in \mathfrak{m} \otimes \Lambda^2\mathfrak{m}$ adapted to $\mathfrak{m}$ and G, is totally skew-symmetric, i.e. $2A^c(X,Y,Z) = T^c(X,Y,Z)$ for any $X, Y, Z \in \mathfrak{m}$.

A special subclass of naturally reductive manifolds $M = G/K$ consists of the so-called *normal* homogeneous Riemannian manifolds. In this case there is an $\mathrm{Ad}(G)$-invariant inner product Q on $\mathfrak{g}$ such that $Q(\mathfrak{k},\mathfrak{m}) = 0$, i.e. $\mathfrak{m} = \mathfrak{k}^{\perp}$ and $Q|_{\mathfrak{m}} = \langle\,,\,\rangle$. Thus, a normal metric is defined by a positive definite bilinear form Q. Notice however that Q can be more general, see [30,13]. If $Q = -B$, where B denotes the Killing form of $\mathfrak{g}$, then the normal metric is called the *Killing* (or *standard*) metric; this is the case if the Lie group G is compact and semi-simple. We mention that in this paper whenever we refer to a naturally reductive space $(M = G/K, g, \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m})$ we shall always assume that G acts effectively on M and that $\mathfrak{g}$ coincides with the ideal $\tilde{\mathfrak{g}} := \mathfrak{m} + [\mathfrak{m},\mathfrak{m}]$. On the level of Lie groups this condition means that G coincides with the transvection group of the associated canonical connection ∇^c. Note that any compact normal homogeneous space satisfies this condition, see [41].

3.2. Properties of invariant connections in the naturally reductive setting

We start with the following simple observation.

Lemma 3.2. *Let $(M = G/K, g)$ be an effective compact homogeneous Riemannian manifold with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ which is not a symmetric space of Type I. Then, there is always an instance of $\mathfrak{m}$ inside $\Lambda^2(\mathfrak{m})$, associated to the restriction of the Lie bracket operation on the reductive complement $\mathfrak{m}$. In particular, this specific copy gives rise to a G-invariant metric connection on M if and only if g is naturally reductive with respect to G and $\mathfrak{m}$.*

Proof. Since $M = G/K$ is not isometric to a symmetric space of Type I, the canonical connection ∇^c has non-trivial torsion $T^c(X,Y) = -[X,Y]_{\mathfrak{m}}$, which gives rise to a non-trivial $\mathrm{Ad}(K)$-equivariant skew-symmetric bilinear mapping $\mathrm{ad}_{\mathfrak{m}} : \Lambda^2\mathfrak{m} \to \mathfrak{m}$. The second statement is apparent due to the naturally reductive property. □

Remark 3.3. If $(M = G/K, g)$ is a Riemannian symmetric space of Type I, then G is a compact simple Lie group and its Killing form B gives rise to a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ such that $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{k}$. Moreover, the restriction $\langle\,,\,\rangle = -B|_{\mathfrak{m}}$ induces a G-invariant metric which is naturally reductive with respect to $\mathfrak{m}$. However, the K-module $\Lambda^2(\mathfrak{m})$ never contains a copy of $\mathfrak{m}$, see also [33]. This is in contrast to a Riemannian symmetric space $(M = G, g = \rho)$ of Type II endowed with a bi-invariant metric ρ, where one can always find a copy of $\mathfrak{g}$ inside $\Lambda^2(\mathfrak{g})$, see also Remark 3.14. Geometrically, this copy represents the existence of 1-parameter family of canonical connections on any compact simple Lie group G (cf. [29,6,17]). The same is true in the more general compact case (cf. [38]).

Lemma 3.4 (*[1,17]*). *Let $(M = G/K, g)$ be an effective naturally reductive manifold with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ such that $\mathfrak{g} = \tilde{\mathfrak{g}}$. Then,*
(i) A G-invariant metric connection ∇ on $(M = G/K, g)$ has totally skew-symmetric torsion $T \in \Lambda^3(\mathfrak{m})$ if and only if $\Lambda(Z)Z = 0$, for any $Z \in \mathfrak{m}$, where Λ is the associated Nomizu map.
(ii) There is a bijective correspondence between $\mathrm{Ad}(K)$-equivariant maps $\Lambda^\alpha : \mathfrak{m} \to \mathfrak{so}(\mathfrak{m})$, defined by $\Lambda^\alpha(X)Y = \frac{1-\alpha}{2}[X, Y]_\mathfrak{m} = (1-\alpha)\Lambda^g(X)Y$ for any $X, Y \in \mathfrak{m}$, and G-invariant metric connections ∇^α with totally skew-symmetric torsion $T^\alpha \in \Lambda^3(\mathfrak{m})$, such that $T^\alpha = \alpha \cdot T^c$ for some parameter α, where T^c is the torsion of the canonical connection ∇^c associated to $\mathfrak{m}$ and $\Lambda^g : \mathfrak{m} \to \mathfrak{so}(\mathfrak{m})$ the Nomizu map associated to the Levi-Civita connection ∇^g.

Let us finally recall the following fundamental result by Olmos and Reggiani.

Theorem 3.5 (*[38, Thm. 1.2], [37, Thm. 2.1]*). *Let (M^n, g) be a simply-connected and irreducible Riemannian manifold that is not isometric to a sphere, nor to a projective space, nor to a compact simple Lie group with a bi-invariant metric or its symmetric dual. Then the canonical connection is unique, i.e. (M^n, g) admits at most one naturally reductive homogeneous structure.*

Combining the observations in Example 2.10 and the results of Lemma 3.4 and Theorem 3.5, in the compact and simply-connected case we obtain the following conclusion about derivations on $\mathfrak{m}$:

Corollary 3.6. *Let $(M = G/K, g)$ be an effective, compact and simply-connected naturally reductive manifold, irreducible as Riemannian manifold, endowed with a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ such that $\mathfrak{g} = \tilde{\mathfrak{g}}$. Assume that $M = G/K$ is not isometric to a symmetric space of Type I, neither to a sphere or to a real projective space. Then, the bilinear mapping $\mathrm{ad}_\mathfrak{m} := [\ ,\ ]_\mathfrak{m}$ gives rise to $\mathrm{Ad}(K)$-equivariant derivation on $\mathfrak{m}$, if and only if M is isometric to a compact simple Lie group G endowed with a bi-invariant metric.*

Proof. A special version of Corollary 3.6 has been proved in [17, Lem. 4.5]. Here we improve this result. We omit some details and only present the main idea. Assume that $\mathrm{ad}_\mathfrak{m} \in \mathrm{Der}_K(\mathfrak{m})$, i.e. $\mathrm{Jac}_\mathfrak{m}(X, Y, Z) = 0$ for any $X, Y, Z \in \mathfrak{m}$ (see Example 2.10). Then, for the family ∇^α of Lemma 3.4, a small computation shows that $\nabla^\alpha T^\alpha = 0$ for any α, see for example [1]. On the other hand, one can easily see that $[X, [Y, Z]_\mathfrak{m}]_\mathfrak{k} + [Y, [Z, X]_\mathfrak{m}]_\mathfrak{k} + [Z, [X, Y]_\mathfrak{m}]_\mathfrak{k} = 0$, for any $X, Y, Z \in \mathfrak{m}$. Combining this identity with $\mathrm{Jac}_\mathfrak{m}(X, Y, Z) = 0$, a long computation certifies that $\nabla^\alpha R^\alpha = 0$ for any α, as well. But then, ∇^α is a 1-parameter family of canonical connections on $M = G/K$ (in the sense of the Ambrose–Singer Theorem) and Theorem 3.5 yields the result. □

Notation: Let $(M = G/K, g)$ be an effective compact naturally reductive Riemannian manifold with a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} = \tilde{\mathfrak{g}}$. If $\chi : K \to \mathrm{Aut}(\mathfrak{m})$ is of real type, we shall denote by $\mathbf{s}$ and $\mathbf{a}$ the multiplicity of $\mathfrak{m}$ inside $\mathrm{Sym}^2(\mathfrak{m})$ and $\Lambda^2(\mathfrak{m})$, respectively (or twice the multiplicity of $\mathfrak{m}$ inside $\mathrm{Sym}^2(\mathfrak{m})$ and $\Lambda^2(\mathfrak{m})$, respectively, if $\chi : K \to \mathrm{Aut}(\mathfrak{m})$ is of complex type). We also set

$$\mathcal{N} = \mathbf{s} + \mathbf{a} := \dim_\mathbb{R} \mathcal{A}ff_G(F(G/K)) = \dim_\mathbb{R} \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$$
$$\mathcal{N}_{\mathrm{mtr}} := \dim_\mathbb{R} \mathcal{M}_G(\mathrm{SO}(G/K)) = \dim_\mathbb{R} \mathrm{Hom}_K(\mathfrak{m}, \Lambda^2(\mathfrak{m})) \leq \mathcal{N}.$$

Since K is compact, and we treat finite dimensional K-representations, we conclude that

Lemma 3.7. *The dimensions of modules $\mathrm{Hom}_K(\Lambda^2\mathfrak{m}, \mathfrak{m})$ and $\mathrm{Hom}_K(\mathfrak{m}, \Lambda^2\mathfrak{m})$ coincide,*

$$\dim_\mathbb{R} \mathrm{Hom}_K(\Lambda^2\mathfrak{m}, \mathfrak{m}) = \dim_\mathbb{R} \mathrm{Hom}_K(\mathfrak{m}, \Lambda^2\mathfrak{m}),$$

or in other words $\mathbf{a} = \mathcal{N}_{\mathrm{mtr}}$.

Remark 3.8. Note that there exists compact Lie groups, e.g. $G = \mathrm{U}_n$, admitting skew-symmetric $\mathrm{Ad}(G)$-equivariant maps $\Lambda^2(\mathfrak{g}) \to \mathfrak{g}$ which do not induce bi-invariant metric connections with respect to the bi-invariant inner product $\langle X, Y\rangle = -\operatorname{tr} XY$ (see also the proof of Theorem 3.15). In fact, below we will show that Lemma 3.7 implies that [6, Lem. 3.1] or [17, Corol. 2.3, Thm. 2.9] are in general false. In particular, the corresponding statements hold only for compact simple Lie groups, but fail for general compact Lie groups.

Next, our aim is to clarify Remark 3.8. For simplicity, given an $\mathrm{Ad}(K)$-equivariant bilinear mapping $\mu : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ associated to a G-invariant connection ∇ on $(M = G/K, g)$ we shall use the same notation for the corresponding K-intertwining map $\mu : \mathfrak{m} \otimes \mathfrak{m} \to \mathfrak{m}$ (and we shall identify them) and denote by $\hat{\mu}$ the contraction of μ with the $\mathrm{Ad}(K)$-invariant inner product $\langle\ ,\ \rangle : \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}$ associated to g, i.e. $\hat{\mu}(X, Y, Z) := \langle \mu(X, Y), Z\rangle$, for any $X, Y, Z \in \mathfrak{m}$. Notice that $\hat{\mu}$ is an $\mathrm{Ad}(K)$-invariant tensor on $\mathfrak{m}$. Initially, it is useful to examine invariant connections related to some $\mu \in \mathrm{Hom}_K(\mathrm{Sym}^2\mathfrak{m}, \mathfrak{m})$. Then, the induced tensor $\hat{\mu} = \hat{\mu}^\mathbf{s}$ is such that $\hat{\mu}(X, Y, Z) = \hat{\mu}(Y, X, Z)$ for any $X, Y, Z \in \mathfrak{m} \cong T_oG/K$ and the corresponding Nomizu map $\Lambda := \Lambda^\mathbf{s} : \mathfrak{m} \to \mathrm{Sym}^2(\mathfrak{m})$ is also symmetric in the sense that $\Lambda(X)Y = \Lambda(Y)X$ (since $\mu(X, Y) = \mu(Y, X)$ for any $X, Y \in \mathfrak{m}$). Next we prove that when $0 \neq \mu \in \mathrm{Hom}_K(\mathrm{Sym}^2\mathfrak{m}, \mathfrak{m})$ is non-trivial, then the induced connection cannot preserve the naturally reductive metric $\langle\ ,\ \rangle$. We start with the non-symmetric case.

Lemma 3.9. *Let $(M = G/K, g)$ be a connected, compact, non-symmetric, naturally reductive space of a compact Lie group G modulo a compact subgroup K. Assume that the transitive G-action is effective and let us denote by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} = \tilde{\mathfrak{g}}$ the associated reductive decomposition. If ∇ is an invariant metric connection induced by some $\mu \in \mathrm{Hom}_K(\mathrm{Sym}^2 \mathfrak{m}, \mathfrak{m})$, then $\mu = 0$ and ∇ coincides with the canonical connection ∇^c associated to $\mathfrak{m}$.*

In fact, the non-existence of an invariant metric connection $\nabla^{\mathbf{s}}$ corresponding to a non-trivial element $0 \neq \mu \in \mathrm{Hom}_K(\mathrm{Sym}^2 \mathfrak{m}, \mathfrak{m})$ can be proved also as follows. By the condition $T^{\mathbf{s}}(X, Y) = -[X, Y]_{\mathfrak{m}}$ and since $\langle\ ,\ \rangle$ is naturally reductive with respect to G, one concludes that $\nabla^{\mathbf{s}}$ is an invariant connection with skew-torsion, which according to Lemma 3.4 is equivalent to say that $\Lambda^{\mathbf{s}}(X)X = 0$ for any $X \in \mathfrak{m}$. But then, it is also $\Lambda^{\mathbf{s}}(X + Y)(X + Y) = 0$ for any $X, Y \in \mathfrak{m}$, i.e. $\Lambda^s(X)Y = -\Lambda^s(Y)X$, which gives rise to a contradiction (since $\mu \neq 0$). Let us now explain also the compact symmetric case.

Lemma 3.10. *Given a connected Riemannian symmetric space $(M = G/K, g)$ of Type I (resp. $(M = (G\times G)/\Delta(G),\ \rho)$ of Type II, for some compact, connected, simple Lie group G with a bi-invariant metric ρ), then the unique G-invariant (resp. bi-invariant) metric connection which is induced by a symmetric $\mathrm{Ad}(K)$-equivariant mapping on $\mathfrak{m}$ (resp, symmetric $\mathrm{Ad}(G)$-equivariant mapping on $\mathfrak{g}$), is the torsion-free Levi-Civita connection.*

Proof. Consider first a symmetric space $(M = G/K, g)$ of Type I, endowed with a G-invariant affine connection ∇^μ associated to an element $\mu \in \mathrm{Hom}_K(\mathrm{Sym}^2 \mathfrak{m}, \mathfrak{m})$. Then, $T^\mu(X, Y) = 0$ for any $X, Y \in \mathfrak{m}$, i.e. $\hat{\mu} \in \mathrm{Sym}^2 \mathfrak{m} \otimes \mathfrak{m}$. Hence, assuming in addition that ∇ is metric with respect to g, the fundamental theorem in Riemannian geometry implies the identification of ∇ with the unique torsion-free metric connection on $(M = G/K, g)$, i.e. the Levi-Civita connection, or the canonical connection associated to $\mathfrak{m}$. In particular, $\mu = 0$ is trivial. The same conclusions, related this time to bi-invariant metric connections corresponding to maps $\mu \in \mathrm{Hom}_G(\mathrm{Sym}^2 \mathfrak{g}, \mathfrak{g})$, hold for a compact, connected, simple Lie group $G \cong (G \times G)/\Delta G$, endowed with a bi-invariant metric ρ. □

Remark 3.11. Laquer proved in [33] the existence of (irreducible) compact symmetric spaces $(M = G/K, g)$ which admit invariant affine connections induced by non-trivial elements $\mu \in \mathrm{Hom}_K(\mathrm{Sym}^2 \mathfrak{m}, \mathfrak{m})$. And indeed, by [6] we know that these G-invariant connections are not metric with respect to $g = -B|_{\mathfrak{m}}$, as it should be according to Lemma 3.10. The same is true for compact simple Lie groups, such as SU_n, see [32,6].

