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Stable sheaves on elliptic fibrations ☆

D. Hernández Ruipérez*, J.M. Muñoz Porras

Departamento de Matemáticas, Universidad de Salamanca, Plaza de la Merced 1/4, 37008 Salamanca, Spain

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Abstract

Let $X \to B$ be an elliptic surface and $\mathcal{M}(a, b)$ the moduli space of torsion-free sheaves on X which are stable of relative degree zero with respect to a polarization of type $aH + b\mu$, H being the section and μ the elliptic fibre ($b \gg 0$). We characterize the open subscheme of $\mathcal{M}(a, b)$ which is isomorphic, via the relative Fourier–Mukai transform, with the relative compactified Simpson–Jacobian of the family of those curves $D \hookrightarrow X$ which are flat over B. This generalizes and completes earlier constructions due to Friedman, Morgan and Witten. We also study the relative moduli scheme of torsion-free and semistable sheaves of rank n and degree zero on the fibres. The relative Fourier–Mukai transform induces an isomorphic between this relative moduli space and the relative nth symmetric product of the fibration. These results are relevant in the study of the conjectural duality between F-theory and the heterotic string.

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1. Elliptic fibrations and relative Fourier–Mukai transform

1.1. Introduction

Recently there has been a growing interest in the moduli spaces of stable vector bundles on elliptic fibrations. Aside from their mathematical importance, these moduli spaces provide

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* Corresponding author.

E-mail addresses: ruiperez@usal.es (D.H. Ruipérez), jmp@usal.es (J.M. Muñoz Porras).

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a geometric background to the study of some recent developments in string theory, notably in connection with the conjectural duality between F-theory and heterotic string theory [4,10,13,14].

In this paper we study such moduli spaces, dealing both with the case of relatively and absolutely stable sheaves. We only consider elliptic fibrations $p: X \to B$ with a section H and geometrically integral fibres.

In the first part, we consider the "dual" elliptic fibration $\hat{p}: \hat{X} \to B$ [5] defined as the compactified relative Jacobian of $X \to B$ (actually, $\hat{X}$ turns out to be isomorphic with X) and we introduce the relative Fourier–Mukai transform and its properties. This allows for a nice description of the spectral cover construction. Given a sheaf $\mathcal{F}$ on $X \to B$ flat over B and fibrewise, torsion-free and semistable of rank n and degree 0, we define its *spectral cover* $C(\mathcal{F}) \hookrightarrow \hat{X}$ as the closed subscheme defined by the 0th Fitting ideal of the first Fourier–Mukai transform $\hat{\mathcal{F}}$. It is finite over B and generically of degree n. When B is a smooth curve, the spectral cover is actually flat of degree n and $\hat{\mathcal{F}}$ is torsion-free and rank one over $C(\mathcal{F})$. Atiyah [2], Tu [23] and Friedman et al. [13] structure theorems for semistable sheaves of degree zero on an elliptic curve play a fundamental role in this section. By the invertibility of the Fourier–Mukai transform, this gives a one-to-one correspondence between fibrewise, torsion-free and semistable sheaves of rank n and degree 0 and torsion-free, rank one sheaves on spectral covers.

The second part is devoted to the study of the relative moduli scheme $\bar{\mathcal{M}}(n, 0)$ of torsion-free and semistable sheaves of rank n and degree 0 on the fibres of $X \to B$. (One should notice that the case of non-zero relative degree is somehow simpler, cf. [6,14].) Using the results of the first section, we prove that the relative Fourier–Mukai induces an isomorphism of B-schemes $\bar{\mathcal{M}}(n, 0) \tilde{\to} \operatorname{Sym}^n_B \hat{X}$ (Theorem 2.1). This isomorphism is probably known to people familiar with the topic, but it cannot be explicitly found and proved elsewhere in the literature. Friedman–Morgan–Witten's theorem on the structure of the moduli $\mathcal{M}(n, \mathcal{O}_X)$ of vector bundles in $\mathcal{M}(n, 0)$ whose determinant is fibrewise trivial is easily derived from our results. As a corollary, we determine the Picard group and the canonical series of the relative moduli scheme $\bar{\mathcal{M}}(n, 0)$.

The third part is devoted to absolute stability of torsion-free sheaves on an elliptic surface with respect to a polarization of the form $aH + b\mu$, where H is the section of $p: X \to B$ and μ the fibre. The main result is that for b big enough (in a way precised in the paper), the stability of a torsion-free sheaf $\mathcal{F}$ on X (fibrewise semistable of rank n and degree 0) is equivalent to the stability of the Fourier–Mukai transform $\hat{\mathcal{F}}$ as a sheaf on the spectral cover $C(\mathcal{F})$. Since non-integral (even non-reduced) spectral covers may occur, we have to consider stability on $C(\mathcal{F})$ with respect a polarization (the one given by the fibre) in the sense of Simpson [22].

We finish the paper with the moduli implications of our results. Let $\mathcal{H}$ be the scheme of all possible spectral covers which are flat of degree n over B. It can be identified with the Hilbert scheme of sections of the projection $\bar{\mathcal{M}}(n, 0) \tilde{\to} \operatorname{Sym}^n_B \hat{X} \to B$. Let $\mathcal{C} \to B \times \mathcal{H}$ be the "universal spectral cover". If we denote by $\mathcal{M}(a, b)$ the moduli space of absolutely stable torsion-free sheaves on X, we prove (Theorem 3.16) that the Fourier–Mukai transform gives rise to an isomorphism between the compactified Jacobian $\mathcal{J}(\mathcal{C}/\mathcal{H})$ of the universal spectral cover and the open subscheme $\mathcal{M}'(a, b)$ of the moduli space $\mathcal{M}(a, b)$ of absolutely stable sheaves on X defined by those sheaves that are semistable on fibres as well. In particular, we obtain that there is a fibration $\pi : \mathcal{M}'(a, b) \to \mathcal{H}$ whose fibres are generalized compactified Jacobians. The generic fibres, for instance, the fibres $\pi^{-1}([C])$ over a point $[C] \in \mathcal{H}$ representing a smooth curve, are abelian varieties, but there are points of $\mathcal{H}$ whose fibres are not abelian varieties.

As before, due to the existence of non-integral spectral covers, the compactified Jacobian of $\mathcal{C} \to \mathcal{H}$ has to be defined as the Simpson moduli scheme of $\mathcal{H}$-flat sheaves on $\mathcal{C}$ whose restriction to every fibre is of pure dimension one, rank one and stable with respect to a fixed polarization. For those sheaves whose spectral covers are integral, we recover the results already proved in [14], but making no assumptions about the generic regularity of the restrictions of the sheaves to the fibres.

The conjectural duality between the heterotic string and F-theory [1,7,18,19,24] could be formulated from a geometrical point of view as the existence of an isomorphism between a moduli space of absolutely stable bundles (of group $E_8 \times E_8$ or $\mathrm{Spin}(32)/2\mathbb{Z}$ in most cases) over a surface X elliptically fibred over $\mathbb{P}^1$ and a moduli space of Calabi–Yau threefold elliptically fibred over a Hirzebruch surface. The knowledge of the structure of the moduli schemes $\mathcal{M}(a, b)$ is then a fundamental step in the understanding of the duality F-theory/heterotic string. We hope that the results in this paper will be useful to the study of such problem.

1.2. Preliminaries

All the schemes considered in this paper are of finite type over an algebraically closed field and all the sheaves are coherent. Let $p : X \to B$ be an elliptic fibration. By this we mean a proper flat morphism of schemes whose fibres are geometrically integral Gorenstein curves of arithmetic genus 1. We also assume that p has a section $e : B \hookrightarrow X$ taking values in the smooth locus $X' \to B$ of p.

We write $H = e(B)$ and we denote by X_t the fibre of p over $t \in B$, and by $i_t : X_t \hookrightarrow X$ the inclusion. We denote by $\mathcal{U} \hookrightarrow B$ be the open subset supporting the smooth fibres of $p : X \to B$. Let us denote by $\omega_{X/B}$ the relative dualizing sheaf. Then $p_*\omega_{X/B}$ is a line bundle $\mathcal{O}_B(E)$ and $\omega_{X/B} \tilde{\to} p^*\mathcal{O}_B(E)$, i.e. $K_{X/B} = p^{-1}E$ is a relative canonical divisor. We denote, as is [9], $\omega = R^1p_*\mathcal{O}_X \tilde{\to} (p_*\omega_{X/B})^*$ so that $\omega = \mathcal{O}_B(-E)$. Adjunction formula for $H \hookrightarrow X$ gives $\mathcal{O}_H = \omega_{H/B} = \omega_{X/B_{|H}} \otimes \mathcal{O}_H(H)$, i.e. $H^2 = -H \cdot p^{-1}E$ as cycles on X.

By [17, Lemma II.4.3], $p : X \to B$ has a Weierstrass form: the divisor $3H$ is relatively very ample and if $V = p_*\mathcal{O}_X(3H) \tilde{\to} \mathcal{O}_B \oplus \omega^{\otimes 2} \oplus \omega^{\otimes 3}$ and $P = \mathrm{Proj}(S^\bullet(V))$ (projective spectrum of the symmetric algebra), then there is a closed immersion of B-schemes $j : X \hookrightarrow P$ such that $j^*\mathcal{O}_P(1) = \mathcal{O}_X(3H)$. Moreover j is locally a complete intersection whose normal sheaf is

$$\mathcal{N}(X/P) \tilde{\to} p^*\omega^{-\otimes 6} \otimes \mathcal{O}_X(9H). \tag{1.1}$$

This follows by relative duality since $\omega_{P/B} = \wedge\Omega_{P/B} \tilde{\to} \bar{p}^*\omega^{\otimes 5}(-3)$, $\bar{p} : P \to B$ being the projection, due to the exact sequence $0 \to \Omega_{P/B} \to \bar{p}^*V(-1) \to \mathcal{O}_P \to 0$. The morphism $p : X \to B$ is then an l.c.i. morphism in the sense of [15, (6.6)] and has a virtual relative tangent bundle $T_{X/B} = [j^*T_{P/B}] - [\mathcal{N}_{X/P}]$ in the K-group $K^\bullet(X)$.

