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On a semi-quasi-Einstein manifold☆

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ABSTRACT

In the present paper we introduce a semi-quasi-Einstein manifold from a semi symmetric metric connection. Among others, the popular Schwarzschild and Kottler spacetimes are shown to possess this structure. Certain curvature conditions are studied in such a manifold with a Killing generator.

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1. Introduction

Ricci curvature plays a significant role both in geometry and in the theory of gravity. If the Ricci curvature of some space satisfies $S(X,Y) = ag(X,Y)$ for scalar a, we call it Einstein manifold as such a space satisfies the Einstein's field equations in vacuum. Einstein space is naturally generalized to a wider range, such as quasi-Einstein space [3], generalized quasi-Einstein space [2], mixed quasi-Einstein space and so on, by adding some curvature constraints to the Ricci tensor.

A quasi-Einstein manifold is the closest and simplest approximation of the Einstein manifold. It evolved as an exact solution of Einstein's field equations and also in geometry while studying quasi-umbilical hypersurfaces. A non-flat Riemannian manifold $(M^n, g)(n > 2)$ is defined to be a quasi-Einstein manifold if the Ricci tensor is not identically zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \forall X, Y \in TM$$

for some scalar functions $a, b \neq 0$, where η is a non-zero 1-form such that $g(X,\xi) = \eta(X)$, η is called the associated 1-form and ξ is called the generator of the manifold.

It is also known that the perfect fluid space–time in general relativity is a 4-dimensional semi-Riemannian quasi-Einstein space. The study of these space–time models can help us better understand the evolution of the universe.

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Quasi-Einstein manifold was defined by M. C. Chaki and R. K. Maity [3], later it was widely studied in [1,4–7,9,13,15]. Z.L. Li [12] discussed the basic properties of quasi-Einstein manifold, obtained some geometric characteristics and proved the non-existence of some quasi-Einstein manifolds. U.C. De and G.C. Ghosh [5] proved a theorem for existence of a QE-manifold and give some examples about QE-manifolds. V.A. Kiosak [10] considered the conformal mappings of quasi-Einstein spaces and proved that they are closed with respect to concircular mappings. M.M. Tripathi and J.S. Kim [17] studied a quasi-Einstein manifold whose generator belongs to the k-nullity distribution $N(k)$ and have proved that conformally quasi-Einstein manifolds are certain $N(k)$-quasi Einstein manifolds.

The purpose of this paper is to investigate quasi-Einstein manifolds from a rather different perspective. By replacing the Levi-Civita connection with a semi-symmetric metric connection [18], we define a semi-quasi-Einstein manifold which generalizes the concept of quasi-Einstein manifolds. Semi-symmetric metric connections are widely used to modify Einstein's gravity theory in form of Einstein–Cartan gravity and also in unifying the gravitational and electromagnetic forces. Recently in [8] the field equations are derived from an action principle which is formed by the scalar curvature of a semi-symmetric metric connection. The derived equations contain the Einstein and Maxwell equations in vacuum. In fact, Schouten criticized Einstein's argument for using a symmetric connection and used the notion of semi-symmetric connections in his approach [16].

We further prove the existence of such structure by analyzing the Schwarzschild and Kottler spacetimes and obtain a necessary and sufficient conditions for such a manifold to be an Einstein space.

In [14] Murathan and Ozgur considered the semi-symmetric metric connection with unit parallel vector field P, and proved that $R.\bar{R} = 0$ if and only if M is semi-symmetric; if $R.\bar{R} = 0$ or $R.\bar{R} - \bar{R}.R = 0$ or M is semi-symmetric and $\bar{R}.\bar{R} = 0$, then M is conformally flat and quasi-Einstein. Motivated by this we further investigate the manifolds for Ricci symmetric and Ricci semi-symmetric criterion under similar ambience.

The paper is organized as follows: Section 2 recalls the form and some curvature properties of semi-symmetric metric connection. Section 3 gives the main definition of semi-quasi-Einstein manifold ($S(QE^n)$) and discusses the relations between $S(QE^n)$, quasi-Einstein and generalized quasi-Einstein manifolds. Section 4 concerns with some physical and geometric characteristics of $S(QE^n)$ manifold under certain curvature conditions. Some interesting examples of $S(QE^n)$ manifold are constructed in Section 5.

2. Preliminaries

Let (M^n, ∇) be a Riemannian manifold of dimension n and ∇ be the Levi-Civita connection compatible to the metric g. A semi-symmetric metric connection is defined by

$$\bar{\nabla}_X Y = \nabla_X Y + \pi(Y)X - g(X,Y)P, \quad \forall X, Y \in TM$$

where π is any given 1-form and P is the associated vector field, $g(X,P) = \pi(X)$.