Let us now consider invariant connections whose torsion is a 3-form. We show that on an effective, non-symmetric, compact, naturally reductive space $(M = G/K, g)$ the G-invariant metric connections whose torsion is a 3-form necessarily correspond to instances of the trivial representation inside the space $\Lambda^3\mathfrak{m}$, and conversely. In particular, the torsion form is a G-invariant 3-form. Let us denote by ℓ the multiplicity of the (real) trivial representation inside $\Lambda^3\mathfrak{m}$.

Lemma 3.12. *Let $(M = G/K, g)$ be a naturally reductive manifold as in Lemma 3.9. Then, the dimension of the affine space of G-invariant metric connections on M which have the same geodesics with the Levi-Civita connection ∇^g, i.e. $\Lambda(X)X = 0$, or equivalent whose torsion form T is a non-trivial G-invariant 3-form, is equal to ℓ. In particular,*

$$1 \leq \ell \leq \mathcal{N}_{\mathrm{mtr}} = \mathbf{a} \leq \mathcal{N}.$$

Proof. First notice that $1 \leq \ell \leq \mathcal{N}_{\mathrm{mtr}}$. This follows since the induced $\mathrm{Ad}(K)$-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{m}$ satisfies the naturally reductive property and hence the torsion of the canonical connection $T^c(X, Y, Z) = -\langle [X, Y]_{\mathfrak{m}}, Z\rangle \neq 0$ is a non-trivial G-invariant 3-form. Then, according to Lemma 3.4, the family $\nabla^\alpha = \nabla^c + \Lambda^\alpha$ induces a 1-parameter family of metric connections with skew-torsion $T^\alpha := \alpha T^c \neq 0$. Now, any instance of the trivial representation inside $\Lambda^3(\mathfrak{m})$ induces a G-invariant (global) 3-form on $M = G/K$, say $0 \neq T \in \Lambda^3(\mathfrak{m})^K$. If $\ell \geq 2$, then we can also assume that $T \neq T^\alpha$. But then, one can define a 1-parameter family of metric connections with skew-torsion, say $2sT$, given by $\nabla^s = \nabla^g + sT$. Obviously, this family is G-invariant and preserves the metric. On the other hand, if $M = G/K$ admits a G-invariant metric connection ∇ with skew-torsion T such that $T \neq T^\alpha$, then T must be a global G-invariant 3-form and hence it corresponds to a new copy of the trivial representation inside $\Lambda^3\mathfrak{m}$. □

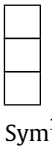
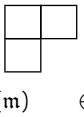
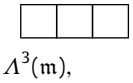
For a complete description of all G-invariant metric connections on $(M = G/K, g)$, one has to encode the "defect"

$$\epsilon := \mathcal{N}_{\mathrm{mtr}} - \ell \geq 0.$$

For this, it is useful to consider the tensor product

$$\otimes^3\mathfrak{m} = \mathfrak{m} \otimes \mathfrak{m} \otimes \mathfrak{m} \cong (\Lambda^2\mathfrak{m} \oplus \mathrm{Sym}^2 \mathfrak{m}) \otimes \mathfrak{m} \cong \left(\Lambda^2\mathfrak{m} \otimes \mathfrak{m}\right) \oplus \left(\mathrm{Sym}^2 \mathfrak{m} \otimes \mathfrak{m}\right)$$

and its decomposition in terms of Young diagrams:

$\otimes^3\mathfrak{m} =$  $\oplus$ 2  $\oplus$ 

$= \mathrm{Sym}^3\,\mathfrak{m} \oplus \mathcal{L}(\mathfrak{m}) \oplus \Lambda^3(\mathfrak{m}),$

where $\mathcal{L}(\mathfrak{m}) := \ker(P_{\mathbf{sym}}) \cap \ker(P_{\mathbf{skew}})$ is the section of the kernels of the equivariant projections

$$P_{\mathbf{sym}} : \otimes^3\mathfrak{m} \to \mathrm{Sym}^3(\mathfrak{m}), \quad P_{\mathbf{skew}} : \otimes^3\mathfrak{m} \to \Lambda^3(\mathfrak{m}).$$

Notice that the kernel of the natural maps $\mathrm{Sym}^2\,\mathfrak{m} \otimes \mathfrak{m} \to \mathrm{Sym}^3\,\mathfrak{m}$ and $\Lambda^2(\mathfrak{m}) \otimes \mathfrak{m} \to \Lambda^3(\mathfrak{m})$ are isomorphic irreducible $\mathrm{GL}(\mathfrak{m})$-modules of real dimension $n(n-1)(n+1)/3$, where $n := \dim_{\mathbb{R}} \mathfrak{m} = \dim_{\mathbb{R}} M$ (for an example see [42, p. 246]). Moreover, there is an equivariant isomorphism

$$\mathcal{L}(\mathfrak{m}) \cong \oplus^2 \ker(\mathfrak{m} \otimes \Lambda^2(\mathfrak{m}) \to \Lambda^3(\mathfrak{m})).$$

The intersection of $\mathcal{L}(\mathfrak{m})$ with the K-module $\mathfrak{m} \otimes \Lambda^2\mathfrak{m}$ consists of metric connections and is isomorphic to the so-called (2,1)-plethysm of the K-representation $\mathfrak{m}$:

$$P_{\mathfrak{m}}(2,1) := \mathcal{L}(\mathfrak{m}) \cap \big(\mathfrak{m} \otimes \Lambda^2\mathfrak{m}\big).$$

Theorem 3.13. *Let $(M^n = G/K, g = \langle\ ,\ \rangle, \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m})$ as in Lemma 3.9. The existence of the trivial representation inside the $n(n-1)(n+1)/3$-dimensional (2, 1)-plethysm $P_{\mathfrak{m}}(2,1)$ of $\mathfrak{m}$, gives rise to a G-invariant metric connection $\nabla = \nabla^{\mu}$ on $M = G/K$ corresponding to a K-intertwining bilinear mapping $\mu : \mathfrak{m} \otimes \mathfrak{m} \to \mathfrak{m}$ which has both non-trivial symmetric and skew-symmetric part, i.e. $\mu = \mu^{\mathrm{skew}} + \mu^{\mathrm{sym}}$, with $0 \neq \mu^{\mathrm{skew}} \in \mathrm{Hom}_K(\Lambda^2\mathfrak{m}, \mathfrak{m})$ and $0 \neq \mu^{\mathrm{sym}} \in \mathrm{Hom}_K(\mathrm{Sym}^2\,\mathfrak{m}, \mathfrak{m})$, respectively. In particular, the torsion of ∇^{μ} is not totally skew-symmetric and the defect $\epsilon := \mathcal{N}_{\mathrm{mtr}} - \ell \geq 0$ coincides with the multiplicity of the trivial representation inside $P_{\mathfrak{m}}(2,1)$.*

Proof. The trivial representation inside $P_{\mathfrak{m}}(2,1)$ induces an $\mathrm{Ad}(K)$-equivariant 3-tensor $\hat{\mu}$ on $\mathfrak{m}$ which is skew-symmetric with respect the last two indices, i.e. $\hat{\mu} \in \mathfrak{m} \otimes \Lambda^2(\mathfrak{m})$. Since the K-module $\mathfrak{m} \otimes \Lambda^2\mathfrak{m}$ corresponds to the set of $\mathfrak{so}(\mathfrak{m})$-valued Nomizu maps on $M = G/K$ with respect to $\langle\ ,\ \rangle$, the induced invariant connection $\nabla = \nabla^{\mu}$ is necessarily metric. In order to prove that its torsion is not a 3-form we rely on the definition of $P_{\mathfrak{m}}(2,1)$ and the orthogonal decomposition

$$\otimes^3\,\mathfrak{m} = \mathrm{Sym}^3\,\mathfrak{m} \oplus \mathcal{L}(\mathfrak{m}) \oplus \Lambda^3(\mathfrak{m}). \tag{3.1}$$

Indeed, since the 3-tensor $\hat{\mu}(X,Y,Z) = \langle \mu(X,Y), Z\rangle$ is induced by the trivial representation inside $P_{\mathfrak{m}}(2,1) := \mathcal{L}(\mathfrak{m}) \cap \big(\mathfrak{m} \otimes \Lambda^2\mathfrak{m}\big)$, the direct sum decomposition (3.1) together with Lemma 3.12, shows that the torsion T^{μ} of ∇^{μ} cannot be totally skew-symmetric. We use now (1.4) and write $\mu = \mu^{\mathrm{skew}} + \mu^{\mathrm{sym}}$ for the corresponding K-intertwining bilinear mapping $\mu \in \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$. Since T^{μ} is not a 3-form, μ^{sym} cannot be trivial, $\mu^{\mathrm{sym}} \neq 0$. Indeed, if $\mu^{\mathrm{sym}} = 0$, then $\mu = \mu^{\mathrm{skew}}$ and hence $\mu(X,X) = 0$ for any $X \in \mathfrak{m}$. But then, using Lemma 3.4, (i) we get a contradiction. Assume now that μ is given by a (non-trivial) symmetric K-intertwining bilinear mapping, i.e $\mu^{\mathrm{skew}} = 0$ and $\mu = \mu^{\mathrm{sym}}$ where $0 \neq \mu^{\mathrm{sym}} : \mathrm{Sym}^2\,\mathfrak{m} \to \mathfrak{m}$. Then, according to Lemma 3.9 our connection ∇^{μ} cannot be metric with respect to $\langle\ ,\ \rangle$, which contradicts to $\hat{\mu} \in \mathfrak{m} \otimes \Lambda^2(\mathfrak{m})$. This shows that $\mu^{\mathrm{skew}} \neq 0$, as well. Now, the identification of the defect $\epsilon := \mathcal{N}_{\mathrm{mtr}} - \ell \geq 0$ with the multiplicity of the trivial representation in $P_{\mathfrak{m}}(2,1)$ is a direct consequence of (3.1) and Lemmas 3.7, 3.12. □

We mention that one cannot drop the naturally reductive assumption in Theorem 3.13, due to the fact that the proof relies on Lemmas 3.9 and 3.12.

Remark 3.14. On a compact simple Lie group G, bi-invariant connections which are compatible with the Killing form are induced by the copy of $\mathfrak{g}$ inside $\Lambda^2(\mathfrak{g})$. Indeed, recall that

$$\mathfrak{so}(\mathfrak{g}) \cong \Lambda^2(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^{\perp}, \quad \mathfrak{g}^{\perp} := \ker \delta_{\mathfrak{g}},$$

where $\delta_{\mathfrak{g}} : \Lambda^2(\mathfrak{g}) \to \mathfrak{g}$ is given by $\delta_{\mathfrak{g}}(X \wedge Y) := [X,Y]$. Since $\delta_{\mathfrak{g}}$ is surjective, $\mathfrak{g}$ always lies inside $\Lambda^2(\mathfrak{g})$. However, the module $P_{\mathfrak{g}}(2,1) := \mathcal{L}(\mathfrak{g}) \cap \big(\mathfrak{g} \otimes \Lambda^2\mathfrak{g}\big)$, where $\mathcal{L}(\mathfrak{g})$ is similarly defined by $\mathcal{L}(\mathfrak{g}) := \oplus^2 \ker\big(\mathfrak{g} \otimes \Lambda^2(\mathfrak{g}) \to \Lambda^3(\mathfrak{g})\big)$, never contains the trivial summand. In contrast, as we noticed in Remark 3.8, for a compact Lie group the case can be different. Let us focus for example on $G = \mathrm{U}_n$ $(n \geq 3)$.

3.3. Bi-invariant metric connections on the compact Lie group U_n

According to Laquer [32], for $n \geq 3$ the space of bi-invariant affine connections on U_n is 6-dimensional. In particular, the following $\mathrm{Ad}(G)$-equivariant bilinear mappings form a basis of $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g})$ for $\mathfrak{g} = \mathfrak{u}(n)$:

$$\left.\begin{array}{lll} \mu_1(X,Y) = [X,Y], & \mu_2(X,Y) = i(XY + YX), & \mu_3(X,Y) = i\,\mathrm{tr}(X)Y \\ \mu_4(X,Y) = i\,\mathrm{tr}\,(Y)X, & \mu_5(X,Y) = i\,\mathrm{tr}\,(XY)\,\mathrm{Id}, & \mu_6(X,Y) = i\,\mathrm{tr}\,(X)\,\mathrm{tr}\,(Y)\,\mathrm{Id} \end{array}\right\}, \tag{3.2}$$

where XY denotes multiplication of matrices and Id is the identity matrix. We also consider the linear combinations

$$\nu(X,Y) := \mu_3(X,Y) - \mu_4(X,Y) = i(\mathrm{tr}(X)Y - \mathrm{tr}\,(Y)X) \in \mathrm{Hom}_G(\Lambda^2\mathfrak{g}, \mathfrak{g}),$$
$$\vartheta(X,Y) := \mu_3(X,Y) + \mu_4(X,Y) = i(\mathrm{tr}(X)Y + \mathrm{tr}\,(Y)X) \in \mathrm{Hom}_G(\mathrm{Sym}^2\,\mathfrak{g}, \mathfrak{g}).$$

Theorem 3.15. *(1) The connection induced by the* $\mathrm{Ad}(\mathfrak{u}(n))$*-equivariant bilinear mapping* $\mu = \mu_4 - \mu_5$*, i.e.* $\mu(X,Y) := i(\mathrm{tr}\,(Y)X - \mathrm{tr}\,(XY)\,\mathrm{Id})$ *for any* $X, Y \in \mathfrak{u}(n)$*, is a bi-invariant metric connection on* U_n $(n \geq 3)$ *with respect to the bi-invariant metric induced by* $\langle\ ,\ \rangle = -\,\mathrm{tr}(XY)$*. The symmetric and skew-symmetric part of* $\mu = \mu^{\mathrm{skew}} + \mu^{\mathrm{sym}}$ *are given by*

$$\mu^{\mathrm{skew}}(X,Y) = -(1/2)\nu(X,Y), \quad \text{and} \quad \mu^{\mathrm{sym}}(X,Y) = (1/2)\vartheta(X,Y) + i\langle X,Y\rangle\,\mathrm{Id},$$

respectively, and its torsion has the form $T^{\mu}(X,Y) = -\nu(X,Y) + T^{c}(X,Y)$*. In particular, the induced tensor* $T^{\mu}(X,Y,Z) = \langle T^{\mu}(X,Y), Z\rangle$ *is not totally skew-symmetric.*
(2) Consequently, for $n \geq 3$ *the Lie group* U_n *carries a 2-dimensional space of bi-invariant metric connections, i.e.* $\mathcal{N}_{\mathrm{mtr}} = \epsilon + \ell = 2$*.*

