



On the r -stability of spacelike hypersurfaces

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ABSTRACT

In this paper we study the strong stability of spacelike hypersurfaces with constant r -th mean curvature in Robertson–Walker spacetimes. In particular, we treat the case in which the ambient spacetime is the de Sitter space.

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1. Introduction

The notion of stability for hypersurfaces of constant mean curvature of Riemannian ambient spaces was first studied by Barbosa and do Carmo in [1], and Barbosa, do Carmo and Eschenburg in [2], where they proved that spheres are the only stable critical points of the area functional for volume-preserving variations. Later, in [3], Barbosa and Colares formulated the notion of stability for hypersurfaces of constant r -th mean curvature, thus extending the results of [2].

In relation to the Lorentz context, in 1993, Barbosa and Olikar [4] obtained an analogous result, proving that constant mean curvature spacelike hypersurfaces in Lorentz manifolds are also critical points of the area functional for variations that keep the volume constant. They also computed the second-variation formula and showed, for the de Sitter space S_1^{n+1} , that spheres maximize the area functional for volume-preserving variations; more precisely, in [4] it is proved that if M^n is a complete spacelike immersed hypersurface in S_1^{n+1} , with constant mean curvature H , then M^n is stable if M^n is compact, $H^2 \geq 1$ or $H^2 < 4(n-1)/n^2$. This result was extended to the case of complete spacelike hypersurfaces with constant r -th mean curvature in de Sitter space by Brasil and Colares, in [5].

More recently, the second author, jointly with Barros and Brasil [6], has studied the problem of strong stability (that is, stability with respect to not necessarily volume-preserving variations) for spacelike hypersurfaces with constant mean

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curvature in a Generalized Robertson–Walker (GRW) spacetime, giving a characterization of maximal and spacelike slices. In [7], Liu and Yang studied the analogous problem for spacelike hypersurfaces with constant scalar curvature.

Here, motivated by the works [6,7], we consider spacelike hypersurfaces with constant r -th mean curvature in Robertson–Walker (RW) spacetimes, that is, GRW spacetimes whose Riemannian fiber has constant sectional curvature, in order to classify the strongly r -stable ones. More precisely, we prove the following result.

Theorem 1.1. *Let $\overline{M}_c^{n+1} = I \times_\phi F^n$ be a Robertson–Walker spacetime of constant sectional curvature c , and $x : M^n \rightarrow \overline{M}_c^{n+1}$ be a closed, strongly r -stable, spacelike hypersurface. If the warping function ϕ is such that $H_r \phi'' \geq \max\{H_{r+1} \phi', 0\}$ and the set where $\phi' = 0$ has empty interior on M^n , then M^n is either r -maximal or a totally umbilical slice $\{s_0\} \times F$.*

As an immediate consequence of the previous result, we obtain the following corollary in the de Sitter space.

Corollary 1.2. *Let $x : M^n \rightarrow \mathbb{S}_1^{n+1}$ be a closed strongly r -stable spacelike hypersurface. Suppose that the set of points in which M^n intersects the equator of $\mathbb{S}_1^{n+1}$ has empty interior. If $H_r \geq \max\{H_{r+1}, 0\}$, then M^n is either r -maximal or an umbilical round sphere.*

2. r -stability of spacelike hypersurfaces

In what follows, $\overline{M}^{n+1}$ denotes a time-oriented Lorentz manifold with Lorentz metric $\bar{g} = \langle, \rangle$, volume element $d\bar{M}$ and semi-Riemannian connection $\bar{\nabla}$. In this context, we consider spacelike hypersurfaces $x : M^n \rightarrow \overline{M}^{n+1}$, namely, isometric immersions from a connected, n -dimensional orientable Riemannian manifold M^n into $\overline{M}$. We let ∇ denote the Levi-Civita connection of M^n .

If $\overline{M}$ is time-orientable and $x : M^n \rightarrow \overline{M}^{n+1}$ is a spacelike hypersurface, then M^n is orientable (cf. [8]) and one can choose a globally defined unit normal vector field N on M^n having the same time orientation of $\overline{M}$. One says that such an N points to the future.

In this setting, let A denote the corresponding shape operator. At each $p \in M^n$, A restricts to a self-adjoint linear map $A_p : T_p M \rightarrow T_p M$. For $1 \leq r \leq n$, let $S_r(p)$ denote the r -th elementary symmetric function on the eigenvalues of A_p ; in this way one gets n smooth functions $S_r : M^n \rightarrow \mathbb{R}$ such that

$$\det(tI - A) = \sum_{k=0}^n (-1)^k S_k t^{n-k},$$

where $S_0 = 1$ by definition. If $p \in M^n$ and $\{e_k\}$ is a basis of $T_p M$ formed by eigenvectors of A_p , with corresponding eigenvalues $\{\lambda_k\}$, one immediately sees that

$$S_r = \sigma_r(\lambda_1, \dots, \lambda_n),$$

where $\sigma_r \in \mathbb{R}[X_1, \dots, X_n]$ is the r -th elementary symmetric polynomial on the indeterminates $X_1, \dots, X_n$.