Proposition 1.1. *The Todd class of the virtual tangent bundle $T_{X/B}$ is*

$$\operatorname{td}(T_{X/B}) = 1 - \tfrac{1}{2} p^{-1}E + H \cdot p^{-1}E + \tfrac{13}{12} p^{-1}E^2 + \textit{terms of higher degree}.$$

Proof. We compute the Todd class from

$$j^* T_{P/B} = \mathcal{O}_X(3H) \oplus \mathcal{O}_X(3H + 2p^*E) \oplus \mathcal{O}_X(3H + 3p^*E),$$

and Eq. (1.1) using $H^2 = -H \cdot p^{-1}E$. □

Let $\operatorname{Pic}^-_{X/B}$ be the functor which to any morphism $f : S \to B$ of schemes associates the space of S-flat sheaves on $p_S : X \times_B S \to S$, whose restrictions to the fibres of p_S are torsion-free, of rank one and degree zero. Two such sheaves $\mathcal{F}, \mathcal{F}'$, are considered to be equivalent if $\mathcal{F}' \simeq \mathcal{F} \otimes p_S^* \mathcal{N}$ for a line bundle $\mathcal{N}$ on S (cf. [3]). Due to the existence of the section e, $\operatorname{Pic}^-_{X/B}$ is a sheaf functor.

By [3], $\operatorname{Pic}^-_{X/B}$ is represented by an algebraic variety $\hat{p} : \hat{X} \to B$ (the Altman–Kleiman compactification of the relative Jacobian). Moreover, the natural morphism of B-schemes $\varpi : X \to \hat{X}$, $x \mapsto \mathfrak{m}_x^* \otimes \mathcal{O}_{X_s}(-e(s))$ is an isomorphism. Here $\mathfrak{m}_x$ is the ideal sheaf of the point x in X_s. The relative Jacobian $J^0 \to B$ of X as a B-scheme is the smooth locus $\hat{X}'$ of $\hat{p} : \hat{X} \to B$ and if $\mathcal{U} \subseteq B$ is the open subset supporting the smooth fibres of p, one has $J^0_{\mathcal{U}} \simeq \hat{X}_{\mathcal{U}}$. As in [5], we denote by $\hat{e} : B \hookrightarrow \hat{X}$ the section $\varpi \circ e$ and by Θ the divisor $\hat{e}(B) = \varpi(H)$. We write $\iota : \hat{X} \to \hat{X}$ for the isomorphism mapping any rank one, torsion-free and zero-degree sheaf $\mathcal{F}$ on a fibre X_s to its dual $\mathcal{F}^*$.

Most of the results in [5] are also true in our more general setting, in some cases just with straightforward modifications.

1.3. Relative Fourier–Mukai transforms

Here we consider an elliptic fibration $p : X \to B$ as above and the associated "dual" fibration $\hat{p} : \hat{X} \to B$. We shall define a relative Fourier–Mukai in this setting by means of the relative universal Poincaré sheaf $\mathcal{P}$ on the fibred product $X \times_B \hat{X}$ normalized so that $\mathcal{P}_{|H \times_B \hat{X}} \simeq \mathcal{O}_{\hat{X}}$ as in [5]. $\mathcal{P}$ is also flat over X, and $\mathcal{P}^*$ enables us to identify $p : X \to B$ with a compactification of the relative Jacobian $\hat{J}^0 \to B$ of $\hat{p} : \hat{X} \to B$.

For every morphism $S \to B$, we denote all objects obtained by base change to S by a subscript S. There is a diagram:

$$\begin{array}{ccc} (X \times_B \widehat{X})_S \simeq X_S \times_S \widehat{X}_S & \xrightarrow{\hat{\pi}_S} & \widehat{X}_S \\ \Big\downarrow{\scriptstyle \pi_S} & & \Big\downarrow{\scriptstyle \hat{p}_S} \\ X_S & \xrightarrow{p_S} & S \end{array}$$

The relative Fourier–Mukai transform is the functor between the derived categories of quasi-coherent sheaves given by

$$\mathbf{S}_S : D(X_S) \to D(\hat{X}_S), \qquad F \mapsto \mathbf{S}_S(F) = R\hat{\pi}_{S*}(\pi_S^* F \otimes \mathcal{P}_S).$$

We then define $\mathbf{S}^i_S(F) = \mathcal{H}^i(\mathbf{S}_S(F))$, $i = 0, 1$ so that $\mathbf{S}^i_S(\mathcal{F}) = R^i\hat{\pi}_{S*}(\pi^*_S\mathcal{F} \otimes \mathcal{P}_S)$ for every sheaf $\mathcal{F}$ on X_S.

There is then a natural notion of WIT_i and IT_i sheaves: we say that a sheaf $\mathcal{F}$ on X_S is WIT_i if $\mathbf{S}^j_S(\mathcal{F}) = 0$ for $j \neq i$ and we say that $\mathcal{F}$ is IT_i if it is WIT_i and $\mathbf{S}^i_S(\mathcal{F})$ is locally free. One easily proves the following proposition.

Proposition 1.2. *Let F be an object in $D^-(\hat{X}_S)$. For every morphism $g : S' \to S$ there is an isomorphism*

$$Lg^*_{\hat{X}}(\mathbf{S}_S(F)) \simeq \mathbf{S}_{S'}(Lg^*_X F)$$

in the derived category $D^-(\hat{X}_{S'})$, where $g_X : X_{S'} \to X_S$, $g_{\hat{X}} : \hat{X}_{S'} \to \hat{X}_S$ are the morphisms induced by g.

Due to this property, we shall very often drop the subscript S and refer only to $X \to B$. Base-change theory gives the following corollary.

Corollary 1.3. *Let $\mathcal{F}$ be a sheaf on X, flat over B.*

1. *The formation of $\mathbf{S}^1(\mathcal{F})$ is compatible with base change, i.e. one has $\mathbf{S}^1(\mathcal{F})_s \simeq \mathbf{S}^1_s(\mathcal{F}_s)$, for every point $s \in B$.*
2. *Assume that $\mathcal{F}$ is WIT_1 and let $\hat{\mathcal{F}} = \mathbf{S}^1(\mathcal{F})$ be its Fourier–Mukai transform. Then for every $s \in B$ there is an isomorphism*

$$\mathrm{Tor}_1^{\mathcal{O}_S}(\hat{\mathcal{F}}, \kappa(s)) \simeq \mathbf{S}^0_s(\mathcal{F}_s)$$

of sheaves over $\hat{X}_s$. In particular $\hat{\mathcal{F}}$ is flat over B if and only if the restriction $\mathcal{F}_s$ to the fibre X_s is WIT_1 for every point $s \in B$.

Corollary 1.4. *Let $\mathcal{F}$ be a sheaf on X, flat over B. There exists an open subscheme $V \subseteq B$ which is the largest subscheme V fulfilling one of the following equivalent conditions hold*:

1. *$\mathcal{F}_V$ is WIT_1 on X_V and the Fourier–Mukai transform $\hat{\mathcal{F}}_V$ is flat over V.*
2. *The sheaves $\mathcal{F}_s$ are WIT_1 for every point $s \in V$.*

There are similar properties for sheaves on $X \times T \to B \times T$ that are only flat over T.

Corollary 1.5. *Let T be a scheme, and $\mathcal{F}$ a sheaf on $X \times T$, flat over T. Assume that $\mathcal{F}$ is WIT_1 and let $\hat{\mathcal{F}} = \mathbf{S}^1_{B\times T}(\mathcal{F})$ be its Fourier–Mukai transform. Then for every morphism $T' \to T$ there is an isomorphism*

$$\mathrm{Tor}_1^{\mathcal{O}_T}(\hat{\mathcal{F}}, \mathcal{O}_{T'}) \simeq \mathbf{S}^0_{B\times T'}(\mathcal{F}_{B\times T'})$$

of sheaves over $\hat{X} \times T'$. In particular $\hat{\mathcal{F}}$ is flat over T if and only if, $\mathcal{F}_{B\times\{t\}}$ is WIT_1 on $X_{B\times\{t\}} \xrightarrow{\sim} X$ for every $t \in T$.

1.4. Fourier–Mukai transform of relatively torsion-free, rank one and degree zero sheaves

Let $\mathcal{L}$ be a sheaf on X_S, flat over S, whose restrictions to the fibres of p_S are torsion-free and have rank one and degree zero. The universal property gives a morphism $\phi : S \to \hat{X}$ so that $(1\times\phi)^*\mathcal{P} \simeq \mathcal{L}\otimes p_S^*\mathcal{N}$ for a certain line bundle $\mathcal{N}$ on S. Let $\Gamma : S \hookrightarrow \hat{X}_S$ be the graph of the morphism $\iota\circ\phi : S \to \hat{X}$. Lemma 2.11 and Corollary 2.12 of [5] now take the following form.

Proposition 1.6. *In the above situation* $\mathbf{S}_S^0(\mathcal{L}) = 0$ *and* $\mathbf{S}_S^1(\mathcal{L})\otimes\hat{p}_S^*\mathcal{N} \simeq \Gamma_*(\omega_S)$. *In particular*:

1. $\mathbf{S}_{\hat{X}}^0(\mathcal{P}) = 0$ *and* $\mathbf{S}_{\hat{X}}^1(\mathcal{P}) \simeq \zeta_*\hat{p}^*\omega$, *where* $\zeta : \hat{X} \hookrightarrow \hat{X}\times_B\hat{X}$ *is the graph of the morphism* ι.
2. $\mathbf{S}_{\hat{X}}^0(\mathcal{P}^*) = 0$ *and* $\mathbf{S}_{\hat{X}}^1(\mathcal{P}^*) \simeq \delta_*\hat{p}^*\omega$, *where* $\delta : \hat{X} \hookrightarrow \hat{X}\times_B\hat{X}$ *is the diagonal immersion.*
3. $\mathbf{S}_S^0(\mathcal{O}_{X_S}) = 0$ *and* $\mathbf{S}_S^1(\mathcal{O}_{X_S}) = \mathcal{O}_\Theta\otimes\hat{p}^*\omega$.

Corollary 1.7. *Let* $\mathcal{L}$ *be a rank one, zero-degree, torsion-free sheaf on a fibre* X_s. *Then*

$$\mathbf{S}_s^0(\mathcal{L}) = 0, \qquad \mathbf{S}_s^1(\mathcal{L}) = \kappa([\mathcal{L}^*]),$$

where $[\mathcal{L}^*]$ *is the point of* $\hat{X}_s$ *defined by* $\mathcal{L}^*$.

