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Spacelike submanifolds of codimension two in de Sitter space

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ABSTRACT

We investigate the differential geometry of spacelike submanifolds of codimension two in de Sitter space and classify the singularities of lightlike hypersurfaces and lightcone Gauss maps in de Sitter 4-space.

1. Introduction

It is known that de Sitter space is a Lorentzian space form with positive curvature. The aim of this paper is to investigate the geometric meanings of the singularities of the lightlike hypersurfaces and the lightcone Gauss maps of spacelike submanifolds in de Sitter space as an application of the Legendrian singularity theory. We will give examples of de Sitter 4-space case. The de Sitter 4-space corresponds to the cosmic model and spacelike surfaces in de Sitter 4-space are submanifolds of codimension two.

Izumiya, Pei and Sano [1] investigated the extrinsic differential geometry of hypersurfaces in hyperbolic space by using the theory of Legendrian singularities. They observed the singularities of lightcone Gauss indicatrices and lightcone Gauss maps, which have geometrical meanings of spacelike hypersurfaces. Izumiya, Kossowski, Pei and Romero Fuster [2] investigated lightlike hypersurfaces of spacelike surfaces in Minkowski 4-space. Moreover, Izumiya and Romero Fuster [3] investigated spacelike submanifold of codimension two in general dimensional Minkowski space. They showed a Gauss–Bonnet type formula in terms of a Gauss–Kronecker curvature with respect to the lightlike normals. Fusho and Izumiya [4] investigated lightlike surfaces of spacelike curves in de Sitter 3-space by using the Frenet–Serret type formula and gave a classification of singularities of lightlike surfaces of generic spacelike curves, which are a cuspidal edge and a swallowtail (Figs. 1 and 2).

In [5] we investigated the singularities of lightcone Gauss images of spacelike hypersurfaces in de Sitter space, which is analogous to the case of hyperbolic space [1]. We are motivated to investigate the differential geometry of spacelike submanifolds of other codimension cases. The normal direction of the spacelike submanifold cannot be chosen uniquely. However, if we consider the codimension two case, we can determine the lightcone normal frames and define two maps called Gauss maps and lightlike hypersurfaces by using the analogous tools in [2,3].

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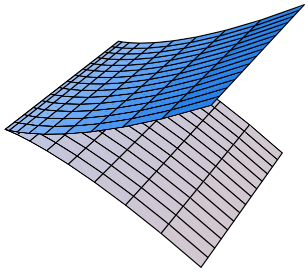

Fig. 1. Cuspidal edge.

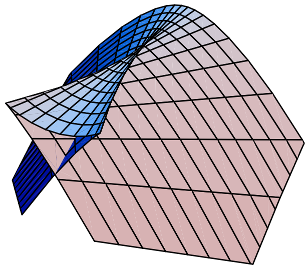

Fig. 2. Swallowtail.

In Section 2 we introduce the notion of the lightcone Gauss map, the normalized lightcone Gauss–Kronecker curvature and principal curvatures. The lightcone Gauss map does not depend on the choice of the future directed normal frame. In Section 3 we introduce the notions of the lightlike hypersurface and a family of functions that is called the Lorentzian distance squared function on the spacelike submanifold. The singular set of the lightlike hypersurface corresponds to the normalized lightcone principal curvatures of the spacelike submanifold, and this can be interpreted as the discriminant set of the family of height functions. In Sections 4 and 5 we discuss the contact of spacelike submanifolds with lightcones in de Sitter space. We apply the theory of Legendrian singularities for the study of lightcone Gauss images of generic spacelike submanifolds. In Sections 6 and 7 we introduce the notion of a family of functions that is called the lightcone height function. The singular set of the normalized lightcone Gauss map corresponds to the normalized lightcone parabolic set on the spacelike submanifold, and this can be interpreted as the discriminant set of the family of lightcone height functions. We discuss the contact of spacelike submanifolds with lightlike cylinders in de Sitter space. In Section 8 we classify the singularities of lightlike hypersurfaces and lightcone Gauss maps of generic spacelike surfaces in de Sitter 4-space, and give some examples which have their singularities.

2. Spacelike submanifolds in de Sitter space

In this section we construct the extrinsic differential geometry of spacelike submanifolds of codimension two in de Sitter space which is analogous to the theory in [3]. Let $\mathbb{R}^{n+1} = \{\mathbf{x} = (x_0, \ldots, x_n) \mid x_i \in \mathbb{R}\ (i = 0, \ldots, n)\}$ be an $(n+1)$-dimensional vector space. For any vectors $\mathbf{x} = (x_0, \ldots, x_n)$, $\mathbf{y} = (y_0, \ldots, y_n)$ in $\mathbb{R}^{n+1}$, the *pseudo-scalar product* of $\mathbf{x}$ and $\mathbf{y}$ is defined by $\langle \mathbf{x}, \mathbf{y} \rangle = -x_0y_0 + \sum_{i=1}^{n} x_iy_i$. We call $(\mathbb{R}^{n+1}, \langle , \rangle)$ a *Minkowski* $(n+1)$*-space* and write $\mathbb{R}^{n+1}_1$ instead of $(\mathbb{R}^{n+1}, \langle , \rangle)$.

We say that a vector $\mathbf{x} \in \mathbb{R}^{n+1}_1 \setminus \{\mathbf{0}\}$ is *spacelike*, *timelike* or *lightlike* if $\langle \mathbf{x}, \mathbf{x} \rangle > 0$, $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ or $\langle \mathbf{x}, \mathbf{x} \rangle < 0$ respectively. The norm of the vector $\mathbf{x} \in \mathbb{R}^{n+1}_1$ is defined by $\|x\| = \sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle|}$. For a vector $\mathbf{v} \in \mathbb{R}^{n+1}_1 \setminus \{\mathbf{0}\}$ and a real number c, we define a *hyperplane with pseudo-normal* $\mathbf{v}$ by $HP(\mathbf{v}, c) = \{\mathbf{x} \in \mathbb{R}^{n+1}_1 \mid \langle \mathbf{x}, \mathbf{v} \rangle = c\}$. We call $HP(\mathbf{v}, c)$ a *spacelike hyperplane*, *timelike hyperplane* or *lightlike hyperplane* if $\mathbf{v}$ is timelike, spacelike or lightlike respectively.

We now respectively define *hyperbolic n-space* and *de Sitter n-space* by

$$H^n_+(-1) = \{\mathbf{x} \in \mathbb{R}^{n+1}_1 \mid \langle \mathbf{x}, \mathbf{x} \rangle = -1, x_0 \geq 1\},$$
$$S^n_1 = \{\mathbf{x} \in \mathbb{R}^{n+1}_1 \mid \langle \mathbf{x}, \mathbf{x} \rangle = 1\}.$$

For any $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in \mathbb{R}^{n+1}_1$, we can define a vector $\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \cdots \wedge \mathbf{x}_n$ with the property $\langle \mathbf{x}, \mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_n \rangle = \det(\mathbf{x}, \mathbf{x}_1, \ldots, \mathbf{x}_n)$, so that $\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_n$ is pseudo-orthogonal to any $\mathbf{x}_i$ (for $i = 1, \ldots, n$) (c.f. [3]).

We also define a set $LC_a = \{\mathbf{x} \in \mathbb{R}^{n+1}_1 \mid \langle \mathbf{x} - \mathbf{a}, \mathbf{x} - \mathbf{a} \rangle = 0\}$, which is called a *closed lightcone* with vertex $\mathbf{a}$. We denote

$$LC^*_\pm = \{\mathbf{x} = (x_0, \ldots, x_n) \in LC_\mathbf{0} \mid x_0 > 0\ (x_0 < 0)\}$$

and call it the *future* (resp. *past*) *lightcone* at the origin.

Let $\mathbf{X} : U \longrightarrow S^n_1$ be an embedding from an open set $U \subset \mathbb{R}^{n-2}$. We say that $\mathbf{X}$ is *spacelike* in S^n_1 if $\{\mathbf{X}_{u_i}(u)\}_{i=1}^{n-2}$ are spacelike, where $u \in U$ and $\mathbf{X}_{u_i} = \partial\mathbf{X}/\partial u_i$. We identify $M = \mathbf{X}(U)$ with U through the embedding $\mathbf{X}$ and call M a *spacelike submanifold of codimension two* in S^n_1.

Since $\langle \mathbf{X}, \mathbf{X}\rangle \equiv 1$, we have $\langle \mathbf{X}_{u_i}, \mathbf{X}\rangle \equiv 0$ (for $i = 1, \ldots, n-2$). In this case, for any $p = \mathbf{X}(u)$, the pseudo-normal space N_pM is a timelike plane. we can choose a *future directed unit normal section* $\mathbf{n}^T(u) \in N_pM$ satisfying $\langle \mathbf{n}^T(u), \mathbf{X}(u)\rangle = 0$. Therefore we can construct a spacelike unit normal section $\mathbf{n}^S(u) \in N_pM$ by

$$\mathbf{n}^S(u) = \frac{\mathbf{n}^T(u) \wedge \mathbf{X}_{u_1}(u) \wedge \cdots \wedge \mathbf{X}_{u_{n-2}}(u)}{\|\mathbf{n}^T(u) \wedge \mathbf{X}_{u_1}(u) \wedge \cdots \wedge \mathbf{X}_{u_{n-2}}(u)\|},$$

and we have $\langle \mathbf{n}^T(u), \mathbf{n}^T(u)\rangle = -1$, $\langle \mathbf{n}^T(u), \mathbf{n}^S(u)\rangle = 0$, $\langle \mathbf{n}^S(u), \mathbf{n}^S(u)\rangle = 1$. Therefore vectors $\mathbf{n}^T(u) \pm \mathbf{n}^S(u)$ are lightlike. We call $(\mathbf{n}^T, \mathbf{n}^S)$ a *future directed normal frame along* $M = \mathbf{X}(U)$. The system $\{\mathbf{X}(u), \mathbf{n}^T(u), \mathbf{n}^S(u), \mathbf{X}_{u_1}(u), \ldots, \mathbf{X}_{u_{n-2}}(u)\}$ is a basis of $T_p\mathbb{R}^{n+1}_1$.