For $1 \leq r \leq n$, one defines the r -th mean curvature H_r of x by

$$\binom{n}{r} H_r = (-1)^r S_r = \sigma_r(-\lambda_1, \dots, -\lambda_n).$$

A spacelike hypersurface $x : M^n \rightarrow \overline{M}^{n+1}$ such that $H_{r+1} = 0$ on M^n is said to be r -maximal.

For $0 \leq r \leq n$ one defines the r -th Newton transformation P_r on M^n by setting $P_0 = I$ (the identity operator) and, for $1 \leq r \leq n$, via the recurrence relation

$$P_r = (-1)^r S_r I + A P_{r-1}. \quad (2.1)$$

A trivial induction shows that

$$P_r = (-1)^r (S_r I - S_{r-1} A + S_{r-2} A^2 - \dots + (-1)^r A^r),$$

so the Cayley–Hamilton theorem gives $P_n = 0$. Moreover, since P_r is a polynomial in A for every r , it is also self-adjoint and commutes with A . Therefore, all bases of $T_p M$ diagonalizing A at $p \in M^n$ also diagonalize all of the P_r at p . Let $\{e_k\}$ be such a basis. Denoting by A_i the restriction of A to $\langle e_i \rangle^\perp \subset T_p \Sigma$, it is easy to see that

$$\det(tI - A_i) = \sum_{k=0}^{n-1} (-1)^k S_k(A_i) t^{n-1-k},$$

where

$$S_k(A_i) = \sum_{\substack{1 \leq j_1 < \dots < j_k \leq n \\ j_1, \dots, j_k \neq i}} \lambda_{j_1} \cdots \lambda_{j_k}.$$

With the above notation, it is also immediately checked that $P_r e_i = (-1)^r S_r(A_i) e_i$, and hence (Lemma 2.1 of [3])

- (a) $S_r(A_i) = S_r - \lambda_i S_{r-1}(A_i)$;
- (b) $\text{tr}(P_r) = (-1)^r \sum_{i=1}^n S_r(A_i) = (-1)^r (n-r) S_r = b_r H_r$;
- (c) $\text{tr}(AP_r) = (-1)^r \sum_{i=1}^n \lambda_i S_r(A_i) = (-1)^r (r+1) S_{r+1} = -b_r H_{r+1}$;
- (d) $\text{tr}(A^2 P_r) = (-1)^r \sum_{i=1}^n \lambda_i^2 S_r(A_i) = (-1)^r (S_1 S_{r+1} - (r+2) S_{r+2})$,

where $b_r = (n-r) \binom{n}{r}$.

Associated with each Newton transformation P_r one has the second-order linear differential operator $L_r : \mathcal{D}(M) \rightarrow \mathcal{D}(M)$, given by

$$L_r(f) = \text{tr}(P_r \text{Hess } f).$$

According to [9], if $\overline{M}^{n+1}$ is of constant sectional curvature, then P_r is a divergence-free and, consequently,

$$L_r(f) = \text{div}(P_r \nabla f).$$

For future use, we recall Lemma 2.6 of [10]: if (a_{ij}) denotes the matrix of A with respect to a certain orthonormal basis $\beta = \{e_k\}$ of $T_p M$, then the matrix (a'_{ij}) of P_r with respect to the same basis is given by

$$a'_{ij} = \frac{(-1)^r}{r!} \sum_{i_1 \dots i_r=1}^n \epsilon_{i_1 \dots i_r i}^{j_1 \dots j_r j} a_{j_1 i_1}, \dots, a_{j_r i_r}, \quad (2.2)$$

where

$$\epsilon_{i_1 \dots i_r}^{j_1 \dots j_r} = \begin{cases} \text{sgn}(\sigma), & \text{if the } i_k \text{ are pairwise distinct and } \sigma = (j_k) \text{ is a permutation of them;} \\ 0, & \text{else.} \end{cases}$$

If x is as above, a *variation* of it is a smooth mapping

$$X : M^n \times (-\epsilon, \epsilon) \rightarrow \overline{M}^{n+1}$$

satisfying the following conditions:

- (1) For $t \in (-\epsilon, \epsilon)$, the map $X_t : M^n \rightarrow \overline{M}^{n+1}$ given by $X_t(p) = X(t, p)$ is a spacelike immersion such that $X_0 = x$.
- (2) $X_t|_{\partial M} = x|_{\partial M}$, for all $t \in (-\epsilon, \epsilon)$.

In all that follows, we let dM_t denote the volume element of the metric induced on M by X_t and N_t the unit normal vector field along X_t .