By direct computations, one can obtain

$$\begin{aligned}
\bar{R}(X,Y)Z &= R(X,Y)Z + g(AX,Z)Y - g(AY,Z)X \\
&\quad + g(X,Z)AY - g(Y,Z)AX,
\end{aligned}$$

$$\begin{aligned}
g(AY,Z) &= (\nabla_Y \pi)(Z) - \pi(Y)\pi(Z) + \frac{1}{2}\pi(P)g(Y,Z) \\
&= g(\nabla_Y P, Z) - \pi(Y)\pi(Z) + \frac{1}{2}\pi(P)g(Y,Z),
\end{aligned}$$

$$g(\bar{\nabla}_Y P, Z) - g(\bar{\nabla}_Z P, Y) = g(\nabla_Y P, Z) - g(\nabla_Z P, Y)$$

$$\begin{aligned}
\bar{S}(Y,Z) &= S(Y,Z) - (n-2)(\nabla_Y \pi)(Z) + (n-2)\pi(Y)\pi(Z) \\
&\quad - \{(n-2)\pi(P) + \sum_i g(\nabla_i P, e_i)\}g(Y,Z),
\end{aligned} \tag{2.1}$$

further one has

$$\begin{aligned}
\bar{R}(X,Y)Z &= -\bar{R}(Y,X)Z, \\
\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y &= (g(AZ,Y) - g(AY,Z))X \\
&\quad + (g(AX,Z) - g(AZ,X))Y \\
&\quad + (g(AY,X) - g(AX,Y))Z.
\end{aligned}$$

It is obvious that the Ricci tensor of the semi-symmetric metric connection is not symmetric unless under certain conditions. For example, if π is closed, then A is symmetric, thus $\bar{S}$ is also symmetric.

In particular, if a unit P satisfies Killing's equation $g(\nabla_X P, Y) + g(\nabla_Y P, X) = 0$, then

$$g(\nabla_P P, Y) = 0, \quad g(\nabla_X P, X) = 0, \quad AX = \nabla_X P - \pi(X)P + \frac{1}{2}X,$$
$$\bar{R}(X, Y)P = R(X, Y)P + \pi(Y)\nabla_X P - \pi(X)\nabla_Y P$$
$$\bar{R}(X, P)Y = R(X, P)Y + g(\nabla_X P, Y) - \pi(Y)\nabla_X P,$$
$$\bar{R}(X, P)P = R(X, P)P + \nabla_X P,$$
$$\bar{S}(Y, Z) = S(Y, Z) - (n-2)g(\nabla_Y P, Z) + (n-2)\pi(Y)\pi(Z) - (n-2)g(Y, Z).$$
$$\bar{S}(Y, P) = S(Y, P) = S(P, Y) = \bar{S}(P, Y).$$

And if P is a unit parallel vector field with respect to ∇,

$$g(AX, Y) = g(AY, X), \quad AX = -\pi(X)P + \frac{1}{2}X,$$
$$\bar{R}(X, Y, Z, W) = -\bar{R}(X, Y, W, Z),$$
$$\bar{R}(X, Y)Z + \bar{R}(Y, Z)X + \bar{R}(Z, X)Y = 0,$$
$$R(P, X, Y, Z) = R(X, Y, P, Z) = 0,$$
$$\bar{R}(X, Y)P = R(X, Y)P = 0,$$
$$\bar{R}(P, X, Y, Z) = \bar{R}(X, Y, P, Z) = 0,$$
$$\bar{S}(Y, Z) = S(Y, Z) + (n-2)\pi(Y)\pi(Z) - (n-2)\pi(P)g(Y, Z) = \bar{S}(Z, Y),$$
$$\bar{S}(Y, P) = S(Y, P) = 0.$$

3. Semi-quasi-Einstein manifolds

In this section, we define a new structure using the semi-symmetric metric connection.

Definition 3.1. A Riemannian manifold $(M^n, \bar{\nabla})$, $n = dimM \geq 3$, is said to be a semi-quasi-Einstein manifold $S(QE^n)$ if the Ricci curvature tensor components are non-zero and of the forms

$$\hat{S}(Y, Z) = sym\bar{S}(Y, Z) = ag(Y, Z) + b\eta(Y)\eta(Z),$$

where a and b are scalars and η is a non-zero 1-form. If $\hat{S}(Y, Z)$ is identically zero, we say the manifold $(M^n, \bar{\nabla})$ is semi-Ricci flat.