The first application is the invertibility of the Fourier–Mukai transform; if we consider the functor

$$\hat{\mathbf{S}}_S : D(\hat{X}_S) \to D(X_S), \qquad G \mapsto \hat{\mathbf{S}}_S(G) = R\pi_{S*}(\hat{\pi}_S^*G\otimes\mathcal{Q}_S),$$

where $\mathcal{Q} = \mathcal{P}^*\otimes\pi^*p^*\omega^{-1}$, then proceeding as in Theorem 3.2 of [5] and taking into account Proposition 1.6, we obtain the following invertibility result (see also [8]).

Proposition 1.8. *For every* $G\in D(\hat{X})_S$, $F\in D(X_S)$ *there are functorial isomorphisms*

$$\mathbf{S}_S(\hat{\mathbf{S}}_S(G)) \simeq G[-1], \qquad \hat{\mathbf{S}}_S(\mathbf{S}_S(F)) \simeq F[-1]$$

in the derived categories $D(\hat{X}_S)$ *and* $D(X_S)$, *respectively*.

The second application is the characterization of relative semistability as the WIT_1 condition. This is a consequence of the properties of semistable torsion-free of degree zero sheaves on a fibre X_s. The structure theorems for those sheaves are essentially due to Atiyah [2] and Tu [23] in the smooth case and to Friedman et al. [14] for Weierstrass curves and locally free sheaves. What we need is the following proposition.

Proposition 1.9. *Every torsion-free semistable sheaf of rank* n *and degree* 0 *on* X_s *is S-equivalent to a direct sum of torsion-free, rank* 1 *and degree* 0 *sheaves*:

$$\mathcal{F} \sim \oplus_{i=0}^r\left(\mathcal{L}_i\oplus\overset{n_i}{\cdots}\oplus\mathcal{L}_i\right).$$

If X_s is smooth all the sheaves $\mathcal{L}_i$ are line bundles. If X_s is singular, at most one of them, say $\mathcal{L}_0$, is non-locally free; the number n_0 of factors isomorphic to $\mathcal{L}_0$ can be zero. Now we have the following proposition.

Proposition 1.10. *Let $\mathcal{F}$ be a zero-degree sheaf of rank $n \geq 1$ on a fibre X_s. Then $\mathcal{F}$ is torsion-free and semistable on X_s if and only if it is* WIT_1.

Proof. Assume first that $\mathcal{F}$ is torsion-free and semistable. The case $n = 1$ is Corollary 1.7. For $n > 1$, we can assume that $\mathcal{F}$ is indecomposable; by Proposition 1.9, there is an exact sequence of torsion-free, degree 0 sheaves $0 \to \mathcal{L} \to \mathcal{F} \to \mathcal{F}' \to 0$, where $\mathcal{L}$ has rank 1 and $\mathcal{F}'$ is semistable. The claim follows by induction on n from the associated exact sequence of Fourier–Mukai transforms. For the converse, if $\mathcal{F}$ is WIT_1, all its subsheaves are WIT_1 as well, and then $\mathcal{F}$ has neither subsheaves supported on dimension zero, nor torsion-free subsheaves of positive degree. □

We go back to our elliptic fibration $p : X \to B$. By Corollary 1.4 and Proposition 1.10, we have obtain the following proposition.

Proposition 1.11. *Let $\mathcal{F}$ be a sheaf on X, flat over B and of fibrewise degree zero. There exists an open subscheme $S(\mathcal{F}) \subseteq B$ which is the largest subscheme of B fulfilling one of the following equivalent conditions*:

1. *$\mathcal{F}_{S(\mathcal{F})}$ is* WIT_1 *and $\hat{\mathcal{F}}_{S(\mathcal{F})}$ is flat over $S(\mathcal{F})$.*
2. *The sheaves $\mathcal{F}_s$ are* WIT_1 *for every point $s \in S(\mathcal{F})$.*
3. *The sheaves $\mathcal{F}_s$ are torsion-free and semistable for every point $s \in S(\mathcal{F})$.*

We shall call $S(\mathcal{F})$ the *relative semistability locus* of $\mathcal{F}$.

Corollary 1.12. *Let $\mathcal{F}$ be a sheaf on X flat over B and fibrewise of degree zero. If $S(\mathcal{F})$ is dense, then $\mathcal{F}$ is* WIT_1.

Proof. By the previous proposition, $\mathcal{F}_{S(\mathcal{F})}$ is WIT_1 and then $\mathbf{S}^0_S(\mathcal{F})_{S(\mathcal{F})} = 0$ because $S(\mathcal{F}) \to S$ is a flat base change. Thus, $\mathbf{S}^0_S(\mathcal{F}) = 0$ since it is flat over S so that $\mathcal{F}$ is WIT_1. □

1.5. The spectral cover

In this section, we give a construction of the spectral cover similar to the one described in [12,14] (Sections 4.3 and 5.1) and [4].

We have seen that the Fourier–Mukai transform of a torsion-free, rank one sheaf $\mathcal{L}$ on a fibre determines a sheaf $\hat{\mathcal{L}} = \kappa(\xi^*)$ concentrated at the point $\xi^* \in \hat{X}_s$ determined by $\mathcal{L}^*$. If we take a higher rank semistable sheaf $\mathcal{F}_s$ of degree zero on X_s, we will see that $\hat{\mathcal{F}}_s$ is concentrated on a finite set of points of $\hat{X}_s$. When $\mathcal{F}_s$ moves in a flat family $\mathcal{F}$ on $X \to B$, the support of $\hat{\mathcal{F}}_s$ moves as well giving a finite covering $C \to B$. One notices, however, that the fibre over s of the support of $\hat{\mathcal{F}}$ may fail to be equal to the support of $\hat{\mathcal{F}}_s$. To circumvent this problem, we consider the closed subscheme defined by the 0th Fitting ideal of $\hat{\mathcal{F}}$ (see, for instance, [21] for a summary of properties of the Fitting ideals). The precise definition is given as follows.

Definition 1.13. Let $\mathcal{F}$ be a sheaf on X. The spectral cover of $\mathcal{F}$ is the closed subscheme $C(\mathcal{F})$ of $\hat{X}$ defined by the 0th Fitting ideal $F_0(\mathbf{S}^1(\mathcal{F}))$ of $\mathbf{S}^1(\mathcal{F})$.

The support of $\mathbf{S}^1(\mathcal{F})$ is contained in the spectral cover $C(\mathcal{F})$ and differs very little from it, in that some embedded components may have been removed. Corollaries 1.3 and 5.1 of [21] give the desired base-change property.

Proposition 1.14. *The spectral cover is compatible with base change, i.e. if $\mathcal{F}$ is a sheaf on X flat over B, then $C(\mathcal{F}_s) = C(\mathcal{F})_s$ as closed subschemes of $\hat{X}_s$ for every point $s \in B$.*

The fibred structure of the spectral cover is a consequence of the following lemma.

Lemma 1.15. *Let $\mathcal{F}$ be a zero-degree torsion-free semistable sheaf of rank $n \geq 1$ on a fibre X_s.*

1. *The 0th Fitting ideal $F_0(\hat{\mathcal{F}})$ of $\hat{\mathcal{F}} = \mathbf{S}^1_s(\mathcal{F})$ only depends on the S-equivalence class of $\mathcal{F}$.*
2. *One has $F_0(\hat{\mathcal{F}}) = \prod_{i=0}^r \mathfrak{m}_i^{n_i}$, where $\mathcal{F} \sim \oplus_{i=0}^r (\mathcal{L}_i \oplus \cdot\cdot\cdot^{n_i} \oplus \mathcal{L}_i)$ is the S-equivalence given by Proposition 1.9 and $\mathfrak{m}_i$ the ideal of the point $\xi_i^* \in \hat{X}_s$ defined by $\mathcal{L}_i^*$. Then, length $(\mathcal{O}_{\hat{X}_t}/F_0(\hat{\mathcal{F}})) \geq n$ with equality if either $n_0 = 0$ or $n_0 = 1$, i.e. if the only possible non-locally free rank 1 torsion-free sheaf of degree 0 occurs at most once.*

Proof.

1. Since the formation of the 0th Fitting ideal is multiplicative over direct sums of arbitrary sheaves [21, (5.1)], we can assume that $\mathcal{F}$ is indecomposable; as in the proof of Proposition 1.10 there is an exact sequence of torsion-free, degree 0 sheaves $0 \to \mathcal{L} \to \mathcal{F} \to \mathcal{F}' \to 0$, where $\mathcal{L}$ has rank 1 and $\mathcal{F}'$ is semistable. The sequence of Fourier–Mukai transforms is $0 \to \kappa[\mathcal{L}^*] \to \hat{\mathcal{F}} \to \hat{\mathcal{F}}' \to 0$ so that it splits and again by (5.1) of [21] we have $F_0(\hat{\mathcal{F}}) = F_0(\kappa[\mathcal{L}^*]) \cdot F_0(\hat{\mathcal{F}}')$. Induction on n gives the result.
2. The description of the Fitting ideal follows from (1) since $F_0(\kappa[\mathcal{L}_i^*]) = \mathfrak{m}_i$. Then length $(\mathcal{O}_{\hat{X}_s}/F_0(\hat{\mathcal{F}})) \geq n$ with equality if and only if either all points ξ_i^* are smooth or the exponent n_0 of the maximal ideal of the singular point ξ_0^* is equal to 1. □

Proposition 1.16. *If $\mathcal{F}$ is relatively torsion-free and semistable of rank n and degree zero on $X \to B$, then the spectral cover $C(\mathcal{F}) \to B$ is a finite morphism with fibres of degree $\geq n$.*

Proof. Since the spectral cover commutes with base changes, $C(\mathcal{F}) \to S$ is quasi-finite with fibres of degree $\geq n$ by Lemma 1.15; then it is finite. □

The most interesting case is when the base B is a *smooth curve* and the generic fibre is *smooth*. Let $\mathcal{F}$ be a sheaf on X flat over B and fibrewise of degree zero. Assume that the restriction of $\mathcal{F}$ to the generic fibre is semistable so that it is $\mathcal{F}$ is WIT_1 by Corollary 1.12. We then have the following proposition.

Proposition 1.17. *Let $V \subseteq B$ be the relative semistability locus of $\mathcal{F}$.*

1. *The spectral cover $C(\mathcal{F}) \to B$ is flat of degree n over V; then $C(\mathcal{F}_V)$ is a Cartier divisor of $\hat{X}_V$.*
2. *If $s \notin V$ is a point such that $\mathcal{F}_s$ is unstable, then $C(\mathcal{F})$ contains the whole fibre $\hat{X}_s$.*

Thus $C(\mathcal{F}) \to B$ is finite (and automatically flat of degree n) if and only if $\mathcal{F}_s$ is semistable for every $s \in B$.