Proof. (1) The module $\mathcal{L}(\mathfrak{g})$ associated to the adjoint representation of $\mathfrak{g} = \mathfrak{u}(n) = \mathbb{R} \oplus \mathfrak{su}(n)$ contains the trivial representation twice. The one copy corresponds to the invariant 3-tensor $\hat{\nu}(X,Y,Z) = \langle \nu(X,Y), Z\rangle$ which is skew-symmetric only with respect to the first two indices, i.e. $\hat{\nu} \in \mathcal{L}(\mathfrak{g}) \cap (\Lambda^2\mathfrak{g} \otimes \mathfrak{g})$ and thus $\nu = \mu_3 - \mu_4$ fails to induce a bi-invariant connection on U_n, preserving $\langle\ ,\ \rangle$. The second copy corresponds to the invariant 3-tensor $\hat{\mu}(X,Y,Z) = \langle \mu(X,Y), Z\rangle$, where $\mu : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ is given by $\mu = \mu_4 - \mu_5$. We will show that $\hat{\mu}$ is indeed inside the $(2,1)$ plethysm $P_{\mathfrak{g}}(2,1) = \mathcal{L}(\mathfrak{g}) \cap (\mathfrak{g} \otimes \Lambda^2\mathfrak{g})$, i.e. $\epsilon = 1$, and hence the associated connection ∇^{μ} gives rise to 1-dimensional family of bi-invariant metric connections on U_n. For simplicity, we set $\mathcal{O}(X,Y,Z) := \langle \mu(X,Y), Z\rangle + \langle Y, \mu(X,Z)\rangle$, for any $X, Y, Z \in \mathfrak{u}(n)$. Then we get that

$$\begin{aligned}\mathcal{O}(X,Y,Z) &= \langle i(\mathrm{tr}\,(Y)X - \mathrm{tr}\,(XY)\,\mathrm{Id}), Z\rangle + \langle Y, i(\mathrm{tr}\,(Z)X - \mathrm{tr}\,(XZ)\,\mathrm{Id})\rangle \\ &= i\big(\mathrm{tr}(Y)\langle X,Z\rangle - \mathrm{tr}(XY)\langle \mathrm{Id}, Z\rangle + \mathrm{tr}(Z)\langle Y,X\rangle - \mathrm{tr}(XZ)\langle Y, \mathrm{Id}\rangle\big) \\ &= i\big(-\,\mathrm{tr}(Y)\,\mathrm{tr}(XZ) + \mathrm{tr}(XY)\,\mathrm{tr}(Z) - \mathrm{tr}(Z)\,\mathrm{tr}(XY) + \mathrm{tr}(XZ)\,\mathrm{tr}(Y)\big) = 0,\end{aligned}$$

for any $X, Y, Z \in \mathfrak{u}(n)$ and this proves our assertion. Now, according to Theorem 3.13, μ has both non-trivial symmetric and skew-symmetric part, namely $\mu^{\mathrm{sym}}(X,Y) = \frac{1}{2}[\mu(X,Y) + \mu(Y,X)]$ and $\mu^{\mathrm{skew}}(X,Y) = \frac{1}{2}[\mu(X,Y) - \mu(Y,X)]$, respectively, and a small computation completes the proof.
(2) For the second statement, the mapping $a\mu_1(X,Y)$ $(a \in \mathbb{R})$ induces a 1-parameter family of bi-invariant metric connections on U_n with skew-torsion and this is the unique family of bi-invariant metric connections with skew-torsion (since the multiplicity of the trivial representation inside $\Lambda^3\mathfrak{g}$ is one, i.e. $\ell = 1$, see also the remark below). Then, according to Theorem 3.13 it must be $\mathcal{N}_{\mathrm{mtr}} = \epsilon + \ell = 1 + 1 = 2$, which also fits with the conclusion that μ is a new bi-invariant metric connection on U_n, and finally also with Lemma 3.7. This induces a 1-parameter family of bi-invariant metric connections ∇^b $(b \in \mathbb{R})$, corresponding to the bilinear mapping $\mu^b(X,Y) := b[i(\mathrm{tr}\,(Y)X - \mathrm{tr}\,(XY)\,\mathrm{Id})] = b\mu(X,Y)$, with $X, Y \in \mathfrak{u}(n)$. The torsion $T^b \in \Lambda^2\mathfrak{u}(n) \otimes \mathfrak{u}(n)$ is not totally skew-symmetric. Indeed, the torsion of the mapping $\mu = \mu_4 - \mu_5$ is given by $T^{\mu}(X,Y) = -\nu(X,Y) + T^{c}(X,Y)$. It is not totally skew-symmetric since for example $\mu(X,X) \neq 0$ and $\langle\ ,\ \rangle$ is a naturally reductive metric. Similarly for μ^b. This finishes the proof. □

Remark 3.16. For a verification of the fact $\ell = 1$ for U_n, one can use the LiE program (and stability arguments), or even apply the following. First, for dimensional reasons notice that

$$\Lambda^3(\mathfrak{g}) = \Lambda^3(\mathfrak{u}(n)) = \Lambda^3(\mathbb{R} \oplus \mathfrak{su}(n)) = \Lambda^3(\mathfrak{su}(n)) \oplus \big(\mathbb{R} \otimes \Lambda^2\mathfrak{su}(n)\big). \tag{$*$}$$

Using (4.3) we also see that $\mathbb{R} \otimes \Lambda^2\mathfrak{su}(n)$ does not contain the trivial representation. For the decomposition of $\Lambda^3(\mathfrak{su}(n))$, recall first that any compact simple lie group $\hat{G}$ admits a non-trivial global $\hat{G}$-invariant 3-from, the so-called Cartan 3-form $\omega_{\hat{\mathfrak{g}}}(X,Y,Z) = B([X,Y],Z)$, where B denotes the Killing form on the Lie algebra $\hat{\mathfrak{g}}$. On the other hand, the $\mathrm{Ad}(\hat{G})$-equivariant differential $d_{\hat{\mathfrak{g}}} : \Lambda^k(\hat{\mathfrak{g}}) \to \Lambda^{k+1}(\hat{\mathfrak{g}})$ on $\hat{\mathfrak{g}}$ is defined by $d_{\hat{\mathfrak{g}}}(\psi \wedge \varphi) = d_{\hat{\mathfrak{g}}}(\psi) \wedge \varphi + (-1)^{\deg\psi}\psi \wedge d_{\hat{\mathfrak{g}}}(\varphi)$ with $d_{\hat{\mathfrak{g}}}(\varphi) = \sum_i (Z_i \lrcorner \omega_{\hat{\mathfrak{g}}}) \wedge (Z_i \lrcorner \varphi)$ for some $(-B)$-orthonormal basis $\{Z_i\}$ of $\hat{\mathfrak{g}}$. In these terms, in [34] it was shown that the splitting $\Lambda^3(\hat{\mathfrak{g}}) = \mathrm{span}_{\mathbb{R}}\{\omega_{\hat{\mathfrak{g}}}\} \oplus \delta_{\hat{\mathfrak{g}}}(\Lambda^4(\mathfrak{g})) \oplus d_{\hat{\mathfrak{g}}}(\hat{\mathfrak{g}}^{\perp})$ defines an equivariant orthogonal decomposition of $\Lambda^3(\hat{\mathfrak{g}})$, where $\delta_{\hat{\mathfrak{g}}}$ is the adjoint operator of $d_{\hat{\mathfrak{g}}}$ with respect to $-B$ (see also Remark 3.14). From this decomposition, one deduces that $\ell = 1$ for any compact simple Lie group $\hat{G}$, and since $\mathfrak{u}(n) = \mathbb{R} \oplus \hat{\mathfrak{g}}$ with $\hat{\mathfrak{g}} = \mathfrak{su}(n)$, by $(*)$ we conclude the same for U_n.

We finally observe that $\mu := \mu_4 - \mu_5$ does not induces a derivation on $\mathfrak{m}$ (apply for example Proposition 2.2 or Theorem 2.8). In particular, [17, Thm. 2.9] holds only for G compact and simple (the direct claim is true even in the compact case, but the converse direction fails for non-simple Lie groups, since [6, Lem. 3.1], or [17, Thm. 2.1], is valid only for a compact simple Lie group).

3.4. Characterization of the types of invariant metric connections

Given an effective naturally reductive Riemannian manifold $(M = G/K, g)$, our aim now is to characterize the possible invariant connections with respect to their torsion type (for skew-torsion, see [1] or Lemma 3.4). We remark that next is not necessary to assume the compactness of $M = G/K$.

Proposition 3.17. *Let* $(M^n = G/K, g)$ *be a homogeneous Riemannian manifold which is naturally reductive with respect to a closed subgroup* $G \subseteq \mathrm{Iso}(M,g)$ *of the isometry group and let* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ *be the associated reductive decomposition. Assume that the transitive G-action is effective,* $\mathfrak{g} = \tilde{\mathfrak{g}}$ *and denote by* $\nabla \equiv \nabla^{\mu}$ *a G-invariant metric connection corresponding to*

$\mu \in \mathrm{Hom}_K(\mathfrak{m}\otimes\mathfrak{m},\mathfrak{m})$. *Set* $\hat{\mu}(X,Y,Z)=\langle\mu(X,Y),Z\rangle$, $A(X,Y)=\nabla_X Y-\nabla^g_X Y$ *and* $A(X,Y,Z)=\langle A(X,Y),Z\rangle$ *for any* $X,Y,Z\in\mathfrak{m}$, *where* ∇^g *is the Levi-Civita connection. Then, the following hold:*
(1) ∇ *is of vectorial type, i.e.* $A\in\mathcal{A}_1$, *if and only if there is a global* G*-invariant 1-form* φ *on* M *such that*

$$\hat{\mu}(X,Y,Z)=\frac{1}{2}\langle[X,Y]_{\mathfrak{m}},Z\rangle+\langle X,Y\rangle\varphi(Z)-\langle X,Z\rangle\varphi(Y),\quad \forall\, X,Y,Z\in\mathfrak{m}.$$

(2) ∇ *is of Cartan type or traceless cyclic, i.e.* $A\in\mathcal{A}_2$, *if any only if the following two conditions are simultaneously satisfied:*

$$(\alpha)\quad \mathfrak{S}_{X,Y,Z}\hat{\mu}(X,Y,Z)=\tfrac{3}{2}\langle[X,Y]_{\mathfrak{m}},Z\rangle,\quad \forall\, X,Y,Z\in\mathfrak{m},$$
$$(\beta)\quad \textstyle\sum_i \mu(Z_i,Z_i)=0,$$

where $Z_1,\dots,Z_n$ *is an arbitrary* $\langle\ ,\ \rangle$*-orthonormal basis of* $\mathfrak{m}$.
(3) ∇ *is cyclic, i.e.* $A\in\mathcal{A}_1\oplus\mathcal{A}_2$, *if and only if* $\mathfrak{S}_{X,Y,Z}\hat{\mu}(X,Y,Z)=\frac{3}{2}\langle[X,Y]_{\mathfrak{m}},Z\rangle$, $\forall\, X,Y,Z\in\mathfrak{m}$.
(4) ∇ *is traceless, i.e.* $A\in\mathcal{A}_2\oplus\mathcal{A}_3$, *if and only if* $\sum_i\mu(Z_i,Z_i)=0$.

Remark 3.18. Before proceeding with the proof, let us first describe a useful formula. Recall that the torsion of ∇ is given by $T(X,Y)=\mu(X,Y)-\mu(Y,X)-[X,Y]_{\mathfrak{m}}$, or in other words $T(X,Y,Z)=\hat{\mu}(X,Y,Z)-\hat{\mu}(Y,X,Z)-\langle[X,Y]_{\mathfrak{m}},Z\rangle$ for any $X,Y,Z\in\mathfrak{m}$. Therefore, a short application of (1.1) gives rise to

$$\begin{aligned}2A(X,Y,Z)&=T(X,Y,Z)-T(Y,Z,X)+T(Z,X,Y)\\&=\hat{\mu}(X,Y,Z)-\hat{\mu}(Y,X,Z)-\langle[X,Y]_{\mathfrak{m}},Z\rangle-\hat{\mu}(Y,Z,Z)+\hat{\mu}(Z,Y,X)+\langle[Y,Z]_{\mathfrak{m}},X\rangle\\&\quad+\hat{\mu}(Z,X,Y)-\hat{\mu}(X,Z,Y)-\langle[Z,X]_{\mathfrak{m}},Y\rangle\\&=2\hat{\mu}(X,Y,Z)-\langle[X,Y]_{\mathfrak{m}},Z\rangle,\end{aligned}$$

since $\hat{\mu}(X,Y,Z)+\hat{\mu}(X,Z,Y)=0$ for any $X,Y,Z\in\mathfrak{m}$ and $\langle\ ,\ \rangle$ is naturally reductive. Thus

$$A(X,Y,Z)=\hat{\mu}(X,Y,Z)-\frac{1}{2}\langle[X,Y]_{\mathfrak{m}},Z\rangle,\quad \forall\, X,Y,Z\in\mathfrak{m}. \tag{3.3}$$

Proof. (1) Assume that $M=G/K$ carries a G-invariant metric connection ∇ whose torsion is of vectorial type. Next we shall identify $\mathfrak{m}\cong T_oM$ and for any $X\in\mathfrak{m}\subset\mathfrak{g}$ we shall write X^* for the (Killing) vector field on M induced by $\exp(-tX)$. Recall that $[X^*,Y^*]_o=-[X,Y]^*_o=-[X,Y]_{\mathfrak{m}}$. Since ∇ is a G-invariant connection, identifying $(\nabla_{X^*}Y^*)_o=\nabla_XY$, we can write

$$\begin{aligned}\langle\nabla_XY,Z\rangle&=\langle\nabla^c_XY,Z\rangle+\langle\mu(X,Y),Z\rangle=\langle\nabla^c_XY,Z\rangle+\langle\Lambda^\mu(X)Y,Z\rangle\\&=-\langle[X,Y]_{\mathfrak{m}},Z\rangle+\hat{\mu}(X,Y,Z),\end{aligned}\tag{$*$}$$

where $(\nabla^c_{X^*}Y^*)_o=\nabla^c_XY=-[X,Y]_{\mathfrak{m}}=[X^*,Y^*]_o$ is the canonical connection with respect to $\mathfrak{m}$ (cf. [38,41]). However, ∇ is of vectorial type, hence there is a 1-form φ on $M=G/K$ such that

$$\mathfrak{m}\otimes\Lambda^2\mathfrak{m}\cong\mathcal{A}\ni A(X,Y,Z)=\langle X,Y\rangle\varphi(Z)-\langle X,Z\rangle\varphi(Y),$$

for any $X,Y,Z\in\mathfrak{m}$. Using that $\langle\ ,\ \rangle$ is naturally reductive with respect to G and $\mathfrak{m}$, we compute $(\nabla^g_{X^*}Y^*)_o=\frac{1}{2}[X^*,Y^*]_o=-\frac{1}{2}[X,Y]_{\mathfrak{m}}$ and

$$\langle\nabla_XY,Z\rangle=\langle\nabla^g_XY,Z\rangle+A(X,Y,Z)=-\frac{1}{2}\langle[X,Y]_{\mathfrak{m}},Z\rangle+\langle X,Y\rangle\varphi(Z)-\langle X,Z\rangle\varphi(Y).$$