The *variational field* associated with the variation X is the vector field $\frac{\partial X}{\partial t} \Big|_{t=0}$. Letting $f = -\langle \frac{\partial X}{\partial t}, N_t \rangle$, we get

$$\frac{\partial X}{\partial t} = f N_t + \left(\frac{\partial X}{\partial t} \right)^\top, \quad (2.3)$$

where $\top$ stands for tangential components.

The *balance of volume* of the variation X is the function $\mathcal{V} : (-\epsilon, \epsilon) \rightarrow \mathbb{R}$ given by

$$\mathcal{V}(t) = \int_{M \times [0, t]} X^*(d\overline{M}),$$

and we say that X is *volume-preserving* if $\mathcal{V}$ is constant. The following lemma is classical (cf. [11]).

Lemma 2.1. *Let $\overline{M}^{n+1}$ be a time-oriented Lorentz manifold and $x : M^n \rightarrow \overline{M}^{n+1}$ a closed spacelike hypersurface. If $X : M^n \times (-\epsilon, \epsilon) \rightarrow \overline{M}^{n+1}$ is a variation of x , then*

$$\frac{d\mathcal{V}}{dt} = \int_M f dM_t.$$

In particular, X is volume-preserving if and only if $\int_M f dM_t = 0$ for all t .

We remark that Lemma 2.2 of [2] remains valid in the Lorentz context, i.e., if $f_0 : M \rightarrow \mathbb{R}$ is a smooth function such that $\int_M f_0 dM = 0$, then there exists a volume-preserving variation of M whose variational field is $f_0 N$. Moreover, if we drop the requirement that the variation is volume-preserving (or, which is the same, that $\int_M f_0 dM = 0$), the argument of that lemma always gives a variation whose variational field is $f_0 N$.

In order to extend [3] to the Lorentz setting, we define the *r-area functional* $\mathcal{A}_r : (-\epsilon, \epsilon) \rightarrow \mathbb{R}$ associated with the variation X by

$$\mathcal{A}_r(t) = \int_M F_r(S_1, S_2, \dots, S_r) dM_t,$$

where $S_r = S_r(t)$ and F_r is recursively defined by setting $F_0 = 1$, $F_1 = -S_1$ and, for $2 \leq r \leq n-1$,

$$F_r = (-1)^r S_r - \frac{c(n-r+1)}{r-1} F_{r-2}.$$

The next step is the Lorentz analogue of Proposition 4.1 of [3]. Since it seems to us that their proof only works on a neighborhood free of umbilics, and in order to keep this work self-contained, we present an alternative one here.

Lemma 2.2. *Let $x : M^n \rightarrow \overline{M}_c^{n+1}$ be a closed spacelike hypersurface of the time-oriented Lorentz manifold $\overline{M}_c^{n+1}$ with constant sectional curvature c , and let $X : M^n \times (-\epsilon, \epsilon) \rightarrow \overline{M}_c^{n+1}$ be a variation of x . Then,*

$$\frac{\partial S_{r+1}}{\partial t} = (-1)^{r+1} [L_r f + c \operatorname{tr}(P_r) f - \operatorname{tr}(A^2 P_r) f] + \left\langle \left(\frac{\partial X}{\partial t} \right)^\top, \nabla S_{r+1} \right\rangle. \quad (2.4)$$

Proof. Formula (2.2) gives

$$(r+1)S_{r+1} = (-1)^r \operatorname{tr}(A P_r) = (-1)^r \sum_{i,j} a_{ji} a_{ij}^r = \frac{1}{r!} \sum_{i,j,k_1 \dots k_r} \epsilon_{i_1 \dots i_r i}^{j_1 \dots j_r j} a_{j_1 i_1} \dots a_{j_r i_r} a_{ji},$$

where the functions S_{r+1} are seen as functions of t . Therefore, differentiation with respect to t gives

$$\begin{aligned} (r+1)S'_{r+1} &= \frac{1}{r!} \sum_{i,j,k_1 \dots k_r} \epsilon_{i_1 \dots i_r i}^{j_1 \dots j_r j} [a'_{j_1 i_1} \dots a_{j_r i_r} a_{ji} + \dots + a_{j_1 i_1} \dots a_{j_r i_r} a'_{ji}] \\ &= \frac{r+1}{r!} \sum_{i,j,k_1 \dots k_r} \epsilon_{i_1 \dots i_r i}^{j_1 \dots j_r j} a'_{ji} a_{j_1 i_1} \dots a_{j_r i_r} \\ &= (-1)^r (r+1) \sum_{i,j} a'_{ji} a_{ij}^r = (-1)^r (r+1) \operatorname{tr} \left(\frac{\partial A}{\partial t} P_r \right). \end{aligned}$$