Proof.

1. $C(\mathcal{F}_V) \to V$ is finite by Proposition 1.16 and V is a smooth curve so that $C(\mathcal{F})_V = C(\mathcal{F}_V) \to V$ is dominant and then it is flat.
2. Let
$$0 \to \mathcal{G} \to \mathcal{F}_s \to K \to 0$$
be a destabilizing sequence, where K is a sheaf on X_s of negative degree. Then K is WIT_1 and $\hat{K}$ is torsion-free (see [6]). Since $\mathbf{S}^1_s(\mathcal{F}_s) \to \mathbf{S}^1_s(K)$ is surjective, $C(\mathcal{F})_s = C(\mathcal{F}_s) = \hat{X}_s$. □

Remark 1.18. By Proposition 1.17, if B is a curve a semistable sheaf $\mathcal{F}_s$ on a singular fibre X_s S-equivalent to $\oplus_{i=0}^r (\mathcal{L}_i \oplus \cdots^{n_i} \oplus \mathcal{L}_i)$ with $n_0 > 1$ cannot be extended to a flat parameterization $\mathcal{F}$ of semistable sheaves on $X \to B$.

2. Moduli of relatively semistable degree zero sheaves on elliptic fibrations

2.1. Moduli of relatively semistable sheaves

In this section, we describe the structure of relatively semistable sheaves on an elliptic fibration $p : X \to B$. If we start with a single fibre X_s, then Proposition 1.9 means that S-equivalence classes of semistable sheaves of rank n and degree 0 on X_s are equivalent to families of n torsion-free, rank one sheaves of degree zero, $\mathcal{F} \sim \oplus_{i=0}^r (\mathcal{L}_i \oplus \cdots^{n_i} \oplus \mathcal{L}_i)$. This gives a one-to-one correspondence

$$\bar{\mathcal{M}}(X_s, n, 0) \leftrightarrow \operatorname{Sym}^n \hat{X}_s, \qquad \mathcal{F} \mapsto n_0\xi_0^* + \cdots + n_r\xi_r^*, \qquad \xi_i^* = [\mathcal{L}_i^*] \tag{2.1}$$

between the moduli space of torsion-free and semistable sheaves of rank n and degree 0 on X_s and the nth symmetric product of the compactified Jacobian $\hat{X}_s$. The reason for taking duals comes from Corollary 1.7 and Lemma 1.15: the skyscraper sheaf $\kappa([\xi_i^*])$ is the Fourier–Mukai transform of $\mathcal{L}_i$, and if $n_0 = 0$ (i.e. if $\mathcal{F}$ is S-equivalent to a direct sum of line bundles), then $n_1\xi_1^* + \cdots + n_r\xi_r^*$ is the spectral cover $C(\mathcal{F})$.

We are now going to extend (2.1) to the whole elliptic fibration $X \to B$ under the assumption that the base scheme B is *normal of dimension bigger than zero* and the generic fibre is *smooth*.

Let $\mathrm{Hilb}^n(\hat{X}/B) \to B$ be the Hilbert scheme of B-flat subschemes of $\hat{X}$ of fibrewise dimension 0 and length n and let $\mathrm{Sym}^n_B \hat{X}$ be the relative symmetric n-product of the fibration $\hat{X} \to B$. The Chow morphism $\mathrm{Hilb}^n(\hat{X}/B) \to \mathrm{Sym}^n_B \hat{X}$ induces an isomorphism $\mathrm{Hilb}^n(\hat{X}'/B) \simeq \mathrm{Sym}^n_B \hat{X}'$, where $\hat{X}' \to B$ is the smooth locus of $\hat{p} : \hat{X} \to B$.

Let us denote by $\bar{\mathcal{M}}(n, 0)$ the (coarse) moduli scheme of torsion-free and semistable sheaves of rank n and degree 0 on the fibres of $X \to B$ and by $\bar{\mathbf{M}}(n, 0)$ the corresponding moduli functor (see [22]). $\mathcal{M}(n, 0)$ will be the open subscheme of $\bar{\mathcal{M}}(n, 0)$ defined by those sheaves on fibres which are S-equivalent to a direct sum of line bundles, and $\mathbf{M}(n, 0)$ the corresponding moduli functor.

If $\mathcal{F}$ is a sheaf on $X \to B$ defining a B-valued point of $\mathbf{M}(n, 0)$, the spectral cover $C(\mathcal{F})$ is flat of degree n over B by Proposition 1.17, and then defines a B-valued point of $\mathrm{Hilb}^n(\hat{X}'/B)$ which depends only on the S-equivalence class of $\mathcal{F}$. This is still true when $\mathcal{F}$ is defined on $X_S \to S$ for an arbitrary base-change $S \to B$ so that we can define a morphism of functors $\mathbf{M}(n, 0) \to \mathrm{Hilb}^n(\hat{X}'/B)$. By definition of the coarse moduli scheme, this results in a morphism of B-schemes

$$\mathbf{C}' : \mathcal{M}(n, 0) \to \mathrm{Hilb}^n(\hat{X}'/B) \simeq \mathrm{Sym}^n_B \hat{X}'$$

defined over geometric points by $\mathbf{C}'([\mathcal{F}]) = C(\mathcal{F})$ where $[\mathcal{F}]$ is the point of $\bar{\mathcal{M}}(n, 0)$ defined by $\mathcal{F}$.

Theorem 2.1.

1. $\mathbf{C}' : \mathcal{M}(n, 0) \to \mathrm{Hilb}^n(\hat{X}'/B) \simeq \mathrm{Sym}^n_B \hat{X}'$ *is an isomorphism.*
2. $\mathbf{C}'$ *extends to an isomorphism of B-schemes* $\mathbf{C} : \bar{\mathcal{M}}(n, 0) \tilde{\to} \mathrm{Sym}^n_B \hat{X}$. *For every geometric point* $\mathcal{F} \sim \oplus_i(\mathcal{L}_i \oplus \cdots^{n_i} \oplus \mathcal{L}_i)$ *the image* $\mathbf{C}([\mathcal{F}])$ *is the point of* $\mathrm{Sym}^n_B \hat{X}$ *defined by* $n_1\xi_1^* + \cdots + n_r\xi_r^*$.

Proof.

1. To see that $\mathbf{C}'$ is an isomorphism we define a morphism $\mathbf{G} : \mathrm{Sym}^n_B \hat{X} \to \bar{\mathcal{M}}(n, 0)$ inducing the inverse isomorphism $\mathbf{G}' : \mathrm{Sym}^n_B \hat{X}' \to \mathcal{M}(n, 0)$. Such a morphism is uniquely determined by an S^n-equivariant functor morphism $\mathfrak{G} : \prod^n_B \hat{X}^\bullet \to \bar{\mathbf{M}}(n, 0)$, where S^n denotes the symmetric group. Let $S \to B$ be a B-scheme and let $\sigma : S \to \prod^n_B \hat{X}$ be a morphism of B-schemes, i.e. a family of points $\sigma_i : S \to \hat{X}$. We then define $\mathfrak{G}(\sigma) = [\oplus_i \mathcal{P}_i^*]$, where $\mathcal{P}_i(1 \times \sigma_i)^*\mathcal{P}$ is the sheaf on X_S defined by σ_i. Since $\mathbf{C}' \circ \mathbf{G}'$ and $\mathbf{G}' \circ \mathbf{C}'$ are the identity on closed points (by (2.1)), $\mathbf{C}'$ is an isomorphism.
2. We know (2.1) that $\mathbf{G}$ is bijective on closed points. If we prove that $\bar{\mathcal{M}}(n, 0)$ is normal, then Zariski's main theorem implies that $\mathbf{G}$ is an isomorphism, and $\mathbf{C} = \mathbf{G}^{-1}$ extends $\mathbf{C}'$. We first notice that $\mathrm{Sym}^n_B \hat{X}$ is a normal because B is normal of dimension greater than one. Since the codimension of $\bar{\mathcal{M}}(n, 0) - \mathcal{M}(n, 0)$ equals to the codimension of $\mathrm{Sym}^n_B \hat{X} - \mathrm{Sym}^n_B \hat{X}'$ which is greater than 1, $\bar{\mathcal{M}}(n, 0)$ is regular in codimension one.

By part (1) and the normality of $\mathrm{Sym}^n_B\,\hat{X}$, we have only to prove that $\bar{\mathcal{M}}(n,0)$ has depth ≥ 2 at every point ξ of $\bar{\mathcal{M}}(n,0) - \mathcal{M}(n,0)$ of codimension bigger than one. The image s of ξ in B is not the generic point because the fibre over the generic point is contained in $\mathcal{M}(n,0)$. Then we are reduced to see that $\bar{\mathcal{M}}(X_s,n,0)$ has depth ≥ 1 at ξ. Since ξ lies in the image of the closed immersion $\bar{\mathcal{M}}(X_s,n-1,0) \hookrightarrow \bar{\mathcal{M}}(X_s,n,0)$ given by $\mathcal{F} \mapsto \mathcal{F} \oplus \mathcal{L}_0$, we finish by induction on n. □

We denote by $J^n \to B$ the relative Jacobian of line bundles on $p : X \to B$ fibrewise of degree n. Similarly, $\hat{J}^n \to B$ is the relative degree n Jacobian of $\hat{p} : \hat{X} \to B$. Let us consider the following isomorphisms: $\tau : \hat{J}^n \tilde{\to} \hat{J}^0$ is the translation $\tau(\mathcal{L}) = \mathcal{L} \otimes \mathcal{O}_{\hat{X}}(-n\Theta)$, $\varpi^* : \hat{J}^0 \tilde{\to} J^0$ is the isomorphism induced by $\varpi : X \tilde{\to} \hat{X}$ and $\iota : J^0 \tilde{\to} J^0$ is the natural involution. Let $\gamma : \hat{J}^n \tilde{\to} J^0$ be the composition $\gamma = \iota \circ \varpi^* \circ \tau$. If $\xi_1 + \cdots + \xi_n$ is a positive divisor in $\hat{X}'_s$, then $\gamma[\mathcal{O}_{\hat{X}_t}(\xi_1 + \cdots + \xi_n)] = [\mathcal{L}_1^* \otimes \cdots \otimes \mathcal{L}_n^*]$, where $\xi_i = [\mathcal{L}_i]$. We have obtain the following theorem.