Lemma 2.1. *Given two future directed unit timelike normal sections* $\mathbf{n}^T(u), \bar{\mathbf{n}}^T(u) \in N_pM$, *the corresponding lightlike normal sections* $\mathbf{n}^T(u) \pm \mathbf{n}^S(u), \bar{\mathbf{n}}^T(u) \pm \bar{\mathbf{n}}^S(u)$ *are parallel.*

The proof is almost the same as that of Lemma 3.1 in [3], so that we omit it. Under the identification of M and U through $\mathbf{X}$, we have the linear mapping provided by the derivative of the *lightlike normal sections* $\mathbf{n}^T \pm \mathbf{n}^S$ at $p \in M$

$$d_p(\mathbf{n}^T \pm \mathbf{n}^S) : T_pM \longrightarrow T_p\mathbb{R}^{n+1}_1 = T_pM \oplus N_pM.$$

Consider two orthonormal projections $\pi^t : T_p\mathbb{R}^{n+1}_1 \longrightarrow T_pM$ and $\pi^n : T_p\mathbb{R}^{n+1}_1 \longrightarrow N_pM$. We define

$$d_p(\mathbf{n}^T \pm \mathbf{n}^S)^t = \pi^t \circ d_p(\mathbf{n}^T \pm \mathbf{n}^S),$$
$$d_p(\mathbf{n}^T \pm \mathbf{n}^S)^n = \pi^n \circ d_p(\mathbf{n}^T \pm \mathbf{n}^S).$$

We respectively call the linear transformation $S^{\pm}_p(\mathbf{n}^T, \mathbf{n}^S) = -d_p(\mathbf{n}^T \pm \mathbf{n}^S)^t$ an $(\mathbf{n}^T, \mathbf{n}^S)$-*shape operator of* $M = \mathbf{X}(U)$ at $p = \mathbf{X}(u)$.

The eigenvalues of $S^{\pm}_p(\mathbf{n}^T, \mathbf{n}^S)$ denoted by $\{\kappa^{\pm}_i(\mathbf{n}^T, \mathbf{n}^S)(p)\}_{i=1}^{n-2}$ are called the *lightcone principal curvatures* with respect to $(\mathbf{n}^T, \mathbf{n}^S)$ at p. Then the *lightcone Gauss–Kronecker curvature* with respect to $(\mathbf{n}^T, \mathbf{n}^S)$ at p is defined as

$$K^{\pm}_{\ell}(\mathbf{n}^T, \mathbf{n}^S)(p) = \det S^{\pm}_p(\mathbf{n}^T, \mathbf{n}^S).$$

We say that a point p is an $(\mathbf{n}^T, \mathbf{n}^S)$-*umbilic point* if all the principal curvatures coincide at p and thus $S^{\pm}_p(\mathbf{n}^T, \mathbf{n}^S) = \kappa^{\pm}\,\mathrm{id}_{T_pM}$ for some $\kappa^{\pm} \in \mathbb{R}$. We say that M is $(\mathbf{n}^T, \mathbf{n}^S)$-*totally umbilic* if all points on M are $(\mathbf{n}^T, \mathbf{n}^S)$-umbilic.

Since $\mathbf{X}_{u_i}$ $(i = 1, \ldots, n-2)$ are spacelike vectors, we have a *Riemannian metric* (or the *first fundamental form*) on M defined by $ds^2 = \sum_{i,j=1}^{n-2} g_{ij}du_idu_j$, where $g_{ij}(u) = \langle \mathbf{X}_{u_i}, \mathbf{X}_{u_j}\rangle$ for any $u \in U$. We also have a *lightcone second fundamental form* (or the *lightcone second fundamental invariant*) with respect to the normal vector field $(\mathbf{n}^T, \mathbf{n}^S)$ defined by $h^{\pm}_{ij}(u) = -\langle (\mathbf{n}^T \pm \mathbf{n}^S)_{u_i}, \mathbf{X}_{u_j}\rangle$ for any $u \in U$.

Lemma 2.2. *We have the following* lightconeWeingartenformula *with respect to* $(\mathbf{n}^T, \mathbf{n}^S)$.

$$(\mathbf{n}^T \pm \mathbf{n}^S)_{u_i} = \pm\langle \mathbf{n}^S, \mathbf{n}^T_{u_i}\rangle(\mathbf{n}^T \pm \mathbf{n}^S) - \sum_{j=1}^{n-2} h^{\pm j}_i(\mathbf{n}^T, \mathbf{n}^S)\mathbf{X}_{u_j},$$

where $(h^{j\pm}_i(\mathbf{n}^T, \mathbf{n}^S))_{ij} = \left(h^{\pm}_{ik}(\mathbf{n}^T, \mathbf{n}^S)\right)_{ik}(g^{kj})_{kj}$ *and* $(g^{kj})_{kj} = (g_{kj})^{-1}$. *Therefore we have*

$$\pi^t \circ (\mathbf{n}^T \pm \mathbf{n}^S)_{u_i} = -\sum_{j=1}^{n-2} h^{j\pm}_i(\mathbf{n}^T, \mathbf{n}^S)\mathbf{X}_{u_j}.$$

The proof is almost the same as that of Proposition 3.2 in [3], so that we omit it. Those formula induce an explicit expression of the lightcone Gauss–Kronecker curvature in terms of the Riemannian metric and the lightcone second fundamental invariant as follows:

$$K^{\pm}_{\ell}(\mathbf{n}^T, \mathbf{n}^S)(p) = \frac{\det(h^{\pm}_{ij}(\mathbf{n}^T, \mathbf{n}^S)(u))}{\det(g_{\alpha\beta})(u)}.$$

We say that a point p is an $(\mathbf{n}^T, \mathbf{n}^S)$-*parabolic point* if $K^{\pm}_{\ell}(\mathbf{n}^T, \mathbf{n}^S)(p) = 0$, and M is an $(\mathbf{n}^T, \mathbf{n}^S)$-*flat point* if p is $(\mathbf{n}^T, \mathbf{n}^S)$-umbilic and $K^{\pm}_{\ell}(\mathbf{n}^T, \mathbf{n}^S)(p) = 0$.

For a lightlike vector $v = (v_0, v_1, \ldots, v_n)$ we define $\widetilde{v} = (1, v_1/v_0, \ldots, v_n/v_0)$. By Lemma 2.1, if we choose another future directed unit timelike normal section $\bar{\mathbf{n}}^T(u)$, then we have $\widetilde{\mathbf{n}^T(u) \pm \mathbf{n}^S(u)} = \widetilde{\bar{\mathbf{n}}^T(u) \pm \bar{\mathbf{n}}^S(u)} \in S^{n-1}_+$. Therefore we define the *lightcone Gauss map* of $M = \mathbf{X}(U)$ as

$$\widetilde{\mathbb{L}}^{\pm} : U \longrightarrow S^{n-1}_{+}, \qquad \widetilde{\mathbb{L}}^{\pm}(u) = \widetilde{\mathbf{n}^{T}(u) \pm \mathbf{n}^{S}(u)}.$$

The lightcone Gauss map is analogous to the Minkowski space which is studied in [3]. This induces a linear mapping $d\widetilde{\mathbb{L}}^{\pm} : T_pM \longrightarrow T_p\mathbb{R}^{n+1}_1$ under the identification of U and M, where $p = \mathbf{X}(u)$. We have the following normalized lightcone Weingarten formula:

$$\pi^t \circ \widetilde{\mathbb{L}}^{\pm}_{u_i} = \frac{1}{\ell^{\pm}_0}(\pi^t \circ \mathbb{L}^{\pm}_{u_i}) = -\sum_{j=1}^{n-2} \frac{1}{\ell^{\pm}_0} h^{\pm j}_i(\mathbf{n}^T, \mathbf{n}^S)\mathbf{X}_{u_j},$$

where $\mathbb{L}^{\pm}(u) = (\ell^{\pm}_0(u), \dots, \ell^{\pm}_n(u))$.

We call linear transformation $S^{\pm}_p = -\pi^t \circ d\widetilde{\mathbb{L}}^{\pm}_p : T_pM \longrightarrow T_pM$ the *normalized lightcone shape operator of* M at p. The eigenvalues $\{\widetilde{\kappa}^{\pm}_i(p)\}_{i=1}^{n-2}$ of $\widetilde{S}^{\pm}_p$ are called *normalized lightcone principal curvatures*. By the above proposition, we have $\widetilde{\kappa}^{\pm}_i(p) = (1/\ell^{\pm}_0(u))\kappa^{\pm}_i(\mathbf{n}^T, \mathbf{n}^S)(p)$. The *normalized lightcone Gauss–Kronecker curvature of* M is defined to be $\widetilde{K}^{\pm}_{\ell}(u) = \det \widetilde{S}^{\pm}_p$. Then we have the following relation between the normalized lightcone Gauss–Kronecker curvature and the lightcone Gauss–Kronecker curvature:

$$\widetilde{K}^{\pm}_{\ell}(u) = \left(\frac{1}{\ell^{\pm}_0(u)}\right)^{n-2} K^{\pm}_{\ell}(\mathbf{n}^T, \mathbf{n}^S)(u).$$

It is clear from the corresponding definitions that the lightcone Gauss map, the normalized lightcone principal curvatures and the normalized lightcone Gauss–Kronecker curvature are independent on the choice of the normal frame $(\mathbf{n}^T, \mathbf{n}^S)$.