Now, it is enough to compute $(-1)^r \operatorname{tr} \left(\frac{\partial A}{\partial t} P_r \right)$. Fix $p \in M$ and let $\{e_k\}$ be a moving frame on a neighborhood $U \subset M$ of p , geodesic at p . Then

$$\begin{aligned} S'_{r+1} &= (-1)^r \operatorname{tr} \left(\frac{\partial A}{\partial t} P_r \right) = (-1)^r \sum_k \left\langle \frac{\partial A}{\partial t} P_r e_k, e_k \right\rangle = \sum_k S_r(A_k) \left\langle \left(\overline{\nabla} \frac{\partial X}{\partial t} A \right) e_k, e_k \right\rangle \\ &= \sum_k S_r(A_k) \left[\left\langle \overline{\nabla} \frac{\partial X}{\partial t} A e_k, e_k \right\rangle - \left\langle A \overline{\nabla} \frac{\partial X}{\partial t} e_k, e_k \right\rangle \right] \\ &= - \sum_k S_r(A_k) \left\langle \overline{\nabla} \frac{\partial X}{\partial t} \overline{\nabla} e_k N, e_k \right\rangle - \sum_k S_r(A_k) \left\langle A \overline{\nabla} e_k \partial X / \partial t, e_k \right\rangle, \end{aligned}$$

where we have used that $[\partial X / \partial t, e_k] = 0$ in the last term.

If $\overline{R}$ denotes the curvature tensor of $\overline{M}$, we have

$$\overline{R}(e_k, \partial X / \partial t) N = \overline{\nabla} \frac{\partial X}{\partial t} \overline{\nabla} e_k N - \overline{\nabla} e_k \overline{\nabla} \frac{\partial X}{\partial t} N + \overline{\nabla} \left[e_k, \frac{\partial X}{\partial t} \right] N.$$

Thus, by using also (2.3),

$$S'_{r+1} = - \sum_k S_r(A_k) \left[\langle \overline{R}(e_k, \partial X / \partial t) N, e_k \rangle + \left\langle \overline{\nabla} e_k \overline{\nabla} \frac{\partial X}{\partial t} N, e_k \right\rangle \right] - \sum_k S_r(A_k) \langle \overline{\nabla} e_k (fN + (\partial X / \partial t)^\top), A e_k \rangle.$$

Since that the ambient spacetime is of constant sectional curvature, this yields that

$$\langle \overline{R}(X, Y) W, Z \rangle = c \{ \langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle \}.$$

Then

$$\begin{aligned} S'_{r+1} &= - \sum_k S_r(A_k) c (\langle e_k, N \rangle \langle \partial X / \partial t, e_k \rangle - \langle e_k, e_k \rangle \langle \partial X / \partial t, N \rangle) \\ &\quad - \sum_k S_r(A_k) \left\langle \overline{\nabla} e_k \overline{\nabla} \frac{\partial X}{\partial t} N, e_k \right\rangle - \sum_k S_r(A_k) \langle A e_k, \overline{\nabla} e_k fN \rangle - \sum_k S_r(A_k) \langle \overline{\nabla} e_k (\partial X / \partial t)^\top, A e_k \rangle \\ &= -c \sum_k S_r(A_k) f - \sum_k S_r(A_k) e_k \left\langle \overline{\nabla} \frac{\partial X}{\partial t} N, e_k \right\rangle + \sum_k S_r(A_k) \left\langle \overline{\nabla} \frac{\partial X}{\partial t} N, \overline{\nabla} e_k e_k \right\rangle \\ &\quad - \sum_k S_r(A_k) \langle A e_k, f \overline{\nabla} e_k N \rangle - \sum_k S_r(A_k) e_k \langle A e_k, (\partial X / \partial t)^\top \rangle + \sum_k S_r(A_k) \langle \overline{\nabla} e_k A e_k, (\partial X / \partial t)^\top \rangle. \end{aligned}$$

On the other hand, the frame $\{e_k\}$ is geodesic at p ; then

$$\left\langle \bar{\nabla}_{\frac{\partial X}{\partial t}} N, \bar{\nabla}_{e_k} e_k \right\rangle = \left\langle \bar{\nabla}_{\frac{\partial X}{\partial t}} N, N \right\rangle \langle N, \bar{\nabla}_{e_k} e_k \rangle.$$