Theorem 2.2. *There is a commutative diagram of B-schemes*

$$\begin{tikzcd} \mathcal{M}(n,0) \arrow[r, "\mathbf{C}", "\sim"'] \arrow[d, "\det"'] & \mathrm{Sym}^n_B(\widehat{X}') \arrow[d, "\phi_n"] \\ J^0 & \widehat{J}^n \arrow[l, "\gamma"', "\sim"] \end{tikzcd}$$

where det *is the "determinant" morphism and* ϕ_n *the Abel morphism of degree n.*

The previous theorem generalizes Theorem 3.14 of [11] and can be considered as a global version of the results obtained in Section 4 of [14] about the relative moduli space of locally free sheaves on $X \to B$ whose restrictions to the fibres have rank n and trivial determinant. Theorem 2.2 leads to these results by using the standard structure theorems for the Abel morphism. The section $\hat{e} : B \hookrightarrow \hat{X}$ induces a section $\hat{e}_n : \mathrm{Sym}^{n-1}_B\,\hat{X} \hookrightarrow \mathrm{Sym}^n_B\,\hat{X}$ and $\tilde{\Theta}_n = \hat{e}_n(\mathrm{Sym}^{n-1}_B\,\hat{X})$ is the natural relative polarization for $\mathrm{Sym}^n_B\,\hat{X}$. Then, $\Theta_{n,0} = \mathbf{C}^{-1}(\hat{\Theta}_n)$ is a natural polarization for the moduli space $\bar{\mathcal{M}}(n,0)$ as a B-scheme. Let $\mathcal{L}_n$ be a universal line bundle over $q : \hat{X} \times_B \hat{J}^n \to \hat{J}^n$. The Picard sheaf $\mathcal{P}_n = R^1 q_*(\mathcal{L}_n^{-1} \otimes \omega_{\hat{X}/B})$ is a locally free sheaf of rank n and then defines a projective bundle $\mathbb{P}(\mathcal{P}_n^*) = \mathrm{Proj}\, S^\bullet(\mathcal{P}_n)$. The following result is well known (see, for instance, Ref. [3]).

Lemma 2.3. *There is a natural immersion of* $\hat{J}^n$*-schemes* $\mathrm{Sym}^n_B\,\hat{X}' \hookrightarrow \mathbb{P}(\mathcal{P}_n^*)$ *such that* $\tilde{\Theta}_n \cap \mathrm{Sym}^n_B\,\hat{X}'$ *is a hyperplane section. Moreover,* $\mathrm{Sym}^n_B\,\hat{X}'$ *is dense in* $\mathbb{P}(\mathcal{P}_n^*)$ *and the above immersion induces an isomorphism* $\mathrm{Sym}^n_{\mathcal{U}}\,\hat{X}_{\mathcal{U}} \tilde{\to} \mathbb{P}(\mathcal{P}^*_{n|\mathcal{U}})$.

If $\tilde{\mathcal{P}}_n = (\gamma^{-1})^* \mathcal{P}_n$, by Theorem 2.2 and Lemma 2.3 one has obtained the following proposition.

Proposition 2.4. *There is a natural immersion of* J^0*-schemes* $\mathcal{M}(n,0) \hookrightarrow \mathbb{P}(\tilde{\mathcal{P}}_n^*)$ *such that* $\Theta_{n,0}$ *is a hyperplane section. Moreover, if* $\mathcal{M}_{\mathcal{U}}(n,0)$ *is the pre-image of* $\mathcal{U}$ *by* $\mathcal{M}(n,0) \to B$, *the above immersion induces an isomorphism* $\mathcal{M}_{\mathcal{U}}(n,0) \tilde{\to} \mathbb{P}(\tilde{\mathcal{P}}^*_{n|\mathcal{U}})$.

Corollary 2.5.

$$\tilde{\mathcal{P}}_n \xrightarrow{\sim} (\det)_* \mathcal{O}_{\mathcal{M}(n,0)}(\Theta_{n,0}).$$

We now obtain the structure theorem proved in [14]: let $\mathcal{M}(n, \mathcal{O}_X) = (\det)^{-1}(\hat{e}(B))$ be the subscheme of those locally free sheaves in $\mathcal{M}(n,0)$ with trivial determinant and $\mathcal{M}_{\mathcal{U}}(n, \mathcal{O}_X) = \mathcal{M}(n, \mathcal{O}_X) \cap \mathcal{M}_{\mathcal{U}}(n,0)$.

Corollary 2.6. *There is a dense immersion of B-schemes $\mathcal{M}(n, \mathcal{O}_X) \hookrightarrow \mathbb{P}(\mathcal{V}_n)$, where $\mathcal{V}_n = p_*(\mathcal{O}_X(nH))$. Moreover, this morphism induces an isomorphism of $\mathcal{U}$-schemes $\mathcal{M}_{\mathcal{U}}(n, \mathcal{O}_X) \xrightarrow{\sim} \mathbb{P}(\mathcal{V}_{n|\mathcal{U}})$.*

Proof. It follows from $\hat{e}^*(\tilde{\mathcal{P}}_n) \xrightarrow{\sim} (p_* \mathcal{O}_X(nH))^*$. □

2.2. The Picard group and the dualizing sheaf of the moduli scheme

Theorem 2.1 and Proposition 2.4 enable us to compute the Picard group and the canonical series of the moduli scheme $\bar{\mathcal{M}}(n,0)$. We are assuming as in the former section that B is *normal* and the generic fibre is *smooth*.

Proposition 2.7. *There is a group immersion η : $\mathrm{Pic}(X) \hookrightarrow \mathrm{Pic}(\bar{\mathcal{M}}(n,0))$ defined by associating to a divisor D in X the closure in $\bar{\mathcal{M}}(n,0)$ of the divisor $(\det)^{-1}(\varpi(D)_{|J^0})$. Moreover, there is an isomorphism*

$$\mathrm{Pic}(\bar{\mathcal{M}}(n,0)) \simeq \eta(\mathrm{Pic}(X)) \oplus \Theta_{n,0} \cdot \mathbb{Z}.$$

Proof. By Theorem 2.2, the complement of $\mathcal{M}(n,0)$ has codimension at least 2 in $\bar{\mathcal{M}}(n,0)$. By Proposition 2.4, $\mathcal{M}(n,0)$ is a subscheme of $\mathbb{P}(\tilde{\mathcal{P}}_n^*)$ whose complement has codimension greater than 1 so that $\mathrm{Pic}(\mathcal{M}(n,0)) \simeq \mathrm{Pic}(\mathbb{P}(\tilde{\mathcal{P}}_n^*))$. Moreover, Corollary 2.5 implies that the class of the relative polarization $\Theta_{n,0}$ in $\mathrm{Pic}(\bar{\mathcal{M}}(n,0))$ goes to the class of $\mathcal{O}_{\mathbb{P}(\tilde{\mathcal{P}}_n^*)}(1)$ in $\mathrm{Pic}(\mathbb{P}(\mathcal{P}_n^*))$. Finally, $\mathrm{Pic}(\hat{X}) \xrightarrow{\sim} \mathrm{Pic}(J^0)$, and the result is now straightforward. □

When B is *smooth*, X (and $\hat{X}$) are Gorenstein. Let K_X be a canonical divisor in X in this case.

Proposition 2.8. *The Cartier divisor $K = \eta(K_X) - n\Theta_{n,0}$ is a canonical divisor of $\mathcal{M}(n,0)$.*

Proof. We have two open immersions $j : \mathcal{M}(n,0) \hookrightarrow \bar{\mathcal{M}}(n,0)$ and $h : \mathcal{M}(n,0) \hookrightarrow \mathbb{P}(\tilde{\mathcal{P}}_n^*)$ (Proposition 2.4), and then, a natural isomorphism between the restrictions of the dualizing sheaves $j^*(\omega_{\bar{\mathcal{M}}(n,0)}) \simeq h^*(\omega_{\mathbb{P}(\tilde{\mathcal{P}}_n^*)})$. Relative duality for the projective bundle $\Phi_n : \mathbb{P}(\tilde{\mathcal{P}}_n^*) \to J^0$ gives $\omega_{\mathbb{P}(\tilde{\mathcal{P}}_n^*)} \simeq \mathcal{O}_{\mathbb{P}(\tilde{\mathcal{P}}_n^*)}(-n) \otimes \Phi_n^*(\omega_{J^0})$, and then

$$h^*(\omega_{\mathbb{P}(\tilde{\mathcal{P}}_n^*)}) \simeq \mathcal{O}_{\mathcal{M}(n,0)}(\Theta_{n,0|\mathcal{M}(n,0)}) \otimes (\det)^{-1}(\omega_{\hat{X}|J^0}).$$

Moreover, since $\mathcal{M}(n, 0)$ is the smooth locus of $\bar{\mathcal{M}}(n, 0) \tilde{\to} \mathrm{Sym}^n_B(\hat{X})$ and this scheme is normal, we have $\omega_{\bar{\mathcal{M}}(n,0)} \simeq j_* j^*(\omega_{\mathcal{M}(n,0)})$, thus finishing the proof. □

3. Absolutely semistable sheaves on an elliptic surface

In this section, we apply the theory so far developed to the study of the moduli space of absolutely stable sheaves on an elliptic surface. The first step is the computation of the Chern character of the Fourier–Mukai transforms. This enables to the study of the preservation of stability. We shall see that stable sheaves on spectral covers transform to absolutely stable sheaves on the surface and prove that in this way one obtains an open subset of the moduli space of absolutely stable sheaves on the surface.

In the whole section, the base B is a *projective smooth curve* and the generic fibre is *smooth*.

3.1. Topological invariants of the Fourier–Mukai transforms

Let us denote by e the degree of the divisor E on B; we have $H \cdot p^*E = e = -H^2$ and $K_{X/B} = p^*E \equiv e\mu$ where μ is the class of a fibre of p. There are similar formulas for $\hat{\pi} : \hat{X} \to B$, namely $\Theta \cdot \hat{p}^*E = e = -\Theta^2$ and $K_{\hat{X}/B} = \hat{p}^*E \equiv e\hat{\mu}$.

By Proposition 1.1, the Todd class of the virtual relative tangent bundle of p is given by

$$\mathrm{td}(T_{X/B}) = 1 - \tfrac{1}{2} p^{-1}E + ew, \tag{3.1}$$

where w is the fundamental class of X. A similar formula holds for $\hat{p}$.