We say that a point $u \in U$ or $p = \mathbf{X}(u)$ is a *lightlike umbilic point* if $\widetilde{S}^{\pm}_p = \widetilde{\kappa}^{\pm}_p(p)\mathrm{id}_{T_pM}$. By the above proposition, p is a lightlike umbilic point if and only if p is a $(\mathbf{n}^T, \mathbf{n}^S)$-umbilic point for any $(\mathbf{n}^T, \mathbf{n}^S)$. We say that M is *totally lightlike umbilic* if all points on M are lightlike umbilic. We also say that p is a *lightlike parabolic point* (briefly $\widetilde{L}^{\pm}$-parabolic) if $\widetilde{K}^{\pm}_{\ell}(u) = 0$. Moreover, p is called a *lightlike flat point* if p is both lightlike umbilic and lightlike parabolic. The spacelike submanifold M in S^n_1 is called *totally lightlike flat* if every point in M is lightlike flat.

3. Lightlike hypersurfaces

In this section we define the Lorentzian distance squared function in order to study the singularities of lightlike hypersurfaces.

We define a hypersurface $LH^{\pm}_M : U \times \mathbb{R} \longrightarrow S^n_1$ by

$$LH^{\pm}_M(u, \mu) = \mathbf{X}(u) + \mu\widetilde{\mathbb{L}}^{\pm}(u).$$

We call $LH^{\pm}_M$ the *lightlike hypersurface along* M. It is analogous to the Minkowski 4-space which is studied in [2], and has been introduced by Izumiya and Fusho [4]. We introduce the notion of Lorentzian distance squared functions on spacelike submanifold of codimension two, which is useful for the study of singularities of lightlike hypersurfaces. We define a family of functions $G : U \times S^n_1 \longrightarrow \mathbb{R}$ on a spacelike submanifold M by

$$G(u, \lambda) = \langle \mathbf{X}(u) - \lambda, \mathbf{X}(u) - \lambda \rangle,$$

where $p = \mathbf{X}(u)$. We call G *Lorentzian distance squared function* on the spacelike submanifold M. For any fixed $\lambda_0 \in S^n_1$, we write $g_{\lambda_0}(u) = G(u, \lambda_0)$ and have following proposition.

Proposition 3.1. *Let M be a spacelike submanifold of codimension two and $G : U \times S^n_1 \longrightarrow \mathbb{R}$ the Lorentzian distance squared function on M. Suppose that $p_0 = \mathbf{X}(u_0) \neq \lambda_0$ and have the following:*

(1) $g_{\lambda_0}(u_0) = \partial g_{\lambda_0}(u_0)/\partial u_i = 0$ *($i = 1, \dots, n-2$) if and only if* $\lambda_0 = LH^{\pm}_M(u_0, \mu)$ *for some* $\mu \in \mathbb{R} \setminus \{\mathbf{0}\}$.
(2) $g_{\lambda_0}(u_0) = \partial g_{\lambda_0}(u_0)/\partial u_i = 0$ *($i = 1, \dots, n-2$) and* $\det \mathrm{Hess}\,(g_{\lambda_0})(u_0) = 0$ *if and only if* $\lambda_0 = LH^{\pm}_M(u_0, \mu_0)$ *for some* $\mu_0 \in \mathbb{R} \setminus \{\mathbf{0}\}$ *and* $-1/\mu_0$ *is one of the non-zero normalized lightcone principal curvatures* $\widetilde{\kappa}^{\pm}_i(p_0)$.

We now naturally interpret the lightlike hypersurface of the spacelike submanifold in S^n_1 as a wave front set in the theory of Legendrian singularities. Let $\pi^{\pm} : PT(S^n_1) \longrightarrow S^n_1$ be the projective cotangent bundles with canonical contact structures. Consider the tangent bundle $\tau^{\pm} : TPT^*(S^n_1) \longrightarrow PT^*(S^n_1)$ and the differential map $d\pi^{\pm} : TPT(S^n_1) \longrightarrow T(S^n_1)$ of $\pi^{\pm}$. For any $X \in TPT^*(S^n_1)$, there exists an element $\alpha \in T^*(S^n_1)$ such that $\tau^{\pm}(X) = [\alpha]$. For an element $V \in T_x(S^n_1)$, the property $\alpha(V) = 0$ does not depend on the choice of representative of the class $[\alpha]$. Thus, we can define the canonical contact structure on $PT^*(S^n_1)$ by

$$K = \{X \in TPT^*(S^n_1) \mid \tau^{\pm}(X)(d\pi^{\pm}(X)) = 0\}.$$

On the other hand, we consider a point $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \dots, \lambda_n) \in S^n_1$, then we have the relation $\lambda_i = \sqrt{\lambda_0^2 - \cdots - \lambda_{i-1}^2 - \lambda_{i+1}^2 - \cdots - \lambda_n^2 + 1} > 0$ for some i. So we adopt the coordinate system $(\lambda_1, \dots, \hat{\lambda}_i, \dots, \lambda_n)$ of the manifold S^n_1. Then we have the trivialization $PT^*(S^n_1) \equiv S^n_1 \times P\mathbb{R}^{n-1}$, and call $((\lambda_0, \dots, \lambda_n), [\xi_1 : \cdots : \xi_n])$ homogeneous coordinates of $PT^*(S^n_1)$, where $[\xi_1 : \cdots : \xi_n]$ are the homogeneous coordinates of the dual projective space $P\mathbb{R}^{n-1}$.

It is easy to show that $X_\bullet \in K^\pm_\bullet$ if and only if $\sum_{i=1}^n \mu_i \xi_i = 0$, where $\bullet = (x, [\xi])$ and $d\pi^\pm_\bullet(X_\bullet) = \sum_{i=1}^n \mu_i \partial/\partial v_i \in T_\bullet S^n_1$. An immersion $i : L \longrightarrow PT^*(S^n_1)$ is said to be a *Legendrian immersion* if $\dim L = n - 1$ and $di_q(T_q L) \subset K_{i(q)}$ for any $q \in L$. The map $\pi \circ i$ is also called the *Legendrian map* and the image $W(i) = \mathrm{image}(\pi \circ i)$, the *wave front* of i. Moreover, i (or the image of i) is called the *Legendrian lift* of $W(i)$.

Let $F : (\mathbb{R}^{n-1} \times \mathbb{R}^k, (u_0, \lambda_0)) \longrightarrow (\mathbb{R}, 0)$ be a function germ. We say that F is a *Morse family* of hypersurfaces if the map germ $\Delta^* F : (\mathbb{R}^{n-1} \times \mathbb{R}^k, (u_0, \lambda_0)) \longrightarrow (\mathbb{R}^n, \mathbf{0})$ defined by

$$\Delta^* F = \left(F, \frac{\partial F}{\partial u_1}, \dots, \frac{\partial F}{\partial u_{n-1}} \right)$$

is nonsingular. In this case, we have a smooth $(k-1)$-dimensional smooth submanifold,

$$\Sigma_*(F) = \left\{ (u, \lambda) \in (\mathbb{R}^{n-1} \times \mathbb{R}^k, (u_0, \lambda_0)) \;\middle|\; F(u, \lambda) = \frac{\partial F}{\partial u_1}(u, \lambda) = \cdots = \frac{\partial F}{\partial u_{n-1}}(u, \lambda) = 0 \right\},$$

and the map germ $\mathcal{L}_F : (\Sigma_*(F), (u_0, \lambda_0)) \longrightarrow PT^*\mathbb{R}^k$ defined by

$$\mathcal{L}_F(u, \lambda) = \left(v, \left[\frac{\partial F}{\partial u_1}(u, \lambda) : \cdots : \frac{\partial F}{\partial u_{n-1}}(u, \lambda) \right] \right)$$

is a Legendrian immersion germ. Then we have the following fundamental theorem of Arnol'd and Zakalyukin [6,7].

Proposition 3.2. *All Legendrian submanifold germs in $PT^*\mathbb{R}^k$ are constructed by the above method.*

We call F a generating family of $\mathcal{L}_F(\Sigma_*(F))$. Therefore the wave front is

$$W(\mathcal{L}_F) = \left\{ \lambda \in \mathbb{R}^k \;\middle|\; \exists u \in \mathbb{R}^{n-1} \text{ such that } F(u, \lambda) = \frac{\partial F}{\partial u_1}(u, \lambda) = \cdots = \frac{\partial F}{\partial u_{n-1}}(u, \lambda) = 0 \right\}.$$

We call it the *discriminant set* of F. By proceeding arguments, the lightlike hypersurface $LH^\pm_M$ is the discriminant set of the Lorentzian distance squared function G, and the singular point set of the lightlike hypersurface is a point $\lambda_0 = LH^\pm_M(u_0, -1/\widetilde{\kappa}^\pm_i(p_0))$. We have the following proposition.

