



Spin geometry of the rational noncommutative torus

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ABSTRACT

We discuss the structure of topologically non-trivial almost-commutative manifold for spectral triples realized on the algebra of smooth functions on the noncommutative torus with rational parameter. This is done by showing isomorphisms with a spectral triple on the algebra of sections of certain bundle of algebras, and with a spectral triple on a certain invariant subalgebra of the product algebra. The isomorphisms intertwine also the grading and real structure. This holds for all four inequivalent spin structures, which are explicitly constructed in terms of double coverings of the noncommutative torus (with arbitrary real parameter). These results are extended also to a class of curved (non flat) spectral triples, obtained as a perturbation of the standard one by eight central elements.

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1. Introduction

The framework of Connes' noncommutative spectral geometry provides a generalization of ordinary Riemannian spin manifolds to noncommutative manifolds. Within this framework, the special case of a (globally trivial) almost-commutative manifold has been shown to describe a (classical) gauge theory over a Riemannian spin manifold (see, e.g., [28]), which ultimately led to a description of the full Standard Model of high energy physics, including the Higgs mechanism and neutrino mixing [9,10,13,14,17,28]. The gauge theories mentioned above are, by construction, topologically trivial, in the sense that the corresponding principal bundles are globally trivial bundles.

Spectral triples constructed on nontrivial bundles, which correspond to non-trivial Yang–Mills gauge theories, have been introduced in [3]. In [5], the general case of a topologically non-trivial almost-commutative manifold is studied via spectral triples, while in [6] the real structure is also included. This kind of spectral triples is further developed in [2], again in connection with Yang–Mills theories.

In this paper, we focus on a particular class of non-trivial almost-commutative spectral triples that are built on a bundle of algebras which, from the topological point of view, is trivial as a vector bundle but not as an algebra bundle. More precisely its algebra of sections is not given just as the tensor product of the algebra of functions on the base space with the algebra of typical fiber, but it is obtained instead through a quotient with respect to the diagonal action of a finite group. This permits a local description of the abstract Dirac operators in terms of the usual differential operators on vector bundles.

Such a situation is known to occur, for example, for the noncommutative torus with rational parameter [24]. In Section 3.2 we take advantage of this bundle of algebras to provide all the other data of a spectral triple, namely the Hilbert space and the analogue of the Dirac operator, as well as grading and real structure.

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We also provide in Section 3.3 a closely related description in which the algebra is realized as a suitable subalgebra of the tensor product of two algebras. It is suited also for infinite dimensional algebras and overcomes the requirement that in the second description vector bundles need to be of finite rank. This will allow us to link in a future work also to the splitting homomorphism of theta deformations of the canonical spectral triple [11] (see also [12]), and to further developments in [7].

In Section 4 we describe four inequivalent spin structures on the noncommutative torus with arbitrary real value of the deformation parameter θ in terms of inequivalent double coverings. We then construct the corresponding standard (flat) spectral triples, which are mutually non-isomorphic already at the level of algebras. They match the classification of spin structures of [27] performed in terms of spectral triples. Note, however, that in contrast with [27], though following a similar scheme, it is claimed in [29] that for some rational values of parameter θ , the number of inequivalent spin structures can be less than four, but the precise correspondence is not clear due to rather different classification criteria of inequivalent spin structures.

In Section 4.3 we indeed analyze the case of rational parameter θ and provide two isomorphic descriptions of spectral triples for all four spin structures in analogy to the case of trivial spin structure presented in Sections 3.2 and 3.3.

Finally, in Section 5 we discuss isomorphic descriptions of spectral triples on the curved (i.e. non flat) rational noncommutative torus. In order to maintain the topologically non-trivial almost-commutative structure, we restrict to modifications of the Dirac operator by the central elements. Then it turns out that, in the description in terms of the bundle of algebras, only the external part of the differential Dirac operator is modified by the corresponding functions on the classical torus, while the internal part remains trivial.

It will be interesting to investigate in future works possible relations of our spectral geometry approach with the approach of [2], which make use of connections on principal bundles, and also with [4], where spectral triples on the noncommutative torus have been employed to study topological insulators. Another direction to be explored concerns the noncommutative coverings and, in particular, spectral triples on self-coverings of the rational noncommutative torus in [1], and the quantum coverings studied in [8] via Hopf algebroids as the noncommutative analogue of groupoids.

2. Preliminaries

The geometry of a compact Riemannian spin manifold M can be encoded [10,24] in terms of its canonical spectral triple, which consists of the algebra of smooth complex functions on M , the Hilbert space $L^2(M, \Sigma)$ of square integrable Dirac spinors on M and the Dirac operator on M . More generally, a *spectral triple* is the datum $(A, \mathcal{H}, D)$ of a unital $*$ -algebra A , a Hilbert space $\mathcal{H}$ carrying a faithful unitary representation $\pi : A \rightarrow B(\mathcal{H})$, and a selfadjoint operator D on $\mathcal{H}$ with compact resolvent, such that the commutators $[D, \pi(a)]$ are bounded operators for any $a \in A$.

A spectral triple is called *even* if there is a $\mathbb{Z}/2$ -grading operator χ commuting with $\pi(a)$ for any $a \in A$ and anticommuting with D . Furthermore, it is called *real* if there is a $\mathbb{C}$ -antiunitary operator J , such that $[a, JbJ] = 0$ for $a, b \in A$, $J^2 = \epsilon$, $JD = \epsilon'DJ$ and $J\chi = \epsilon''\chi J$. The three signs $\epsilon, \epsilon', \epsilon''$ determine the so called KO-dimension of the spectral triple.

Definition 2.1. We call two spectral triples $(A_1, \mathcal{H}_1, D_1)$ and $(A_2, \mathcal{H}_2, D_2)$ isomorphic if and only if there exists an isomorphism of algebras $\gamma : A_1 \rightarrow A_2$ together with unitary operator $T : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ such that

$$Ad_T \pi_1 = \pi_2 \gamma \quad (1)$$

and

$$TD_1 = D_2 T. \quad (2)$$

In case of spectral triples that are even, real or both, we require in addition that

$$T\chi_1 = \chi_2 T, \quad (3)$$

$$TJ_1 = J_2 T, \quad (4)$$

or both.

Note that (1) can be equivalently stated in terms of $\check{\pi} : A \times \mathcal{H} \rightarrow \mathcal{H}$, $\check{\pi}(a, h) := \pi(a)h$, as

$$T\check{\pi}_1 = \check{\pi}_2(\gamma \times T). \quad (5)$$

We remark that when two even real spectral triples are isomorphic then they must have the same KO-dimension by (2) and (4).

Recall that *product* of two (even) spectral triples $(A_1, \mathcal{H}_1, D_1)$ with $(A_2, \mathcal{H}_2, D_2)$, which corresponds to the usual direct product of two (even dimensional) Riemannian spin manifolds, is given by

$$(A_1 \otimes A_2, \mathcal{H}_1 \otimes \mathcal{H}_2, D_1 \otimes 1 + \chi_1 \otimes D_2), \quad (6)$$

where we assume that the algebraic tensor product of algebras can be suitably completed, and we use the usual tensor product of Hilbert spaces. Equivalently one can take $D_1 \otimes \chi_2 + 1 \otimes D_2$ as Dirac operator. Furthermore, in the case of even real spectral triples the grading is $\chi_1 \otimes \chi_2$, and the real structure is $J_1 \otimes J_2$.

By *almost-commutative spectral triple*, we mean the product of the canonical spectral triple $(C^\infty(M), L^2(M, \Sigma), \not{D})$ with a finite spectral triple, i.e. one with finite dimensional Hilbert space $\mathcal{H}_2 = \mathbb{C}^n$. Thus, taking advantage of the isomorphism $L^2(M, \Sigma) \otimes \mathbb{C}^n \approx L^2(M, \Sigma \times (M \times \mathbb{C}^n))$ we see that the Hilbert space consists of n -copies of Dirac spinors (globally), and the algebra of smooth A_2 -valued functions on M . A generalization of this notion is given when one allows the Hilbert space to consist of L^2 sections of the product of Σ with a (locally trivial) vector bundle with a typical fiber $\mathbb{C}^n$, and correspondingly the algebra to consist of smooth sections of some (locally trivial) bundle of finite dimensional $*$ -algebras.

Definition 2.2. A topologically non-trivial almost-commutative manifold is a spectral triple of the form $(C^\infty(M, F), L^2(M, \Sigma \otimes E), D)$, where F is an algebra sub-bundle of endomorphisms of a (locally trivial) finite rank hermitian vector bundle E on M and the operator D has locally an almost product form

$$D = \not{D} \otimes \text{id}_E + \chi \otimes D_E, \quad (7)$$

where $D_E \in \text{End}(E)$.