Let $\mathcal{F}$ be an object of $D(X)$. The topological invariants of the Fourier–Mukai transform $\mathbf{S}(\mathcal{F}) = R\hat{\pi}_*(\pi^*\mathcal{F} \otimes \mathcal{P})$ are computed by using the singular Riemann–Roch theorem for $\hat{\pi}$. This is allowed because $\hat{\pi}$ is an l.c.i. morphism since it is obtained from p by base change. By [15, Corollary 18.3.1], we have

$$\mathrm{ch}\,\mathbf{S}(\mathcal{F}) = \hat{\pi}_*[\pi^*(\mathrm{ch}\,\mathcal{F}) \cdot \mathrm{ch}(\mathcal{P})\,\mathrm{td}(T_{X/B})].$$

The Todd class $\mathrm{td}(T_{X/B})$ is readily determined from Eq. (3.1). The Chern character of $\mathcal{P}$ is computed from

$$\mathcal{P} = \mathcal{I} \otimes \pi^*\mathcal{O}_X(H) \otimes \hat{\pi}^*\mathcal{O}_{\hat{X}}(\Theta) \otimes q^*\omega^{-1},$$

where $\mathcal{I}$ is the ideal of the graph $\gamma : X \hookrightarrow X \times_B \hat{X}$ of $\varpi : X \tilde{\to} \hat{X}$ and $q = p \circ \pi = \hat{p} \circ \hat{\pi}$.

Lemma 3.1. *The Chern character of* $\mathcal{I}$ *is*

$$\mathrm{ch}(\mathcal{I}) = 1 - \gamma_*(1) - \tfrac{1}{2}\gamma_*(p^*E) + e\gamma_*(w).$$

Proof. $\mathcal{I} = (1 \times \varpi^{-1})^*\mathcal{I}_\Delta$ where $\mathcal{I}_\Delta$ is the ideal of the diagonal immersion $\delta : X \hookrightarrow X \times_B X$. We are then reduced to prove that $\mathrm{ch}(\mathcal{I}_\Delta) = 1 - \Delta - 1/2\delta_*(p^*E) + e\delta_*(w)$. We have $\mathrm{ch}(\mathcal{I}_\Delta) = 1 - \mathrm{ch}(\delta_*\mathcal{O}_X)$. Since δ is a perfect morphism [15, Corollary 18.3.1], singular Riemann–Roch gives $\mathrm{ch}(\delta_*\mathcal{O}_X) \cdot \mathrm{Td}(X \times_B X) = \delta_*(\mathrm{Td}(X))$. Moreover $X \times_B X$ is l.c.i.

because B is smooth and the corresponding virtual tangent bundle is $T_{X\times_B X}=\pi_2^*T_X+T_{\pi_2}$. Then

$$\mathrm{Td}(X)=\mathrm{td}(T_X)=1-\tfrac{1}{2}K_X+ew,$$
$$\mathrm{Td}(X\times_B X)=\mathrm{td}(T_{X\times_B X})=(1-\tfrac{1}{2}\pi_2^*K_X+e\pi_2^*w)\cdot(1-\tfrac{1}{2}q^*(E)+e\pi_1^*w)$$

by the same reference. A standard computation gives the formula. □

Proposition 3.2. *Let $\mathcal{F}$ be in $D(X)$. The Chern character of the Fourier–Mukai transform $\mathbf{S}(\mathcal{F})$ is*

$$\mathrm{ch}(\mathbf{S}(\mathcal{F}))=\hat{\pi}_*[\pi^*(\mathrm{ch}\,\mathcal{F})\cdot(1-\gamma_*(1)-\tfrac{1}{2}\gamma_*(p^*E)+e\gamma_*(w)) \cdot(1+\pi^*H-\tfrac{1}{2}ew)\cdot(1-\tfrac{1}{2}p^*E+ew)]\cdot(1+\Theta-\tfrac{1}{2}\hat{w})\cdot(1+e\hat{\mu}).$$

Corollary 3.3. *The first Chern characters of $\mathbf{S}(\mathcal{F})$ are*

$$\mathrm{ch}_0(\mathbf{S}(\mathcal{F}))=d,\quad \mathrm{ch}_1(\mathbf{S}(\mathcal{F}))=-\varpi(c_1(\mathcal{F}))+d\hat{p}^*E+(d-n)\Theta+(c-\tfrac{1}{2}ed+s)\hat{\mu},$$
$$\mathrm{ch}_2(\mathbf{S}(\mathcal{F}))=(-c-de+\tfrac{1}{2}ne)\hat{w},$$

where $n=\mathrm{ch}_0(\mathcal{F})$, $d=c_1(\mathcal{F})\cdot\mu$ is the relative degree, $c=c_1(\mathcal{F})\cdot H$ and $\mathrm{ch}_2(\mathcal{F})=sw$.

Similar calculations can be done for the inverse Fourier–Mukai transform.

Corollary 3.4. *Let $\mathcal{G}$ be in $D(\hat{X})$. The first Chern characters of $\hat{\mathbf{S}}(\mathcal{G})$ are*

$$\mathrm{ch}_0(\hat{\mathbf{S}}(\mathcal{G}))=\hat{d},$$
$$\mathrm{ch}_1(\hat{\mathbf{S}}(\mathcal{G}))=\varpi^{-1}(c_1(\mathcal{G}))-\hat{n}p^*E-(\hat{d}+\hat{n})H+(\hat{s}+\hat{n}e-\hat{c}-\tfrac{1}{2}e\hat{d})\mu,$$
$$\mathrm{ch}_2(\hat{\mathbf{S}}(\mathcal{G}))=-(\hat{c}+\hat{d}e+\tfrac{1}{2}\hat{n}e)w,$$

where $\hat{n}=\mathrm{ch}_0(\mathcal{G})$, $\hat{d}=c_1(\mathcal{G})\cdot\hat{\mu}$ is the relative degree, $\hat{c}=c_1(\mathcal{G})\cdot\Theta$ and $\mathrm{ch}_2(\mathcal{G})=\hat{s}\hat{w}$.

3.2. Pure dimension one sheaves on spectral covers

We know that if $S=B\times T$ and $\mathcal{F}$ is an S-flat sheaf on $X_S\to S$, fibrewise, torsion-free and semistable of rank n and degree 0, then $\mathcal{F}$ is WIT_1 and the spectral cover $C(\mathcal{F})\to S$ is finite of degree n and contains the support of the Fourier–Mukai transform $\hat{\mathcal{F}}$ (Proposition 1.17). We consider the spectral cover as a family of curves $C(\mathcal{F})_t\hookrightarrow\hat{X}$ $(t\in T)$ flat of degree n over B. As the curves $C(\mathcal{F})_t$ may fail to be integral we need to choose a polarization in them to be able to define rank, degree and stability.

We first consider the case of a single Cartier divisor C in $\hat{X}$ finite of degree n over B. The fibres of $\hat{p}$ define a polarization $\mu_C=\hat{\mu}\cap C$ on C.

Definition 3.5. The rank and the degree (with respect to μ_C) of a sheaf $\mathcal{G}$ on C are the rational numbers $r_C(\mathcal{G})$ and $d_C(\mathcal{G})$ determined by the Hilbert polynomial

$$P(\mathcal{G},m)=\chi(C,\mathcal{G}(m\mu_C))=r_C(\mathcal{G})n\cdot m+d_C(\mathcal{G})+r_C(\mathcal{G})\chi(C).$$

With this definition, rank and degree coincide with the standard ones when the curve is integral. Stability and semistability considered in terms of the slope $d_C(\mathcal{G})/r_C(\mathcal{G})$ are clearly equivalent with Simpson's [22].

In the relative case, given a Cartier divisor $C \hookrightarrow \hat{X} \times T$ such that $C \to B \times T$ is finite and flat of degree n, the relative curve $C \to T$ admits a relative polarization μ_C of relative degree n given by the fibres of $\hat{p}$. We define the relative rank and degree of a T-flat sheaf $\mathcal{G}$ on C as above.

Proposition 3.6. *Let g be the genus of B.*

1. *Let $\mathcal{G}$ be a rank n' sheaf on X. Assume that $\mathcal{G}$ is* WIT_1 *and that the support of $\hat{\mathcal{G}}$ is contained in C. Then $c_1(\mathcal{G}) \cdot \mu = 0$ and $\hat{\mathcal{G}}$ has rank n'/n on C and degree*
$$d_C(\hat{\mathcal{G}}) = c' - n'e + n'(1-g) - \frac{n'}{n}\chi(C)$$
with respect to μ_D, where $c' = c_1(\mathcal{G}) \cdot H$.
2. *Let $\mathcal{F}$ be a sheaf on $X \to B$ flat over B, fibrewise, torsion-free and semistable of rank n and degree* 0. *As a sheaf on the spectral cover $C(\mathcal{F})$, the Fourier–Mukai transform $\hat{\mathcal{F}}$ has pure dimension one, rank one and degree*
$$d_{C(\mathcal{F})}(\hat{\mathcal{F}}) = c - ne + n(1-g) - \chi(C(\mathcal{F})).$$

Proof.

1. By Corollary 3.3, we have
$$\mathrm{ch}(\hat{\mathcal{G}}(m\hat{\mu})) = [\varpi(c_1(\mathcal{G}) + n'm\hat{\mu} + n'\Theta - (c'+s')\hat{\mu}] + (c' - \tfrac{1}{2}n'e + n'm)\hat{w},$$
where $\mathrm{ch}_2(\mathcal{G}) = s'w$, and then $\chi(\hat{\mathcal{G}}(m\hat{\mu})) = n' \cdot m + c' + n'(1-g) - n'e$.
2. If there is a subsheaf $\mathcal{G}$ of $\hat{\mathcal{F}}$ concentrated on a zero-dimensional subscheme of $C(\mathcal{F})$, then $\mathcal{G}$ is WIT_0 as a sheaf on $\hat{X}$ and $\hat{\mathbf{S}}^0(\mathcal{G})$ is a subsheaf of $\mathcal{F}$ concentrated topologically on some fibres which is absurd. Then $\hat{\mathcal{F}}$ is of pure dimension 1. By 1, $\hat{\mathcal{F}}$ has rank one on $C(\mathcal{F})$ and degree $c - ne + n(1-g) - \chi(C(\mathcal{F}))$. □

Let $C \hookrightarrow \hat{X}$ be a Cartier divisor flat of degree n over B. We write $p = 1 - \chi(C)$ and $\ell = C \cdot \Theta$.