Theorem 3.1. *If the generator P of manifold $(M^n, \bar{\nabla})$ is a Killing vector field with respect to ∇, then an Einstein manifold (M^n, ∇) is a $S(QE^n)$manifold.*

Proof. In view of the relation (2.1),

$$\begin{aligned}\hat{S}(Y, Z) &= \frac{1}{2}\{\bar{S}(Y, Z) + \bar{S}(Z, Y)\} \\ &= S(Y, Z) - (n-2)[(\nabla_Y \pi)(Z) + (\nabla_Z \pi)(Y)] + (n-2)\pi(Y)\pi(Z) \\ &\quad - \{(n-2)\pi(P) + \sum_i g(\nabla_i P, e_i)\}g(Y, Z).\end{aligned}$$

Notice that $(\nabla_Y \pi)(Z) + (\nabla_Z \pi)(Y) = g(\nabla_Y P, Z) + g(\nabla_Z P, Y)$, if P is a Killing vector field with respect to ∇, then one has

$$\hat{S}(Y, Z) = S(Y, Z) + (n-2)\pi(Y)\pi(Z) - (n-2)\pi(P)g(Y, Z). \tag{3.1}$$

Therefore, if (M^n, ∇) is Einstein, then $(M^n, \bar{\nabla})$ is a $S(QE^n)$manifold. □

Remark 3.1. The above result is also true if we replace the Einstein manifold with a Ricci-flat manifold.

Remark 3.2. If the generator P is a unit parallel vector, $\hat{S}(X, Y) = \bar{S}(X, Y)$.

Theorem 3.2. *For a unit vector field P, $(M^n, \bar{\nabla})$ is a $S(QE^n)$ manifold, if for some scalar ρ*

$$\hat{S}(Y, Z)\hat{S}(X, W) - \hat{S}(X, Z)\hat{S}(Y, W) = \rho(g(Y, Z)g(X, W) - g(X, Z)g(Y, W)). \tag{3.2}$$

Proof. Taking $X = W = P$ in (3.2) we have

$$\hat{S}(Y, Z)\hat{S}(P, P) - \hat{S}(P, Z)\hat{S}(Y, P) = \rho\{g(Y, Z) - g(P, Y)g(P, Z)\}$$

We denote by $\alpha = \hat{S}(P,P)$, $\hat{S}(P,Z) = g(\hat{Q}Z,P) = \pi(\hat{Q}Z) = \eta(Z)$, then

$$\hat{S}(Y,Z) = \frac{\rho}{\alpha}g(Y,Z) - \frac{\rho}{\alpha}\pi(Y)\pi(Z) + \frac{1}{\alpha}\eta(Y)\eta(Z). \tag{3.3}$$

On the other hand, putting $X = P$ in (3.2) we have

$$\hat{S}(Y,Z)\eta(W) - \hat{S}(Y,W)\eta(Z) = \rho\{g(Y,Z)\pi(W) - g(Y,W)\pi(Z)\} \tag{3.4}$$

Substituting (3.3) into (3.4) we obtain

$$\begin{aligned}&\frac{\rho}{\alpha}\{g(Y,Z)\eta(W) - g(Y,W)\eta(Z)\} - \frac{\rho}{\alpha}\{\eta(W)\eta(Z)\pi(Y) - \pi(W)\pi(Y)\eta(Z)\}\\ &= \rho\{g(Y,Z)\pi(W) - g(Y,W)\pi(Z)\}\end{aligned} \tag{3.5}$$

Now in (3.5) , put $Y = Z = e_i$ and take sum about i, we get

$$\eta(W) = \alpha\pi(W) \tag{3.6}$$

we further obtain by substituting (3.6) into (3.3)

$$\hat{S}(Y,Z) = \frac{\rho}{\alpha}g(Y,Z) + (\alpha - \frac{\rho}{\alpha})\pi(Y)\pi(Z).$$

The proof is finished. □

4. $(M^n, \bar{\nabla})$-manifold with unit generator

In this section we consider a $(M^n, \bar{\nabla})$-manifold with unit generator $\|P\| = 1$.

Definition 4.1. $(M^n, \bar{\nabla})$ is said to be $\bar{\nabla}$-symmetric if the curvature tensor satisfies $\bar{\nabla}\bar{R} = 0$; $\bar{\nabla}$-Ricci symmetric if $\bar{\nabla}\hat{S} = 0$; semi-symmetric if $\bar{R}\cdot\bar{R} = 0$; Ricci semi-symmetric if $\bar{R}\cdot\hat{S} = 0$.

If H is a $(0,k)$-type tensor field, we define the operation $R.H$ by

$$\begin{aligned}(R(X,Y)\cdot H)(W_1,\ldots,W_k) = &-H(R(X,Y)W_1,\ldots,W_k)\\ &-\cdots - H(W_1,\ldots,R(X,Y)W_k).\end{aligned}$$

If θ is a symmetric $(0,2)$-type tensor, the following formula can define a $(0,k+2)$-type tensor

$$\begin{aligned}Q(\theta,H)(W_1,\ldots,W_k;X,Y) = &-H((X\wedge_\theta Y)W_1,\ldots,W_k)\\ &-\cdots - H(W_1,\ldots,(X\wedge_\theta Y)W_k),\end{aligned} \tag{4.1}$$

where $X\wedge_\theta Y$ is given by $(X\wedge_\theta Y)Z = \theta(Y,Z)X - \theta(X,Z)Y$.