Hence, a small combination with $(*)$ gives rise to

$$\hat{\mu}(X,Y,Z)=\frac{1}{2}\langle[X,Y]_{\mathfrak{m}},Z\rangle+\langle X,Y\rangle\varphi(Z)-\langle X,Z\rangle\varphi(Y).\tag{$**$}$$

However, $\hat{\mu}$ is an $\mathrm{Ad}(K)$-invariant tensor (or in other words, it corresponds to a G-invariant tensor field on $M=G/K$), and hence by $(**)$ we conclude that φ must be a (global) G-invariant 1-form on M. This proves the one direction. Assume now that $(M=G/K,g)$ is endowed with a G-invariant tensor $\hat{\mu}\in\mathfrak{m}\otimes\Lambda^2\mathfrak{m}$ satisfying $(**)$ for some G-invariant 1-form φ on M and let us denote by ∇ the associated G-invariant metric connection. Then, a combination of (3.3) and $(**)$ yields that $A\in\mathcal{A}_1$, which completes the proof of (1).
(2) Assume that $M=G/K$ carries a G-invariant metric connection ∇ which is traceless cyclic. This means that the invariant tensor $A(X,Y,Z)$ must satisfy the conditions

$$\mathfrak{S}_{X,Y,Z}A(X,Y,Z)=0\ \text{ and }\ \sum_i A(Z_i,Z_i,Z)=0,\tag{$\dagger$}$$

where $\{Z_i\}$ is an orthonormal basis of $\mathfrak{m}$ with respect to $\langle\ ,\ \rangle$. By (3.3) we see that

$$\sum_i A(Z_i,Z_i,Z)=0\ \Leftrightarrow\ \sum_i\hat{\mu}(Z_i,Z_i,Z)=0.$$

However, $\sum_i \hat{\mu}(Z_i, Z_i, Z) = \sum_i \langle \mu(Z_i, Z_i), Z\rangle = \langle \sum_i \mu(Z_i, Z_i), Z\rangle = \langle \sum_i \Lambda(Z_i)Z_i, Z\rangle$, where $\Lambda \equiv \Lambda^\mu : \mathfrak{m} \to \mathfrak{so}(\mathfrak{m})$ is the associated connection map. Thus, the traceless condition in (†) holds if and only if $\sum_i \mu(Z_i, Z_i) = \sum_i \Lambda(Z_i)Z_i = 0$. Now, for the cyclic condition in (†), using (3.3) we obtain the relation

$$\mathfrak{S}_{X,Y,Z} A(X, Y, Z) = \mathfrak{S}_{X,Y,Z} \hat{\mu}(X, Y, Z) - \frac{3}{2}\langle [X, Y]_{\mathfrak{m}}, Z\rangle$$

and in this way we conclude the second stated relation. In fact, this follows also by the cyclic sum $\mathfrak{S}_{X,Y,Z} T(X, Y, Z) = 0$, where T is the torsion of ∇.
(3) Parts (3) and (4) are immediate due to the description given in (2) and the definition of the classes $\mathcal{A}_1 \oplus \mathcal{A}_2$, and $\mathcal{A}_2 \oplus \mathcal{A}_3$. □

Remark 3.19. If $(M = G/K, g)$ is an effective Riemannian symmetric space endowed with a G-invariant metric connection $\nabla \equiv \nabla^\mu$ corresponding to some $\mu \in \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$, then the conclusions in Proposition 3.17 are simplified, i.e. for the tensor $A = \nabla^\mu - \nabla^g$ we deduce that

- $A \in \mathcal{A}_1$, i.e. ∇ is vectorial, if and only if $\exists$ a global G-invariant 1-form φ on M such that

$$\hat{\mu}(X, Y, Z) = \langle X, Y\rangle\varphi(Z) - \langle X, Z\rangle\varphi(Y), \quad \forall\, X, Y, Z \in \mathfrak{m}.$$

- $A \in \mathcal{A}_2$, i.e. ∇ is traceless cyclic, if and only if $\mathfrak{S}_{X,Y,Z}\hat{\mu}(X, Y, Z) = 0$ and $\sum_i \Lambda(Z_i)Z_i = 0$.
- $A \in \mathcal{A}_1 \oplus \mathcal{A}_2$, i.e. ∇ is cyclic, if and only if $\mathfrak{S}_{X,Y,Z}\hat{\mu}(X, Y, Z) = 0$ for any $X, Y, Z \in \mathfrak{m}$.

Because on a compact Riemannian symmetric space $(M = G/K, g)$ of Type I, the G-invariant metric connections are exhausted by the torsion-free canonical connection $\nabla^c = \nabla^g$ associated to $\mathfrak{m}$, in the compact case the above conditions are of particular interest for compact connected (non-simple) Lie groups endowed with a bi-invariant metric, where $A = \nabla^\mu - \nabla^g$ can be non-trivial. For example, below we apply these considerations for the Lie group U_n. Finally notice that considering a naturally reductive space as in Proposition 3.17 (or even a symmetric space as above), it is easy to certify that any G-invariant metric connection of type $\mathcal{A}_3$ is also of type $\mathcal{A}_2 \oplus \mathcal{A}_3$, any G-invariant metric connection of type $\mathcal{A}_1$ it is also of type $\mathcal{A}_1 \oplus \mathcal{A}_2$, etc.

Proposition 3.20. *For $n \geq 3$, the bi-invariant metric connection ∇^μ on $(\mathrm{U}_n, \langle\ ,\ \rangle)$ induced by the map $\mu := \mu_4 - \mu_5$ of Theorem 3.15 has torsion of vectorial type.*

Proof. The Lie group U_n has 1-dimensional centre Z; hence the quotient $(\mathrm{U}_n \times \mathrm{U}_n)/\Delta\mathrm{U}_n$ is not yet effective, but the expression $(\mathrm{U}_n/Z)/(\Delta\mathrm{U}_n/\Delta Z)$ gives rise to an effective coset space. From now on we shall identify $\mathrm{U}_n \cong (\mathrm{U}_n \times \mathrm{U}_n)/\Delta\mathrm{U}_n \cong (\mathrm{U}_n/Z)/(\Delta\mathrm{U}_n/\Delta Z)$ and write $\mathfrak{u}(n) \oplus \mathfrak{u}(n) = \Delta\mathfrak{u}(n) \oplus \mathfrak{m}$ for the associated symmetric reductive decomposition, where

$$\Delta\mathfrak{u}(n) := \{(X, X) \in \mathfrak{u}(n) \oplus \mathfrak{u}(n) : X \in \mathfrak{u}(n)\}, \quad \mathfrak{m} := \{(X, -X) \in \mathfrak{u}(n) \oplus \mathfrak{u}(n) : X \in \mathfrak{u}(n)\}$$

are both isomorphic to $\mathfrak{u}(n)$ as U_n-modules. Because any compact connected Lie group G endowed with a bi-invariant metric is a compact normal homogeneous space and moreover a compact symmetric space, the condition $\mathfrak{g} = \tilde{\mathfrak{g}}$ of Proposition 3.17 is satisfied and we can apply the considerations of Remark 3.19. Consider the Lie algebra $\mathfrak{u}(n)$ endowed with the bilinear mapping $\mu(X, Y) = i(\mathrm{tr}(Y)X - \mathrm{tr}(XY)\,\mathrm{Id})$, given in Theorem 3.15. Since $\langle X, Y\rangle = -\mathrm{tr}(XY)$ we conclude that

$$\begin{aligned}\hat{\mu}(X, Y, Z) := \langle \mu(X, Y), Z\rangle &= i\,\mathrm{tr}(Y)\langle X, Z\rangle - i\,\mathrm{tr}(XY)\langle \mathrm{Id}, Z\rangle \\ &= i\,\mathrm{tr}(Y)\langle X, Z\rangle + i\langle X, Y\rangle\langle \mathrm{Id}, Z\rangle \\ &= i\,\mathrm{tr}(Y)\langle X, Z\rangle - i\,\mathrm{tr}(Z)\langle X, Y\rangle, \end{aligned} \tag{3.4}$$

for any $X, Y, Z \in \mathfrak{u}(n)$. Consider now the 1-form $\varphi : \mathfrak{u}(n) \to \mathbb{R}$, $Y \mapsto \varphi(Y) := -i\,\mathrm{tr}(Y)$. It is easy to see that φ is a U_n-invariant 1-form with kernel $\mathfrak{su}(n)$. But then, based on (3.4) we obtain that

$$\hat{\mu}(X, Y, Z) = -\langle X, Z\rangle\varphi(Y) + \langle X, Y\rangle\varphi(Z),$$

for any $X, Y, Z \in \mathfrak{u}(n)$ and using Remark 3.19 we conclude that $\mathcal{A}^\mu := \nabla^\mu - \nabla^g \in \mathcal{A}_1$. □

Remark 3.21. By Theorem 3.15, the group U_n $(n \geq 3)$ is equipped with a two dimensional space of bi-invariant metric connections ∇^f, given by the bilinear map $f := a\mu_1 + b\mu$ $(a, b \in \mathbb{R})$ where μ_1 and μ are given by (3.2) and Theorem 3.15, respectively. In general, ∇^f is of mixed type $\mathcal{A}_1 \oplus \mathcal{A}_3$, but the conditions that the type of ∇^f is either purely $\mathcal{A}_3$ or purely $\mathcal{A}_1$, naturally defines the one dimensional subfamilies $a\mu_1$ and $b\mu$, respectively. Thus, we can express the space of bilinear mappings inducing bi-invariant metric connections on U_n as a direct sum of these families.

3.5. The curvature tensor and the Ricci tensor

Let us now examine the curvature tensor.

Proposition 3.22. *Let $(M = G/K, g)$ be a naturally reductive Riemannian manifold as in Proposition 3.17. Then, the curvature tensor $R^{\nabla^\mu} \equiv R^\nabla$ associated to a G-invariant metric connection $\nabla \equiv \nabla^\mu$ on $M = G/K$, induced by some $\mu \in \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$, satisfies the following relation*

$$\begin{aligned} R^\nabla(X,Y)Z &= R^g(X,Y)Z + A(X,\mu(Y,Z)) - A(Y,\mu(X,Z)) - A([X,Y]_\mathfrak{m},Z) \\ &\quad + \frac{1}{2}\Big([X,A(Y,Z)]_\mathfrak{m} - [Y,A(X,Z)]_\mathfrak{m}\Big), \end{aligned}$$

for any $X, Y, Z \in \mathfrak{m}$, where the tensor A is defined by the difference $A = \nabla - \nabla^g$ and R^g is the Riemannian curvature tensor. If $(\mathfrak{g}, \mathfrak{k})$ is a symmetric pair, then the last three terms in the previous relation are cancelled.

Proof. The proof relies on a straightforward computation using the formulas

$$R^\nabla(X,Y)Z = \mu(X,\mu(Y,Z)) - \mu(Y,\mu(X,Z)) - \mu([X,Y]_\mathfrak{m})Z - [[X,Y]_\mathfrak{k},Z],$$

and $A(X,Y) = \mu(X,Y) - \mu^g(X,Y) = \Lambda(X)Y - \Lambda^g(X)Y$ where $\mu^g(X,Y) = \Lambda^g(X)Y = \frac{1}{2}[X,Y]_\mathfrak{m}$ is the bilinear map associated to the Levi-Civita connection on $M = G/K$, see also (3.3). The last conclusion relies on the symmetric reductive decomposition, in particular (3.3) reduces to $A(X,Y,Z) = \hat{\mu}(X,Y,Z)$ for any $X, Y, Z \in \mathfrak{m}$. □

Consider now a G-invariant metric connection ∇ of vectorial type. Let us denote by φ the associated $\mathrm{Ad}(K)$-invariant 1-form on $\mathfrak{m}$ and by $\xi \in \mathfrak{m}$ the dual vector with respect to $\langle\ ,\ \rangle$. If $\|\xi\|^2 \neq 0$, then ∇ is called a *G-invariant connection of non-degenerate vectorial type*. In this case, by applying [7, Corol. 3.1] or by a direct calculation based on Proposition 3.22, we get that

Corollary 3.23. *Let $(M = G/K, g)$ be an n-dimensional naturally reductive manifold as in Proposition 3.17, endowed with a G-invariant metric connection $\nabla \equiv \nabla^\mu$ of non-degenerate vectorial type. Then*
(1) For any $X, Y \in \mathfrak{m}$, the Ricci tensor Ric^∇ associated to ∇ satisfies the relation

$$\mathrm{Ric}^\nabla(X,Y) = \mathrm{Ric}^g(X,Y) + (n-2)\langle X,\xi\rangle\langle Y,\xi\rangle + (2-n)\|\xi\|^2\langle X,Y\rangle + \frac{2-n}{2}\langle [X,Y]_\mathfrak{m},\xi\rangle. \tag{3.5}$$

(2) Ric^∇ is symmetric if and only if $(\mathfrak{g},\mathfrak{k})$ is a symmetric pair and this is equivalent to say that φ is a closed invariant 1-form.

Proof. We prove only the second claim. By (3.5) it follows that

$$\mathrm{Ric}^\nabla(X,Y) - \mathrm{Ric}^\nabla(Y,X) = (n-2)\langle [X,Y]_\mathfrak{m},\xi\rangle, \quad \forall\, X,Y \in \mathfrak{m}.$$

Hence, Ric^∇ is symmetric if and only if $\langle [X,Y]_\mathfrak{m},\xi\rangle = 0$. But since $\xi \neq 0$, this is equivalent to say that $(\mathfrak{g},\mathfrak{k})$ is a symmetric pair, i.e. $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{k}$. By the definition of the differential of an invariant form (cf. [47, pp. 248–250]), or by [7, Prop. 3.2] we get the last correspondence. □

Specializing to the Lie group U_n we conclude that

Corollary 3.24. *Consider the Lie group U_n ($n \geq 3$) endowed with the bi-invariant metric connection ∇^μ induced by the map $\mu = \mu_4 - \mu_5$, as described in Theorem 3.15. Then, the Ricci tensor Ric^μ associated to ∇^μ is given by the following symmetric invariant bilinear form on $\mathfrak{u}(n)$:*

$$\mathrm{Ric}^\mu(X,Y) = \frac{1}{2}\Big\{(2n^2 - n - 4)\,\mathrm{tr}\,XY + (5-2n^2)\,\mathrm{tr}\,X\,\mathrm{tr}\,Y\Big\},$$

for any $X, Y \in \mathfrak{m} \cong \mathfrak{u}(n)$.