In particular we will be interested in the case where, even though F is trivial from the topological point of view, it is globally non-trivial as algebra bundle.

A special case of the above definition occurs when M is a quotient of another Riemannian spin manifold $\tilde{M}$ by a group of isometries G (for simplicity assumed to be discrete, or even finite, and preserving the spin structure). Then our spectral triples can be built from the canonical spectral triple $(C^\infty(\tilde{M}), L^2(\Sigma), \not{D})$ on $\tilde{M}$ and a finite G -equivariant noncommutative spectral triple (A, H, D) . More precisely, the relevant algebra will be given by the G -invariant subalgebra of $C^\infty(\tilde{M}) \otimes A$, the Hilbert space given by the G -invariant Hilbert subspace of $L^2(\Sigma) \otimes H$ and the Dirac-type operator given by $\not{D} \otimes 1 + \chi \otimes D$, where χ is the chiral grading of $L^2(\Sigma)$. This set-up lends itself however to a generalization to not necessarily finite spectral triples (A, H, D) (i.e. to infinite dimensional H) and to fully-fledged nontrivial bundles F , which would require the use of Hilbert modules rather than sections of finite rank vector bundles as in Definition 2.2, and possibly internal KK-products of spectral triples (see, e.g., [26]).

3. Rational noncommutative torus

In this section, we establish an isomorphism between the standard spectral triple on the rational noncommutative torus, that is $(C^\infty(\mathbb{T}_{p/q}^2), L^2(\mathbb{T}_{p/q}^2) \otimes \mathbb{C}^2, D_{p/q})$, and two other spectral triples. The first one is $(\Gamma^\infty(F), L^2(F) \otimes \mathbb{C}^2, D_F)$, where F is an algebra bundle of $q \times q$ matrices over the torus $\mathbb{T}^2$, and D_F is certain differential operator on F . This construction originates from a known (see, e.g., [24]) isomorphism between the algebras $C(\mathbb{T}_{p/q}^2)$ and the continuous sections of F , which as a bundle of algebras is not a product bundle. It is an example of Definition 2.2 with F regarded as self-endomorphisms consisting of fiber-wise left multiplication. The second one, denoted as $(\mathcal{A}_{p/q}, \mathcal{H}_{p/q}, \mathcal{D}_{p/q})$, is built on $\mathcal{A}_{p/q}$ which is the subalgebra of $\mathbb{Z}_q \times \mathbb{Z}_q$ -invariants of $C(\mathbb{T}^2) \otimes M_q$, $\mathcal{H}_{p/q}$ is a Hilbert subspace of $\mathbb{Z}_q \otimes \mathbb{Z}_q$ -invariants in $L^2(\mathbb{T}^2) \otimes M_q \otimes \mathbb{C}^2$, and $\mathcal{D}_{p/q}$ is the canonical Dirac operator $\not{D}$ on $\mathbb{T}^2$ tensor the identity on M_q .

The Hilbert spaces of both these spectral triples are given as completions of the corresponding two algebras with respect to certain norms, while the Dirac operators are defined in such a way that their action on the respective spaces satisfy (2).

3.1. The standard spectral triple

We recall the definition of the standard spectral triple on the noncommutative torus $\mathbb{T}_\theta$ with a parameter θ .

The algebra

Definition 3.1. Let U, V be two unitary generators with the commutation relation

$$UV = \lambda VU, \quad (8)$$

where $\lambda = e^{2\pi i \theta}$, $0 \leq \theta \leq 1$. The algebra $C^\infty(\mathbb{T}_\theta^2)$ of smooth complex valued functions on the noncommutative torus consists of the series:

$$a = \sum_{(m,n) \in \mathbb{Z}_2} a_{mn} U^m V^n, \quad (9)$$

where the double sequence of $a_{mn} \in \mathbb{C}$ satisfies

$$\|a\|_k := \sup_{(m,n) \in \mathbb{Z}^2} (1 + m^2 + n^2)^k |a_{mn}|^2 < \infty, \quad \forall k \in \mathbb{N}. \quad (10)$$

Clearly, when $\theta = 0$ the algebra $C^\infty(\mathbb{T}_0^2)$ is isomorphic to the algebra $C^\infty(\mathbb{T}^2)$ of smooth complex functions on the classical torus $\mathbb{T}^2 := \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| = 1 = |z_2|\}$, generated by the identity functions on the two factors $S^1 \subset \mathbb{C}$ denoted (with a slight abuse of notation) by z_1 and z_2 and called coordinate functions on $\mathbb{T}^2$. It should be clear from the context if we regard z_1 and z_2 as numbers or as functions.

The Hilbert space $L^2(\mathbb{T}_\theta^2) \otimes \mathbb{C}^2$

Denote by $\mathfrak{t}$ the following tracial state on $C^\infty(\mathbb{T}_\theta^2)$:

$$\mathfrak{t}\left(\sum a_{mn}U^mV^n\right) = a_{00}, \quad (11)$$

where a_{00} is the coefficient of 1. We will refer to $\mathfrak{t}$ as the *trace*.

The trace defines a sesquilinear form on $C^\infty(\mathbb{T}_\theta^2)$ by

$$\langle a \mid b \rangle = \mathfrak{t}(a^*b) \quad (12)$$

and a norm:

$$\|a\| = \sqrt{\mathfrak{t}(a^*a)}. \quad (13)$$

Denote by $L^2(\mathbb{T}_\theta^2)$ the Hilbert space obtained by completion of $C^\infty(\mathbb{T}_\theta^2)$ with respect to this norm. It carries a $*$ -representation of $C^\infty(\mathbb{T}_\theta^2)$ by left multiplication:

$$\pi(a) : b \mapsto ab, \quad (14)$$

i.e. $L^2(\mathbb{T}_\theta^2)$ is a left $*$ -module over $C^\infty(\mathbb{T}_\theta^2)$. The elements of $L^2(\mathbb{T}_\theta^2)$ are analogues of Weyl spinors on the noncommutative torus.

As the full Hilbert space of analogues of Dirac spinors on the noncommutative torus we take $L^2(\mathbb{T}_\theta^2) \otimes \mathbb{C}^2$, with the diagonal $*$ -module structure over $C^\infty(\mathbb{T}_\theta^2)$.

The Dirac operator D_θ

The trace (11) is invariant under the actions of the torus group $\mathbb{T}^2$ on $C^\infty(\mathbb{T}_\theta^2)$ by automorphisms defined by

$$U \mapsto z_1 U, \quad V \mapsto z_2 V, \quad \forall (z_1, z_2) \in \mathbb{T}^2. \quad (15)$$

These actions are infinitesimally generated by the two commuting derivations

$$\delta_1 U = iU, \quad \delta_1 V = 0 \quad (16)$$

$$\delta_2 U = 0, \quad \delta_2 V = iV. \quad (17)$$

The canonical flat Dirac operator on the Hilbert space $L^2(\mathbb{T}_\theta^2) \otimes \mathbb{C}^2$ is a contraction of derivations δ_ℓ with Pauli matrices σ_ℓ (Clifford multiplication):

$$D_\theta = i(\sigma_1 \delta_1 + \sigma_2 \delta_2) := \begin{pmatrix} 0 & i\delta_1 + \delta_2 \\ i\delta_1 - \delta_2 & 0 \end{pmatrix}. \quad (18)$$

Recall that the spectral triple $(C^\infty(\mathbb{T}_\theta^2), L^2(\mathbb{T}_\theta^2) \otimes \mathbb{C}^2, D_\theta)$ is even, with the grading $\chi_\theta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ that commutes with every $a \in C^\infty(\mathbb{T}_\theta^2)$ and anticommutes with D_θ .

This spectral triple is also real, by taking as real structure

$$J_\theta = -iJ_\theta^0 \otimes (\sigma_2 \circ c.c.), \quad (19)$$

where $J_\theta^0 : \mathcal{H}_\theta^0 \rightarrow \mathcal{H}_\theta^0$ is the Tomita conjugation:

$$J_\theta^0(a) = a^*. \quad (20)$$

It is immediately seen that for $\theta = 0$ the derivations δ_ℓ become the coordinate derivatives that can be expressed also as $\partial_\ell = z_\ell \frac{\partial}{\partial z_\ell}$. Furthermore the standard spectral triple described above is just the canonical spectral triple and in particular

$$D_0 = i(\sigma_1 \partial_1 + \sigma_2 \partial_2) = \begin{pmatrix} 0 & i\partial_1 + \partial_2 \\ i\partial_1 - \partial_2 & 0 \end{pmatrix} \quad (21)$$

is the Dirac operator constructed from the (flat) Levi-Civita connection. It should be however mentioned that this corresponds to a particular choice of a spin structure on the noncommutative torus; we will describe the other spin structures in Section 4.