When P is a unit Killing field, based on the relation (3.1) we get

$$\hat{S}(Y,Z) = S(Y,Z) + (n-2)\pi(Y)\pi(Z) - (n-2)g(Y,Z).$$

By direct calculations we obtain

$$\begin{aligned}(\bar{\nabla}_X\hat{S})(Y,Z) = &(\nabla_X S)(Y,Z) + (n-2)\{(\nabla_X\pi)(Y)\pi(Z) + (\nabla_X\pi)(Z)\pi(Y)\}\\ &-\pi(Y)[S(X,Z) + (n-2)\pi(X)\pi(Z) - (n-2)g(X,Z)]\\ &-\pi(Z)[S(Y,X) + (n-2)\pi(X)\pi(Y) - (n-2)g(X,Y)]\\ &+\pi(QY)g(X,Z) + \pi(QZ)g(X,Y)\\ = &(\nabla_X S)(Y,Z)\\ &+g(X,Z)\bar{S}(Y,P) - g(Y,P)\bar{S}(X,Z) + g(X,Y)\bar{S}(Z,P) - g(Z,P)\bar{S}(X,Y)\end{aligned} \tag{4.2}$$

which derives the following:

Theorem 4.1. *If the generator P is unit Killing, (M^n,∇) is Ricci symmetric if and only if*

$$(\bar{\nabla}_X\hat{S})(Y,Z) = g(X,Z)\bar{S}(Y,P) - g(Y,P)\bar{S}(X,Z) + g(X,Y)\bar{S}(Z,P) - g(Z,P)\bar{S}(X,Y).$$

Corollary 4.2. *Let P be a unit parallel vector field, then (M^n,∇) is Ricci symmetric if and only if*

$$(\bar{\nabla}_X\hat{S})(Y,Z) = -\pi(Y)\bar{S}(X,Z) - \pi(Z)\bar{S}(X,Y).$$

Alternately, for a specific form of the Ricci curvature $S(X, Y)$ we have the following:

Theorem 4.3. *Let $S(Y, Z) = (n-2)g(Y, Z) - (n-2)\pi(Y)\pi(Z)$, $\|P\| = 1$. If the manifold (M^n, ∇) is Ricci symmetric then $(M^n, \bar{\nabla})$ is $\bar{\nabla}$-Ricci symmetric.*

Proof. Let $S(Y, Z) = (n-2)g(Y, Z) - (n-2)\pi(Y)\pi(Z)$ and $(\nabla_X S)(Y, Z) = 0$. These together imply $(\nabla_X \pi)(Y)\pi(Z) + (\nabla_X \pi)(Z)\pi(Y) = 0$ which means $\nabla P = 0$ for a unit vector field P. Hence, $\bar{S}(Y, P) = 0$. Therefore, using (4.2) we conclude

$$\begin{aligned}(\bar{\nabla}_X \hat{S})(Y, Z) = (\nabla_X S)(Y, Z) &- \pi(Y)[S(X, Z) + (n-2)\pi(X)\pi(Z) - (n-2)g(X, Z)] \\ &- \pi(Z)[S(Y, X) + (n-2)\pi(X)\pi(Y) - (n-2)g(X, Y)]\end{aligned}$$

Hence the result. □

Theorem 4.4. *In general, if $(M^n, \bar{\nabla})$ is a $\bar{\nabla}$-Ricci symmetric $S(QE^n)$ manifold, $\hat{S}(X, Y) = ag(X, Y) + b\eta(X)\eta(Y)$ with a unit vector ξ associated to the one-form η, then $a + b =$ constant.*

Proof. We have

$$\begin{aligned}(\bar{\nabla}_X \hat{S})(Y, Z) = {} & X(a)g(Y, Z) + X(b)\eta(Y)\eta(Z) \\ & + b[(\bar{\nabla}_X \eta)(Y)\eta(Z) + (\bar{\nabla}_X \eta)(Z)\eta(Y)].\end{aligned} \tag{4.3}$$

Putting $Y = Z = \xi$ in (4.3), we finish the proof. □

Theorem 4.5. *Let the generator P in $(M^n, \bar{\nabla})$ be a unit parallel vector field and $\bar{R} \cdot \bar{S} = -Q(g, \bar{S})$, if (M^n, ∇) is Ricci semi-symmetric then M is a quasi-Einstein manifold of the form*

$$S(X, Y) = (n-2)g(X, Y) - (n-2)\pi(X)\pi(Y).$$

Conversely, if the Ricci curvature satisfies $S(X, Y) = (n-2)g(X, Y) - (n-2)\pi(X)\pi(Y)$, then M is Ricci semi-symmetric.