Proposition 3.3. *Let G be the Lorentzian distance squared function on M. For any point $(u, \lambda) \in \Delta^* G^{-1}(\mathbf{0})$, G is a Morse family of hypersurfaces around (u, λ).*

Proof. For $\lambda = (\lambda_0, \dots, \lambda_n) \in S^n_1, \lambda_i \neq 0$ for some i. Without loss of generality, we assume that $\lambda_n > 0$ and local coordinates around λ in de Sitter space S^n_1 is given by $\lambda = (\lambda_0, \dots, \widehat{\lambda_k}, \dots, \lambda_{n-1})$, where $\lambda_n = \sqrt{1 + \lambda_0^2 - \lambda_1^2 - \cdots - \lambda_{n-1}}$. Jacobian of $\Delta^* G$ is given by

$$B(u, \lambda) = \begin{pmatrix} \left(-X_j(u) + \frac{X_n(u)}{\lambda_n} \lambda_j \right)_{j=0,\dots,n-1} \\ \hline \left(X_{j,u_i}(u) - \frac{X_{n,u_i}(u)}{\lambda_n} \lambda_j \right)_{\substack{j=0,\dots,n-1 \\ i=1,\dots,n-2}} \end{pmatrix}$$

where $\mathbf{X}(u) = (X_0(u), \dots, X_n(u)), \mathbf{X}_{u_i} = (X_{0,u_i}(u), \dots, X_{n,u_i}(u))$ for $(i = 1, \dots, n-1)$. On the other hand, $\lambda, \mathbf{X}(u), \mathbf{X}_{u_1}(u), \dots, \mathbf{X}_{u_{n-2}}$ are linearly independent on $(u, \lambda) \in \Delta^* G^{-1}(0)$, so that rank of $n \times (n-1)$ matrix

$$\begin{pmatrix} \lambda_0 & -\lambda_1 & \cdots & -\lambda_{n-1} & -\lambda_n \\ X_0(u) & -X_1(u) & \cdots & -X_{n-1}(u) & -X_n(u) \\ X_{0,u_1}(u) & -X_{1,u_1}(u) & \cdots & -X_{n-1,u_1}(u) & -X_{n,u_1}(u) \\ \vdots & \vdots & & \vdots & \vdots \\ X_{0,u_{n-2}}(u) & -X_{1,u_{n-2}}(u) & \cdots & -X_{n-1,u_{n-2}}(u) & -X_{n,u_{n-2}}(u) \end{pmatrix}$$

is n. We subtract the first row multiplied by $\mathbf{X}_n(u)/\lambda_n$ from the second row, and then subtract the first row multiplied by $\mathbf{X}_{n,u_k}(u)/\lambda_n$ from the $(2+k)$th row for $k = 1, \dots, n-2$. We have

$$\left(\begin{array}{c|c} \lambda_0 \ -\lambda_1 \ \cdots \ -\lambda_{n-1} & -\lambda_n \\ \hline & 0 \\ \mathrm{B}(u, \lambda) & \vdots \\ & 0 \end{array} \right).$$

Therefore $\operatorname{rank} B(u, \lambda) = n - 1$. This completes the proof. $\square$

Since G is a Morse family of hypersurfaces, we have the Legendrian immersion $\mathcal{L}_G^{\pm} : \Sigma_*(G) \longrightarrow PT^*(S_1^n)$ defined by

$$\mathcal{L}_G^{\pm}(u,\lambda) = \left(\lambda, \left[\frac{\partial G}{\partial \lambda_1}(u,\lambda) : \cdots : \widehat{\frac{\partial G}{\partial \lambda_k}}(u,\lambda) : \cdots : \frac{\partial G}{\partial \lambda_n}(u,\lambda)\right]\right)$$

where $\lambda = (\lambda_0, \ldots, \lambda_n)$ and $\Sigma_*(G) = (\Delta^* G)^{-1}(0) = \{(u,\lambda) \in U \times S_1^n \mid \lambda = LH_M^{\pm}(u,\mu),\ \mu \in \mathbb{R}\}$. We observe that G is a generating family of the Legendrian immersion $\mathcal{L}_G^{\pm}$ whose wave front set is the image of $LH_M^{\pm}$.

4. Contact with lightcones

In this section we use the theory of contacts between submanifolds due to Montaldi [8]. We define a set $LC(S_1^n)_{\lambda_0} = LC_{\lambda_0} \cap S_1^n$ and call it a de Sitter lightcone.

Proposition 4.1. *Let $\lambda_0 \in S_1^n$ and M be a spacelike submanifold of codimension two without umbilic points satisfying $\widetilde{K}_\ell \neq 0$. Then $M \subset LC(S_1^n)_{\lambda_0}$ if and only if λ_0 is an isolated singular value of the lightlike hypersurface $LH_M^{\pm}$ and $LH_M^{\pm}(U \times \mathbb{R}) \subset LC(S_1^n)_{\lambda_0}$.*

Proof. We assume that $M \subset LC(S_1^n)_{\lambda_0}$. By Proposition 3.1, there exists a smooth function $\mu : U \longrightarrow \mathbb{R}$ such that $\mathbf{X}(u) = \lambda_0 + \mu(u) \cdot (\widetilde{\mathbf{n}^T \pm \mathbf{n}^S})(u)$. Therefore, $LH_M^{\pm}(U \times \mathbb{R}) \subset LC(S_1^n)_{\lambda_0}$.

We now show that λ_0 is isolated singularity. It follows that

$$\frac{\partial LH_M^{\pm}}{\partial t}(u,t) = (\widetilde{\mathbf{n}^T + \mathbf{n}^S})(u)$$
$$\frac{\partial LH_M^{\pm}}{\partial u_i}(u,t) = \mu_{u_i}(u)(\widetilde{\mathbf{n}^T + \mathbf{n}^S})(u) + (t+\mu(u))(\widetilde{\mathbf{n}^T + \mathbf{n}^S})_{u_i}(u) \quad (i = 1, \ldots, n-2).$$

Then, we have

$$\begin{aligned} P(u) &:= \mathbf{X}(u) \wedge \frac{\partial LH_M^{\pm}}{\partial t}(u,t) \wedge \frac{\partial LH_M^{\pm}}{\partial u_1}(u,t) \wedge \cdots \wedge \frac{\partial LH_M^{\pm}}{\partial u_{n-2}}(u,t) \\ &= (t+\mu(u))^{n-2} \cdot \mathbf{X}(u) \wedge (\widetilde{\mathbf{n}^T + \mathbf{n}^S})(u) \wedge (\widetilde{\mathbf{n}^T + \mathbf{n}^S})_{u_1}(u) \wedge \cdots \wedge (\widetilde{\mathbf{n}^T + \mathbf{n}^S})_{u_{n-2}}(u). \end{aligned}$$

On the other hand, $\mathbf{X}(u) - \lambda_0 = \mu(u) \cdot (\widetilde{\mathbf{n}^T + \mathbf{n}^S})(u) \neq 0$ is a lightlike vector and T_pM are spacelike, so that $\mathbf{X}(u), \mathbf{X}(u) - \lambda_0, \mathbf{X}_{u_1}(u), \ldots, \mathbf{X}_{u_{n-2}}(u)$ are linearly independent. Therefore we have

$$\begin{aligned} \mathbf{0} &\neq \mathbf{X}(u) \wedge (\mathbf{X}(u) - \lambda_0) \wedge \mathbf{X}_{u_1}(u) \wedge \cdots \wedge \mathbf{X}_{u_{n-2}}(u) \\ &= \mu(u)^{n-1} \cdot \mathbf{X}(u) \wedge (\widetilde{\mathbf{n}^T + \mathbf{n}^S})(u) \wedge (\widetilde{\mathbf{n}^T + \mathbf{n}^S})_{u_1}(u) \wedge \cdots \wedge (\widetilde{\mathbf{n}^T + \mathbf{n}^S})_{u_{n-2}}(u) \end{aligned}$$

so that $\mathbf{X}(u) \wedge (\widetilde{\mathbf{n}^T + \mathbf{n}^S})(u) \wedge (\widetilde{\mathbf{n}^T + \mathbf{n}^S})_{u_1}(u) \wedge \cdots \wedge (\widetilde{\mathbf{n}^T + \mathbf{n}^S})_{u_{n-2}}(u) \neq \mathbf{0}$. Therefore $P(u) = 0$ if and only if $t + \mu(u) = 0$. This means that λ_0 is an isolated singular value of $LH_M^{\pm}$. The converse is trivial. □

We remark that this proposition is generalization of Proposition 4.1 in [2]. We now consider the contact of spacelike submanifolds of codimension two with lightcones due to Montaldi's result [8]. Let X_i and Y_i $(i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. We say that *the contact* of X_1 and Y_1 at y_1 is *the same type* as *the contact* of X_2 and Y_2 at y_2 if there is a diffeomorphism germ $\Phi : (\mathbb{R}^n, y_1) \longrightarrow (\mathbb{R}^n, y_2)$ such that $\Phi(X_1) = X_2$ and $\Phi(Y_1) = Y_2$. In this case we write $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$.

Two function germs $g_1, g_2 : (\mathbb{R}^n, a_i) \longrightarrow (\mathbb{R}, 0)$ $(i = 1, 2)$ are *$\mathcal{K}$-equivalent* if there are a diffeomorphism germ $\Phi : (\mathbb{R}^n, a_1) \longrightarrow (\mathbb{R}^n, a_2)$, and a function germ $\lambda : (\mathbb{R}^n, a_1) \longrightarrow \mathbb{R}$ with $\lambda(a_1) \neq 0$ such that $f_1 = \lambda \cdot (g_2 \circ \Phi)$. In [8] Montaldi has shown the following theorem.

Theorem 4.2. *(Montaldi [8]) Let X_i and Y_i (for $i = 1, 2$) be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. Let $g_i : (X_i, x_i) \longrightarrow (\mathbb{R}^n, y_i)$ be immersion germs and $f_i : (\mathbb{R}^n, y_i) \longrightarrow (\mathbb{R}^p, \mathbf{0})$ be submersion germs with $(Y_i, y_i) = (f_i^{-1}(\mathbf{0}), y_i)$. Then $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$. if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are $\mathcal{K}$-equivalent.*

Returning to lightlike hypersurfaces, we now consider the function $\mathcal{G} : S_1^n \times S_1^n \longrightarrow \mathbb{R}$ defined by $\mathcal{G}(x, \lambda) = \langle x - \lambda, x - \lambda \rangle$. For a given $\lambda_o \in S_1^n$, we denote $\mathfrak{g}_{\lambda_0}(x) = \mathcal{G}(x, \lambda_0)$, then we have $\mathfrak{g}_{\lambda_0}^{-1}(0) = LC(S_1^n)_{\lambda_0}$. For any $u_0 \in U$, we take the point $\lambda_0^{\pm} = \mathbf{X}(u_0) + \mu_0 \widetilde{L^{\pm}}(u_0)$ and have

$$(\mathfrak{g}_{\lambda_0^{\pm}} \circ \mathbf{X})(u_0) = \mathcal{G} \circ (\mathbf{X} \times \mathrm{id}_{S_1^n})(u_0, \lambda_0^{\pm}) = G(u_0, \lambda_0^{\pm}) = 0,$$

where $p_0 = \mathbf{X}(u_0)$ and $\mu_0 = -1/\widetilde{\kappa}_i^{\pm}(u_0)$, $(i = 1, \ldots, n-1)$. We also have

$$\frac{\partial(\mathfrak{g}_{\lambda_0^{\pm}} \circ \mathbf{X})}{\partial u_i}(u_0) = \frac{\partial G}{\partial u_i}(u_0, \lambda_0^{\pm}) = 0.$$

It follows that the lightcone $\mathfrak{g}_{\lambda_0^{\pm}}^{-1}(0) = LC(S_1^n)_{\lambda_0}$ is tangent to M at $p_0 = \mathbf{X}(u_0)$. In this case, we call each $LC_{\lambda_0^{\pm}}$ a *tangent lightcone* of M at p_0.