In the following two subsections we will focus on the case when θ is a rational number, so unless stated differently from now on:

$$\theta = p/q, \quad \text{i.e.,} \quad \lambda = e^{2\pi i p/q}, \quad (22)$$

where $0 < p < q \in \mathbb{Z}$ are relatively prime. In this case the center $\mathcal{Z}_{p/q}$ of $C^\infty(\mathbb{T}_{p/q}^2)$ is generated by U^q and V^q , and is just the invariant subalgebra for the finite subgroup $G \approx \mathbb{Z}_q \times \mathbb{Z}_q$ of (15) of pairs of q th roots of 1. The center $\mathcal{Z}_{p/q}$ is isomorphic to $C^\infty(\check{\mathbb{T}}^2)$, where $\check{\mathbb{T}}^2$ is the quotient of $\mathbb{T}^2$ by the free action κ of G given by

$$\kappa_{m,n}(z_1, z_2) = (\lambda^m z_1, \lambda^n z_2). \quad (23)$$

Clearly $\mathbb{T}^2$ is a q^2 -fold covering of $\check{\mathbb{T}}^2$ (a principal G -bundle), but $\check{\mathbb{T}}^2$ is also diffeomorphic to a torus. We denote by $[z_1, z_2]_\kappa$ the κ -equivalence classes (orbits of κ). From the metric point of view we will equip $\mathbb{T}^2$ first with the standard flat Riemannian metric, and then also with some other G -invariant ones. They descend to $\check{\mathbb{T}}^2$ so that π is an isometric submersion. Then $\check{\mathbb{T}}^2$ is actually isometric to $\mathbb{T}^2$ when the latter one is equipped with the original metric rescaled by q^2 . These metric properties reflect themselves via certain invariance properties of D_0 in expression (18). Namely it commutes with the derivations δ_ℓ and with the torus group action (15) they generate; thus, in particular, it is invariant under the subgroup G .

3.2. First isomorphic spectral triple

The algebra $\Gamma^\infty(F)$

As it is well known the C^* -algebra $C(\mathbb{T}_{p/q}^2)$ of the rational noncommutative torus is isomorphic to the algebra of continuous sections of certain vector bundle F of $q \times q$ matrix algebras, over a 2-torus. The same holds of course also on the smooth level, as we will present now in full detail.

Denote by M_q the algebra of $q \times q$ complex matrices, and define $R, S \in M_q$ by

$$R = \begin{pmatrix} 1 & & & 0 \\ & \lambda & & \\ & & \lambda^2 & \\ \dots & \dots & \dots & \dots \\ & & & \lambda^{q-1} \end{pmatrix}, \quad S = \begin{pmatrix} 0 & & & 1 \\ 1 & 0 & & \\ & 1 & & \\ \dots & \dots & \dots & 1 \\ & & & 0 \end{pmatrix}. \quad (24)$$

We have $R^q = S^q = \mathbb{1}$ and $RS = \lambda SR$. Consider another action τ of G on $\mathbb{T}^2 \times M_q$:

$$\tau_{m,n}(z_1, z_2, A) = (\lambda^m z_1, \lambda^n z_2, R^m S^n A S^{-n} R^{-m}), \quad \forall A \in M_q.$$

We denote by $[z_1, z_2, A]_\tau$ the τ -equivalence classes (orbits). The space F of orbits of τ forms a vector bundle over $\check{\mathbb{T}}^2$ with typical fiber M_q , and (well defined) projection

$$\pi_F : F \rightarrow \check{\mathbb{T}}^2, \quad \pi_F : [z_1, z_2, A]_\tau \mapsto [z_1, z_2]_\kappa.$$

We remark that the bundle F is associated to the principal G -bundle $\mathbb{T}^2$ over $\check{\mathbb{T}}^2$, via the representation $\rho : G \rightarrow \text{End}(M_q)$, given by

$$\rho_{m,n}(A) = R^m S^n A S^{-n} R^{-m}, \quad A \in M_q.$$

Indeed, the assignment

$$F \ni [z_1, z_2, A]_\tau \mapsto [z_1, z_2, A]_\rho \in \mathbb{T}^2 \times_\rho M_q,$$

is well defined since $[\cdot, \cdot, \cdot]_\rho$ are the equivalence classes of the relation

$$(\lambda^m z_1, \lambda^n z_2, A) \sim (z_1, z_2, \rho_{m,n}^{-1} A),$$

and is an isomorphism. $\diamond$

The smooth sections of F form a $*$ -algebra $\Gamma^\infty(F)$ with respect to the point-wise multiplication and point-wise hermitian conjugation (of matrices). Its obvious completion is the C^* -algebra of continuous sections. With these observations we can state:

Lemma 3.2. *The map Q defined on the generators by*

$$U \mapsto \xi_U, \quad V \mapsto \xi_V, \quad (25)$$

where

$$\xi_U : \check{\mathbb{T}}^2 \rightarrow F, \quad [z_1, z_2]_\kappa \mapsto [z_1, z_2, z_1 S]_\tau,$$

$$\xi_V : \check{\mathbb{T}}^2 \rightarrow F, \quad [z_1, z_2]_\kappa \mapsto [z_1, z_2, z_2 R^{-1}]_\tau$$

extends to a $*$ -isomorphism of algebras $Q : C^\infty(\mathbb{T}_{p/q}^2) \rightarrow \Gamma^\infty(F)$.

Proof. It is straightforward to check that ξ_U and ξ_V are well defined and (25) extends to a $*$ -isomorphism due to the properties of the Fourier coefficients of a smooth function and the exchange rule $\xi_U \xi_V = \lambda \xi_V \xi_U$. $\square$

Note that F is trivial as a vector bundle but it is nontrivial as the bundle of algebras. Indeed the monomials $\xi_U^m \xi_V^n$ define a basis of $\Gamma^\infty(F)$ over $C^\infty(\check{\mathbb{T}}^2)$ and any $\xi \in \Gamma^\infty(F)$ can be written as

$$\xi = \sum_{m,n=1}^q f_{mn} \xi_U^m \xi_V^n, \quad (26)$$

where $f_{mn} \in C^\infty(\check{\mathbb{T}}^2)$, $\forall 1 \leq m, n \leq q$. By viewing the coefficients f_{mn} as a $q \times q$ matrix of functions in $C^\infty(\check{\mathbb{T}}^2)$ we can write an isomorphism of vector bundles

$$F \approx \check{\mathbb{T}}^2 \times M_q, \quad \xi([z_1, z_2]_k) \mapsto ([z_1, z_2]_k, f_{mn}([z_1, z_2]_k)).$$

However, for $\theta = p/q \neq 0$ this is not an isomorphism of algebra bundles since the multiplication of sections ξ does not correspond to the matrix multiplication of f_{mn} .

As a matter of fact, the bundle F is the bundle of all vertical (or based) endomorphisms of another complex vector bundle E of rank q . Namely, E is the orbit space of another free action of G this time on $\mathbb{T}^2 \times \mathbb{C}^q$, given by

$$(z_1, z_2, r) \mapsto (\lambda^m z_1, \lambda^n z_2, R^m S^n r),$$

where $r \in \mathbb{C}^q$. In fact, the (completed) algebras $C(\mathbb{T}_{p/q}^2)$ and $C(\check{\mathbb{T}}^2)$ are strongly Morita equivalent via the $C(\check{\mathbb{T}}^2) - C(\mathbb{T}_{p/q}^2)$ bimodule of continuous sections of E . The bundle E will not play however any role in the definition of the Hilbert space representation of $\Gamma^\infty(F)$, for which we shall employ the bundle F itself, with $\Gamma^\infty(F)$ acting on itself by left multiplication (left regular representation).

Observe also that the center of $\Gamma^\infty(F)$ is generated by $\xi_U^q = \xi_U^q$ and $\xi_V^q = \xi_V^q$ and is isomorphic to the center $\mathcal{Z}_{p/q}$ of $C^\infty(\mathbb{T}_{p/q}^2)$, and thus also to $C^\infty(\check{\mathbb{T}}^2)$, in turn identified with the G -invariant subalgebra $C^\infty(\mathbb{T}^2)^G$ of $C^\infty(\mathbb{T}^2)$, via the map that sends $\xi_U^q \mapsto z_1^q$ and $\xi_V^q \mapsto z_2^q$. Of course, over this isomorphism $\Gamma^\infty(F)$ and $C^\infty(\mathbb{T}_{p/q}^2)$ are isomorphic as modules over their centers.