Proof. We use the notation of Proposition 3.20 and view U_n as an effective symmetric space endowed with the bi-invariant metric induced by $\langle X,Y\rangle = -\mathrm{tr}(XY)$. Consider the Nomizu map

$$\Lambda^\mu(X)Y := i(\mathrm{tr}\,(Y)X - \mathrm{tr}\,(XY)\,\mathrm{Id}), \quad \forall\, X,Y \in \mathfrak{m} \cong \mathfrak{u}(n).$$

By Proposition 3.20 we know that the bi-invariant metric connection $\nabla^\mu_X Y = \nabla^c_X Y + \Lambda^\mu(X)Y$ has torsion of vectorial type, associated to the U_n-invariant linear form $\varphi(Z) = -i\,\mathrm{tr}(Z) = i\langle \mathrm{Id}, Z\rangle$. The dual vector $\xi \in \mathfrak{m}$ is defined by $\varphi(Z) = \langle Z,\xi\rangle$ for any $Z \in \mathfrak{m}$ and hence we conclude that $\xi = i\,\mathrm{Id}$, in particular $0 \neq \langle \xi,\xi\rangle = 1 = \|\xi\|^2$. Thus, the vectorial structure is non-degenerate and we can apply Corollary 3.23, i.e.

$$\begin{aligned} \mathrm{Ric}^\mu(X,Y) &= \mathrm{Ric}^g(X,Y) + (n^2-2)\Big(\langle X,\xi\rangle\langle Y,\xi\rangle - \langle X,Y\rangle\Big) \\ &= \mathrm{Ric}^g(X,Y) + (n^2-2)\Big(\mathrm{tr}(XY) - \mathrm{tr}\,X\,\mathrm{tr}\,Y\Big). \end{aligned}$$

Now, $\langle\ ,\ \rangle$ is a bi-invariant inner product and hence $\mathrm{Ric}^g(X,Y) = -\frac{1}{4}B(X,Y)$ for any $X, Y \in \mathfrak{u}(n)$, where $B(X,Y) = 2n\,\mathrm{tr}\,XY - 2\,\mathrm{tr}\,X\,\mathrm{tr}\,Y$ is the Killing form of U_n (cf. [13,10] where the statement is given for a compact semi-simple Lie group, but notice that Ric^g satisfies the same formula for any bi-invariant metric g on a Lie group G). Thus, a small computation in combination with the formula given above yields the result. □

4. Classification of invariant connections on non-symmetric SII spaces

4.1. Strongly isotropy irreducible spaces (SII)

Consider a compact, connected, effective, non-symmetric SII homogeneous space $M = G/K$. Since G is a compact simple Lie group (see [46, p. 62]), any such manifold is a standard homogeneous Riemannian manifold. Passing to a covering $\tilde{G}$ of G, if G/K is not simply-connected but G is connected, then $\tilde{G}$ acts transitively on the universal covering of G/K with connected isotropy group, say K', and it turns out that G/K is SII if and only if $\tilde{G}/K'$ is. Hence, whenever necessary we can assume that G/K is a compact, connected and simply-connected, effective, non-symmetric SII space, with G being compact, connected and simple and $K \subset G$ compact and connected. In this setting, the strongly isotropy irreducible condition is equivalent to an (almost effective) irreducible action of the Lie algebra $\mathfrak{k} = T_eK$ on $\mathfrak{m} \cong T_oG/K$. For a list of non-symmetric SII spaces we refer to [13, Tables 5, 6, p. 203]. We remark however that there are misprints in Table 6 of [13], related to SII homogeneous spaces $M = G/K$ of $G = \mathrm{Sp}_n$ (compare for example with [46, Thm. 7.1]). We correct these errors in our Table 5.

Proposition 4.1. *Let $(M = G/K, g = -B|_{\mathfrak{m}})$ be an effective, non-symmetric (compact) SII homogeneous Riemannian manifold, endowed with a G-invariant affine connection ∇^{μ} compatible with the Killing metric $\langle\ ,\ \rangle = -B|_{\mathfrak{m}}$, where $\mu \in \mathrm{Hom}_K(\mathfrak{m}\otimes\mathfrak{m}, \mathfrak{m})$. Then, the torsion T^{μ} of ∇^{μ} does not carry a component of vectorial type.*

Proof. Assume that $M = G/K$ carries a G-invariant metric connection ∇ whose torsion is of vectorial type and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be the reductive decomposition with respect to the Killing metric. Then, by Proposition 3.17(1), we have that $\hat{\mu}(X, Y, Z) = \frac{1}{2}\langle [X, Y]_{\mathfrak{m}}, Z\rangle + \langle X, Y\rangle\varphi(Z) - \langle X, Z\rangle\varphi(Y)$, for some G-invariant 1-form φ on $M = G/K$. However, $\mathfrak{m}$ is a self-dual and (strongly) irreducible K-module over $\mathbb{R}$; thus global G-invariant 1-forms do not exist, since dually the isotropy representation needs to preserve some vector field ξ and hence a 1-dimensional subspace of $\mathfrak{m}$, spanned by ξ. □

Corollary 4.2. *Let $(M = G/K, g)$ be an effective, non-symmetric (compact) SII homogeneous Riemannian manifold, endowed with a non torsion-free G-invariant metric connection ∇. Then, the torsion $0 \neq T$ of ∇ is totally skew-symmetric, $T \in \mathcal{A}_3 \cong \Lambda^3 TM$, or traceless cyclic $T \in \mathcal{A}_2$, or of mixed type $T \in \mathcal{A}_2 \oplus \mathcal{A}_3$, i.e. traceless.*

4.2. An application in the spin case

Consider an effective, non-symmetric (compact) SII homogeneous Riemannian manifold $(M^n = G/K, g)$. Assume that $M = G/K$ admits a G-invariant spin structure, i.e. a G-homogeneous $\mathrm{Spin}(\mathfrak{m})$-principal bundle $P \to M$ and a double covering morphism $\Lambda : P \to \mathrm{SO}(M, g)$ compatible with the principal groups' actions. Recall that an invariant spin structure corresponds to a lift of the isotropy representation χ into the spin group $\mathrm{Spin}(\mathfrak{m}) \equiv \mathrm{Spin}_n$, i.e. a homomorphism $\widetilde{\chi} : K \to \mathrm{Spin}(\mathfrak{m})$ such that $\chi = \lambda \circ \widetilde{\chi}$, where $\lambda : \mathrm{Spin}(\mathfrak{m}) \to \mathrm{SO}(\mathfrak{m})$ is the double covering of $\mathrm{SO}(\mathfrak{m}) \equiv \mathrm{SO}_n$. We shall denote by $\kappa_n : \mathrm{C}\ell^{\mathbb{C}}(\mathfrak{m}) \xrightarrow{\sim} \mathrm{End}(\Delta_{\mathfrak{m}})$ the Clifford representation and by $\mathrm{C}\ell(X\otimes\phi) := \kappa_n(X)\psi = X\cdot\psi$ the Clifford multiplication between vectors and spinors, see [1] for more details. Set $\rho := \kappa \circ \widetilde{\chi} : K \to \mathrm{Aut}(\Delta_{\mathfrak{m}})$, where $\kappa = \kappa_n|_{\mathrm{Spin}(\mathfrak{m})}$: $\mathrm{Spin}(\mathfrak{m}) \to \mathrm{Aut}(\Delta_{\mathfrak{m}})$ is the spin representation. The spinor bundle $\Sigma \to G/K$ is the homogeneous vector bundle associated to $P := G \times_{\widetilde{\chi}} \mathrm{Spin}(\mathfrak{m})$ via the representation ρ, i.e. $\Sigma = G \times_{\rho} \Delta_{\mathfrak{m}}$. Therefore we may identify sections of Σ with smooth functions $\varphi : G \to \Delta_{\mathfrak{m}}$ such that $\varphi(gk) = \kappa\big(\widetilde{\chi}(k^{-1})\big)\varphi(g) = \rho(k^{-1})\varphi(g)$ for any $g \in G$, $k \in K$.

Choose a G-invariant metric connection ∇ on G/K, corresponding to a connection map $\Lambda \in \mathrm{Hom}_K(\mathfrak{m}, \mathfrak{so}(\mathfrak{m}))$. The lift $\widetilde{\Lambda} := \lambda_*^{-1} \circ \Lambda : \mathfrak{m} \to \mathfrak{spin}(\mathfrak{m})$ induces a covariant derivative on spinor fields (which we still denote by the same symbol) $\nabla : \Gamma(\Sigma) \to \Gamma(T^*(G/K) \otimes \Sigma)$, given by $\nabla_X\psi = X(\psi) + \widetilde{\Lambda}(X)\psi$. Here, the vector $X \in \mathfrak{m}$ is considered as a left-invariant vector field in G and $\widetilde{\Lambda}(X)\psi$ as an equivariant function $\widetilde{\Lambda}(X)\psi : G \to \mathfrak{m}$. The Dirac operator $D := \mathrm{C}\ell \circ \nabla : \Gamma(\Sigma) \to \Gamma(\Sigma)$ associated to ∇ is defined as follows (cf. [1]):

$$D(\psi) := \sum_i \kappa_n(Z_i)\{Z_i(\psi) + \widetilde{\Lambda}(Z_i)\psi\} = \sum_i Z_i \cdot \{Z_i(\psi) + \widetilde{\Lambda}(Z_i)\psi\},$$

where Z_i denotes a $\langle\ ,\ \rangle$-orthonormal basis of $\mathfrak{m}$.

Remark 4.3. Given a spin Riemannian manifold (M, g) endowed with a metric connection ∇, basic properties of the induced Dirac operator $D = \mathrm{C}\ell \circ \nabla$ are reflected in the type of the torsion of ∇. For example, by a result of Th. Friedrich [24] (see also [26,40]), it is known that the formal self-adjointness of the Dirac operator $D = \mathrm{C}\ell \circ \nabla$ is equivalent to the condition $A \in \mathcal{A}_2 \oplus \mathcal{A}_3$, where $A = \nabla - \nabla^g$. Hence, in our case as an immediate consequence of Corollary 4.2 we obtain that

Corollary 4.4. *Let $(M = G/K, g)$ be an effective, non-symmetric (compact) SII homogeneous Riemannian manifold, endowed with a G-invariant metric connection ∇ and a G-invariant spin structure. Then, the Dirac operator D associated to ∇ is formally self-adjoint.*

Note that the classification of invariant spin structures on non-symmetric SII spaces is an open problem (see [15] for invariant spin structures on symmetric spaces and [8] for a more recent study of spin structures on reductive homogeneous spaces).

4.3. Classification results on invariant connections

For the presentation of the classification results, we use the notation of [39, p. 299]. In particular, for a compact simple Lie group G we shall denote by $R(\pi)$ the complex irreducible representation of highest weight π. We mention that the isotropy representation of a compact, non-symmetric, effective SII space turns out to be of either real or complex type. In fact, fixing a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, whenever the complexification $\mathfrak{m}^{\mathbb{C}}$ splits into two complex submodules, these are never equivalent representations (see also [46]). Hence, by Schur's lemma we have the identification $\mathrm{Hom}_K(\mathfrak{m},\mathfrak{m}) = \mathbb{C}$ for complex type and $\mathrm{Hom}_K(\mathfrak{m},\mathfrak{m}) = \mathbb{R}$ for real type. In the first case, the endomorphism J induced by $i \in \mathbb{C}$ makes G/K a homogeneous almost complex manifold. Note that the same conclusions are true for a symmetric space, see [32,33] (recall that the adjoint representation of a compact simple Lie group is always of real type).

Remark 4.5. The multiplicities that we describe below have also been presented in the PhD thesis [19] (see Tables I.3.1–I.3.4, pp. 77–79), for a different however aim, namely the description of the components of the intrinsic torsion associated to (irreducible) G-structures over non-symmetric compact SII spaces (see also [20]). We remark that there are a few errors/omissions in [19], related with some low-dimensional cases, namely:

- the case $p = 2$, $q \geq 3$ of the family $\mathrm{SU}_{pq}/\mathrm{SU}_p \times \mathrm{SU}_q$,
- the case $n = 5$ of the family $\mathrm{SU}_{\frac{n(n-1)}{2}}/\mathrm{SU}_n$,
- the case $n = 6$ of the family $\mathrm{SO}_{\frac{(n-1)(n+2)}{2}}/\mathrm{SO}_n$ (due to isomorphism $\mathfrak{so}(6) = \mathfrak{su}(4)$).

In these mentioned cases, the general decompositions of $\Lambda^2\mathfrak{m}$ or $\mathrm{Sym}^2\,\mathfrak{m}$ change and most times this affects to multiplicities that we are interested in. Notice also that for the manifold $\mathrm{SO}_{4n}/\mathrm{Sp}_n \times \mathrm{Sp}_1$ the enumeration in [46,13] starts for $n \geq 2$ (as we do), but in [19] it is written $n \geq 3$. We correct these errors in our Table 4 (they are indicated by an asterisk). Notice finally that the author of this thesis uses the LiE program (as we do) and for infinite families he is based on stability arguments, see [19, Rem. I.3.9]. Below we also give examples of how such families can be treated even without the aid of a computer.

Remark 4.6. Given a reductive homogeneous space $M = G/K$ of a classical simple Lie group G, there is a simple method for the computation of the associated isotropy representation $\chi : K \to \mathrm{Aut}(\mathfrak{m})$, given as follows. Let us denote by $\rho_n : \mathrm{SO}_n \to \mathrm{Aut}(\mathbb{R}^n)$, $\mu_n : \mathrm{SU}_n \to \mathrm{Aut}(\mathbb{C}^n)$ and $\nu_n : \mathrm{Sp}_n \to \mathrm{Aut}(\mathbb{H}^n)$ be the standard representations of SO_n, SU_n (or U_n), and Sp_n, respectively. Recall that the complexified adjoint representation $\mathrm{Ad}_G^{\mathbb{C}} = \mathrm{Ad}_G \otimes \mathbb{C}$, satisfies

$$\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SO}_n} = \Lambda^2\rho_n, \quad \mathrm{Ad}^{\mathbb{C}}_{\mathrm{U}_n} = \mu_n \otimes \mu_n^*, \quad \mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_n} \oplus 1 = \mu_n \otimes \mu_n^*, \quad \mathrm{Ad}^{\mathbb{C}}_{\mathrm{Sp}_n} = \mathrm{Sym}^2\nu_n,$$

where μ_n^* is the dual representation of μ_n and 1 denotes the trivial 1-dimensional representation. Let G be one of the Lie groups SO_n, SU_n, Sp_n and let $\pi : K \to G$ be an (almost) faithful representation of a compact connected subgroup K. Using the identity $\mathrm{Ad}\big|_K = \mathrm{Ad}_K \oplus \chi$, we see that the isotropy representation χ of $G/\pi(K)$ is determined by $\Lambda^2\pi = \mathrm{ad}_{\mathfrak{k}} \oplus \chi$ in the orthogonal case, by $\pi \otimes \pi^* = 1 \oplus \mathrm{ad}_{\mathfrak{k}} \oplus \chi$ in the unitary case and finally by $\mathrm{Sym}^2\,\pi = \mathrm{ad}_{\mathfrak{k}} \oplus \chi$ in the symplectic case (see [10,46]).