We now review some notions of Legendrian singularity theory to study the contact between hypersurfaces and de Sitter hyperhorospheres. We say that Legendrian immersion germs $i_j : (U_j, u_j) \longrightarrow (PT^*\mathbb{R}^n, p_j)$ $(j = 1, 2)$ are *Legendrian equivalent* if there exists a contact diffeomorphism germ $H : (PT^*\mathbb{R}^n, p_1) \longrightarrow (PT^*\mathbb{R}^n, p_2)$ such that H preserves fibers of π and $H(U_1) = U_2$. A Legendrian immersion germ at a point is said to be *Legendrian stable* if for every map with the given germ there are a neighborhood in the space of Legendrian immersions (in the Whitney C^∞-topology) and a neighborhood of the original point such that each Legendrian map belonging to the first neighborhood has in the second neighborhood a point at which its germ is Legendrian equivalent to the original germ.

Proposition 4.3 (*Zakalyukin [9]*). *Let i_1, i_2 be Legendrian immersion germs such that regular sets of $\pi \circ i_1$ and $\pi \circ i_2$ are respectively dense. Then i_1, i_2 are Legendrian equivalent if and only if corresponding wave front sets $W(i_1)$ and $W(i_2)$ are diffeomorphic as set germs.*

Let $F_i : (\mathbb{R}^n \times \mathbb{R}^k, (a_i, b_i)) \longrightarrow (\mathbb{R}, c)$ $(k = 1, 2)$ be k-parameter unfoldings of function germs f_i, we say F_1 and F_2 are $\mathcal{P}$–$\mathcal{K}$-equivalent if there exists a diffeomorphism germ $\Phi : (\mathbb{R}^n \times \mathbb{R}^k, (a_1, b_1)) \longrightarrow (\mathbb{R}^n \times \mathbb{R}^k, (a_2, b_2))$ of the form $\Phi(u, x) = (\phi_1(u, x), \phi_2(x))$ for $(u, x) \in \mathbb{R}^n \times \mathbb{R}^k$ and a function germ $\lambda : (\mathbb{R}^n \times \mathbb{R}^k, (a_1, b_1)) \longrightarrow \mathbb{R}$ such that $\lambda(a_1, b_1) \neq 0$ and $F_1(u, x) = \lambda(u, x) \cdot (F_2 \circ \Phi)(u, x)$.

Theorem 4.4 (*Arnol'd, Zakalyukin [6,7]*). *Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ be Morse families and denote the corresponding Legendrian immersion germs by $\mathcal{L}_F, \mathcal{L}_G$. Then*

(1) *$\mathcal{L}_F$ and $\mathcal{L}_G$ are Legendrian equivalent if and only if F and G are $\mathcal{P}$–$\mathcal{K}$-equivalent.*
(2) *$\mathcal{L}_F$ is Legendrian stable if and only if F is $\mathcal{K}$-versal deformation of f.*

Let $LH_{M,i}^{\pm} : (U, u_i) \longrightarrow (S_1^n, \lambda_i^{\pm})$ (for $i = 1, 2$) be lightlike hypersurface germs of $\mathbf{X}_i : (U, u_i) \longrightarrow (S_1^n, \lambda_i)$. We say that $LH_{M,1}^{\pm}$ and $LH_{M,2}^{\pm}$ are $\mathcal{A}$-equivalent if and only if there exist diffeomorphism germs $\phi : (U, u_1) \longrightarrow (U, u_2)$ and $\Phi : (S_1^n, \lambda_1^{\pm}) \longrightarrow (S_1^n, \lambda_2^{\pm})$ such that $\Phi \circ \mathbb{L}_1^{\pm} = \mathbb{L}_2^{\pm} \circ \phi$. We denote $g_{i,\lambda_i^{\pm}} : (U, u_i) \longrightarrow (\mathbb{R}, \mathbf{0})$ by $g_{i,\lambda_i^{\pm}}(u) = G_i(u, \lambda_i^{\pm})$. Then we have $g_{i,\lambda_i^{\pm}}(u) = (\mathfrak{g}_{i,\lambda_i^{\pm}} \circ \mathbf{X}_i)(u)$. By Theorem 4.2,

$$K(\mathbf{X_1}(U), LC_{\lambda_1^{\pm}}; \lambda_1^{\pm}) = K(\mathbf{X_2}(U), LC_{\lambda_2^{\pm}}; \lambda_2^{\pm})$$

if and only if $g_{1,\lambda_1^{\pm}}$ and $g_{2,\lambda_2^{\pm}}$ are $\mathcal{K}$-equivalent.

Let $Q^{\pm}(\mathbf{X}, u_0)$ be the local ring of the function germ $g_{\lambda_0^{\pm}} : (U, u_0) \longrightarrow \mathbb{R}$ defined by

$$Q^{\pm}(\mathbf{X}, u_0) = C_{u_0}^{\infty}(U)/\langle g_{\lambda_0^{\pm}} \rangle_{C_{u_0}^{\infty}(U)},$$

where $\lambda_0 = LH_M^{\pm}(u_0, \mu_0)$ and $C_{u_0}^{\infty}(U)$ is the local ring of function germs at u_0 with the unique maximal ideal $\mathfrak{M}$.

Proposition 4.5. *Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ be Morse families. Suppose that Legendrian immersion germs $\mathcal{L}_F$ and $\mathcal{L}_G$ are Legendrian stable, then the following conditions are equivalent:*

(1) *$(W(\mathcal{L}_F), \lambda)$ and $(W(\mathcal{L}_G), \lambda')$ are diffeomorphic as set germs.*
(2) *$\mathcal{L}_F$ and $\mathcal{L}_G$ are Legendrian equivalent.*
(3) *$Q(f)$ and $Q(g)$ are isomorphic as $\mathbb{R}$-algebras, where $f = F|_{\mathbb{R}^k \times \{\mathbf{0}\}}$ and $g = G|_{\mathbb{R}^k \times \{\mathbf{0}\}}$.*

The proof is almost the same as that of Theorem 6.3 in [1], so that we omit it. By the above propositions, we have following theorem.

Theorem 4.6. *Let $\mathbf{X}_i : (U, u_i) \longrightarrow (S_1^n, p_i)$ (for $i = 1, 2$) be spacelike submanifold germs such that the corresponding Legendrian immersion germs are Legendrian stable. Then the following conditions are equivalent:*

(1) *Lightlike hypersurface germs $LH_{M,1}^{\pm}$ and $LH_{M,2}^{\pm}$ are $\mathcal{A}$-equivalent.*
(2) *Legendrian immersion germs $\mathcal{L}_1^{\pm}$ and $\mathcal{L}_2^{\pm}$ are Legendrian equivalent.*
(3) *Lorentzian distance squared function germs G_1 and G_2 are $\mathcal{P}$–$\mathcal{K}$-equivalent.*
(4) *$g_{1,\lambda_1}^{\pm}$ and $g_{2,\lambda_2}^{\pm}$ are $\mathcal{K}$-equivalent.*
(5) $K(\mathbf{X_1}(U), LC_{\lambda_1^{\pm}}; p_1) = K(\mathbf{X_2}(U), LC_{\lambda_2^{\pm}}; p_2)$
(6) *Local rings $Q^{\pm}(\mathbf{X}_1, u_1)$ and $Q^{\pm}(\mathbf{X}_2, u_2)$ are isomorphic as $\mathbb{R}$-algebras.*

Proof. Since $LH^{\pm}_{M,1}$ and $LH^{\pm}_{M,2}$ are Legendrian stable, regular sets of $LH^{\pm}_{M,1}$ and $LH^{\pm}_{M,2}$ are respectively dense, by Proposition 4.3, the conditions (1) and (2) are equivalent. And we apply Theorem 4.4, the conditions (2) and (3) are equivalent. By the previous arguments from Theorem 4.2, the conditions (4) and (5) are equivalent. If we assume the condition (3), then $\mathcal{P}$–$\mathcal{K}$-equivalence preserves the $\mathcal{K}$-equivalence, so that the condition (4) holds. Since the local ring $Q^{\pm}(\mathbf{X}_i, u_i)$ is $\mathcal{K}$-invariant, this means that the condition (6) holds. By Proposition 4.5, the condition (6) implies the condition (2). □

In the next section, we will prove that the assumption of the Theorem 4.6 is a generic property in the case when $n \le 6$. In general we have the following proposition.

Proposition 4.7. *Let $\mathbf{X}_i : (U, u_i) \longrightarrow (S^n_1, p_i)$ (for $i = 1, 2$) be spacelike submanifold germs and regular sets of their lightlike surfaces $LH^{\pm}_{M,i}$ are dense in U. If lightlike hypersurface germs $LH^{\pm}_{M,1}$ and $LH^{\pm}_{M,2}$ are $\mathcal{A}$-equivalent, then*

$$K(\mathbf{X}_1(U), LC_{\lambda_1^{\pm}}; p_1) = K(\mathbf{X}_2(U), LC_{\lambda_2^{\pm}}; p_2).$$

In this case, $(\mathbf{X}_1^{-1}(LC_{\lambda_1^{\pm}}), u_1)$ and $(\mathbf{X}_2^{-1}(LC_{\lambda_2^{\pm}}), u_2)$ are diffeomorphic as set germs.