Proof. We have

$$\begin{aligned}-(\bar{R}(X, Y) \cdot \bar{S})(Z, W) &= \bar{S}(\bar{R}(X, Y)Z, W) + \bar{S}(Z, \bar{R}(X, Y)W) \\ &= S(R(X, Y)Z, W) + S(Z, R(X, Y)W) \\ &\quad + g(AX, Z)S(Y, W) + g(AX, W)S(Y, Z) \\ &\quad - g(AY, Z)S(X, W) - g(AY, W)S(X, Z) \\ &\quad + g(X, Z)S(AY, W) + g(X, W)S(AY, Z) \\ &\quad - g(Y, Z)S(AX, W) - g(Y, W)S(AX, Z) \\ &\quad + (n-2)[\pi(W)\bar{R}(X, Y, Z, P) + \pi(Z)\bar{R}(X, Y, W, P)] \\ &= -(R(X, Y) \cdot S)(Z, W) + Q(g, S)(Z, W; X, Y) \\ &\quad + \pi(Y)\pi(W)S(X, Z) - \pi(X)\pi(W)S(Y, Z) \\ &\quad + \pi(Y)\pi(Z)S(X, W) - \pi(X)\pi(Z)S(Y, W) \\ &= -(R(X, Y) \cdot S)(Z, W) + Q(g, \bar{S})(Z, W; X, Y) \\ &\quad + [S(X, Z) - (n-2)g(X, Z)]\pi(Y)\pi(W) \\ &\quad - [S(Y, Z) - (n-2)g(Y, Z)]\pi(X)\pi(W) \\ &\quad + [S(X, W) - (n-2)g(X, W)]\pi(Y)\pi(Z) \\ &\quad - [S(Y, W) - (n-2)g(Y, W)]\pi(X)\pi(Z),\end{aligned}$$

where the last equality follows from

$$\begin{aligned}Q(g, \bar{S})(Z, W; X, Y) &= -\bar{S}((X \wedge Y)Z, W) - \bar{S}(Z, (X \wedge Y)W) \\ &= Q(g, S)(Z, W; X, Y) + (n-2)[g(X, Z)\pi(Y)\pi(W) \\ &\quad - g(Y, Z)\pi(X)\pi(W) + g(X, W)\pi(Y)\pi(Z) \\ &\quad - g(Y, W)\pi(X)\pi(Z)].\end{aligned}$$

Since $\bar{R} \cdot \bar{S} = -Q(g, \bar{S})$, we have

$$\begin{aligned}(R(X, Y) \cdot S)(Z, W) = {} & [S(X, Z) - (n-2)g(X, Z)]\pi(Y)\pi(W) \\ & - [S(Y, Z) - (n-2)g(Y, Z)]\pi(X)\pi(W)\end{aligned}$$

$$+ [S(X,W) - (n-2)g(X,W)]\pi(Y)\pi(Z)$$
$$- [S(Y,W) - (n-2)g(Y,W)]\pi(X)\pi(Z). \tag{4.4}$$

Since $R \cdot S = 0$, using $Y = W = P$ in the above equation we get the result. The converse part is quite obvious at this point. □

5. Some examples

Example 5.1. The first example we consider is the popular four dimensional Schwarzschild spacetime M with the Ricci-flat metric

$$g = \left(\frac{2m}{r} - 1\right) \mathrm{d}t \otimes \mathrm{d}t + \left(-\frac{1}{\frac{2m}{r} - 1}\right) \mathrm{d}r \otimes \mathrm{d}r + r^2 \mathrm{d}\theta \otimes \mathrm{d}\theta + r^2 \sin(\theta)^2 \, \mathrm{d}\phi \otimes \mathrm{d}\phi,$$

and a Killing vector $\frac{\partial}{\partial t}$ in this spacetime.

The corresponding nontrivial Levi-Civita connection components are:

$$\begin{cases} \Gamma^{t}_{tr} = -\frac{m}{2mr-r^2}, \Gamma^{r}_{tt} = -\frac{2m^2-mr}{r^3}, \Gamma^{r}_{rr} = \frac{m}{2mr-r^2}, \Gamma^{r}_{\theta\theta} = 2m - r, \Gamma^{\phi}_{\theta\phi} = \frac{\cos(\theta)}{\sin(\theta)}, \\ \Gamma^{r}_{\phi\phi} = (2m - r)\sin(\theta)^2, \Gamma^{\theta}_{r\theta} = \frac{1}{r}, \Gamma^{\theta}_{\phi\phi} = -\cos(\theta)\sin(\theta), \Gamma^{\phi}_{r\phi} = \frac{1}{r}. \end{cases}$$