The Hilbert space $L^2(F) \otimes \mathbb{C}^2$

Now we look for a Hilbert space which can serve as a codomain of the isometric extension of Q to $L^2(\mathbb{T}_{p/q}^2)$. For that define on $\Gamma(F)$ a tracial state $\mathfrak{t}_F := Q\mathfrak{t}Q^{-1}$, i.e.

$$\mathfrak{t}_F(\xi) = \int_{\check{\mathbb{T}}^2} f_{00} \quad (27)$$

for ξ as in (26), where $\int_{\check{\mathbb{T}}^2}$ is the normalized integral. The corresponding sesquilinear form reads

$$\langle \xi | \xi' \rangle = \mathfrak{t}_F(\xi^* \xi'), \quad (28)$$

where

$$\xi^* = \sum_{m,n=1}^q \bar{f}_{mn} \xi_V^{-n} \xi_U^{-m}. \quad (29)$$

We define first the Hilbert space $L^2(F)$ as the completion of $\Gamma(F)$ with respect to the norm defined by the scalar product (28). It carries a $*$ -representation of $\Gamma(F)$ by left multiplication, i.e. it is a $\Gamma(F)$ -module (and similarly for $\Gamma^\infty(F)$). Then as the full Hilbert space we take $L^2(F) \otimes \mathbb{C}^2$. Taking advantage of the (inverse) isomorphism Q given by (25) and its Hilbert space amplification we have

Lemma 3.3. *The $*$ -representation of $\Gamma^\infty(F)$ on $L^2(F) \otimes \mathbb{C}^2$ is unitarily equivalent to the $*$ -representation of $C^\infty(\mathbb{T}_{p/q}^2)$ on $L^2(\mathbb{T}_{p/q}^2) \otimes \mathbb{C}^2$.*

The Dirac operator D_F

We define the derivations algebra $\Gamma^\infty(F)$ by $\partial_\ell^F := Q\delta_\ell Q^{-1}$, so that their actions on the generators of $\Gamma^\infty(F)$ are just

$$\partial_1^F \xi_U = i\xi_U, \quad \partial_1^F \xi_V = 0, \quad \partial_2^F \xi_U = 0, \quad \partial_2^F \xi_V = i\xi_V. \quad (30)$$

The Dirac operator D_F which satisfies (2) for $D_1 = D_\theta$ and $D_2 = D_F$ is then

$$D_F = i(\sigma_1 \partial_1^F + \sigma_2 \partial_2^F). \quad (31)$$

We call a (locally defined) M_q -valued function $\tilde{\xi}$ on $\mathbb{T}^2$ local components of ξ when

$$\xi([z_1, z_2]_k) = [z_1, z_2, \tilde{\xi}(z_1, z_2)]_\tau.$$

In particular, the local components of ξ_U and ξ_V are respectively $z_1 S$ and $z_2 R^{-1}$. Next, we call $\tilde{T}$ local components of an operator T on $\Gamma^\infty(F)$ when $\tilde{T}\xi = T\xi$, and similarly for operators on $L^2(F) \otimes \mathbb{C}^2$. In particular, the local components of the differential operators ∂_ℓ^F are simply the coordinate derivatives

$$\tilde{\partial}_\ell^F = \partial_\ell. \quad (32)$$

Thus, the local components of D_F are

$$\tilde{D}_F = i(\sigma_1 \partial_1 + \sigma_2 \partial_2). \quad (33)$$

Note that $\tilde{D}_F$ looks quite like the canonical Dirac operator (21) on the torus constructed from the (flat) Levi-Civita connection of the standard metric on $\mathbb{T}^2$, and D_F in fact is unitary equivalent to (21).

The isomorphism

By using Lemmata 3.2 and 3.3 and the above discussion, we obtain:

Proposition 3.4. *The spectral triple $(\Gamma^\infty(F), L^2(F) \otimes \mathbb{C}^2, D_F)$ is isomorphic to the standard spectral triple $(C^\infty(\mathbb{T}_{p/q}^2), L^2(\mathbb{T}_{p/q}^2) \otimes \mathbb{C}^2, D_{p/q})$, where $D_{p/q}$ is given by (18) for $\theta = p/q$.*

Moreover we can equip $(\Gamma^\infty(F), L^2(F) \otimes \mathbb{C}^2, D_F)$ with a grading and real structure and enhance the isomorphism to an isomorphism of even real spectral triples.

The suitable grading χ is just given by $\text{id} \otimes \text{diag}(1, -1)$. Furthermore, it is evident that (4) holds for the following real structure:

$$J_F = -iJ_F^0 \otimes (\sigma_2 \circ c.c.), \quad (34)$$

where J_F^0 acts on a section $\xi : \mathbb{T}^2 \rightarrow F$ by hermitian conjugation, that is:

$$J_F^0 \left(\sum_{m,n=1}^q f_{mn} \xi_U^m \xi_V^n \right) = \sum_{m,n=1}^q \bar{f}_{mn} \lambda^{-mn} \xi_U^{-m} \xi_V^{-n}. \quad (35)$$

Notice that J_F admits a decomposition along the infinite and finite dimensional components of its Hilbert space:

$$J_F = J \otimes h.c. \quad (36)$$

where $J : L^2(\mathbb{T}^2, \Sigma) \otimes \mathbb{C}^2 \rightarrow L^2(\mathbb{T}^2, \Sigma) \otimes \mathbb{C}^2$ is the charge conjugation on the spinor bundle of the commutative torus and $h.c.$ denotes fiber-wise hermitian conjugation on the matrix algebra M_q .

3.3. Second isomorphic spectral triple

The algebra $\mathcal{A}_{p/q}$

Now we pass to another description of $C^\infty(\mathbb{T}_{p/q}^2)$. The starting point is the natural bijective identification of an arbitrary smooth section of the bundle F with a smooth function $\varphi : \mathbb{T}^2 \rightarrow M_q$ that is κ - ρ -equivariant, i.e. $\varphi \circ \kappa_{m,n} = \rho_{m,n} \circ \varphi$, or more explicitly

$$\varphi(\lambda^m z_1, \lambda^n z_2) = R^m S^n \varphi(z_1, z_2) S^{-n} R^{-m}, \quad (37)$$

via the algebra isomorphism

$$\varphi \mapsto \xi_\varphi,$$

where

$$\xi_\varphi : \check{\mathbb{T}}^2 \rightarrow F, \quad [z_1, z_2]_\kappa \mapsto [z_1, z_2, \varphi(z_1, z_2)]_\tau.$$

Next, we observe that a smooth function $\varphi : \mathbb{T}^2 \rightarrow M_q$ is κ - ρ -equivariant as in (37), exactly when it is invariant under the pullback of the τ -action of G , which is

$$\tau_{m,n}^*(\varphi) := \rho_{m,n} \circ \varphi_{m,n} \circ \kappa_{m,n}.$$

Furthermore, under the standard identification

$$C^\infty(\mathbb{T}^2, M_q) = C^\infty(\mathbb{T}^2) \otimes M_q \quad (38)$$

the subalgebra $C^\infty(\mathbb{T}^2, M_q)^\tau$ of τ^* -invariant functions corresponds to the subalgebra

$$\mathcal{A}_{p/q} := (C^\infty(\mathbb{T}^2) \otimes M_q)^{\kappa \otimes \rho}$$

of invariant elements under the action of the tensor product representation $\kappa \otimes \rho$ of G .

With these observations we can state:

Lemma 3.5. The map T defined on the generators by

$$U \mapsto u := z_1 \otimes S, \quad V \mapsto v := z_2 \otimes R^{-1}, \quad (39)$$

where z_ℓ is the ℓ th coordinate function on $\mathbb{T}^2$, extends to a $*$ -isomorphism from the algebra $C^\infty(\mathbb{T}_{p/q}^2)$ to the algebra $\mathcal{A}_{p/q} = (C^\infty(\mathbb{T}^2) \otimes M_q)^{\kappa \otimes \rho}$.

Proof. By a straightforward check using the properties of the Fourier coefficients of a smooth function and noting that $u = z_1 \otimes S$ and $v = z_2 \otimes R^{-1}$ are $\kappa \otimes \rho$ -invariant, unitary and satisfy $uv = \lambda vu$.

Next, it is easily seen that any element in $\mathcal{A}_{p/q}$ can be written as:

$$\sum_{r,s=0}^{q-1} f_{rs}(z_1, z_2) u^r v^s, \quad (40)$$

where, for any $(r, s) \in (\mathbb{Z}/q)^2$, $f_{rs}(z_1, z_2)$ are Schwartz functions on $\mathbb{T}^2$. Since u and v are invariant such an element is $\kappa \otimes \rho$ -invariant if and only if each f_{rs} is κ -invariant, that is defines a function on $\check{\mathbb{T}}^2$. Thus the set $\{u^m v^n\}$ for $m, n \in \mathbb{Z}$ is a basis of $\mathcal{A}_{p/q}$ over $C^\infty(\check{\mathbb{T}}_{p/q}^2)$. This shows surjectivity of T and concludes the proof. $\square$

Lemma 3.5 and its direct proof refine to smooth algebras the $*$ -isomorphism in [25] for the rational rotation algebra, where the classification theory of C^* -algebras admitting an ergodic action of $\mathbb{T}^2$ is used.