Proof. By Proposition 4.3, if $LH^{\pm}_{M,1}$ and $LH^{\pm}_{M,1}$ are $\mathcal{A}$-equivalent, then $\mathcal{L}^{\pm}_1$ and $\mathcal{L}^{\pm}_2$ are Legendrian equivalent. By Theorem 4.4, G_1 and G_2 are $\mathcal{P}$–$\mathcal{K}$-equivalent, so that $g_{1,\lambda_1^{\pm}}$ and $g_{2,\lambda_2^{\pm}}$ are $\mathcal{K}$-equivalent. Applying Theorem 4.2, the first assertion holds. On the other hand, $g^{-1}_{i,\lambda_i^{\pm}}(0) = (\mathbf{X}_i^{-1}(LC_{\lambda_i^{\pm}}), u_i)$ and $\mathcal{K}$-equivalence preserves the zero level sets, so that $(\mathbf{X}_1^{-1}(LC_{\lambda_1^{\pm}}), u_1)$ and $(\mathbf{X}_2^{-1}(LC_{\lambda_2^{\pm}}), u_2)$ are diffeomorphic as set germs. □

5. Generic properties

In this section we consider generic properties of spacelike submanifolds in S^n_1. We consider the space of spacelike embeddings Sp-Emb(U, S^n_1) with Whitney C^∞-topology. We define a function $\mathcal{G} : S^n_1 \times S^n_1 \longrightarrow \mathbb{R}$ by $\mathcal{G}(x, \lambda) = \langle x, \lambda\rangle$, and denote $\mathfrak{g}_x(\lambda) = \mathcal{G}(x, \lambda)$. Then $\mathfrak{g}_x$ is a submersion at $x \ne \lambda$ for any $\lambda \in S^n_1$. For any spacelike submanifolds $x \in$ Sp-Emb(U, S^n_1), we have $G = \mathcal{G} \circ (x \times \mathrm{id}_{S^n_1})$. We also have the ℓ-jet extension $j^\ell_1 G : U \times S^n_1 \longrightarrow J^\ell(U, \mathbb{R})$ defined by $j^\ell_1 G(x, \lambda) = j^\ell g_\lambda(u)$. We consider the trivialization $J^\ell(U, \mathbb{R}) \equiv U \times \mathbb{R} \times J^\ell(n-1, 1)$. For any submanifold $Q \subset J^\ell(n-1, 1)$, we denote $\widetilde{Q} = U \times \{0\} \times Q$. Then we have the following proposition as a corollary of Lemma 6 of Wassermann [10].

Proposition 5.1. *Let Q be a submanifold of $J^\ell(n-1, 1)$. Then the set*

$$T_Q = \{x \in \textit{Sp-Emb}(U, S^n_1) \mid j^\ell_1 G \textit{ is transversal to } \widetilde{Q}\}$$

is a residual subset of Sp-Emb(U, S^n_1). If Q is a closed subset, then T_Q is open.

We remark that if the corresponding Lorentzian distance squared function g_{λ_0} is ℓ-deter- mined relative to $\mathcal{K}$, then G is a $\mathcal{K}$-versal deformation if and only if $j^\ell_1 G$ is transversal to $\widetilde{\mathcal{K}}^\ell_{g,\lambda_0}$, where $\mathcal{K}^\ell_{g,\lambda_0}$ is the $\mathcal{K}$-orbit through $j^\ell g_{\lambda_0}(\mathbf{0}) \in J^\ell(n-1, 1)$. Applying Theorem 4.4, this condition is equivalent to the condition that the corresponding Legendrian immersion germ is Legendrian stable. From the previous arguments and the Appendix of [2], we have the following proposition. (See also [6].)

Theorem 5.2. *if $n \le 6$, there exists an open subset $\mathcal{O} \subset$ Sp-Emb(U, S^n_1) such that for any $x \in \mathcal{O}$, the corresponding Legendrian immersion germ $\mathcal{L}$ is Legendrian stable.*

6. Lightcone Gauss maps and lightcone height functions

In this section, we define the lightcone height function whose wave front set is the image of the lightcone Gauss map.

We define a *lightcone height function* $H : U \times S^{n-1}_+ \longrightarrow \mathbb{R}$ by $H(u, v) = \langle X(u), v\rangle$. For $v_0 \in S^{n-1}_+$, we write $h_{v_0}(u) = H(u, v_0)$ and have following proposition.

Proposition 6.1. *Let H be the lightcone height function of spacelike submanifold $\mathbf{X}$, then we have the following:*

(1) *$H(u_0, v_0) = H_{u_i}(u_0, v_0) = 0$ ($i = 1, \ldots, n-2$) if and only if $v_0 = \widetilde{\mathbb{L}}^{\pm}(u_0)$.*

(2) *$H(u_0, v_0) = H_{u_i}(u_0, v_0) = 0$ ($i = 1, \ldots, n-2$) and* $\det \mathrm{Hess}\,(h_{v_0})(u_0)$ *if and only if $v_0 = \widetilde{\mathbb{L}}^{\pm}(u_0)$ and $\widetilde{K}^{\pm}_\ell(u_0) = 0$.*

Proof. Let $v_0 = \lambda \mathbf{X}(u_0) + \eta^T \mathbf{n}^T(u_0) + \eta^S \mathbf{n}^S(u_0) + \sum_{j=1}^{n-2} \xi_j \mathbf{X}_j(u_0)$ for some $\lambda, \eta^T, \eta^S, \xi_j \in \mathbb{R}$. By the assumption, we have $\lambda = 0$, $|\eta^T| = |\eta^S|$ and $\bar{\mathbf{H}}'(u_0, v_0) = (g_{ij}(u_0))\,\bar{\boldsymbol{\xi}}$, where $\bar{\mathbf{H}}' = {}^t(H_{u_1}, \ldots, H_{u_{n-2}})$, $\bar{\boldsymbol{\xi}} = {}^t(\xi_1, \ldots, \xi_{n-2})$ and (g_{ij}) is the first fundamental form on M. Since $(g_{ij}(u_0))$ is regular, $\bar{\mathbf{H}}'(u_0, v_0) = \mathbf{0}$ if and only if $\bar{\boldsymbol{\xi}} = \mathbf{0}$. Therefore we have $v_0 = \widetilde{\mathbb{L}}^{\pm}(u_0)$. The converse of (1) is trivial. By the calculation,

$$\left(\frac{\partial^2 H}{\partial u_i \partial u_j}(u_0, v_0)\right)_{ij} = \left(\langle \mathbf{X}_{u_i u_j}(u_0), \widetilde{\mathbb{L}}^{\pm}(u_0)\rangle\right)_{ij} = \frac{1}{\ell^{\pm}_0(u_0)}\left(h^{\pm}_{ij}(u_0)\right),$$

where $\ell_0^{\pm}(u_0)$ is the first component of $\widetilde{\mathbb{L}}^{\pm}(u_0)$ and $(h_{ij}^{\pm}(u_0))$ is the lightcone second fundamental form with respect to the lightcone normal frame $(\mathbf{n}^T, \mathbf{n}^S)$. Therefore Hess $H(u_0, v_0)$ is degenerate if and only if u_0 is a lightcone parabolic point. This completes the proof. □

By the above proposition, the discriminant set of the lightcone height function is given by

$$D_H = \left\{ v \in S_+^{n-1} \,\middle|\, v = \widetilde{\mathbb{L}}^{\pm}(u), u \in U \right\}$$

which is the image of the lightcone Gauss map of M. The singular set of the lightcone Gauss map is the normalized lightcone parabolic set of M.

Proposition 6.2. *Let H is the lightcone height function on M. Then H is a Morse family of hypersurfaces around $(u, v) \in \Delta^* H^{-1}(0)$.*

Proof. We denote that $\mathbf{X}(u) = (X_0(u), \dots, X_n(u))$, $\mathbf{X}_{u_i}(u) = (X_{0,u_i}(u), \dots, X_{n,u_i}(u))$ and $v = (v_0, \dots, v_n)$. Without the loss of generality, we assume that $v_n > 0$. Therefore we denote a matrix B and C by

$$\mathrm{B} = \left(\begin{array}{c} \left(X_j(u) - \frac{v_j}{v_n} X_n(u) \right)_{j=1,\dots,n-1} \\ \hline \left(X_{j,u_i}(u) - \frac{v_j}{v_n} X_{n,u_i}(u) \right)_{\substack{j=1,\dots,n-1 \\ i=1,\dots,n-2}} \end{array} \right), \qquad \mathrm{C} = \begin{pmatrix} 1 \quad 0 \ \cdots \quad 0 \\ \widetilde{\mathbb{L}}^{\pm}(u) \\ \mathbf{X}(u) \\ \mathbf{X}_{u_1}(u) \\ \vdots \\ \mathbf{X}_{u_{n-2}}(u) \end{pmatrix}.$$

Then we have $J(\Delta^* H) = (*|B)$ and $\det B = (-1)^{n-2} \det C / v_n$.