We define a semi-symmetric metric connection corresponding to the one-form $\pi = (\frac{2m-r}{r})dt$ associated to the Killing vector $\frac{\partial}{\partial t}$ by

$$\bar{\Gamma}^{k}_{ij} = \Gamma^{k}_{ij} + \pi_j \delta^{k}_{i} - g_{ij}\pi^{k}, \pi^{k} = g^{ik}\pi_i.$$

We calculate the nontrivial components as:

$$\begin{cases} \bar{\Gamma}^{t}_{tr} = -\frac{m}{2mr-r^2}, \bar{\Gamma}^{t}_{rt} = -\frac{m}{2mr-r^2}, \bar{\Gamma}^{t}_{rr} = \frac{r}{2m-r}, \bar{\Gamma}^{t}_{\theta\theta} = -r^2, \bar{\Gamma}^{t}_{\phi\phi} = -r^2\sin(\theta)^2, \\ \bar{\Gamma}^{r}_{tt} = -\frac{2m^2-mr}{r^3}, \bar{\Gamma}^{r}_{rt} = \frac{2m-r}{r}, \bar{\Gamma}^{r}_{rr} = \frac{m}{2mr-r^2}, \bar{\Gamma}^{r}_{\theta\theta} = 2m - r, \bar{\Gamma}^{r}_{\phi\phi} = (2m - r)\sin(\theta)^2, \\ \bar{\Gamma}^{\theta}_{r\theta} = \frac{1}{r}, \bar{\Gamma}^{\theta}_{\theta t} = \frac{2m-r}{r}, \bar{\Gamma}^{\theta}_{\theta r} = \frac{1}{r}, \bar{\Gamma}^{\theta}_{\phi\phi} = -\cos(\theta)\sin(\theta), \bar{\Gamma}^{\phi}_{r\phi} = \frac{1}{r}, \bar{\Gamma}^{\phi}_{\theta\phi} = \frac{\cos(\theta)}{\sin(\theta)}, \\ \bar{\Gamma}^{\phi}_{\phi t} = \frac{2m-r}{r}, \bar{\Gamma}^{\phi}_{\phi r} = \frac{1}{r}, \bar{\Gamma}^{\phi}_{\phi\theta} = \frac{\cos(\theta)}{\sin(\theta)}. \end{cases}$$

The corresponding Ricci curvature components $\bar{S}_{ij}$ are:

$$\begin{pmatrix} 0 & -\frac{m}{r^2} & 0 & 0 \\ \frac{m}{r^2} & 1 & 0 & 0 \\ 0 & 0 & r^2 - 2mr & 0 \\ 0 & 0 & 0 & -(2mr - r^2)\sin(\theta)^2 \end{pmatrix}$$

For $a = \frac{r-2m}{r}$ and $b = 1$, we check that:

$$\begin{aligned} 0 = \hat{S}_{tt} &= ag_{tt} + b\pi_t\pi_t \\ 1 = \hat{S}_{rr} &= ag_{rr} + b\pi_r\pi_r \\ r^2 - 2mr = \hat{S}_{\theta\theta} &= ag_{\theta\theta} + b\pi_\theta\pi_\theta \\ -(2mr - r^2)\sin^2\theta = \hat{S}_{\phi\phi} &= ag_{\phi\phi} + b\pi_\phi\pi_\phi \end{aligned}$$

Hence, $(M, \bar{\nabla})$ is a $S(QE^n)$ manifold.

Example 5.2. We can further consider more complicated 4-dimensional Kottler spacetime with metric [11]

$$g = \left(\frac{1}{3}\Lambda r^2 + \frac{2m}{r} - 1\right) \mathrm{d}t \otimes \mathrm{d}t + \left(-\frac{3}{\Lambda r^2 + \frac{6m}{r} - 3}\right) \mathrm{d}r \otimes \mathrm{d}r + r^2 \mathrm{d}\theta \otimes \mathrm{d}\theta + r^2 \sin(\theta)^2 \, \mathrm{d}\phi \otimes \mathrm{d}\phi$$

as an example of an Einstein space carrying a $S(QE^n)$ structure with a Killing generator. This spacetime satisfies Einstein's field equations of general relativity with positive cosmological constant for a vacuum space around a center of spherical symmetry. The manifold is also called the Schwarzschild-de Sitter spacetime and provides us with a two-parameter family of static spacetime with compact spacelike slices, which are locally (but not globally) conformally flat. The Levi-Civita components of this metric are:

$$\begin{cases} \Gamma^{t}_{tr} = \frac{\Lambda r^3 - 3m}{\Lambda r^4 + 6mr - 3r^2}, \Gamma^{r}_{tt} = \frac{\Lambda^2 r^6 + 3\Lambda m r^3 - 3\Lambda r^4 - 18m^2 + 9mr}{9r^3}, \Gamma^{r}_{rr} = -\frac{\Lambda r^3 - 3m}{\Lambda r^4 + 6mr - 3r^2} \\ \Gamma^{r}_{\theta\theta} = \frac{1}{3}\Lambda r^3 + 2m - r, \Gamma^{r}_{\phi\phi} = \frac{1}{3}\left(\Lambda r^3 + 6m - 3r\right)\sin(\theta)^2, \Gamma^{\theta}_{r\theta} = \frac{1}{r} \\ \Gamma^{\theta}_{\phi\phi} = -\cos(\theta)\sin(\theta), \Gamma^{\phi}_{r\phi} = \frac{1}{r}, \Gamma^{\phi}_{\theta\phi} = \frac{\cos(\theta)}{\sin(\theta)} \end{cases}$$