Therefore, by using the expression for the trace of the operator P_r , we get

$$\begin{aligned} S'_{r+1} &= -(-1)^r c \operatorname{tr}(P_r) f + \sum_k (-1)^r P_r e_k \left\langle N, \bar{\nabla}_{\frac{\partial X}{\partial t}} e_k \right\rangle \\ &\quad - \sum_k S_r(A_k) \left\langle \bar{\nabla}_{\frac{\partial X}{\partial t}} N, N \right\rangle \langle N, \bar{\nabla}_{e_k} e_k \rangle + f \sum_k S_r(A_k) \langle Ae_k, Ae_k \rangle \\ &\quad - \sum_k (-1)^r P_r e_k \langle Ae_k, (\partial X / \partial t)^\top \rangle + \sum_k S_r(A_k) \langle \nabla_{e_k} Ae_k, (\partial X / \partial t)^\top \rangle \\ &= -(-1)^r c \operatorname{tr}(P_r) f + \sum_k (-1)^r P_r e_k \langle N, \bar{\nabla}_{e_k} \partial X / \partial t \rangle + f \sum_k (-1)^r \langle AP_r e_k, Ae_k \rangle \\ &\quad - \sum_k (-1)^r P_r e_k \langle Ae_k, \partial X / \partial t \rangle + \sum_k (-1)^r \langle \nabla_{P_r e_k} Ae_k, (\partial X / \partial t)^\top \rangle \\ &= -(-1)^r c \operatorname{tr}(P_r) f - \sum_k (-1)^r (P_r e_k e_k(f) + P_r e_k \langle \bar{\nabla}_{e_k} N, \partial X / \partial t \rangle) \\ &\quad + (-1)^r \operatorname{tr}(A^2 P_r) f - \sum_k (-1)^r P_r e_k \langle Ae_k, \partial X / \partial t \rangle + (-1)^r \left\langle \sum_k \nabla_{P_r e_k} Ae_k, (\partial X / \partial t)^\top \right\rangle. \end{aligned}$$

Now, by using Codazzi's equation, we have

$$\begin{aligned} \sum_k \nabla_{P_r e_k} Ae_k &= \sum_k (\nabla_{e_k} AP_r e_k + A[P_r e_k, e_k]) = \sum_k \nabla_{e_k} ((-1)^r S_{r+1} I + P_{r+1}) e_k + \sum_k A (\nabla_{P_r e_k} e_k - \nabla_{e_k} P_r e_k) \\ &= \sum_k (-1)^r e_k (S_{r+1}) e_k + \sum_k \nabla_{e_k} P_{r+1} e_k - A \nabla_{e_k} P_r e_k \\ &= (-1)^r \operatorname{div} S_{r+1} + \operatorname{div}(P_{r+1}) - A(\operatorname{div} P_r) = (-1)^r \nabla S_{r+1}, \end{aligned}$$

since that the operators P_r are free of divergence. Hence

$$S'_{r+1} = (-1)^{r+1} (c \operatorname{tr}(P_r) f + L_r f - \operatorname{tr}(A^2 P_r) f) + \langle \nabla S_{r+1}, (\partial X / \partial t)^\top \rangle. \quad \square$$

The previous lemma allows us to compute the first variation of the r -area functional.

Proposition 2.3. Under the hypotheses of Lemma 2.2, if X is a variation of x , then

$$\mathcal{A}'_r(t) = \int_M [(-1)^{r+1} (r+1) S_{r+1} + c_r] f \, dM_t, \quad (2.5)$$

where $c_r = 0$ if r is even and $c_r = -\frac{n(n-2)(n-4)\cdots(n-r+1)}{(r-1)(r-3)\cdots 2} (-c)^{(r+1)/2}$ if r is odd.

Proof. We make an inductive argument. The case $r = 0$ is well known, and for the case $r = 1$ we use the classical formula (see Lemma 4.2 in [3])

$$\frac{\partial}{\partial t} dM_t = [-S_1 f + \operatorname{div}(\partial X / \partial t)^\top] dM_t$$

to get

$$\begin{aligned} \mathcal{A}'_1 &= \int_M F'_1 dM_t + \int_M F_1 \frac{\partial}{\partial t} dM_t \\ &= - \int_M S'_1 dM_t - \int_M F_1 [-S_1 f + \operatorname{div}(\partial X / \partial t)^\top] dM_t \\ &= \int_M [\Delta f - (S_1^2 - 2S_2) f + nc f - \langle (\partial X / \partial t)^\top, \nabla S_1 \rangle + S_1^2 f - S_1 \operatorname{div}(\partial X / \partial t)^\top] dM_t \\ &= \int_M 2S_2 f dM_t + nc \int_M f dM_t - \int_M \operatorname{div}(S_1 (\partial X / \partial t)^\top) dM_t \\ &= \int_M (2S_2 + nc) f dM_t, \end{aligned}$$

where in the last equality we have used that M is closed.