On the other hand, determinant of a matrix

$$\mathrm{C} \begin{pmatrix} -1 & 0 & & \\ 0 & 1 & & \mathrm{O} \\ & & \ddots & 0 \\ \mathrm{O} & & 0 & 1 \end{pmatrix} {}^t\mathrm{C} = \left(\begin{array}{cc|ccc} -1 & -1 & * & \cdots & * \\ -1 & 0 & 0 & \cdots & 0 \\ \hline * & 0 & & & \\ \vdots & \vdots & 1 & \mathrm{O} & \\ * & 0 & \mathrm{O} & (g_{ij}) & \end{array} \right)$$

equals to $-\det(g_{ij}) \neq 0$, where (g_{ij}) is the first fundamental form on M. This implies that both B and C are regular, therefore $\operatorname{rank} J(\Delta^* H) = n - 1$. This completes the proof. □

By Proposition 3.2 and the above proposition, we have the Legendrian immersion $\mathcal{L}_H^{\pm} : \Sigma_*(H) \longrightarrow PT^*(S_+^{n-1})$ defined by

$$\mathcal{L}_H^{\pm}(u, v) = \left(\lambda, \left[\frac{\partial H}{\partial v_1}(u, v) : \cdots \widehat{\frac{\partial H}{\partial v_k}}(u, v) : \cdots \frac{\partial H}{\partial v_n}(u, v) \right] \right)$$

where $v = (v_0, v_1, \dots, v_n) \in S_+^{n+1}$ and $\Sigma_*(H) = \{(u, v) \in U \mid v = \widetilde{\mathbb{L}}^{\pm}(u),\ \widetilde{K}_\ell^{\pm}(u_0) = 0\}$. The lightcone height function H is the generating family of the Legendrian immersion $\mathcal{L}_H^{\pm}$ whose wave front set is the image of lightcone Gauss map $\widetilde{\mathbb{L}}^{\pm}$.

7. Contact with lightlike cylinders

In this section we describe contacts of submanifolds with lightlike cylinders by applying Montaldi's theory.

For any $v \in S_+^{n-1}$, we define a *lightlike cylinder* along v by $HP(v, 0) \cap S_1^n$. It is an $(n-1)$-dimensional submanifold in S_1^n which is isomorphic to $S^{n-2} \times \mathbb{R}$. We observe that its tangent space at each point has lightlike directions.

Proposition 7.1. *Let $\widetilde{\mathbb{L}}^{\pm}$ be a lightcone Gauss map of $\mathbf{X}$. Then $\widetilde{\mathbb{L}}^{\pm}$ is a constant map if and only if M is a part of lightlike cylinder $HP(v, 0) \cap S_1^n$ for some $v \in S_+^{n-1}$.*

Proof. Necessity is trivial, so we prove sufficient condition. If $M \subset HP(v, 0) \cap S_1^n$, then $v = \alpha(u)\mathbf{n}^T(u) + \beta(u)\mathbf{n}^S(u)$ for some functions $\alpha, \beta : U \longrightarrow \mathbb{R}$. Since v is lightlike, we have $\alpha = |\beta| > 0$. Therefore $v = \widetilde{\mathbb{L}}^{\pm}(u)$ for all $u \in U$. This completes the proof. □

We now consider the function $\mathscr{H} : S_1^n \times S_+^{n-1} \longrightarrow \mathbb{R}$ defined by $\mathscr{H}(x, v) = \langle x, v \rangle$. Given $v_0 \in S_+^{n-1}$, we denote $\mathfrak{h}_{v_0}(x) = \mathscr{H}(x, v_0)$, so that we have $\mathfrak{h}_{v_0}^{-1}(0) = HP(v_0, 0) \cap S_1^n$. For any $u_0 \in U$, we take the point $v_0^{\pm} = \widetilde{\mathbb{L}}^{\pm}(u_0)$ and have

$$(\mathfrak{h}_{v_0} \circ \mathbf{X})(u_0) = \mathscr{H} \circ (\mathbf{X} \times \mathrm{id}_{S_+^{n-1}})(u_0, v_0^{\pm}) = H(u_0, v_0^{\pm}) = 0,$$

where $p_0 = \mathbf{X}(u_0)$. We also have

$$\frac{\partial(\mathfrak{h}_{v_0^\pm} \circ \mathbf{X})}{\partial u_i}(u_0) = \frac{\partial H}{\partial u_i}(u_0, v_0^\pm) = 0.$$

It follows that the lightcone $\mathfrak{h}_{v_0^\pm}^{-1}(0) = LC_{v_0}$ is tangent to M at $p_0 = \mathbf{X}(u_0)$. In this case, we call $LC_{v_0^\pm}$ a *tangent lightlike cylinder* of M at p_0.

Theorem 7.2. *$\mathbf{X}_i : (U, u_i) \longrightarrow (S_1^n, p_i)$ $(i = 1, 2)$ be spacelike submanifold germs and $v_i = \widetilde{\mathbb{L}}_i^\pm(u_i)$. If the corresponding Legendrian immersion germs are Legendrian stable. Then the following conditions are equivalent:*

(1) *Lightcone Gauss map germs $\widetilde{\mathbb{L}}_1^\pm$ and $\widetilde{\mathbb{L}}_2^\pm$ are $\mathcal{A}$-equivalent.*
(2) *Legendrian immersion germs $\mathcal{L}_1^\pm$ and $\mathcal{L}_2^\pm$ are Legendrian equivalent.*
(3) *Lightcone height function germs H_1 and H_2 are $\mathcal{P}$–$\mathcal{K}$-equivalent.*
(4) *$h_{1,v_1}^\pm$ and $h_{2,v_2}^\pm$ are $\mathcal{K}$-equivalent.*
(5) $K(\mathbf{X_1}(U), HP(v_1, 0) \cap S_1^n; p_1) = K(\mathbf{X_2}(U), HP(v_2, 0) \cap S_1^n; p_2)$

Proof. This proof is similar to the proof of Theorem 4.6. □

We observe that the assumption of the Theorem 7.2 is a generic property in the case when $n \le 6$.

Proposition 7.3. *Let $\mathbf{X}_i$ (for $i = 1, 2$) be spacelike submanifold germs and regular sets of their lightcone Gauss maps $\widetilde{\mathbb{L}}_i^\pm$ are dense in U. If lightcone Gauss map germs $\widetilde{\mathbb{L}}_1^\pm$ and $\widetilde{\mathbb{L}}_2^\pm$ are $\mathcal{A}$-equivalent, then we have*

$$K(\mathbf{X_1}(U), HP(v_1^\pm, 0) \cap S_1^n; p_1) = K(\mathbf{X_2}(U), HP(v_2^\pm, 0) \cap S_1^n; p_2)$$

In this case, $(\mathbf{X_1}^{-1}(HP(v_1^\pm, 0) \cap S_1^n), u_1)$ and $(\mathbf{X_2}^{-1}(HP(v_2^\pm, 0) \cap S_1^n), u_2)$ are diffeomorphic as set germs.

The proof of this proposition is almost the same as Proposition 6.5 in [1], so that we omit it. We call $(\mathbf{X_i}^{-1}(HP(v_i^\pm, 0) \cap S_1^n), u_i)$ a *tangent lightlike cylindrical indicatrix germ* of M_i at p_0.

8. Classification in de Sitter 4-space

In this section we consider the case of $n = 4$ and classify singularities of lightlike hypersurface and lightcone Gauss map. We also give some examples of spacelike surfaces in de Sitter 4-space.

We now define $\mathcal{K}$-invariants of spacelike surfaces in de Sitter space. For open subset $U \subset \mathbb{R}^2$ and spacelike submanifold $X : U \longrightarrow S_1^4$, we define the $\mathcal{K}$-codimension (or Tyurina number) of the function germs $h_{v_0^\pm}, g_{\lambda_0^\pm}$ and corank of $h_{v_0^\pm}, g_{\lambda_0^\pm}$ by

$$\text{H-ord}^\pm(\mathbf{X}, u_0) = \dim C_{u_0}^\infty / \langle h_{v_0^\pm}(u_0), \partial h_{v_0^\pm}(u_0)/\partial u_i \rangle_{C_{u_0}^\infty},$$

$$\text{H-corank}^\pm(\mathbf{X}, u_0) = 2\text{-rank Hess}\,(h_{v_0^\pm}(u_0)),$$

$$\text{G-ord}^\pm(\mathbf{X}, u_0) = \dim C_{u_0}^\infty / \langle g_{\lambda_0^\pm}(u_0), \partial g_{\lambda_0^\pm}(u_0)/\partial u_i \rangle_{C_{u_0}^\infty},$$

$$\text{G-corank}^\pm(\mathbf{X}, u_0) = 2\text{-rank Hess}\,(g_{\lambda_0^\pm}(u_0)),$$

where $v_0^\pm = \widetilde{\mathbb{L}}^\pm(u_0)$ and $\lambda_0^\pm = \mathbf{X}(u_0) + t_0$.

Theorem 8.1. *Let Sp-Emb(U, S_1^n) be the set of spacelike submanifolds. We have open dense subset $\mathcal{O} \subset$ Sp-Emb(U, S_1^n) such that for $\mathbf{X} \in \mathcal{O}$, $\mathbf{v}_0^\pm = \mathbb{L}^\pm(u_0)$ and $\lambda_0^\pm = LH_M^\pm(u_0, t_0)$, we have the following:*

(1) *$\lambda_0^\pm$ is an singular value of $LH_M^\pm$ if and only if G-corank$^\pm(\mathbf{X}, u_0) = 1$ or 2.*
(2) *If G-corank$^\pm(\mathbf{X}, u_0) = 1$ then there are distinct principal curvatures $\widetilde{\kappa}_1^\pm, \widetilde{\kappa}_2^\pm$ such that $\widetilde{\kappa}_1^\pm \ne 0$, $t_0 = -1/\widetilde{\kappa}_1^\pm$ and $LH_M^\pm$ has the $\mathcal{A}_k$-type singularity $(k = 2, 3, 4)$ at (u_0, t_0). In this case we have G-ord$^\pm(\mathbf{X}, u_0) = k$.*
(3) *If G-corank$^\pm(\mathbf{X}, u_0) = 2$ then u_0 is an non-flat umbilic point and $t_0 = -1/\widetilde{\kappa}_1^\pm$. In this case, $LH_M^\pm$ has the $\mathcal{D}_4^+$-type or $\mathcal{D}_4^-$-type singularity at (u_0, t_0). In this case we have G-ord$^\pm(\mathbf{X}, u_0) = 4$.*

where the singular type of $LH_M^\pm$ is $\mathcal{A}$-equivalent to one of the map germs $f : (\mathbb{R}^3, \mathbf{0}) \longrightarrow (\mathbb{R}^4, \mathbf{0})$ in the following list:

$(\mathcal{A}_2)$ $\quad f(u_1, u_2, u_3) = (3u_1^2, 2u_1^3, u_1, u_2)$

$(\mathcal{A}_3)$ $\quad f(u_1, u_2, u_3) = (4u_1^3 + 2u_1u_2, 3u_1^4 + u_2u_1^2, u_2, u_3)$

$(\mathcal{A}_4)$ $\quad f(u_1, u_2, u_3) = (5u_1^4 + 3u_2u_1^2 + 2u_1u_3, 4u_1^5 + 2u_2u_1^3 + u_3u_1^2, u_2, u_3)$

$(\mathcal{D}_4^+)$ $\quad f(u_1, u_2, u_3) = (2(u_1^3 + u_2^3) + u_1u_2u_3, 3u_1^2 + u_2u_3, 3u_2^2 + u_1u_3, u_3)$

$(\mathcal{D}_4^-)$ $\quad f(u_1, u_2, u_3) = (2(u_1^3 - u_1u_2^2) + (u_1^2 + u_2^2)u_3, u_2^2 - 3u_1^2 - 2u_1u_3, u_1u_2 - u_2u_3, u_3)$.