The Ricci tensor S_{ij} is calculated as

$$\begin{pmatrix} -\frac{\Lambda^2 r^3+6\Lambda m-3\Lambda r}{3r} & 0 & 0 & 0 \\ 0 & \frac{3\Lambda r}{\Lambda r^3+6m-3r} & 0 & 0 \\ 0 & 0 & -\Lambda r^2 & 0 \\ 0 & 0 & 0 & -\Lambda r^2\sin(\theta)^2 \end{pmatrix}$$

Clearly $S_{ij} = -\Lambda g_{ij}$ in this spacetime. Hence it is an Einstein space.

The semi-symmetric metric connection corresponding to the one-form $\pi = (\frac{\Lambda r^3+6m-3r}{3r})dt$ associated to the Killing vector $\frac{\partial}{\partial t}$ is given by

$$\bar{\Gamma}^k_{ij} = \Gamma^k_{ij} + \pi_j\delta^k_i - g_{ij}\pi^k,\ \pi^k = g^{ik}\pi_i.$$

We calculate its nontrivial components as:

$$\begin{cases} \bar{\Gamma}^{\ t}_{t\ r} = \frac{\Lambda r^3-3m}{\Lambda r^4+6mr-3r^2},\ \bar{\Gamma}^{\ t}_{r\ t} = \frac{\Lambda r^3-3m}{\Lambda r^4+6mr-3r^2},\ \bar{\Gamma}^{\ t}_{r\ r} = \frac{3r}{\Lambda r^3+6m-3r},\ \bar{\Gamma}^{\ t}_{\theta\ \theta} = -r^2,\ \bar{\Gamma}^{\ t}_{\phi\ \phi} = -r^2\sin(\theta)^2 \\ \bar{\Gamma}^{\ r}_{t\ t} = \frac{\Lambda^2 r^6+3\Lambda mr^3-3\Lambda r^4-18m^2+9mr}{9r^3},\ \bar{\Gamma}^{\ r}_{r\ t} = \frac{\Lambda r^3+6m-3r}{3r},\ \bar{\Gamma}^{\ r}_{r\ r} = -\frac{\Lambda r^3-3m}{\Lambda r^4+6mr-3r^2},\ \bar{\Gamma}^{\ \phi}_{\phi\ \theta} = \frac{\cos(\theta)}{\sin(\theta)} \\ \bar{\Gamma}^{\ r}_{\theta\ \theta} = \frac{1}{3}\Lambda r^3 + 2m - r,\ \bar{\Gamma}^{\ r}_{\phi\ \phi} = \frac{1}{3}\left(\Lambda r^3 + 6m - 3r\right)\sin(\theta)^2,\ \bar{\Gamma}^{\ \theta}_{r\ \theta} = \frac{1}{r},\ \bar{\Gamma}^{\ \theta}_{\theta\ t} = \frac{\Lambda r^3+6m-3r}{3r} \\ \bar{\Gamma}^{\ \theta}_{\phi\ \phi} = -\cos(\theta)\sin(\theta),\ \bar{\Gamma}^{\ \phi}_{r\ \phi} = \frac{1}{r},\ \bar{\Gamma}^{\ \phi}_{\theta\ \phi} = \frac{\cos(\theta)}{\sin(\theta)},\ \bar{\Gamma}^{\ \phi}_{\phi\ t} = \frac{\Lambda r^3+6m-3r}{3r},\ \bar{\Gamma}^{\ \phi}_{\phi\ r} = \frac{1}{r},\ \bar{\Gamma}^{\ \theta}_{\theta\ r} = \frac{1}{r} \end{cases}$$

The corresponding Ricci curvature components $\bar{S}_{ij}$ are:

$$\begin{pmatrix} -\frac{\Lambda^2 r^3+6\Lambda m-3\Lambda r}{3r} & \frac{\Lambda r^3-3m}{3r^2} & 0 & 0 \\ -\frac{\Lambda r^3-3m}{3r^2} & \frac{\Lambda r^3+3(\Lambda-1)r+6m}{\Lambda r^3+6m-3r} & 0 & 0 \\ 0 & 0 & -\frac{\Lambda r^4+3(\Lambda-1)r^2+6mr}{3} & 0 \\ 0 & 0 & 0 & -\frac{(\Lambda r^4+3(\Lambda-1)r^2+6mr)}{3}\sin(\theta)^2 \end{pmatrix}$$