Lemma 3.7. *Let $\mathcal{L}$ be a sheaf on C of pure dimension one, rank one and degree r. As a sheaf on $\hat{X}$, $\mathcal{L}$ is* WIT_0 *and the inverse Fourier–Mukai transform $\hat{\mathcal{L}}$ is a B-flat sheaf on $X \to B$ fibrewise of rank n, torsion-free, of degree zero and semistable whose Chern character is $(n, \Delta(n,r,p,\ell), s)$, where $\Delta(n,r,p,\ell) = \varpi^{-1}(C) - nH + (r - p + 1 + n(g-1) - \ell)\mu$ and $s = s(n,\ell) = -(ne+\ell)w$.*

Proof. $\mathcal{L}$ is WIT_0 as a sheaf on $\hat{X}$ since it is concentrated on points. Moreover $\mathcal{L}$ is flat over B since B is a smooth curve. Thus $\hat{\mathcal{L}} = \hat{\mathbf{S}}^0(\mathcal{L})$ is a sheaf on X flat over B. Since the Chern characters of $\mathcal{L}$ as a sheaf on $\hat{X}$ are $\mathrm{ch}_0(\mathcal{L}) = 0$, $\mathrm{ch}_1(\mathcal{L}) = C$, $\mathrm{ch}_2(\mathcal{L}) = r - \frac{1}{2}C^2$, the formula for $\mathrm{ch}(\hat{\mathcal{L}})$ now follows from Corollary 3.4 and Proposition 3.6. Then $\hat{\mathcal{L}}$ has rank n and its relative degree is zero. Semistability follows from Proposition 1.11. □

3.3. Preservation of absolute stability

Let $C \hookrightarrow \hat{X}$ be a Cartier divisor flat of degree n over B.

Proposition 3.8. *Given $a > 0$, there exists $b_0 \geq 0$ depending only on $p = 1 - \chi(C)$ and $\ell = C \cdot \Theta$ such that for every $b \geq b_0$ and every sheaf $\mathcal{L}$ on C of pure dimension one, rank one, degree r and semistable with respect to μ_C, the Fourier–Mukai transform $\hat{\mathcal{L}}$ is semistable on X with respect to the polarization $aH + b\mu$. Moreover, if $\mathcal{L}$ is stable on C, then $\hat{\mathcal{L}}$ is stable as well on X.*

Proof. If the statement is not true, given a and b there exists a destabilizing sequence with respect to $H' = aH + b\mu$,

$$0 \to \mathcal{G} \to \hat{\mathcal{L}} \to \mathcal{E} \to 0, \tag{3.2}$$

where $\mathcal{G}$ is torsion-free of rank $n' < n$, $\mathcal{E}$ the torsion-free and H'-semistable and $[nc_1(\mathcal{G}) - n'c_1(\hat{\mathcal{L}})] \cdot H' > 0$. Let us write $c = c_1(\hat{\mathcal{L}}) \cdot H$, $c' = c_1(\mathcal{G}) \cdot H$, $c'' = c_1(\mathcal{E}) \cdot H$ and $d' = c_1(\mathcal{G}) \cdot \mu$. We have $d - d' \geq 0$ since $\hat{\mathcal{L}}$ is fibrewise semistable by Lemma 3.7; then $d' \leq 0$.

Assume first that $d' < 0$ and let ρ be the maximum of the integers $nc_1(\mathcal{F}) \cdot H - \mathrm{rk}(\mathcal{F})c$ for all non-zero subsheaves $\mathcal{F}$ of $\hat{\mathcal{L}}$. Then $[nc_1(\mathcal{G}) - n'c_1(\hat{\mathcal{L}})] \cdot H' = nac' - n'ac + nbd' \leq a\rho + nbd'$ is strictly negative for b sufficiently large, which is absurd.

Then $d' = 0$ and the destabilizing condition is $nc' > n'c$. We will get a contradiction by applying the Fourier–Mukai transform to Eq. (3.2). The sheaf $\mathcal{G}$ is WIT_1 since it is a subsheaf of $\hat{\mathcal{L}}$; $\mathcal{E}$ is WIT_1 as well by Proposition 1.11 because $\mathcal{E}_s$ is torsion-free and semistable of degree zero for every point $s \in B$. We then have an exact sequence of Fourier–Mukai transforms

$$0 \to \hat{\mathcal{G}} \to \mathcal{L} \to \hat{\mathcal{E}} \to 0.$$

By Proposition 3.6, $\hat{\mathcal{G}}$ has rank n'/n and degree $d_C(\mathcal{G}) = c' - n'e + n'(1-g) - \chi(C)n'/n$ on C and we have $r = c - ne + n(1-g) - \chi(C)$. The semistability of $\mathcal{L}$ implies $d_C(\hat{\mathcal{G}})/(n'/n) \leq r$; we then obtain $nc' \leq n'c$ which is absurd. The same argument proves the stability statement. □

Corollary 3.9. *In the situation of the previous proposition, if C is integral, then for every sheaf $\mathcal{L}$ on C of pure dimension one, rank one and degree r, the Fourier–Mukai transform $\hat{\mathcal{L}}$ is stable on X with respect to the polarization $aH + b\mu$.*

Proof. Every torsion-free, rank one sheaf on an integral curve is stable. □

Remark 3.10. In the case of non-integral spectral covers $C \to B$, the stability condition for the sheaf $\mathcal{L}$ on C is essential because even line bundles may fail to be semistable. One may then have *line bundles on C whose Fourier–Mukai transform is unstable*. Let us consider, for instance, the exact sequence

$$0 \to \hat{p}^*\omega \otimes \mathcal{O}_\Theta \to \hat{p}^*\omega^2 \otimes \mathcal{O}_C \to \hat{p}^*\omega^2 \otimes \mathcal{O}_\Theta \to 0,$$

where $C = 2\Theta$. The Fourier–Mukai transform of this sequence is the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{F} \to p^*\omega \to 0,$$

where $\mathcal{F}$ is the rank 2 vector bundle on X obtained as the Fourier–Mukai transform of the line bundle $\mathcal{L} = \hat{p}^*\omega^2 \otimes \mathcal{O}_C$ on C. One sees that $\mathcal{F}$ is unstable with respect to every polarization of the form $aH+b\mu$ unless $e = 0$. But according to Definition 3.5, one checks that the slope of $\mathcal{L}$ as a sheaf on C is $-4e$, whereas the slope of $\hat{p}^*\omega \otimes \mathcal{O}_\Theta$ is $-3e$. This proves that $\mathcal{L}$ is *unstable* on C, again unless $e = 0$, which agrees with Proposition 3.8. Actually, the structure sheaf $\mathcal{O}_C$ is unstable as well.

Friedman [11, Theorem 3.3], Friedman and Morgan [12] and O'Grady [20, Proposition I.1.6] have proved that for vector bundles of positive relative degree there exists a polarization on the surface such that absolute stability with respect to it is equivalent to the stability of the restriction to the generic fibre. For degree 0, the result is no longer true, but if we consider semistability instead of stability we can adapt O'Grady's proof to show the following.

Lemma 3.11. *Let us fix a Mukai vector (n, Δ, s) with $\Delta \cdot \mu = 0$. For every $a > 0$, there exists b_0 such that for every $b \geq b_0$ and every sheaf $\mathcal{F}$ on X with Chern character (n, Δ, s) and semistable with respect to the polarization $aH+b\mu$, the restriction of $\mathcal{F}$ to the generic fibre X_ν is semistable (ν is the generic point of B). In particular $\mathcal{F}$ is* WIT_1 (Corollary 1.12).

Proof. If the restriction $\mathcal{F}_\nu = \mathcal{F}_{|X_\nu}$ to the generic fibre is unstable, there exists a subsheaf $\mathcal{G}$ of $\mathcal{F}$ of rank $n' \leq n$ of fibrewise positive degree, $d' > 0$. Then there exists b_0 such that if $b > b_0$, $nc_1(\mathcal{G})\cdot(aH+b\mu) - n'c_1(\mathcal{F})\cdot(aH+b\mu) = a(nc_1(\mathcal{G}) - n'c_1(\mathcal{F}))\cdot H + bnd'$ is strictly positive, and $\mathcal{F}$ is unstable as well. Moreover, we can choose the integer b_0 independent of $\mathcal{F}$. Since we are considering sheaves with fixed Hilbert polynomial, there is only a finite number of possibilities for the Hilbert polynomials of the subsheaves $\mathcal{G}$ of the sheaves $\mathcal{F}$ with respect to a given polarization, and then there is also a finite number of possibilities for $c_1(\mathcal{G}) \cdot H$ and $d' = c_1(\mathcal{G}) \cdot \mu$. □

Let us write $\Delta = \Delta(n, r, p, \ell)$ and let $aH + b\mu$ be a polarization of X of the type considered in Lemma 3.11 for (n, Δ, s). Let $\mathcal{F}$ be a sheaf on X flat over B with Chern character (n, Δ, s) and *semistable* with respect to $aH + b\mu$. We assume $n > 1$. Then $\mathcal{F}$ is WIT_1 by Corollary 1.12 and the spectral cover $C(\mathcal{F})$ is finite over the open subset of the points $s \in B$ for which $\mathcal{F}_s$ is semistable (Proposition 1.17).

Proposition 3.12. *If the spectral cover $C(\mathcal{F})$ of $\mathcal{F}$ is finite over B, then $\hat{\mathcal{F}}$ is of pure dimension one, rank one, degree r and semistable on $C(\mathcal{F})$. Moreover, if $\mathcal{F}$ is stable on X, $\hat{\mathcal{F}}$ is stable on $C(\mathcal{F})$ as well.*

Proof. Let

$$0 \to \mathcal{G} \to \hat{\mathcal{F}} \to K \to 0 \tag{3.3}$$

be a destabilizing exact sequence on $C(\mathcal{F})$. We have an exact sequence of Fourier–Mukai transforms $0 \to \hat{\mathcal{G}} \to \mathcal{F} \to \hat{K} \to 0$. If we write $c = c_1(\mathcal{F}) \cdot H$, $c' = c_1(\hat{\mathcal{G}}) \cdot H$ and $n' = \mathrm{rk}(\hat{\mathcal{G}})$, then by the semistability of $\mathcal{F}$ with respect to $aH + b\mu$, we have $c'n \leq cn'$. By Proposition 3.6, $\hat{\mathcal{F}}$ has rank one on $C(\mathcal{F})$ and degree $c - ne + n(1-g) - \chi(C(\mathcal{F})) = r$ and $\mathcal{G}$ has rank n'/n on $C(\mathcal{F})$ and degree $c' - n'e + n'(1-g) - \chi(C(\mathcal{F}))n'/n$. The destabilizing condition for Eq. (3.3) now reads $nc' > n'c$, which is absurd. The proof of the stability is the same. □

Very recently, Jardim and Maciocia [16] and Yoshioka [25] have obtained stability results related with those in this section.