Now, if $r \geq 2$, the induction hypothesis and (2.4) give

$$\begin{aligned}
 \mathcal{A}'_r &= \int_M F'_r dM_t + \int_M F_r \frac{\partial}{\partial t} dM_t \\
 &= \int_M \left[(-1)^r S'_r - \frac{c(n-r+1)}{r-1} F'_{r-2} \right] dM_t + \int_M \left((-1)^r S_r - \frac{c(n-r+1)}{r-1} F_{r-2} \right) \frac{\partial}{\partial t} dM_t \\
 &= \int_M (-1)^r \{ S'_r + S_r [-S_1 f + \operatorname{div}(\partial X / \partial t)^T] \} dM_t - \frac{c(n-r+1)}{r-1} \left[\int_M F'_{r-2} dM_t + \int_M F_{r-2} \frac{\partial}{\partial t} dM_t \right] \\
 &= \int_M (-1)^r \{ S'_r - S_1 S_r f + S_r \operatorname{div}(\partial X / \partial t)^T \} dM_t - \frac{c(n-r+1)}{r-1} \mathcal{A}'_{r-2} \\
 &= \int_M [c \operatorname{tr}(P_{r-1})f + L_{r-1}f - \operatorname{tr}(A^2 P_{r-1})f + (-1)^r \langle \nabla S_r, (\partial X / \partial t)^T \rangle] dM_t \\
 &\quad + (-1)^r \int_M (-S_1 S_r f + S_r \operatorname{div}(\partial X / \partial t)^T) dM_t - \frac{c(n-r+1)}{r-1} \mathcal{A}'_{r-2} \\
 &= \int_M [c(-1)^{r-1}(n-r+1)S_{r-1}f - (-1)^{r-1}(S_1 S_r - (r+1)S_{r+1})f] dM_t \\
 &\quad + (-1)^r \int_M \langle \nabla S_r, (\partial X / \partial t)^T \rangle dM_t + \int_M ((-1)^{r+1}S_1 S_r f + (-1)^r S_r \operatorname{div}(\partial X / \partial t)^T) dM_t \\
 &\quad - \frac{c(n-r+1)}{r-1} \int_M [(-1)^{r-1}(r-1)S_{r-1} + c_{r-2}]f dM_t \\
 &= \int_M \left[(-1)^{r+1}(r+1)S_{r+1} - \frac{c(n-r+1)}{r-1} c_{r-2} \right] f dM_t + (-1)^r \int_M \operatorname{div}(S_r (\partial X / \partial t)^T) dM_t.
 \end{aligned}$$

It now suffices to apply the divergence theorem and note that $c_r = -\frac{c(n-r+1)}{r-1} c_{r-2}$. $\square$

In order to characterize spacelike immersions of constant $(r+1)$ -th mean curvature, let λ be a real constant and $\mathcal{J}_r : (-\epsilon, \epsilon) \rightarrow \mathbb{R}$ be the *Jacobi functional* associated with the variation X , i.e.,

$$\mathcal{J}_r(t) = \mathcal{A}_r(t) - \lambda \mathcal{V}(t).$$

As an immediate consequence of (2.5) we get

$$\mathcal{J}'_r(t) = \int_M [b_r H_{r+1} + c_r - \lambda] f dM_t,$$

where $b_r = (r+1) \binom{n}{r+1}$. Therefore, if we choose $\lambda = c_r + b_r \bar{H}_{r+1}(0)$, where

$$\bar{H}_{r+1}(0) = \frac{1}{\mathcal{A}_0(0)} \int_M H_{r+1}(0) dM$$

is the mean of the $(r+1)$ -th curvature $H_{r+1}(0)$ of M , we arrive at

$$\mathcal{J}'_r(t) = b_r \int_M [H_{r+1} - \bar{H}_{r+1}(0)] f dM_t.$$

Hence, a standard argument (cf. [1]) shows that M is a critical point of $\mathcal{J}_r$ for all variations of x if and only if M has constant $(r+1)$ -th mean curvature.

We wish to study spacelike immersions $x : M^n \rightarrow \bar{M}^{n+1}$ that maximize $\mathcal{J}_r$ for all variations X of x . The above discussion shows that M must have constant $(r+1)$ -th mean curvature and, for such an M , leads us naturally to compute the second variation of $\mathcal{J}_r$. This, in turn, motivates the following:

Definition 2.4. Let $\bar{M}_c^{n+1}$ be a Lorentz manifold of constant sectional curvature c , and $x : M^n \rightarrow \bar{M}^{n+1}$ be a closed spacelike hypersurface having constant $(r+1)$ -th mean curvature. We say that x is strongly r -stable if, for every smooth function $f : M \rightarrow \mathbb{R}$, one has $\mathcal{J}''_r(0) \leq 0$.

The sought formula for the second variation of $\mathcal{J}_r$ is another straightforward consequence of Proposition 2.3.