For $a = -\Lambda - \frac{\Lambda r^2+6m-3r}{3r}$ and $b = 1$, we check that the nontrivial $\bar{S}_{ij}$ components satisfy:

$$\begin{aligned} \frac{\Lambda^2 r^3 + 6\Lambda m - 3\Lambda r}{3r} &= \hat{S}_{tt} = ag_{tt} + b\pi_t\pi_t \\ \frac{\Lambda r^3 + 3(\Lambda-1)r + 6m}{\Lambda r^3 + 6m - 3r} &= \hat{S}_{rr} = ag_{rr} + b\pi_r\pi_r \\ \frac{\Lambda r^4 + 3(\Lambda-1)r^2 + 6mr}{3} &= \hat{S}_{\theta\theta} = ag_{\theta\theta} + b\pi_\theta\pi_\theta \\ \frac{\Lambda r^4 + 3(\Lambda-1)r^2 + 6mr}{3}\sin^2\theta &= \hat{S}_{\phi\phi} = ag_{\phi\phi} + b\pi_\phi\pi_\phi \end{aligned}$$

Hence, $(M, \bar{\nabla})$ is a $S(QE^n)$ manifold.

Example 5.3. We consider a three-dimensional manifold M^3 endowed with a metric

$$g_{ij}dx^i dx^j = e^{x^1}[(dx^1)^2 - (dx^2)^2] + (dx^3)^2$$

and a non-zero 1-form $\pi = e^{\frac{1}{2}x^1 - \frac{1}{2}x^2}dx^1 - e^{\frac{1}{2}x^1-\frac{1}{2}x^2}dx^2 + dx^3$.

The non-zero components of the corresponding Levi-Civita connection are $\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^1_{22} = \frac{1}{2}$ and the manifold is Ricci flat. Furthermore, π is a parallel one-form with respect to this connection.

The semi-symmetric metric connection corresponding to the 1-form π is given by

$$\bar{\Gamma}^k_{ij} = \Gamma^k_{ij} + \pi_j\delta^k_i - g_{ij}\pi^k,$$

and the non-zero coefficients are

$$\begin{cases} \bar{\Gamma}^1_{11} = \frac{1}{2},\ \bar{\Gamma}^2_{11} = -e^{\frac{1}{2}x^1-\frac{1}{2}x^2},\ \bar{\Gamma}^3_{11} = -e^{x^1},\ \bar{\Gamma}^1_{12} = -e^{\frac{1}{2}x^1-\frac{1}{2}x^2},\ \bar{\Gamma}^2_{12} = \frac{1}{2},\ \bar{\Gamma}^1_{13} = 1, \\ \bar{\Gamma}^2_{23} = 1,\ \bar{\Gamma}^2_{21} = \frac{1}{2} + e^{\frac{1}{2}x^1-\frac{1}{2}x^2},\ \bar{\Gamma}^1_{22} = \frac{1}{2} + e^{\frac{1}{2}x^1-\frac{1}{2}x^2},\ \bar{\Gamma}^3_{22} = e^{x^1},\ \bar{\Gamma}^3_{31} = e^{\frac{1}{2}x^1-\frac{1}{2}x^2}, \\ \bar{\Gamma}^3_{32} = -e^{\frac{1}{2}x^1-\frac{1}{2}x^2},\ \bar{\Gamma}^1_{33} = -e^{-\frac{1}{2}x^1-\frac{1}{2}x^2} = \bar{\Gamma}^2_{33}. \end{cases}$$

The Ricci curvature components $\bar{S}_{ij}$ satisfies

$$\begin{cases} \hat{S}_{11} = -e^{x^1} + e^{x^1-x^2} = -g_{11} + \pi_1\pi_1, \\ \hat{S}_{12} = -e^{x^1-x^2} = -g_{12} + \pi_1\pi_2, \\ \hat{S}_{13} = e^{\frac{1}{2}x^1-\frac{1}{2}x^2} = -g_{13} + \pi_1\pi_3, \\ \hat{S}_{22} = e^{x^1} + e^{x^1-x^2} = -g_{22} + \pi_2\pi_2, \\ \hat{S}_{23} = -e^{\frac{1}{2}x^1-\frac{1}{2}x^2} = -g_{23} + \pi_2\pi_3, \\ \hat{S}_{33} = 0 = -g_{33} + \pi_3\pi_3, \end{cases}$$

now we take $a = -1$ and $b = 1$, then

$$\bar{S}_{ij} = ag_{ij} + b\pi_i\pi_j, i, j = 1, 2, 3$$

therefore $(M^3, \bar{\nabla})$ is a $S(QE^n)$ manifold.

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