The Hilbert space $\mathcal{H}_{p/q}$

Let $\mathcal{H}_{p/q}^0 := (L^2(\mathbb{T}^2) \otimes M_q)^{\kappa \otimes \rho}$ be the Hilbert subspace of invariant elements in $L^2(\mathbb{T}^2) \otimes M_q$ under the (extension of the bounded) action of the tensor product representation $\kappa \otimes \rho$ of G , which also is the same as the obvious completion of $\mathcal{A}_{p/q}$. For the spectral triple on $\mathcal{A}_{p/q}$ we take as Hilbert space

$$\mathcal{H}_{p/q} = \mathcal{H}_{p/q}^+ \oplus \mathcal{H}_{p/q}^- = \mathcal{H}_{p/q}^0 \otimes \mathbb{C}^2, \quad (41)$$

where the superscripts $+$ and $-$ are just to mark which copy of $\mathcal{H}_{p/q}^0$ is in the ± 1 eigenspace of the grading operator $\gamma_{p/q} = \text{diag}(1, -1)$. Then, taking advantage of the (inverse) isomorphism T given by (39) and its Hilbert space amplification, it is clear that:

Lemma 3.6. The Hilbert modules $(\mathcal{A}_{p/q}, \mathcal{H}_{p/q})$ and $(C^\infty(\mathbb{T}_{p/q}^2), L^2(\mathbb{T}_{p/q}^2) \otimes \mathbb{C}^2)$ are $*$ -isomorphic.

The Dirac operator $\mathcal{D}_{p/q}$

Now we are going to select the Dirac operator $\mathcal{D}_{p/q}$ on $\mathcal{H}_{p/q}$ in such a way that (2) is satisfied for $D_1 = D_\theta$ and $D_2 = \mathcal{D}_{p/q}$. For sake of simplicity, we start with Weyl spinors (of grade $+1$) on the noncommutative torus. By linearity, it is enough to check (2) on each vector $u^m v^n = T(U^m V^n)$ of the basis, which in view of

$$(\delta_1 + i\delta_2) U^m V^n = i(m + in) U^m V^n \quad (42)$$

requires that

$$\begin{aligned} \mathcal{D}_{p/q} u^m v^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= i(m + in) u^m v^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= i(\partial_1 + i\partial_2) \otimes 1 (z^m w^n \otimes R^m S^n) \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned} \quad (43)$$

Thus the appropriate Dirac operator on $\mathcal{H}_{p/q}$ reads (modulo exchange of the tensor factors in $M_q \otimes \mathbb{C}^2$):

$$\mathcal{D}_{p/q} = \not{D} \otimes \mathbb{1}_q, \quad (44)$$

that has the usual product form (6) though with a vanishing second term and acting not on the full tensor product of Hilbert spaces but only on its subspace of $\kappa \otimes \rho$ -invariants.

The isomorphism

Similarly to the treatment of the first isomorphic spectral triple, by means of 3.5 and 3.6 and the above discussion, we obtain:

Proposition 3.7. The spectral triple $(\mathcal{A}_{p/q}, \mathcal{H}_{p/q}, \mathcal{D}_{p/q})$ is isomorphic to the standard spectral triple $(C^\infty(\mathbb{T}_{p/q}^2), L^2(\mathbb{T}_{p/q}^2) \otimes \mathbb{C}^2, D_{p/q})$, where $D_{p/q}$ is given by (18) for a fractional $\theta = p/q$.

Furthermore, this isomorphism becomes an isomorphism of even real spectral triples if we equip $(\mathcal{A}_{p/q}, \mathcal{H}_{p/q}, \mathcal{D}_{p/q})$ with the grading $\gamma_{p/q}$ as above and a real structure satisfying (4) given by the $\mathbb{C}$ -antiunitary operator

$$\mathcal{J}_{p/q} = -i\mathcal{J}_{p/q}^0 \otimes (\sigma_2 \circ c.c.), \quad (45)$$

where $\mathcal{J}_{p/q}^0$ acts by component-wise conjugation:

$$\mathcal{J}_{p/q}^0 (f \otimes A) = \bar{f} \otimes A^*. \quad (46)$$

It should be mentioned that the second isomorphic realization as a suitable subalgebra of the tensor product of two algebras of this subsection echoes the splitting homomorphism of theta deformations of the canonical spectral triple [11] (see also [12]), and in particular the spectral triple of Proposition 3.7 is a concrete example for the class of spectral triples obtained in theorem 5.14 of [7] through a refinement of the aforesaid splitting homomorphism for the specific case of Connes–Landi deformations of commutative spectral triples when the deformation is performed via a rational parameter θ . Such realization is in fact suited also to infinite dimensional “internal” algebras and overcomes the requirement that in the second description, vector bundles need to be of finite rank. This will allow us, in a future work to make connections with the Hilbert modules (see, e.g., [26]) also for the first isomorphic realization corresponding to the splitting homomorphism itself, or (dually) to ‘gluing’ a noncommutative torus along the action of the usual torus on a manifold.

4. Inequivalent spin structures

On the noncommutative torus the inequivalent spin structures correspond in a natural manner to double coverings. This is most easily seen classically since $\mathbb{T}^2$ is parallelizable. Thus the structure group of its bundle of oriented orthonormal frames can be reduced to the trivial (one element) group, and so the total space of such a reduced bundle is just a copy of $\mathbb{T}^2$ itself, with a projection on the base being the identity map. In a similar way, the whole spin structure can be reduced: the structure group $Spin(2)$ to its two element subgroup $\mathbb{Z}_2$, the total space of the principal $Spin(2)$ -bundle to a double cover of $\mathbb{T}^2$ and the spin structure map to the double covering map. The fully fledged (non-reduced) spin structure can be reconstructed from the double cover as the bundle associated with the natural action of $\mathbb{Z}_2$ on $Spin(2)$ (as a subgroup). It is also a matter of straightforward checking that two such spin structures are equivalent precisely if and only if the double coverings are equivalent.

Furthermore, in the so reduced setting the Weyl spinors are just sections of the bundle associated with the faithful representation of $\mathbb{Z}_2$ on $\mathbb{C}$, or equivalently $\mathbb{Z}_2$ -equivariant complex valued smooth functions on the double cover of $\mathbb{T}^2$, or what is the same, (-1) -eigenfunctions of the generator of $\mathbb{Z}_2$. Then, of course, the Dirac spinors are just two copies of Weyl spinors.

All that makes sense also in the noncommutative realm by working dually in terms of algebras. The appropriate language is actually that of noncommutative double coverings, interpreted as noncommutative principal $\mathbb{Z}_2$ -bundles. For our purposes this will essentially mean that we consider C^* algebras that contain $C^\infty(\mathbb{T}_{p/q}^2)$ as a subalgebra of index 2. More precisely, we formulate it as follows.

Definition 4.1. Let A and B be C^* algebras. We say that B is a noncommutative double coverings of A if B is a graded algebra $B = B^0 \oplus B^1$, such that $B^0 B^1 = B^1 = B^{1*}$, $B^1 B^1 = B^0$ and B^0 is isomorphically identified with A . Two double noncommutative coverings B and B' of A are said to be equivalent if and only if there is a $*$ -isomorphism from B to B' that is identity on A .

This definition extends easily to suitable pre- C^* algebras of B and of A , and in particular to the case of smooth noncommutative torus.

4.1. Inequivalent double coverings

In the classical case, there exist four inequivalent double coverings $c_{j,k}: \widetilde{\mathbb{T}}_{j,k}^2 \rightarrow \mathbb{T}^2$ labeled by a pair of indices j, k that take the values 0 or 1 (see Table 1).

Table 1

j, k	$\widetilde{\mathbb{T}}_{j,k}^2$	$c_{j,k}$
0, 0	$\mathbb{T}^2 \times \mathbb{Z}_2$	$(z, \pm 1) \mapsto z$
1, 0	$\mathbb{T}^2$	$(w, z) \mapsto (w^2, z)$
0, 1	$\mathbb{T}^2$	$(w, z) \mapsto (w, z^2)$
1, 1	$\mathbb{T}^2 / \mathbb{Z}_2^{diag}$	$[(w, z)] \mapsto (w^2, z^2)$

(47)

The double coverings of noncommutative torus have been studied in [18]. Just like in the classical case there exist four inequivalent double coverings, and thus four inequivalent spin structures for arbitrary parameter θ . They can be again labeled by $j, k = 0, 1$ and to simplify the notation of the double covers $C^\infty(\widetilde{\mathbb{T}}_{j,k}^2)_\theta$ we shall denote them by $C_{j,k}$ and correspondingly denote their even and odd part respectively by $C_{j,k}^0$ and $C_{j,k}^1$. The concrete form of the algebras $C_{j,k}$ is shown in the first column of Table 2.