3.4. *Moduli of absolutely stable sheaves and compactified Jacobian of the universal spectral cover*

In this section, we shall prove that there exists a universal spectral cover over a Hilbert scheme and that the Fourier–Mukai transform embeds the compactified Jacobian of the universal spectral cover as an open subspace the moduli space of absolutely stable sheaves on the elliptic surface. Most of what is needed has been proven in the preceding section.

In this section, the base B is always a *smooth projective curve*. We start by describing the spectral cover of a relatively semistable sheaf in terms of the isomorphism $\bar{\mathcal{M}}(n,0) \tilde{\to} \mathrm{Sym}^n_B \hat{X}$ provided by Theorem 2.1. There is a "universal" subscheme

$$C \hookrightarrow \hat{X} \times_B \mathrm{Sym}^n_B \hat{X}$$

defined as the image of the closed immersion $\hat{X} \times_B \mathrm{Sym}^{n-1}_B \hat{X} \hookrightarrow \hat{X} \times_B \mathrm{Sym}^n_B \hat{X}$, $(\xi, \xi_1 + \cdots + \xi_{n-1}) \mapsto (\xi, \xi + \xi_1 + \cdots + \xi_{n-1})$. The natural morphism $g: C \to \mathrm{Sym}^n_B \hat{X}$ is finite and generically of degree n. Let $A: S \to \mathrm{Sym}^n_B \hat{X}$ be a morphism of B-schemes and let $C(A) = (1 \times A)^{-1}(C) \hookrightarrow \hat{X}_S$ be the closed subscheme of $\hat{X}_S$ obtained by pulling the universal subscheme back by the graph $1 \times A : \hat{X}_S \hookrightarrow \hat{X} \times_B \mathrm{Sym}^n_B \hat{X}$ of A. There is a finite morphism $g_A : C(A) \to S$ induced by g.

By Theorem 2.1, an S-flat sheaf $\mathcal{F}$ on X_S fibrewise, torsion-free and semistable of rank n and degree 0 defines a morphism $A: S \to \mathrm{Sym}^n_S(\hat{X}_S)$; we easily see from Lemma 1.15 the following proposition.

Proposition 3.13. *$C(A)$ is the spectral cover associated to $\mathcal{F}$, $C(A) = C(\mathcal{F})$.*

When $S = B$, A is merely a section of $\mathrm{Sym}^n_B \hat{X} \simeq \bar{\mathcal{M}}(n,0) \to B$. In this case, $\mathcal{C}(A) \to B$ is flat of degree n because it is finite and B is a smooth curve (see also Proposition 1.17). The same happens when the base scheme is of the form $S = B \times T$, where T is an arbitrary scheme.

Proposition 3.14. *For every morphism $A: B \times T \to \operatorname{Sym}^n_B \hat{X}$ of B-schemes, the spectral cover projection $g_A : C(A) \to B \times T$ is flat of degree n.*

If the section A takes values in $\operatorname{Sym}^n_B \hat{X}' \simeq \mathcal{M}(n,0) \to B$, then $g_A : C(A) \to B$ coincides with the spectral cover constructed in [14].

Let $\mathcal{H}$ be the Hilbert scheme of sections of the projection $\hat{\pi}_n : \operatorname{Sym}^n_B \hat{X} \to B$. If T is a k-scheme, a T-valued point of $\mathcal{H}$ is a section $B \times T \hookrightarrow \operatorname{Sym}^n_B \hat{X} \times T$ of the projection $\hat{\pi}_n \times 1 : \operatorname{Sym}^n_B \hat{X} \times T \to B \times T$, i.e. a morphism $B \times T \to \operatorname{Sym}^n_B \hat{X}$ of B-schemes. There is a universal section $\mathcal{A} : B \times \mathcal{H} \to \operatorname{Sym}^n_B \hat{X}$. It gives rise to a *"universal" spectral cover* $\mathcal{C}(\mathcal{A}) \hookrightarrow \hat{X} \times \mathcal{H}$. By Proposition 3.14, the "universal" spectral cover projection $g_A : \mathcal{C}(A) \to B \times \mathcal{H}$ is flat of degree n. It is endowed with a relative polarization $\Xi = g_{\mathcal{A}}^{-1}(\{s\} \times \mathcal{H})\ \ (s \in B)$.

Let $\bar{\mathbf{J}}^r \to \mathcal{H}$ be the functor of sheaves of pure dimension one, rank one, degree r (cf. Definition 3.5), and semistable with respect to Ξ on the fibres of the flat family of curves $\rho : \mathcal{C}(\mathcal{A}) \to \mathcal{H}$. A T-valued point of $\bar{\mathbf{J}}^r$ is then a pair $(A, |\mathcal{L}|)$ where A is a T-valued point of $\mathcal{H}$ (i.e. a morphism $A : B \times T \hookrightarrow \operatorname{Sym}^n_B \hat{X}$ of B-schemes) and $|\mathcal{L}|$ is the class of a sheaf $\mathcal{L}$ on the spectral cover $C(A)$, flat over T, and whose restrictions to the fibres of $\rho_T : C(A) \to T$ have pure dimension one, rank one, degree r and are semistable. Two such sheaves $\mathcal{L}, \mathcal{L}'$ are equivalent if $\mathcal{L}' \tilde{\to} \mathcal{L} \otimes \rho_T^* \mathcal{N}$, where $\mathcal{N}$ is a line bundle on T.

Let $\mathcal{H}_{p,\ell}$ be the subscheme of those points $h \in \mathcal{H}$ such that the Euler characteristic of $\rho^{-1}(h)$ is $1-p$ and $\rho^{-1}(h) \cdot \Theta = \ell$. The subscheme $\mathcal{H}_{p,\ell}$ is a disjoint union of connected components of $\mathcal{H}$ and then we can decompose ρ as a union of projections $\rho_{p,\ell} : \mathcal{C}(\mathcal{A})_{p,\ell} \to \mathcal{H}_{p,\ell}$. We decompose $\bar{\mathbf{J}}^r$ accordingly into functors $\bar{\mathbf{J}}^r_{p,\ell}$.

By Theorem 1.21 of [22] there exists a coarse moduli scheme $\bar{\mathcal{J}}^r_{p,\ell}$ for $\bar{\mathbf{J}}^r_{p,\ell}$ in the category of $\mathcal{H}_{p,\ell}$-schemes. It is projective over $\mathcal{H}_{p,\ell}$ and can be considered as a "compactified" relative Jacobian of the universal spectral cover $\rho_{p,\ell} : \mathcal{C}(\mathcal{A})_{p,\ell} \to \mathcal{H}_{p,\ell}$. The open subfunctor $\mathbf{J}^r_{p,\ell}$ of $\bar{\mathbf{J}}^r_{p,\ell}$ corresponding to stable sheaves has a fine moduli space $\mathcal{J}^r_{p,\ell}$ and it is an open subscheme of $\bar{\mathcal{J}}^r_{p,\ell}$.

On the other side, we can consider the coarse moduli scheme $\bar{\mathcal{M}}(a,b)$ torsion-free sheaves on X that are semistable with respect to $aH + b\mu$ and have Chern character (n, Δ, s) and the corresponding moduli functor $\bar{\mathbf{M}}(a,b)$ (see again [22]). Let $\mathcal{M}(a,b) \subset \bar{\mathcal{M}}(a,b)$ the open subscheme defined by the stable sheaves. It is a fine moduli scheme for its moduli functor $\mathbf{M}(a,b)$.

Given $a > 0$, let us fix b_0 so that Proposition 3.8 holds for p and ℓ and Lemma 3.11 holds for $(n, \Delta = \Delta(n,r,p,\ell), s)$, and take $b > b_0$.

Lemma 3.15. *The Fourier–Mukai transform induces morphisms of functors*

$$\hat{\mathbf{S}}^0 : \bar{\mathbf{J}}^r_{p,\ell} \hookrightarrow \bar{\mathbf{M}}(a,b), \qquad \hat{\mathbf{S}}^0 : \mathbf{J}^r_{p,\ell} \hookrightarrow \mathbf{M}(a,b)$$

that are representable by open immersions.

Proof. If T is a k-scheme and $(A, [\mathcal{L}])$ is a T-valued point of $\bar{\mathbf{J}}^r_{p,\ell}$, then $\hat{\mathbf{S}}^0_S(\mathcal{L})\ \ (S = B \times T)$ is a T-valued point of $\bar{\mathbf{M}}_{p,\ell}(a,b)$ by Proposition 3.8. Moreover, by the invertibility of the

Fourier–Mukai transform (Proposition 1.8), Proposition 3.12 and Corollary 1.5, $\hat{\mathbf{S}}^0_S$ is an isomorphism of $\bar{\mathbf{J}}^r_{p,\ell}$ with the subfunctor $\bar{\mathbf{M}}'_{p,\ell}(a,b)$ of those points of $\bar{\mathbf{M}}(a,b)$ whose spectral cover C is finite over $S = B \times T$ and verifies $\chi(C_t) = 1 - p$, $C_t \cdot \Theta = \ell$ for every $t \in T$. By Corollary 1.12, $\bar{\mathbf{M}}'_{p,\ell}(a,b)$ parameterizes precisely those semistable sheaves whose restriction to every fibre if semistable; $\bar{\mathbf{M}}'_{p,\ell}(a,b)$ is then an open subfunctor of $\bar{\mathbf{M}}(a,b)$ (Proposition 1.11). By Proposition 3.8, $\hat{\mathbf{S}}^0$ preserves stability and the statement for the stable case follows. □

Theorem 3.16. *The Fourier–Mukai transform gives a morphism $\hat{\mathbf{S}}^0 : \bar{\mathcal{J}}^r_{p,\ell} \to \bar{\mathcal{M}}(a,b)$ of schemes that induces an isomorphism*

$$\hat{\mathbf{S}}^0 : \mathcal{J}^r_{p,\ell} \xrightarrow{\sim} \mathcal{M}'_{p,\ell}(a,b),$$

where $\mathcal{M}'_{p,\ell}(a,b)$ is the open subscheme of those sheaves in $\mathcal{M}(a,b)$ whose spectral cover is finite over $S = B \times T$ and verifies $\chi(C_t) = 1 - p$, $C_t \cdot \Theta = \ell$ for every $t \in T$.

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