Proposition 2.5. Let $x : M^n \rightarrow \bar{M}_c^{n+1}$ be a closed spacelike hypersurface of the time-oriented Lorentz manifold $\bar{M}_c^{n+1}$, having constant $(r+1)$ -mean curvature H_{r+1} . If $X : M^n \times (-\epsilon, \epsilon) \rightarrow \bar{M}_c^{n+1}$ is a variation of x , then

$$\mathcal{J}''_r(0) = (r+1) \int_M [L_r(f) + c \operatorname{tr}(P_r)f - \operatorname{tr}(A^2 P_r)f] f dM. \quad (2.6)$$

3. r -stable spacelike hypersurfaces in RW spacetimes

As in the previous section, let $\bar{M}^{n+1}$ be a Lorentz manifold. A vector field V on $\bar{M}^{n+1}$ is said to be *conformal* if

$$\mathcal{L}_V \langle \cdot, \cdot \rangle = 2\psi \langle \cdot, \cdot \rangle \quad (3.1)$$

for some function $\psi \in C^\infty(\bar{M})$, where $\mathcal{L}$ stands for the Lie derivative of the Lorentz metric of $\bar{M}$. The function ψ is called the *conformal factor* of V .

Since $\mathcal{L}_V(X) = [V, X]$ for all $X \in \mathfrak{X}(\bar{M})$, it follows from the tensorial character of $\mathcal{L}_V$ that $V \in \mathfrak{X}(\bar{M})$ is conformal if and only if

$$\langle \bar{\nabla}_X V, Y \rangle + \langle X, \bar{\nabla}_Y V \rangle = 2\psi \langle X, Y \rangle, \quad (3.2)$$

for all $X, Y \in \mathfrak{X}(\bar{M})$. In particular, V is a Killing vector field relative to $\bar{g}$ if and only if $\psi \equiv 0$.

Any Lorentz manifold $\bar{M}^{n+1}$ possessing a globally defined, timelike conformal vector field is said to be a *conformally stationary spacetime*.

In what follows we need a formula first derived in [9]. As stated below, it is the Lorentz version of the one stated and proved in [12].

Lemma 3.1. *Let $\bar{M}_c^{n+1}$ be a conformally stationary Lorentz manifold having constant sectional curvature c and conformal vector field V . Let also $x : M^n \rightarrow \bar{M}_c^{n+1}$ be a spacelike hypersurface of $\bar{M}_c^{n+1}$ and N be a future-pointing, unit normal vector field globally defined on M^n . If $\eta = \langle V, N \rangle$, then*

$$L_r \eta = \text{tr}(A^2 P_r) \eta - c \text{tr}(P_r) \eta - b_r H_r N(\psi) + b_r H_{r+1} \psi + \frac{b_r}{r+1} \langle V, \nabla H_{r+1} \rangle, \quad (3.3)$$

where $\psi : \bar{M}^{n+1} \rightarrow \mathbb{R}$ is the conformal factor of V , H_j is the j -th mean curvature of x and ∇H_j stands for the gradient of H_j on M .

A particular class of conformally stationary spacetimes is that of *generalized Robertson–Walker* spacetimes, or *GRW* for short (cf. [13]), namely, warped products $\bar{M}^{n+1} = I \times_\phi F^n$, where $I \subset \mathbb{R}$ is an interval with the metric $-ds^2$, F^n is an n -dimensional Riemannian manifold and $\phi : I \rightarrow \mathbb{R}$ is positive and smooth. For such a space, let $\pi_I : M^{n+1} \rightarrow I$ denote the canonical projection onto the I -factor. Then the vector field

$$V = (\phi \circ \pi_I) \frac{\partial}{\partial s}$$

is a conformal, timelike and closed (in the sense that its dual 1-form is closed) one, with conformal factor $\psi = \phi'$, where the prime denotes differentiation with respect to s . Moreover (cf. [14]), for $s_0 \in I$, the (spacelike) leaf $M_{s_0}^n = \{s_0\} \times F^n$ is totally umbilical, with umbilicity factor $-\frac{\phi'(s_0)}{\phi(s_0)}$ with respect to the future-pointing unit normal vector field N .

If $\bar{M}^{n+1} = I \times_\phi F^n$ is a GRW and $x : M^n \rightarrow \bar{M}^{n+1}$ is a complete spacelike hypersurface of $\bar{M}^{n+1}$, such that $\phi \circ \pi_I$ is limited on M , then $\pi_F|_M : M^n \rightarrow F^n$ is necessarily a covering map (cf. [13]). In particular, if M^n is closed then F^n is automatically closed.

Also, recall (cf. [8]) that a GRW as above has constant sectional curvature c if and only if F has constant sectional curvature k and the warping function ϕ satisfies the ODE

$$\frac{\phi''}{\phi} = c = \frac{(\phi')^2 + k}{\phi^2}.$$

In such cases, $\bar{M}^{n+1}$ is simply a *Robertson–Walker (RW) spacetime* of constant sectional curvature.

We can now state and prove our main result, which generalizes the main theorems of [6,7].