Proof. By Proposition 3.1, if $\lambda_0^{\pm}$ is singular value then G-corank$^{\pm}(\mathbf{X}, u_0) \leq 2$. By Theorem 5.2, there exists an open subset $\mathcal{O} \subset \text{Sp-Emb}(U, S_1^n)$ such that for any $\mathbf{X} \in \mathcal{O}$, corresponding Lorentzian distance squared function G is a versal deformation of $g_{\lambda_0}^{\pm}$. By Thom's classification of function germs, $g_{\lambda_0}^{\pm}$ is $\mathcal{K}$-equivalent to $\mathcal{A}_k$-type germ ($k = 2, 3, 4$) or $\mathcal{D}_4^{\pm}$-type function germ, so that we have G-corank$^{\pm}(\mathbf{X}, u_0) \geq 1$, therefore (1) holds. If $g_{\lambda_0}^{\pm}$ has $\mathcal{A}_k$-type singularity, then it is $\mathcal{K}$-equivalent to $f(u_1, u_2) = u_1^2 \pm u_2^{k+1}$ and G-ord$^{\pm}(\mathbf{X}, u_0) = k$. Since the corresponding lightlike hypersurface $LH_M^{\pm}$ is the discriminant set of the Lorentzian distance squared function G, therefore (2) holds. If $g_{\lambda_0}^{\pm}$ has $\mathcal{D}_k^{\pm}$-type singularity, then it is $\mathcal{K}$-equivalent to $f(u_1, u_2) = u_1^3 \pm u_1 u_2^2$ and G-ord$^{\pm}(\mathbf{X}, u_0) = 4$. This completes the proof. $\square$

We remark that corresponding tangent lightcone indicatrix germ is diffeomorphic to the following list:

($\mathcal{A}_2$) $\{(u_1, u_2) \in (\mathbb{R}^2, \mathbf{0}) \mid u_1^2 + u_2^3 = 0\}$ (ordinary cusp)
($\mathcal{A}_3$) $\{(u_1, u_2) \in (\mathbb{R}^2, \mathbf{0}) \mid u_1^2 \pm u_2^4 = 0\}$ (tacnode or a point)
($\mathcal{A}_4$) $\{(u_1, u_2) \in (\mathbb{R}^2, \mathbf{0}) \mid u_1^2 + u_2^5 = 0\}$ (rhamphoid cusp)
($\mathcal{D}_4^+$) $\{(u_1, u_2) \in (\mathbb{R}^2, \mathbf{0}) \mid u_1 + u_2 = 0\}$ (a line)
($\mathcal{D}_4^-$) $\{(u_1, u_2) \in (\mathbb{R}^2, \mathbf{0}) \mid u_1^3 - u_1 u_2^2 = 0\}$ (triple point).

For normalized Gauss maps, we have following results.

Theorem 8.2. *There exists an open dense subset $\mathcal{O}' \subset$ Sp-Emb(U, S_1^n) such that for any $\mathbf{X} \in \mathcal{O}'$, the following conditions hold.*

(1) *u_0 is an $\widetilde{L}^{\pm}$-parabolic point if and only if H-corank$^{\pm}(\mathbf{X}, u_0) = 1$ (that is, u_0 is not a flat point).*
(2) *The $\widetilde{L}^{\pm}$-parabolic set $\widetilde{K}_\ell^{-1}(0)$ is a regular curve. Along the curve $\widetilde{\mathbb{L}}^{\pm}$ has cuspidal edge points except at isolated points. At this points $\widetilde{\mathbb{L}}^{\pm}$ has swallowtail points.*
(3) *If $\widetilde{\mathbb{L}}^{\pm}$ has the cuspidal edge points, then $h_{\mathbf{v}_0^{\pm}}$ is $\mathcal{K}$-equivalent to $(u_1^2 + u_2^3) : (\mathbb{R}^2, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ and H-ord$^{\pm}(\mathbf{X}, u_0) = 2$. In this case, the tangent lightlike cylindrical indicatrix germ is an ordinary cusp.*
(4) *If $\widetilde{\mathbb{L}}^{\pm}$ has the swallowtail points, then $h_{\mathbf{v}_0^{\pm}}$ is $\mathcal{K}$-equivalent to $(u_1^2 \pm u_2^4) : (\mathbb{R}^2, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ and H-ord$^{\pm}(\mathbf{X}, u_0) = 3$. In this case, the tangent lightlike cylindrical indicatrix germ is a tacnode or a point.*

where $\mathbb{L}^{\pm}$ has cuspidal edge point if $\mathbb{L}^{\pm}$ is $\mathcal{A}$-equivalent to $(3u_1^2, 2u_1^3, u_1) : (\mathbb{R}^2, \mathbf{0}) \longrightarrow (\mathbb{R}^3, \mathbf{0})$, and $\mathbb{L}^{\pm}$ has swallowtail point if $\mathbb{L}^{\pm}$ is $\mathcal{A}$-equivalent to $(4u_1^3 + 2u_1u_2, 3u_1^4 + u_2u_1^2, u_2)$.

Proof. By Proposition 6.1, the condition that $\mathbf{v}_0^{\pm}$ is singular value is equivalent to the condition H-corank$^{\pm}(\mathbf{X}, u_0) \geq 1$. By Theorem 5.2, there exists an open subset $\mathcal{O}' \subset$ Sp-Emb(U, S_1^n) such that for any $\mathbf{X} \in \mathcal{O}$, corresponding lightcone height function H is a versal deformation of $h_{\mathbf{v}_0^{\pm}}$. By Thom's classification of function germs, $h_{\mathbf{v}_0^{\pm}}$ has $\mathcal{A}_k$-type singularity ($k = 2, 3$) and H-corank$^{\pm}(\mathbf{X}, u_0) = 1$, therefore (1) holds. On the other hand, the condition H-corank$^{\pm}(\mathbf{X}, u_0) = 1$ means that the parabolic set $\widetilde{K}_\ell^{-1}(0)$ is a part of curves. If $h_{\mathbf{v}_0^{\pm}}$ has $\mathcal{A}_2$-type singularity, then it is $\mathcal{K}$-equivalent to $f(u_1, u_2) = u_1^2 + u_2^3$ and H-ord$^{\pm}(\mathbf{X}, u_0) = 2$. Since the corresponding lightcone Gauss map $\mathbb{L}^{\pm}$ is the discriminant set of the lightcone height function H, therefore (3) holds. If $h_{\mathbf{v}_0^{\pm}}$ has $\mathcal{A}_3$-type singularity, then it is $\mathcal{K}$-equivalent to $f(u_1, u_2) = u_1^3 \pm u_2^3$ and H-ord$^{\pm}(\mathbf{X}, u_0) = 3$, therefore (4) holds. On the other hand, the swallowtail points are isolated points, therefore (2) holds. This completes the proof. $\square$

Example 8.3. Let $f : (U, \mathbf{0}) \longrightarrow \mathbb{R}, f(\mathbf{0}) = f_{u_i}(\mathbf{0}) = 0$ and spacelike submanifold $M = \mathbf{X}(U)$ in S_1^n by

$$\mathbf{X}_f(u_1, u_2) = \left(f(u), 0, \sqrt{1 + f(u)^2 - u_1^2 - u_2^2}, u_1, u_2\right).$$

If $f = \frac{1}{2}(u_1^2 - u_2^2 + 2u_1^{k+1})$ for some $k = 2, 3, 4$, then LH_M^+ and LH_M^- have $\mathcal{A}_k$-type singularities at $\lambda_0^{\pm} = LH_M^{\pm}(0, 1)$. In this case, the corresponding tangent lightcone indicatrix germs $(\mathbf{X}_f^{-1}(LC_{\lambda_0^{\pm}}), \mathbf{0})$ are $\{(u_1, u_2) \mid u_2^2 + u_1^{k+1} = 0\}$.

If $f = \frac{1}{2}(u_1^2 + u_2^2 + u_1^3 \pm u_1u_2^2)$, then LH_M^+ and LH_M^- have $\mathcal{D}_4^{\pm}$-type singularities at $\lambda_0^+ = LH_M^+(0, -1)$, $\lambda_0^- = LH_M^-(0, -1)$. The corresponding tangent lightcone indicatrix germs are $\{(u_1, u_2) \mid u_1^3 \pm u_1u_2^2 = 0\}$.

If $f = \frac{1}{2}u_1^2 - \frac{1}{k}u_2^{k+1}$ for some $k = 2, 3$, then both $\mathbb{L}^+$ and $\mathbb{L}^-$ have $\mathcal{A}_k$-type singularities at the origin. The corresponding tangent lightlike cylindrical indicatrix germs are ordinal cusp ($k = 2$) and tacnode ($k = 3$).

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