Theorem 4.7. *Let $(M = G/K, g = -B|_{\mathfrak{m}})$ be an effective, non-symmetric (compact) SII homogeneous space. Consider the B-orthogonal reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Then, the complexified isotropy representation $\mathfrak{m}^{\mathbb{C}}$ and the multiplicities* **a**, **s**, $\mathcal{N}$ *and* ℓ *are given in Tables 4 and 5.*

4.4. On the Theorems A.1, A.2 and B—Conclusions

The results in Tables 4 and 5 allow us to deduce that several non-symmetric SII spaces are carrying *new* families of invariant metric connections, in the sense that they are different from the Lie bracket family $\eta^\alpha(X,Y) = \frac{1-\alpha}{2}[X,Y]_{\mathfrak{m}}$ (see Lemma 3.4). In combination with Lemma 3.12, we also certify the existence of compact, effective, non-symmetric SII quotients $M = G/K$ which are endowed with additional families of G-invariant metric connections with skew-torsion, besides η^α. In full details, this occurs in the following two situations:

- when $2 \leq \ell \leq \mathbf{a}$ and the isotropy representation is not of complex type (since for complex type we may have $\ell = 2 = \mathbf{a}$, but due to Schur's lemma all these invariant connections must be exhausted by the family $\eta^\alpha(X,Y) = \frac{1-\alpha}{2}[X,Y]_{\mathfrak{m}}$ with $\alpha \in \mathbb{C}$), or
- when the isotropy representation is of complex type but ℓ (and hence **a**) is strictly greater than 2.

This observation, in combination with Lemmas 3.7, 3.12 and the results in Tables 4 and 5, yields Theorems A.1 and A.2. Theorem B it is also a direct conclusion of the multiplicity **s** given in Tables 4 and 5 and Lemma 3.9. In fact, for affine connections induced by symmetric elements $\mu \in \mathrm{Hom}_K(\mathrm{Sym}^2\,\mathfrak{m},\mathfrak{m})$, we also conclude that

Corollary 4.8. *Let $(M = G/K, g = -B|_{\mathfrak{m}})$ be an effective, non-symmetric, SII homogeneous space associated to the Lie group $G = \mathrm{SU}_n$. Then, there is always a copy of $\mathfrak{m}$ inside $\mathrm{Sym}^2\,\mathfrak{m}$, induced by the restriction of the $\mathrm{Ad}(\mathrm{SU}_n)$-invariant symmetric bilinear*

mapping

$$\eta : \mathfrak{su}(n) \times \mathfrak{su}(n) \to \mathfrak{su}(n), \quad \eta(X, Y) := i\Big\{XY + YX - \frac{2}{n}\operatorname{tr}(XY)\,\mathrm{Id}\Big\}$$

on the corresponding reductive complement $\mathfrak{m}$. *If* M *is isometric to one of the manifolds*

$$\mathrm{SU}_{10}/\mathrm{SU}_5, \quad \mathrm{SU}_{2q}/\mathrm{SU}_2 \times \mathrm{SU}_q\ (q \geq 3), \quad \mathrm{SU}_{16}/\mathrm{Spin}_{10},$$

then the 1-parameter family of SU_n*-invariant affine connections on* $M = \mathrm{SU}_n/K$ *associated to the restriction* $\eta|_{\mathfrak{m}}$: $\mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$, *exhausts all* SU_n*-invariant affine connections induced by some* $0 \neq \mu \in \mathrm{Hom}_K(\mathrm{Sym}^2\,\mathfrak{m}, \mathfrak{m})$.

Proof. The first part is based on [33, Thm. 6.1]. Notice that η is known by [32, p. 550]. Now, using the results of Tables 4 and 5 about the multiplicity **s** of $\mathfrak{m}$ inside $\mathrm{Sym}^2\,\mathfrak{m}$, we obtain the result. □

4.5. Some explicit examples

Let us now compute the desired multiplicities **a**, **s** and ℓ, for general families of (non-symmetric) SII spaces, without the aid of computer. For this we need first to recall some preliminaries of representation theory (for more details we refer to [14,42,19]).

If π is a complex representation of a compact Lie algebra $\mathfrak{k}$, then $\overline{\pi} \cong \pi^*$, where $\overline{\pi}$ denotes the complex conjugate representation and π^* the dual representation. If π is a complex representation of $\mathfrak{k}$ on V, then there is a symmetric (resp. skew-symmetric) non-degenerate bilinear form on V invariant under π, if and only if there is an anti-linear intertwining map τ with $\tau^2 = \mathrm{Id}$ (resp. $\tau^2 = -\mathrm{Id}$) [14, Prop. 6.4.]. If π is irreducible then Schur's lemma ensures the uniqueness of such a bilinear form. A complex representation carrying a conjugate linear intertwining map τ with $\tau^2 = \mathrm{Id}$ (resp. $\tau^2 = -\mathrm{Id}$) is called of real type (resp. quaternionic type). Finally we call a complex representation $\pi : \mathfrak{k} \to \mathfrak{gl}(V)$ of complex type if it is not self dual, i.e. $V \ncong V^*$.

Let (π, V) and (π', W) be representations of a connected (not necessarily compact) Lie group H on two vector spaces V and W, respectively. It is important to note that even if π and π' are irreducible, then the tensor product representation $V \otimes W$, defined by $\pi \otimes \pi' : H \to \mathrm{Aut}(V \otimes W)$, $(\pi \otimes \pi')(h)(u \otimes w) = \pi(h)u \otimes \pi'(h)w$, is always reducible. Let us denote by $\Lambda^k\pi$ and $\mathrm{Sym}^k\,\pi$ the kth exterior power and kth symmetric power, respectively. For $k = 2$, it is easy to prove that

$$\begin{cases} \Lambda^2(V \oplus W) &= \Lambda^2 V \oplus (V \otimes W) \oplus \Lambda^2 W, \\ \mathrm{Sym}^2(V \oplus W) &= \mathrm{Sym}^2\,V \oplus (V \otimes W) \oplus \mathrm{Sym}^2\,W. \end{cases} \tag{4.1}$$

If (π, V) and (π', W) are representations of two connected Lie groups H and H', respectively, then the vector space $V \otimes W$ carries a representation of the product group $H \times H'$, say $(\pi \hat{\otimes} \pi', V \otimes W)$, given by $\pi \hat{\otimes} \pi'(h, h')(u \otimes w) = \pi(h)u \otimes \pi'(h')w$. This representation is called the external tensor product of π and π'. In the finite-dimensional case, $\pi \hat{\otimes} \pi'$ is an irreducible representation of $H \times H'$, if and only π and π' are both irreducible. In particular, if H, H' are compact Lie groups, then a representation of $H \times H'$ in $\mathrm{GL}(\mathbb{C}^n)$ is irreducible if and only if it is the tensor product of an irreducible representation of H with one of H'. Finally, one has the following equivariant isomorphisms:

$$\begin{cases} (V \hat{\otimes} W) \otimes (V' \hat{\otimes} W') &= (V \otimes V') \hat{\otimes} (W \otimes W') \\ \Lambda^2(V \hat{\otimes} W) &= (\Lambda^2 V \hat{\otimes} \mathrm{Sym}^2\,W) \oplus (\mathrm{Sym}^2\,V \hat{\otimes} \Lambda^2 W) \\ \mathrm{Sym}^2(V \hat{\otimes} W) &= (\mathrm{Sym}^2\,V \hat{\otimes} \mathrm{Sym}^2\,W) \oplus (\Lambda^2 V \hat{\otimes} \Lambda^2 W) \end{cases} \tag{4.2}$$

We finally remark that if V, W are complex irreducible representations of two compact Lie groups H and H' respectively, then $V \hat{\otimes} W$ is of real type if V, W are both of real type or both of quaternionic type, $V \hat{\otimes} W$ is if complex type if at least one of V, W are of complex type and finally, $V \hat{\otimes} W$ is of quaternionic type if one of V, W is of real type and the other one of quaternionic type.

Lemma 4.9. *Consider the homogeneous space* $M_{p,q} := G/K = \mathrm{SU}_{pq}/\mathrm{SU}_p \times \mathrm{SU}_q$ *with* $p, q > 2$, $p+q > 4$. *Then, the multiplicities of the isotropy representation* $\mathfrak{m} = \mathrm{Ad}_{\mathrm{SU}_p} \hat{\otimes} \mathrm{Ad}_{\mathrm{SU}_q}$ *inside* $\Lambda^2\mathfrak{m}$ *and* $\mathrm{Sym}^2\,\mathfrak{m}$ *are given as follows: for* $p = 2$, $q \geq 3$ *it is* $\mathbf{a} = \mathbf{s} = 1$, *while for* $p, q \geq 3$ *it is* $\mathbf{a} = \mathbf{s} = 2$. *Moreover, the dimension of the trivial submodule* $(\Lambda^3\mathfrak{m})^K$ *is* $\ell = 1$ *for* $p = 2$, $q \geq 3$ *and* $\ell = 2$ *for* $p, q \geq 3$.

Proof. The inclusion $\pi : K \to G$ is given by the external tensor product of the standard representations μ_p and μ_q of SU_p and SU_q, respectively. Thus, a short application of Remark 4.6 in combination with the relation $\mathrm{Ad}^{\mathbb{C}}_{A_n} = R(\pi_1 + \pi_n)$, yields that

$$\mathfrak{m}^{\mathbb{C}} = (\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_p} \hat{\otimes} \mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}) = R(\pi_1 + \pi_{p-1}) \hat{\otimes} R(\pi_1 + \pi_{q-1}).$$

Consequently, the isotropy representation $\mathfrak{m} = \mathrm{Ad}_{\mathrm{SU}_p} \hat{\otimes} \mathrm{Ad}_{\mathrm{SU}_q}$ of $M_{p,q} = \mathrm{SU}_{pq}/\mathrm{SU}_p \times \mathrm{SU}_q$ is irreducible over $\mathbb{R}$ and is of real type, since it is the external tensor product of two representations of real type. Now, by [39] we also know that

$$\Lambda^2\,\mathrm{Ad}^{\mathbb{C}}_{A_n} = R(2\pi_1 + \pi_{n-1}) \oplus R(\pi_2 + 2\pi_n) \oplus \mathrm{Ad}^{\mathbb{C}}_{A_n}, \quad n \geq 3 \tag{4.3}$$

$$\operatorname{Sym}^2 \operatorname{Ad}^{\mathbb{C}}_{A_n} = \begin{cases} R(2\pi_1+2\pi_n)\oplus R(\pi_2+\pi_{n-1})\oplus \operatorname{Ad}^{\mathbb{C}}_{A_n}\oplus 1, & \text{if } n\geq 3,\\ R(2\pi_1+2\pi_2)\oplus \operatorname{Ad}^{\mathbb{C}}_{A_n}\oplus 1, & \text{if } n=2. \end{cases} \tag{4.4}$$

Certainly, for $\mathfrak{su}_2=\mathfrak{so}_3=\mathfrak{sp}_1$ one gets a 3-dimensional irreducible representation $\Lambda^2(\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_2})\cong \operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_2}=R(2\pi_1)$. Moreover, it is $\operatorname{Sym}^2(\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_2})=R(4\pi_1)\oplus 1$. Notice also that $\Lambda^2(\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_3})=R(3\pi_1)\oplus R(3\pi_2)\oplus \operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_3}$, since $\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_3}=R(\pi_1+\pi_2)$. Due to this small speciality of SU_2 and the different decomposition of $\operatorname{Sym}^2\operatorname{Ad}^{\mathbb{C}}_{A_n}$ (for $n=2$ and $n\geq 3$, respectively) one has to separate the examination into two cases:

Case A: $p=2$, $q\geq 3$. Then we have $M_{2q}=\mathrm{SU}_{2q}/\mathrm{SU}_2\times \mathrm{SU}_q$ and

$$\mathfrak{m}^{\mathbb{C}}=\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_2}\hat{\otimes}\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}=R(2\pi_1)\hat{\otimes}R(\pi_1+\pi_{q-1}),\quad (q\geq 3).$$

Hence, a combination of (4.2), (4.3), (4.4) and $\Lambda^2(\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_2})=\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_2}=R(2\pi_1)$ shows that

$$\begin{aligned}\Lambda^2(\mathfrak{m}^{\mathbb{C}}) &= \big(R(2\pi_1)\hat{\otimes}\operatorname{Sym}^2\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\big)\oplus\big((R(4\pi_1)\oplus 1)\hat{\otimes}\Lambda^2\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\big)\\ &= \big(R(2\pi_1)\hat{\otimes}R(2\pi_1+2\pi_{q-1})\big)\oplus\big(R(2\pi_1)\hat{\otimes}R(\pi_2+\pi_{q-2})\big)\oplus\big(\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_2}\hat{\otimes}\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\big)\oplus R(2\pi_1)\\ &\quad\oplus\big(R(4\pi_1)\hat{\otimes}R(2\pi_1+\pi_{q-2})\big)\oplus\big(R(4\pi_1)\hat{\otimes}R(\pi_2+2\pi_{q-1})\big)\oplus\big(R(4\pi_1)\hat{\otimes}\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\big)\\ &\quad\oplus R(2\pi_1+\pi_{q-2})\oplus R(\pi_2+2\pi_{q-1})\oplus\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}.\end{aligned}$$

We deduce that $\mathfrak{m}^{\mathbb{C}}$ appears once inside $\Lambda^2(\mathfrak{m}^{\mathbb{C}})$ and since $\mathfrak{m}$ is of real type, it follows that $\mathbf{a}=1$.