Table 2

j, k	$C_{j,k}$	$h_{j,k}(U_\theta)$	$h_{j,k}(V_\theta)$	
0, 0	$C^\infty(\mathbb{T}_\theta^2) \otimes \mathbb{C}^2$	$U_\theta \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$V_\theta \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}$	(48)
1, 0	$C^\infty\left(\mathbb{T}_{\frac{\theta}{2}}^2\right)$	$U_{\frac{\theta}{2}}^2$	$V_{\frac{\theta}{2}}^2$	
0, 1	$C^\infty\left(\mathbb{T}_{\frac{\theta}{2}}^2\right)$	$U_{\frac{\theta}{2}}$	$V_{\frac{\theta}{2}}^2$	
1, 1	$C^\infty\left(\mathbb{T}_{\frac{\theta}{4}}^2\right)^{\mathbb{Z}'_2}$	$U_{\frac{\theta}{4}}^2$	$V_{\frac{\theta}{4}}^2$	

The corresponding embedding homomorphisms

$$h_{j,k} : C_{j,k} \hookrightarrow C^\infty(\mathbb{T}_\theta^2)$$

of $C^\infty(\mathbb{T}_\theta^2)$ as subalgebras of index 2, send the generators $U_\theta, V_\theta \in C^\infty(\mathbb{T}_\theta^2)$ to the elements listed respectively in the third and fourth columns, where we also introduce a label on the generators to indicate the parameter of the corresponding noncommutative torus. In the last row the generator of $\mathbb{Z}'_2$ acts by

$$U_{\frac{\theta}{4}}^m V_{\frac{\theta}{4}}^n \mapsto (-)^{m+n} U_{\frac{\theta}{4}}^m V_{\frac{\theta}{4}}^n$$

and thus the generators of $\mathbb{Z}'_2$ -invariant subalgebra are $U_{\frac{\theta}{4}}^m V_{\frac{\theta}{4}}^n$ with $m+n$ even.

It is not difficult to see that the isomorphic images of $C^\infty(\mathbb{T}_\theta^2)$ under the embeddings $h_{j,k}$ are the subalgebras of $\mathbb{Z}_2$ -fixed elements

$$(C_{j,k})^{\mathbb{Z}_2} = C^\infty(\mathbb{T}_\theta^2).$$

Here the generator of $\mathbb{Z}_2$ acts by

$$a \otimes \begin{pmatrix} w \\ z \end{pmatrix} \mapsto a \otimes \begin{pmatrix} z \\ w \end{pmatrix}, \text{ if } j, k = 0, 0,$$

$$U_{\frac{\theta}{2}}^m V_{\frac{\theta}{2}}^n \mapsto (-)^m U_{\frac{\theta}{2}}^m V_{\frac{\theta}{2}}^n, \text{ if } j, k = 1, 0,$$

$$U_{\frac{\theta}{2}}^m V_{\frac{\theta}{2}}^n \mapsto (-)^n U_{\frac{\theta}{2}}^m V_{\frac{\theta}{2}}^n, \text{ if } j, k = 0, 1,$$

$$U_{\frac{\theta}{4}}^m V_{\frac{\theta}{4}}^n \mapsto (-)^{mn} U_{\frac{\theta}{4}}^m V_{\frac{\theta}{4}}^n, \text{ if } j, k = 1, 1,$$

where in the last case $m+n$ is even. Note that although in the maximally twisted case the fourth root of λ is involved in $C^\infty(\mathbb{T}_{\frac{\theta}{4}}^2)$, only the square root of λ really matters in $C^\infty(\mathbb{T}_{\frac{\theta}{4}}^2)^{\mathbb{Z}'_2}$. Anyhow a kind of ‘transmutation’ occurs: the more twisted the spin structure is, the more the commutative parameter λ is involved, namely $\lambda, \lambda^{1/2}, \lambda^{1/4}$.

It turns out that the four spin structures described above are inequivalent and the only four possible if assumed projectively $\mathbb{T}^2$ -equivariant:

Proposition 4.2. *The four spin structures on the noncommutative 2-torus $\mathbb{T}_\theta^2$ for arbitrary θ represented by the noncommutative double coverings $C_{j,k}$ are pairwise inequivalent in the sense of Definition 4.1. They are the only possible if assumed to be equivariant under one of the four double coverings of $\mathbb{T}^2$ as listed in (48).*

Proof. It is straightforward to check that the lifts of $\mathbb{T}^2$ as two-parameter group of automorphisms of $C_{j,k}$ indeed form the four inequivalent usual double coverings $\widetilde{\mathbb{T}}_{j,k}^2$ of $\mathbb{T}^2$, as listed in (48). This can be most easily seen on the odd part $C_{j,k}^1$ of the algebra $C_{j,k}$. However, these double covers of $\mathbb{T}^2$ would be in fact equivalent if the noncommutative double coverings in question were equivalent, which shows the first statement.

Next assume that a double covering $B = B^0 \oplus B^1$ of the noncommutative torus algebra A admits a continuous projective action of $\mathbb{T}^2$ by automorphisms, or what is the same, a continuous action of some double covering $\widetilde{\mathbb{T}}_{j,k}^2$ of the torus group

$\mathbb{T}^2$ (15) as listed in (48). By the saturation property of B in Definition 4.1 it can be seen that each subspace of a fixed $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_2$ -grade if $(j, k) = (0, 0)$ or $\mathbb{Z} \times \mathbb{Z}$ -grade if $(j, k) = (1, 0), (0, 1)$ or $(1, 1)$, is complex one-dimensional. Clearly a non-zero element b in B^1 of $\tilde{\mathbb{T}}_{j,k}^2$ weight $(0, 0, 1 \bmod 2), (1, 0), (0, 1)$ or $(1, 1)$ respectively when $(j, k) = (0, 0), (1, 0), (0, 1)$ or $(1, 1)$ generates B^1 as a A -module. Then, the assignment to b of respectively $1 \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}, U_{\frac{\theta}{2}}, V_{\frac{\theta}{2}}$ or $U_{\frac{\theta}{4}} V_{\frac{\theta}{4}}$, extends to a $*$ -isomorphism with one of the quantum coverings listed in (49), which concludes the proof. $\square$

We note in passing that the inequivalent double coverings of the group $\mathbb{T}^2$ regarded as lifts to inequivalent spin structures of the canonical action of $\mathbb{T}^2$ on $\mathbb{T}^2$ appeared already in [16].

Some comments are now in order. We wonder if the uniqueness of the four spin structures could be shown without the equivariance assumption. The fact that they correspond to (trivial) noncommutative principal bundles with the structure group $\mathbb{Z}_2$ (or more precisely the commutative Hopf algebra $C(\mathbb{Z}_2) \approx \mathbb{C}^2$) signifies that we have to do not just with (a reduction) of merely a spin_c structure but in fact of a spin structure. This will be reflected in the next subsection via the appearance of a real structure. By constructing the related full real spectral triples the results of [27] in fact confirm that the double coverings given in (49) are the only four possible, i.e. any other one is equivalent to one of them (cf. also the comment at the end of the next subsection).

4.2. Inequivalent spectral triples

Now we shall construct a spectral triple for each spin structure. In analogy to the commutative case, Dirac operators of these spin structures are obtained by lifting the Dirac operator on the base space $\mathbb{T}_\theta^2$.

Algebra

For every spectral triple as its algebra datum we take the even part $C_{j,k}^0 \approx C^\infty(\mathbb{T}_\theta^2)$ of the algebra of functions on the covering space. Instead, as discussed above, the smooth Dirac spinors are direct sum of smooth Weyl spinors, which are just those elements of $C_{j,k}$ that change sign under the action of the generator of $\mathbb{Z}_2$, that is the elements of the odd part $C_{j,k}^1$. Therefore the space of smooth (two-component) Dirac spinors is just $C_{j,k}^1 \otimes \mathbb{C}^2$.

Hilbert space

Next to obtain a suitable Hilbert space we use the completion $\overline{C_{j,k}^1}$ of $C_{j,k}^1$ with respect to the norm as in (13), and take as square integrable Weyl spinors the elements of its odd part $C_{j,k}^1$ while as the Hilbert space of Dirac spinors we take

$$\mathcal{H}_{j,k} := \overline{C_{j,k}^1} \otimes \mathbb{C}^2.$$

Dirac operator

To construct the Dirac operator $D_{j,k}$ we start by lifting the derivations on the noncommutative torus to its double coverings, or more precisely, extending the derivations δ_ℓ , $\ell = 1, 2$ to derivations $\tilde{\delta}_\ell$ of the algebras $C_{j,k}$. In other words, we require the commutativity of the diagram:

$$\begin{array}{ccc} C_{j,k} & \xrightarrow{\tilde{\delta}_\ell} & C_{j,k} \\ \uparrow & & \uparrow \\ C^\infty(\mathbb{T}_\theta^2) & \xrightarrow{\delta_\ell} & C^\infty(\mathbb{T}_\theta^2) \end{array} \quad (49)$$

As easily seen $\tilde{\delta}_\ell$ are nothing but the usual derivations on the noncommutative tori with the modified parameters $\theta, \theta/2, \theta/2$ and $\theta/4$, respectively as in Table (49), times a factor $1/2$ every time the ℓ th entry of the spin structure label j_k equals 1. Then, the action of $\tilde{\delta}_\ell$ on $C_{j,k}^1$ extends to unbounded densely defined operators on the completions $\overline{C_{j,k}^1}$ and (diagonally in $\mathbb{C}^2$) on the Hilbert spaces $\mathcal{H}_{j,k}$.