Let us treat the decomposition of the second symmetric power. We start with the low dimensional case $p=2$, $q=3$, i.e. $\mathfrak{m}^{\mathbb{C}}=\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_2}\hat{\otimes}\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_3}$. A combination of (4.2), (4.3) and (4.4), yields

$$\begin{aligned}\operatorname{Sym}^2(\mathfrak{m}^{\mathbb{C}}) &= \Big(\big(R(4\pi_1)\oplus 1\big)\hat{\otimes}\big(R(2\pi_1+2\pi_2)\oplus\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_3}\oplus 1\big)\Big)\oplus\Big(\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_2}\hat{\otimes}\big(R(3\pi_1)\oplus R(3\pi_2)\oplus\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_3}\big)\Big)\\ &= \big(R(4\pi_1)\hat{\otimes}R(2\pi_1+2\pi_2)\big)\oplus\big(R(4\pi_1)\hat{\otimes}\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_3}\big)\oplus R(4\pi_1)\oplus R(2\pi_1+2\pi_2)\oplus\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_3}\oplus 1\\ &\quad\oplus\big(\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_2}\hat{\otimes}R(3\pi_1)\big)\oplus\big(\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_2}\hat{\otimes}R(3\pi_2)\big)\oplus\big(\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_2}\hat{\otimes}\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_3}\big).\end{aligned}$$

Hence, there is a copy of $\mathfrak{m}^{\mathbb{C}}$ inside $\operatorname{Sym}^2(\mathfrak{m}^{\mathbb{C}})$ and as above we conclude that $\mathbf{s}=1$. In a similar way, for $p=2$ and $q\leq 4$, we get that

$$\begin{aligned}\operatorname{Sym}^2(\mathfrak{m}^{\mathbb{C}}) &= \big(R(4\pi_1)\oplus 1\big)\hat{\otimes}\big(R(2\pi_1+2\pi_{q-1})\oplus R(\pi_2+\pi_{q-2})\oplus\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\oplus 1\big)\\ &\quad\oplus\Big(\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_2}\hat{\otimes}\big(R(2\pi_1+\pi_{q-2})\oplus R(\pi_2+2\pi_{q-1})\oplus\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\big)\Big)\\ &= \big(R(4\pi_1)\hat{\otimes}R(2\pi_1+2\pi_{q-1})\big)\oplus\big(R(4\pi_1)\hat{\otimes}R(\pi_2+\pi_{q-2})\big)\oplus\big(R(\pi_2+\pi_{q-2})\hat{\otimes}\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\big)\\ &\quad\oplus R(4\pi_1)\oplus R(2\pi_1+2\pi_{q-1})\oplus R(\pi_2+\pi_{q-2})\oplus\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\oplus 1\\ &\quad\oplus\big(\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_2}\hat{\otimes}R(2\pi_1+\pi_{q-2})\big)\oplus\big(\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_2}\hat{\otimes}R(\pi_2+2\pi_{q-1})\big)\oplus\big(\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_2}\hat{\otimes}\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\big).\end{aligned}$$

Thus again we conclude $\mathbf{s}=1$.

Case B: $3\leq p\leq q$. In this case, a combination of (4.2), (4.3) and (4.4) yields the following decomposition for any $p\geq 3$ and $q\geq p$:

$$\begin{aligned}\Lambda^2(\mathfrak{m}^{\mathbb{C}}) &= \big(\Lambda^2R(\pi_1+\pi_{p-1})\hat{\otimes}\operatorname{Sym}^2R(\pi_1+\pi_{q-1})\big)\oplus\big(\operatorname{Sym}^2R(\pi_1+\pi_{p-1})\hat{\otimes}\Lambda^2R(\pi_1+\pi_{q-1})\big)\\ &= \big(R(2\pi_1+\pi_{p-2})\oplus R(\pi_2+2\pi_{p-1})\oplus\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_p}\big)\hat{\otimes}\big(R(2\pi_1+2\pi_{q-1})\oplus R(\pi_2+\pi_{q-2})\oplus\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\oplus 1\big)\\ &\quad\oplus\big(R(2\pi_1+2\pi_{p-1})\oplus R(\pi_2+\pi_{p-1})\oplus\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_p}\oplus 1\big)\hat{\otimes}\big(R(2\pi_1+\pi_{q-2})\oplus R(\pi_2+2\pi_{q-1})\oplus\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\big).\end{aligned}$$

These two external tensor products each contain one copy of $\mathfrak{m}^{\mathbb{C}}$, hence we have $\mathbf{a}=2$ in this case. Passing to the second symmetric power and working in the same way we get that

$$\begin{aligned}\operatorname{Sym}^2(\mathfrak{m}^{\mathbb{C}}) &= \big(\operatorname{Sym}^2\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_p}\hat{\otimes}\operatorname{Sym}^2\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\big)\oplus\big(\Lambda^2\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_p}\hat{\otimes}\Lambda^2\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\big)\\ &= \big(R(2\pi_1+2\pi_{p-1})\oplus R(\pi_2+\pi_{p-2})\oplus\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_p}\oplus 1\big)\hat{\otimes}\big(R(2\pi_1+2\pi_{q-1})\oplus R(\pi_2+\pi_{q-2})\oplus\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\oplus 1\big)\\ &\quad\oplus\big(R(2\pi_1+\pi_{p-2})\oplus R(\pi_2+2\pi_{p-1})\oplus\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_p}\big)\hat{\otimes}\big(R(2\pi_1+\pi_{q-2})\oplus R(\pi_2+2\pi_{q-1})\oplus\operatorname{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\big).\end{aligned}$$

It follows that there are two instances of $\mathfrak{m}^{\mathbb{C}}$ inside $\operatorname{Sym}^2(\mathfrak{m}^{\mathbb{C}})$, i.e. $\mathbf{s}=2$.

To compute the dimension of the space of invariant three-forms, consider the additional equivariant isomorphism

$$\Lambda^3(V\otimes W)=\big(\Lambda^3V\otimes\operatorname{Sym}^3W\big)\oplus\big(P_V(2,1)\otimes P_W(2,1)\big)\oplus\big(\operatorname{Sym}^3V\otimes\Lambda^3W\big),$$

where $P_V(2,1)$ and $P_W(2,1)$ are the (2,1)-plethysms of V and W, respectively. From this we deduce that any invariant 3-form on $V\otimes W$ can be projected to component forms induced from the following cases:

- An invariant 3-form on V and an invariant symmetric 3-tensor on W
- The product of two invariant elements of the (2,1)-plethysms of V and of W
- An invariant symmetric 3-tensor on V and an invariant 3-form on W

Let us first compute such objects for $V := \mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_p} = R(\pi_1 + \pi_{p-1})$. We have

$$\Lambda^3 R(\pi_1 + \pi_{p-1}) \oplus P_{R(\pi_1+\pi_{p-1})}(2,1) = \Lambda^2 R(\pi_1 + \pi_{p-1}) \otimes R(\pi_1 + \pi_{p-1}),$$

and by (4.3) we have $\Lambda^2 \mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_p} = \Lambda^2 R(\pi_1 + \pi_{p-1}) = R(\pi_1 + \pi_{p-1}) \oplus R(2\pi_1 + \pi_{p-2}) \oplus R(\pi_2 + 2\pi_{p-1})$. The first part of the computation is

$$\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_p} \otimes \mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_p} = R(\pi_1 + \pi_{p-1}) \otimes R(\pi_1 + \pi_{p-1}) = R(2\pi_1 + 2\pi_{p-1}) \oplus R(\pi_1 + \pi_{p-1}) \oplus \mathbb{R},$$

where the trivial term $\mathbb{R}$ corresponds to the tensor of the Killing form of $\mathfrak{su}(p)$ (see Remark 3.16). This yields the first invariant 3-form on $R(\pi_1 + \pi_{p-1})$. Now we are left with the task of computing unsymmetrized tensor products of irreducible SU_p-modules, and this can be done via the Littlewood–Richardson rule. The result is that the products $R(2\pi_1+\pi_{p-2})\otimes R(\pi_1+\pi_{p-1})$ and $R(\pi_2 + 2\pi_{p-1}) \otimes R(\pi_1 + \pi_{p-1})$, do not contain the trivial representation. Hence the (2,1)-plethysm of $R(\pi_1 + \pi_{p-1})$ also does not admit any trivial modules. By (4.4) we also deduce that the multiplicity of V in $\mathrm{Sym}^2 R(\pi_1 + \pi_{p-1})$ is 0 for $p = 2$, or 1 for $p \geq 3$. Furthermore, we have

$$\mathrm{Sym}^3 R(\pi_1 + \pi_{p-1}) \oplus P_{R(\pi_1+\pi_{p-1})}(2,1) = \mathrm{Sym}^2 R(\pi_1 + \pi_{p-1}) \otimes R(\pi_1 + \pi_{p-1})$$

and the tensor product between $R(\pi_1 + \pi_{p-1}) \subset \mathrm{Sym}^2 R(\pi_1 + \pi_{p-1})$ and the rightmost factor $R(\pi_1 + \pi_{p-1})$ contains a trivial submodule, corresponding as before to the tensor of the Killing form. This trivial submodule is contained in $\mathrm{Sym}^3 R(\pi_1 + \pi_{p-1})$, since we have already shown that $P_{R(\pi_1+\pi_{p-1})}(2,1)$ contains no such. Note that the computations for $W := \mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_q} = R(\pi_1 + \pi_{q-1})$ are identical (except for switching the label p to q). Therefore, we finally get a one dimensional trivial submodule in $\Lambda^3(V \otimes W)$ for the case $p = 2$ (or q=2), and when $p, q \geq 3$ we get a two dimensional trivial submodule in $\Lambda^3(V \otimes W)$. In our notation, this means $\ell = 1$ for $p = 2$, $q \geq 3$ and $\ell = 2$ for $p, q \geq 3$. This completes the proof. □

Lemma 4.10. *Consider the homogeneous space $M = G/K = \mathrm{Sp}_n/(\mathrm{SO}_n \times \mathrm{Sp}_1)$ with $n \geq 3$. Then, the isotropy representation $\mathfrak{m} = R(2\pi_1)\hat{\otimes}\mathfrak{sp}(1)$ has multiplicity $\mathbf{a} = 1$ inside $\Lambda^2\mathfrak{m}$ and multiplicity $\mathbf{s} = 0$ inside $\mathrm{Sym}^2\mathfrak{m}$. Moreover, the dimension of trivial submodule $(\Lambda^3\mathfrak{m})^K$ is $\ell = 1$.*

Proof. An embedding of a compact Lie group K into Sp_n is equivalent to a (faithful) representation $\phi : K \to \mathrm{GL}(\mathbb{H}^n)$. This is a representation of real dimension $4n$ with an invariant quaternionic structure. Since $K = \mathrm{SO}_n \times \mathrm{Sp}_1$ is compact, the image of ϕ will be inside some conjugacy class of Sp_n. We are looking for the unique isotropy irreducible embedding, which means that ϕ should be an irreducible representation. Let $R(\omega_1, \omega_2) = R(\omega_1)_{\mathrm{SO}_n}\hat{\otimes}_{\mathbb{C}}R(\omega_2)_{\mathrm{Sp}_1}$ denotes the associated real irreducible representation. The obvious candidate is $\phi = R(\pi_1, \pi_1) = \mathbb{R}^n\hat{\otimes}_{\mathbb{R}}\mathbb{H} = \mathbb{R}^n\hat{\otimes}_{\mathbb{R}}\mathbb{C}^2$. This irreducible representation is obviously of quaternionic type. Recall now the adjoint representation of Sp_n is the real submodule inside $\mathrm{Ad}^{\mathbb{C}}_{\mathrm{Sp}_n} = \mathrm{Sym}^2\,\nu_n = \mathrm{Sym}^2_{\mathbb{C}}\,\mathbb{H}^n$. Thus we must take into account the complex structure on ϕ, which is defined by its action on the right tensor factor $\mathbb{H} \simeq \mathbb{C}^2$. By applying (4.2), we compute

$$\begin{aligned}\mathrm{Sym}^2_{\mathbb{C}}\,\phi &= \mathrm{Sym}^2_{\mathbb{C}}(\mathbb{R}^n\hat{\otimes}_{\mathbb{R}}\mathbb{C}^2) = (\mathrm{Sym}^2_{\mathbb{R}}\,\mathbb{R}^n\hat{\otimes}_{\mathbb{R}}\,\mathrm{Sym}^2_{\mathbb{C}}\,\mathbb{C}^2) \oplus (\Lambda^2_{\mathbb{R}}\mathbb{R}^n\hat{\otimes}\Lambda^2_{\mathbb{C}}\mathbb{C}^2)\\ &= (R(2\pi_1) \oplus R(0)) \otimes \mathrm{Ad}^{\mathbb{C}}_{\mathrm{Sp}_1} \oplus \mathrm{Ad}^{\mathbb{C}}_{\mathrm{SO}_n}.\end{aligned}$$

This immediately yields the isotropy representation

$$\mathfrak{m} = \mathfrak{sp}(n)/(\mathfrak{so}(n) \oplus \mathfrak{sp}(1)) = R(2\pi_1)\hat{\otimes}\mathfrak{sp}(1) = R(2\pi_1, 2\pi_1),$$

which is irreducible. Since $\mathfrak{m}$ has real type and its tensor factors also have real type, we can apply complex representation theory without performing any extra complexifications. We proceed with the decomposition of $\Lambda^2\mathfrak{m}$ and $\mathrm{Sym}^2\,\mathfrak{m}$. For any $n > 4$ note the following decompositions of SO_n-modules: $\Lambda^2 R(2\pi_1) = R(2\pi_1 + \pi_2) \oplus R(\pi_2)$ and $\mathrm{Sym}^2 R(2\pi_1) = R(4\pi_1)\oplus R(2\pi_1)\oplus R(2\pi_2)\oplus R(0)$. For Sp_1 we have that $\Lambda^2\mathfrak{sp}(1) = \mathfrak{sp}(1)$ and $\mathrm{Sym}^2\,\mathfrak{sp}(1) = R(4\pi_1)\oplus R(0) = R(4\pi_1)\oplus 1$. Hence we conclude that only those terms in the tensor square that contain a factor of $\mathrm{Sym}^2 R(2\pi_1)$ and $\Lambda^2\mathfrak{sp}(1)$ will yield copies of $\mathfrak{m}$. In particular, the decomposition

$$\Lambda^2\mathfrak{m} = \Lambda^2\big(R(2\pi_1)\hat{\otimes}\mathfrak{sp}(1)\big) = \big(\Lambda^2 R(2\pi_1)\hat{\otimes}\,\mathrm{Sym}^2\,\mathfrak{sp}(1)\big) \oplus \big(\mathrm{Sym}^2 R(2\pi_1)\hat{\otimes}\Lambda^2\mathfrak{sp}(1)\big)$$

contains precisely one instance of $\mathfrak{m}$, i.e. $\mathbf{a} = 1$. One the other hand,

$$\mathrm{Sym}^2\,\mathfrak{m} = \mathrm{Sym}^2\big(R(2\pi_1)\hat{\otimes}\mathfrak{sp}(1)\big) = \big(\Lambda^2 R(2\pi_1)\hat{\otimes}\Lambda^2\mathfrak{sp}(1)\big) \oplus \big(\mathrm{Sym}^2 R(2\pi_1)\hat{\otimes}\,\mathrm{Sym}^2\,\mathfrak{sp}(1)\big),$$

hence $\mathbf{s} = 0$. This proves the claim for $n > 4$. For completeness we examine the low-dimensional cases. Let first $n = 3$. The defining representation ϕ of $K = \mathrm{SO}_3 \times \mathrm{Sp}_1$ must be of real dimension $12 = 3 \times 4$ and hence the only irreducible possibility is $\phi = \mathbb{R}^3\hat{\otimes}\mathbb{H} = R(2\pi_1)\hat{\otimes}R(\pi_1)$. Thus we get

$$\mathrm{Sym}^2_{\mathbb{C}}\,\phi = \mathrm{Sym}^2_{\mathbb{C}}(\mathbb{R}^3\hat{\otimes}_{\mathbb{R}}\mathbb{C}^2) = \big(R(4\pi_1)\hat{\otimes}\mathfrak{sp}(1)\big)^{\mathbb{C}} \oplus \big(\mathfrak{sp}(1) \oplus \mathfrak{so}(3)\big)^{\mathbb{C}}.$$

Hence in this case $\mathfrak{m} = R(4\pi_1)\hat{\otimes}\mathfrak{sp}(1)$. As $\mathfrak{so}(3)$-modules, we have that

$$\Lambda^2 R(4\pi_1) = R(6\pi_1) \oplus \mathfrak{so}(3), \quad \operatorname{Sym}^2 R(4\pi_1) = R(8\pi_1) \oplus R(4\pi_1) \oplus R(0).$$

Therefore, only products of $\operatorname{Sym}^2 R(4\pi_1)$ and $\Lambda^2\mathfrak{sp}(1)$ yield copies of $\mathfrak{m}$. Consequently, the result is the same as above, the multiplicity of $\mathfrak{m}$ is one in $\Lambda^2\mathfrak{m}$ and zero in $\operatorname{Sym}^2\mathfrak{m}$.

Assume now that $n = 4$. The defining representation of $K = \mathrm{SO}_4 \times \mathrm{Sp}_1$ is $\phi = \mathbb{R}^4\hat{\otimes}\mathbb{H}$, but $\mathbb{R}^4 = R(\pi_1 + \pi_2)$ in terms of highest weights, instead of being $R(\pi_1)$ as before, because SO_4 is non-simple. We get

$$\operatorname{Sym}^2_{\mathbb{C}}\phi = \operatorname{Sym}^2_{\mathbb{C}}(\mathbb{R}^4\hat{\otimes}_{\mathbb{R}}\mathbb{C}^2) = \big(R(2\pi_1 + 2\pi_2)\hat{\otimes}\mathfrak{sp}(1)\big)^{\mathbb{C}} \oplus \big(\mathfrak{sp}(1) \oplus \mathfrak{so}(4)\big)^{\mathbb{C}}$$

and thus $\mathfrak{m} = R(2\pi_1 + 2\pi_2)\hat{\otimes}\mathfrak{sp}(1)$ in this case. As $\mathfrak{so}(4)$-modules, we see that

$$\begin{aligned}\Lambda^2 R(2\pi_1 + 2\pi_2) &= R(2\pi_1 + 4\pi_2) \oplus R(4\pi_1 + 2\pi_2) \oplus \mathfrak{so}(3),\\ \operatorname{Sym}^2 R(2\pi_1 + 2\pi_2) &= R(4\pi_1 + 4\pi_2) \oplus R(4\pi_1) \oplus R(4\pi_2) \oplus R(2\pi_1 + 2\pi_2) \oplus R(0),\end{aligned}$$

and the same argument as previously yields that $\mathbf{a} = 1$ and $\mathbf{s} = 0$.

Now, our assertion for $(\Lambda^3\mathfrak{m})^K$ can be deduced very easily as follows: Any invariant element of $\Lambda^3\mathfrak{m}$ induces an equivariant map in $\operatorname{Hom}_K(\mathfrak{m}, \Lambda^2\mathfrak{m})$. For any $n \geq 3$ we have shown that $\mathbf{a} = \dim_{\mathbb{R}} \operatorname{Hom}_K(\mathfrak{m}, \Lambda^2\mathfrak{m}) = 1$. Thus, $\dim_{\mathbb{R}}(\Lambda^3\mathfrak{m})^K \leq 1$, but by Lemma 3.12 we also get $\dim_{\mathbb{R}}(\Lambda^3\mathfrak{m})^K \geq 1$ and the result follows. Note that this method for the computation of the multiplicity ℓ, applies on any non-symmetric SII space $M = G/K$ whose isotropy representation is of real type and has $\mathbf{a} = 1$ (or whose isotropy representation is of complex type and has $\mathbf{a} = 2$, e.g. $\mathrm{S}^6 \cong \mathrm{G}_2/\mathrm{SU}_3$). □

5. Classification of homogeneous ∇-Einstein structures on SII spaces

5.1. Homogeneous ∇-Einstein structures

Similarly with invariant Einstein metrics on homogeneous Riemannian manifolds, on triples (M^n, g, T) consisting of a homogeneous Riemannian manifold $(M^n = G/K, g)$ endowed with a (non-trivial) invariant 3-form T, on may speak of *homogeneous ∇-Einstein structures*. In particular,

Definition 5.1. A triple (M^n, g, T) of a connected Riemannian manifold (M, g) carrying a (non-trivial) 3-form $T \in \Lambda^3 T^*M$, is called a *G-homogeneous ∇-Einstein manifold* (with skew-torsion) if there is a closed subgroup $G \subseteq \mathrm{Iso}(M, g)$ of the isometry group of (M, g), which acts transitively on M and a G-invariant connection ∇ compatible with g and with skew-torsion T, whose Ricci tensor satisfies the condition (1.2).

In this case, g is a G-invariant metric, the Levi-Civita connection ∇^g is a G-invariant metric connection and since $2(\nabla - \nabla^g) = T$, the torsion T of ∇ is given necessarily by a G-invariant 3-form $0 \neq T \in \Lambda^3(\mathfrak{m})^K$, where $\mathfrak{m} \cong T_oM$ is a reductive complement of $M = G/K$ with $K \subset G$ being the (closed) isotropy group. In particular, the ∇-Einstein condition (1.2) is $\mathrm{Ad}(K)$-invariant, in the sense that the Ricci tensor Ric^{∇} is a G-invariant covariant 2-tensor which is described by an $\mathrm{Ad}(K)$-invariant bilinear form on $\mathfrak{m}$, and the same for its symmetric part. Moreover,

Proposition 5.2. *On a homogeneous Riemannian manifold $(M = G/K, g)$ carrying a G-invariant (non-trivial) 3-form $T \in \Lambda^3(\mathfrak{m})^K$, the scalar curvature $\mathrm{Scal} = \mathrm{Scal}^{\nabla}$ associated to the G-invariant metric connection $\nabla := \nabla^g + \frac{1}{2}T$ is a constant function on M.*

Proof. It is well-known that on a reductive homogeneous space, the scalar curvature Scal^g of the Levi-Civita connection (related to a G-invariant Riemannian metric g, or the corresponding $\mathrm{Ad}(K)$-invariant inner product $\langle\ ,\ \rangle$ on the reductive complement $\mathfrak{m}$) is independent of the point, i.e. it is a constant function on M [13,35]. Let ∇ be a G-invariant metric connection on $(M = G/K, g)$ whose skew-torsion coincides with the invariant 3-form $0 \neq T \in \Lambda^3(\mathfrak{m})^K$. Due to the identity $\mathrm{Scal} = \mathrm{Scal}^g - \frac{3}{2}\|T\|^2$ it is sufficient to prove that $\|T\|^2$ is constant, which is obvious since T corresponds to a G-invariant tensor field. Consequently, $\mathrm{Scal}^{\nabla} : G/K \to \mathbb{R}$ is constant. □

5.2. On the proofs of Theorems C–E

Let us focus now on an effective, non-symmetric (compact) strongly isotropy irreducible homogeneous Riemannian manifold $(M = G/K, g = -B|_{\mathfrak{m}})$, where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is the associated B-orthogonal decomposition. We denote by $\mathcal{M}^{sk}_G(\mathrm{SO}(G/K, g)) \subseteq \mathcal{M}_G(\mathrm{SO}(G/K, g)) \subseteq \mathcal{A}ff_G(F(G/K))$ the space of G-invariant affine connections on $M = G/K$ which are compatible with the Killing metric $g = -B|_{\mathfrak{m}}$ and have invariant 3-forms $0 \neq T \in (\Lambda^3\mathfrak{m})^K$ as their torsion tensors. For the corresponding set of homogeneous ∇-Einstein structures, we will write $\mathcal{E}^{sk}_G(\mathrm{SO}(G/K, g))$. As stated in the introduction, Lemma 3.12 and Schur's lemma allow us to parametrize $\mathcal{E}^{sk}_G(\mathrm{SO}(G/K, g))$ by the space of global G-invariant 3-forms. Hence, this finally yields the identification $\mathcal{E}^{sk}_G(\mathrm{SO}(G/K, g)) = \mathcal{M}^{sk}_G(\mathrm{SO}(G/K, g))$. With the aim to clarify this identification and give explicit proofs of Theorems C–E in introduction, let us recall first the following important result of [17].

Theorem 5.3 ([17, Thm. 4.7]). *Let $(M^n = G/K, g)$ be an effective, compact and simply-connected, isotropy irreducible standard homogeneous Riemannian manifold $(M^n = G/K, g)$ of a compact connected simple Lie group G, which is not a symmetric space of Type I. Then, $(M^n = G/K, g)$ is a ∇^α-Einstein manifold for any parameter $\alpha \neq 0$, where $\nabla^\alpha = \nabla^g + \frac{1}{2}T^\alpha = \nabla^c + \Lambda^\alpha$ is the 1-parameter family of G-invariant metric connections on M, with skew-torsion $0 \neq T^\alpha = \alpha \cdot T^c$ (see Lemma 3.4).*

Note that for a symmetric space $M = G/K$ of Type I, the associated space of G-invariant affine *metric* connections is always a point, i.e. $\nabla^\alpha \equiv \nabla^c \equiv \nabla^g$ and no torsion appears. On the other hand, Theorem 5.3 generalizes the well-known fact that a compact simple Lie group G is ∇^α-Einstein with (non-trivial) *parallel torsion* for any $0 \neq \alpha \in \mathbb{R}$, with the flat ± 1-connections of Cartan–Schouten being the trivial members (see for example [3, Lemma 1.8] or [17, Thm. 1.1]). Notice however, that if $M = G/K$ is not isometric to a compact simple Lie group, then the ∇^α-Einstein structures described in Theorem 5.3 have parallel torsion only for $\alpha = 1$. We finally remark that both $S^6 = G_2/SU_3, S^7 = G_2/\mathrm{Spin}_7$ are (strongly) isotropy irreducible and non-symmetric, hence they are ∇^α-Einstein manifolds with skew-torsion, for any $0 \neq \alpha \in \mathbb{C}$, $0 \neq \alpha \in \mathbb{R}$, respectively (due to the type of their isotropy representation). The same applies for any compact, non-symmetric, effective SII homogeneous Riemannian manifold and this gives rise to Theorem C which is an immediate consequence of Theorem A.1 and Theorem 5.3.

Let us proceed now with a proof of Theorems D and E.

Proof of Theorem D. If $M = G/K$ is a manifold in Table 2 whose isotropy representation is of real type, different than SO_{10}/Sp_2, then Theorem A.2(i) ensures the existence of a second (real) 1-parameter family of G-invariant connections $\nabla^t \neq \nabla^a$, compatible with the Killing metric and with skew-torsion T^t such that $T^t \neq T^c \sim T^\alpha$ (for any $t, \alpha \in \mathbb{R}$), where T^c is the torsion of the (unique) canonical connection corresponding to $\mathfrak{m}$. Thus, we can write $\nabla^t = \nabla^g + \frac{1}{2}T^t$ with $\nabla^t \neq \nabla^\alpha$. Since $\mathrm{Ric}^{\nabla^t} \equiv \mathrm{Ric}^t$ is G-invariant, the same is true for $\delta^g T^t = \delta^{\nabla^t} T^t$, in particular we can view the codifferential of $T^t \in (\Lambda^3\mathfrak{m})^K$ as a G-invariant 2-form. However, χ is of real type, hence the trivial representation $\mathbb{R}$ does not appear in $\Lambda^2\mathfrak{m}$, i.e. $(\Lambda^2\mathfrak{m})^K = 0$. Hence, we deduce that $\delta^g T^t = 0 = \delta^t T^t$ and since $\mathfrak{m}$ is (strongly) isotropy irreducible over $\mathbb{R}$ and the Ricci tensor Ric^t is symmetric, by Schur's lemma it must be a multiple of the Killing metric, i.e. $(M = G/K, -B|_{\mathfrak{m}}, \nabla^t)$ is ∇^t-Einstein with skew-torsion. Our final claim follows now in combination with Theorem 5.3. □

It is well-known that an effective SII homogeneous space $M = G/K$ admits an (integrable) G-invariant complex structure if and only if is a Hermitian symmetric space [46]. Moreover, the existence of an invariant *almost* complex structure $J \in \mathrm{End}(\mathfrak{m})$ on a strongly isotropy irreducible space implies that the isotropy representation is not of real type, hence $\chi = \phi \oplus \overline{\phi}$ for some irreducible complex representation with $\phi \not\cong \overline{\phi}$. Consequently, any manifold which appears in Tables 4, 5 and whose isotropy representation is of complex type, is a G-homogeneous almost complex manifold (see also [46, Cor. 13.2]). Notice also

Lemma 5.4. *Let $\mathfrak{k}$ be a compact Lie algebra and let $\rho : \mathfrak{k} \to \mathrm{End}(\mathfrak{m})$ be a faithful (irreducible) representation of $\mathfrak{k}$ over $\mathbb{R}$, endowed with an invariant inner product $B_{\mathfrak{m}}$. Assume that $\dim \mathfrak{m} \geq 2$. If $\mathfrak{m}$ admits an $\mathrm{ad}(\mathfrak{k})$-invariant complex structure J (as a vector space), then $\Lambda^2\mathfrak{m}$ contains the trivial representation $\mathbb{R}$.*

Proof. We only mention that since $\mathfrak{m}$ is an irreducible complex type representation of a compact Lie algebra $\mathfrak{k}$, it is unitary, therefore the $\mathrm{ad}(\mathfrak{k})$-invariant Kähler form $\omega(X, Y) = B_{\mathfrak{m}}(JX, Y)$ gives rise to an $\mathrm{ad}(\mathfrak{k})$-invariant element inside $\Lambda^2\mathfrak{m}$. □

Consider now the spaces SO_{n^2-1}/SU_n $(n \geq 4)$ and E_6/SU_3. Since their isotropy representation is of complex type, Lemma 5.4 certifies the existence of G-invariant 2-forms. Thus, in contrast to Theorem D, we cannot deduce that the Ricci tensor of all predicted G-invariant metric connections with skew-torsion must be necessarily symmetric (although this is the case always for Ric^α). However, since we are considering the isotropy irreducible case, we obtain Theorem E as follows:

Proof of Theorem E. Assume that $(M = G/K, g = -B|_{\mathfrak{m}})$ is one of the manifolds SO_{n^2-1}/SU_n $(n \geq 4)$ or E_6/SU_3. By Theorem A.2(ii) (see also Table 2) we know that $M = G/K$ admits a 4-dimensional space of G-invariant metric connections with skew-torsion. Now, the ∇-Einstein condition is related only with the symmetric part of the Ricci tensor associated to any such connection. Since this tensor is G-invariant, Schur's lemma ensures that the ∇-Einstein equation is satisfied for any available G-invariant metric connection ∇ with skew-torsion. Therefore, the space of G-invariant ∇-Einstein structures has the same dimension with the space of G-invariant metric connections with skew-torsion. This proves Theorem E. □

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