Remark. It is not difficult to see that these operators are precisely the infinitesimal generators of the lifted two-parameter groups of automorphisms forming the four inequivalent double coverings of $\mathbb{T}^2$ as in the Proof of Proposition 4.2. $\diamond$

Next we contract these operators (extended derivations) with the Pauli matrices (Clifford multiplication) to get an operator which acts on Dirac spinors:

$$D_{j,k} = i(\sigma_1 \tilde{\delta}_1 + \sigma_2 \tilde{\delta}_2). \quad (50)$$

In Table 3 in the first column we list all $D_{j,k}$'s in terms of the usual derivations δ_1 and δ_2 defined on each covering noncommutative torus $C_{j,k}$ (with parameters as in (49)), while in the second column we report their respective spectra $\text{Spec}(D_{j,k})$ as operators on $\mathcal{H}_{j,k}$.

Table 3

j, k	D_{jk}	$\text{Spec}(D_{j,k})$	
0, 0	$i\sigma_1\delta_1 + i\sigma_2\delta_2$	$\pm\sqrt{m^2 + n^2}$	(51)
1, 0	$\frac{i}{2}\sigma_1\delta_1 + i\sigma_2\delta_2$	$\pm\sqrt{(m + \frac{1}{2})^2 + n^2}$	
0, 1	$i\sigma_1\delta_1 + \frac{i}{2}\sigma_2\delta_2$	$\pm\sqrt{m^2 + (n + \frac{1}{2})^2}$	
1, 1	$\frac{i}{2}\sigma_1\delta_1 + \frac{i}{2}\sigma_2\delta_2$	$\pm\sqrt{(m + \frac{1}{2})^2 + (n + \frac{1}{2})^2}$	

As it should, the spectral triple for the first spin structure agrees with the one given in Section 3.1, since in that case both the even and odd subspaces (of functions and of Weyl spinors) are isomorphic with the algebra $C^\infty(\mathbb{T}_\theta^2)$. Note also that for all the four inequivalent spin structures $D_{j,k}$ are isospectral deformations (have the same spectra) of the classical case $\theta = 0$. Furthermore, the four spectral triples defined above are even and real with grading and real structure defined as in Section 3.1 for an appropriate value of the parameter θ .

A few remarks are in order. For any (in particular rational) θ we recalled after [18] the four inequivalent spin structures as double coverings of the noncommutative torus in the sense of noncommutative principal bundles. For any spin structure we constructed a corresponding real spectral triple. In fact, these spectral triples have an additional property of being equivariant under the Lie algebra actions of $u(1) \approx i\mathbb{R}$ (c.f. our lifts of the derivations δ_j), or equivalently, under one of the four inequivalent double covers of the automorphism group of $\mathbb{T}^2$. Thus they fit the classification scheme of [27], where it was also shown that up to a unitary equivalence such real spectral triples are the only four possible. Since it is not difficult to see that an equivalence of double coverings would lead to a unitary equivalence of the related equivariant real spectral triples, it provides an independent proof that the four double coverings listed in (49) are the only four possible. Note however that in contrast with [27] it is claimed in [29] that some of the four spin structures can be equivalent precisely in the case of rational parameter θ , however it is unclear to us what is a relation between the notions and classifications in [27] and [29].

4.3. Isomorphic spectral triples

We now assume that $\theta = p/q$ and study isomorphic images of these spectral triples under both of the isomorphisms presented in Sections 3.2 and 3.3.

As just mentioned, the case $(j, k) = (0, 0)$ is identical to what we told in previous sections. Furthermore, for all the spin structures the algebra of smooth functions is isomorphic to $C^\infty(\mathbb{T}_\theta^2)$ and hence both to $\Gamma(F)$ and $\mathcal{A}_\theta$. Thus we shall just take care of the Hilbert spaces and Dirac operators, in the three nontrivial cases when $(j, k) \neq (0, 0)$.

In the following we let

$$\begin{aligned} G^{10} &= G^{01} = \mathbb{Z}_{2q} \times \mathbb{Z}_{2q}, & \lambda^{10} &= \lambda^{01} = e^{2\pi i \frac{p}{2q}}, \\ G^{11} &= \mathbb{Z}_{4q} \times \mathbb{Z}_{4q}, & \lambda^{11} &= e^{2\pi i \frac{p}{4q}}. \end{aligned}$$

We denote by $\check{\mathbb{T}}^2$ the quotient of $\mathbb{T}^2$ by the free action κ of G^{jk} , given by $\kappa_{m,n}(z_1, z_2) = ((\lambda^{jk})^m z_1, (\lambda^{jk})^n z_2)$, similarly as in Section 3.2.

Now let $(j, k) = (1, 0)$ and take $R, S \in M_{2q}$ as in (24) and define an action τ of G^{10} on $\mathbb{T}^2 \times M_{2q}$ given by

$$\tau_{m,n}(z_1, z_2, A) = (\lambda^m z_1, \lambda^n z_2, R^m S^n A S^{-n} R^{-m}),$$

where $A \in M_{2q}$. We let F' to be the orbit space of τ , that is a vector bundle over $\check{\mathbb{T}}^2$, with typical fiber M_{2q} . The space of functions $C_{1,0}^+$ is regarded as the $*$ -subalgebra of $\Gamma(F')$ generated by ξ_U^2 and ξ_V , where

$$\xi_U : \check{\mathbb{T}}^2 \rightarrow F', \quad [z_1, z_2]_\kappa \mapsto [z_1, z_2, z_1 S]_\tau,$$

$$\xi_V : \check{\mathbb{T}}^2 \rightarrow F', \quad [z_1, z_2]_\kappa \mapsto [z_1, z_2, z_2 R^{-1}]_\tau.$$

The odd subalgebra $C_{1,0}^-$ is isomorphic to the linear span of $U^{2m+1} V^n_{(m,n) \in \mathbb{Z}^2}$ with coefficients f_{mn} which are smooth functions on the torus. The Hilbert space of Weyl spinors is isomorphic with its closure with respect to the norm defined by the scalar product

$$(g, f) = \sum_{(m,n) \in \mathbb{Z}^2} \int_{\check{\mathbb{T}}^2} \bar{g}_{-m-n} f_{mn}. \quad (52)$$

The Hilbert space $\mathcal{H}_{1,0}$ of Dirac spinors is as usual a direct sum of two Hilbert spaces of Weyl spinors.

Finally, with $\partial_1^F, \partial_2^F$ defined in (31), the Dirac operator $D_{1,0}$ is unitarily equivalent to:

$$D_{1,0}^F = i(\sigma_1 \partial_1^F + \sigma_2 \partial_2^F). \quad (53)$$

Concerning the second isomorphic spectral triple, we have to regard $\mathcal{A}_{\frac{p}{q}}$ as the subalgebra of $\mathcal{A}_{\frac{p}{2q}}$ generated by $u_{1,0} = z_1^2 \otimes S$ and $v_{1,0} = z_2 \otimes R^{-1}$, while the Hilbert space $\mathcal{H}_{10}$ is isomorphic to its orthogonal complement, completed with respect to the scalar product defined by

$$(g, f) = \sum_{(m,n) \in \mathbb{Z}^2} \bar{g}_{-2m-n-1} f_{2m+n+1}. \quad (54)$$

Then the Dirac operator D_{10} is unitarily equivalent to the restriction of $\mathcal{D}_{p/q} = \not{D} \otimes \mathbb{1}_q$ to $\mathcal{H}_{10} \otimes \mathbb{C}^2$ (modulo the exchange of the tensor factors as in (44)).

Both these descriptions are similar for the spin structure $(0, 1)$ provided that one exchanges the roles of U and V , while for the fourth spin structure $(1, 1)$ one has to replace every M_{2q} by M_{4q} and repeat the constructions above.

5. Curved rational noncommutative torus

So far, we established isomorphisms between spectral triples by selecting Dirac operators, both for the first and the second case, which act trivially as the identity on the finite part of the respective Hilbert spaces. In order to push further the analogy with almost-commutative manifolds, hence with standard model of particles [2,28], it would be interesting to get a Dirac operator whose action on the “internal” degrees of freedom is non trivial. In quantum field theory, internal degrees of freedom of a single, isolated fermion change whenever it moves in a space–time region with a gauge field (e.g. electromagnetic) whose field strength is different from zero. From a more mathematical point of view [21,22,28], having a non zero field strength corresponds to consider a Dirac operator which contains a non flat connection acting on the finite part of the Hilbert space. Roughly speaking, to have a Dirac operator which changes internal degrees of freedom of particles, one should consider the one that describes the analogue of a curved geometry of the internal directions.

Thus we are going to investigate some curved geometries on the rational noncommutative torus in view of the generalizations of the spectral triple on $\mathbb{T}_{p/q}^2$ to the case of spectral triples on a non necessarily flat topologically non-trivial almost-commutative manifold. The study of curvature on noncommutative torus was initiated in [15], where conformal rescaling of the standard Dirac operator are considered, that have been later generalized to arbitrary conformal class in [23]. A different perspective, that we are going to adopt, can be found in [19] and [20], where the kind of perturbations employed preserves the boundedness of the commutator of the Dirac operators with any element of the algebra and relies on the real structure J of the noncommutative torus.

For simplicity we focus on the case of trivial spin structure but the other ones can be dealt with analogously. In principle we would like to introduce wide class of perturbations that include the rescalings with a conformal factor in $J\mathcal{C}^\infty(\mathbb{T}_\theta^2)J^{-1}$ and the transformations studied in [19] and [20]

$$D^{(k)} = i \sum_{j,\ell=1,2} \sigma^j k_j^\ell \delta_\ell k_j^{\ell'} + h.c., \quad (55)$$

where $k_j^\ell, k_j^{\ell'} \in J\mathcal{C}^\infty(\mathbb{T}_{p/q}^2)J^{-1}$ for $j, \ell \in \{1, 2\}$ are assumed such that $D^{(k)}$ has compact resolvent. With this choice of $k_j^\ell, k_j^{\ell'}$ the operator $D^{(k)}$ maintains the bounded commutators with the algebra elements, while the choice of $k_j^\ell, k_j^{\ell'}$ from the algebra $\mathcal{C}^\infty(\mathbb{T}_{p/q}^2)$ would lead to a spectral triple with twisted commutators.

To see whether for rational $\theta = p/q$ the counterparts of $D^{(k)}$ on $L^2(F) \otimes \mathbb{C}^2$ and $\mathcal{H}_{p/q}$ will present some non trivial action on the finite part of the respective Hilbert spaces we consider in the following two simple examples the special case

$$D^k = i\sigma^1 \delta_1 + i\sigma^2 k \delta_2, \quad \text{where } 0 < k \in JAJ. \quad (56)$$

Example 5.1. Let $k = J(U + U^*)J^{-1} + t$, where $t > 2$. The action of D^k on the basis of left handed Weyl spinors is given by

$$D^k \begin{pmatrix} U^m V^n \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ D^{k^+}(U^m V^n) \end{pmatrix}, \quad (57)$$

where

$$D^{k^+}(U^m V^n) = mU^m V^n + n(\lambda U^{m+1} V^n + \lambda^{-1} U^{m-1} V^n + tU^m V^n), \quad (58)$$

whose isomorphic image under a similarity with the map Q given by (25) is given by

$$D_F^{k^+}(\xi_U^m \xi_V^n) = m\xi_U^m \xi_V^n + n(\lambda \xi_U^{m+1} \xi_V^n + \lambda^{-1} \xi_U^{m-1} \xi_V^n + t\xi_U^m \xi_V^n). \quad (59)$$

Hence the operator D^{k^+} is unitarily equivalent to

$$D_F^{k^+} = i\partial_1^F + A\partial_2^F : L^2(F) \rightarrow L^2(F), \quad (60)$$

where $A \in \Gamma^\infty(\text{End}(F))$ has local components given by

$$\tilde{A}(z_1, z_2) = \lambda z_1 S + \lambda^{-1} z_1^{-1} S^{-1} + t \mathbb{1}_q. \quad (61)$$

From this Example we see that in general the operators $D_F^{(k)}$ on the Hilbert space $L^2(F) \otimes \mathbb{C}^2$ and $\mathcal{D}_{p/q}^{(k)}$ on $\mathcal{H}_{p/q}$, do not admit the decomposition $D_1 \otimes \mathbb{1}_q + \chi_1 \otimes D_2$. However this is not a surprise since even classically a product manifold need not have such a structure on the metric level. Indeed in the noncommutative setting both for the description of the spectral triple on the associated vector bundle F and for the description as subalgebra of $C^\infty(\mathbb{T}^2) \otimes M_q$, topologically non-trivial almost-commutative spectral triples have been modeled on the invariant elements of the respective algebras under some action of $\mathbb{Z}_q \times \mathbb{Z}_q$ which intertwines the external and internal degrees of freedom. Thus, for these spectral triples, we are not able to write every admissible Dirac operator by simply joining together a Dirac operator on the standard spectral triple of the commutative torus with a Dirac operator on the finite space M_q .

Next we use an element k in the center of $C^\infty(\mathbb{T}_{p/q}^2)$ to transform the Dirac operator.

Example 5.2. Let $k = U^q + U^{-q} + t$, where $t > 2$. We consider again the action of D^k on a basis of left-handed Weyl spinor as in the previous example and similar computations lead to

$$D_F^{k+} \xi_U^m \xi_V^n = m \xi_U^m \xi_V^n + n(\xi_U^{m+q} \xi_V^n + \xi_U^{m-q} \xi_V^n + t \xi_U^m \xi_V^n). \quad (62)$$

Hence, with the exchange of tensor factors as in (44), D^k is unitarily equivalent to

$$D_F^k = i\sigma^1 \partial_1^F + A \sigma^2 \partial_2^F : L^2(F) \otimes \mathbb{C}^2 \rightarrow L^2(F) \otimes \mathbb{C}^2, \quad (63)$$

where $A \in \Gamma^\infty(\text{End}(F \otimes \mathbb{C}^2))$ has local components given by

$$\tilde{A}(z_1, z_2) = z_1^q + z_1^{-q} + t. \quad (64)$$

This Example shows that the situation is different if we assume that in (55) the elements k_j^ℓ and $k_j^{\ell'}$ belong to the center $\mathcal{Z}_{p/q}$ of $C^\infty(\mathbb{T}_{p/q}^2)$ (so in fact to the center of $J C^\infty(\mathbb{T}_{p/q}^2) J^{-1}$ too). We prove the following proposition for transformations (55) of the isomorphic spectral triple $(\Gamma(F), L^2(F) \otimes \mathbb{C}^2, D_F)$, the proof for the corresponding transformations of the spectral triple $(\mathcal{A}_{p/q}, \mathcal{H}_{p/q}, \mathcal{D}_{p/q})$ is similar. We adopt the notation of Section 3.2 for the local components of various partial differential operators.

Proposition 5.3. Let $k_j^\ell, k_j^{\ell'} \in \mathcal{Z}_{p/q}$ and let $\check{k}_j^\ell, \check{k}_j^{\ell'} \in C^\infty(\mathbb{T}^2)^G$ be the corresponding elements via the composed isomorphism $\mathcal{Z}_{p/q} = C^\infty(\check{\mathbb{T}}^2) = C^\infty(\mathbb{T}^2)^G$ that sends $U^q \mapsto z_1^q$ and $V^q \mapsto z_2^q$. The isomorphic image $D_F^{(k)}$ of the operator $D^{(k)}$ defined as in (55) has a local expression with vanishing second term:

$$\widetilde{D_F^{(k)}} = \mathcal{D}^{(\check{k})} \mathbb{1}_q, \quad (65)$$

where

$$\mathcal{D}^{(\check{k})} = i \sum_{j,\ell=1}^2 \sigma^j \check{k}_j^\ell \partial_\ell \check{k}_j^{\ell'} + h.c. \quad (66)$$

is the accordingly transformed canonical Dirac operator on the classical torus $\mathbb{T}^2$.

Proof. If k_j^ℓ and $k_j^{\ell'}$ are in the center of $C^\infty(\mathbb{T}_{p/q}^2)$, then their isomorphic images regarded as (scalar) multiplication operators on $L^2(F) \otimes \mathbb{C}^2$ have local expressions given by $\check{k}_j^\ell$ and $\check{k}_j^{\ell'}$. The statement then follows by (32). $\square$

This result demonstrates that modifications of the Dirac operator on the rational noncommutative torus by the central elements exhibit in the second description (in terms of bundle F) only the ‘external’ part given by the usual flat differential Dirac operator appropriately modified by the corresponding functions on the classical torus, while the ‘internal’ part remains trivial. This of course applies also to the third description.

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