Theorem 3.2. *Let $\bar{M}_c^{n+1} = I \times_\phi F^n$ be a Robertson–Walker spacetime of constant sectional curvature c , and $x : M^n \rightarrow \bar{M}_c^{n+1}$ be a closed strongly r -stable spacelike hypersurface. If the warping function ϕ is such that $H_r \phi'' \geq \max\{H_{r+1} \phi', 0\}$ and the set where $\phi' = 0$ has empty interior on M , then M^n is either r -maximal or a totally umbilical slice $\{s_0\} \times F$.*

Proof. Since M^n is strongly r -stable then

$$0 \geq \mathcal{J}_r''(0) = (r+1) \int_M [L_r(f) + \text{ctr}(P_r)f - \text{tr}(A^2 P_r)f] f dM$$

for all smooth $f : M \rightarrow \mathbb{R}$. In particular, if $f = \eta$, where (as in Lemma 3.1) $\eta = \langle V, N \rangle = \phi \left(\frac{\partial}{\partial s}, N \right)$, and we take into account that H_{r+1} is constant on M , then

$$L_r \eta + c \text{tr}(P_r) \eta - \text{tr}(A^2 P_r) \eta = -b_r H_r N(\phi') + b_r H_{r+1} \phi',$$

and so

$$\int_M [-H_r N(\phi') + H_{r+1} \phi'] \phi \left\langle \frac{\partial}{\partial s}, N \right\rangle dM \leq 0. \quad (3.4)$$

Now, observe that $\bar{\nabla} \phi' = -\langle \bar{\nabla} \phi', \frac{\partial}{\partial s} \rangle \frac{\partial}{\partial s} = -\phi'' \frac{\partial}{\partial s}$, and hence

$$N(\phi') = \langle N, \bar{\nabla} \phi' \rangle = -\phi'' \left\langle N, \frac{\partial}{\partial s} \right\rangle = \phi'' \cosh \theta,$$

where θ is the hyperbolic angle between N and $\frac{\partial}{\partial s}$. Substituting the above into (3.4), we finally arrive at

$$\int_M [H_r \phi'' \cosh \theta - H_{r+1} \phi'] \phi \cosh \theta dM \leq 0.$$

Arguing as at the end of the proof of Theorem 1.1 of [6], we get

$$H_r \phi'' (\cosh \theta - 1) = 0 \quad \text{and} \quad H_r \phi'' = H_{r+1} \phi'$$

on M . Since H_{r+1} is constant on M , either M is r -maximal or $H_{r+1} \neq 0$ on M . If this last case happens, the condition on the zero set of ϕ on M , together with the above, gives $H_r \phi'' \neq 0$ in a dense subset of M , and hence $\cosh \theta = 1$ on this set. By continuity, $\cosh \theta = 1$ on M , so M is a slice. $\square$

The above result has an interesting application in the case in which $\bar{M}_c^{n+1}$ is the de Sitter space of constant sectional curvature 1. For this, we make a brief description of this spacetime. Let $\mathbb{L}^{n+2}$ denote the $(n+2)$ -dimensional Lorentz–Minkowski space ($n \geq 2$), that is, the real vector space $\mathbb{R}^{n+2}$, endowed with the Lorentz metric

$$\langle v, w \rangle = \sum_{i=1}^{n+1} v_i w_i - v_{n+2} w_{n+2},$$

for all $v, w \in \mathbb{R}^{n+2}$. We define the $(n+1)$ -dimensional de Sitter space $\mathbb{S}_1^{n+1}$ as the following hyperquadric of $\mathbb{L}^{n+2}$:

$$\mathbb{S}_1^{n+1} = \{p \in \mathbb{L}^{n+2}; \langle p, p \rangle = 1\}.$$

From the above definition it is easy to show that the metric induced from $\langle, \rangle$ turns $\mathbb{S}_1^{n+1}$ into a Lorentz manifold with constant sectional curvature 1.

Choose a unit timelike vector $a \in \mathbb{L}^{n+2}$; then $V(p) = a - \langle p, a \rangle p$, $p \in \mathbb{S}_1^{n+1}$ is a conformal and closed timelike vector field. It foliates the de Sitter space by means of umbilical round spheres $M_\tau = \{p \in \mathbb{S}_1^{n+1} : \langle p, a \rangle = \tau\}$, $\tau \in \mathbb{R}$. The level set given by $\{p \in \mathbb{S}_1^{n+1} : \langle p, a \rangle = 0\}$ defines a round sphere of radius 1 which is a totally geodesic hypersurface in $\mathbb{S}_1^{n+1}$. We will refer to that sphere as the equator of $\mathbb{S}_1^{n+1}$ determined by a .

In the context of warped products, the de Sitter space can be thought of as the following RW:

$$\mathbb{S}_1^{n+1} = -\mathbb{R} \times_{\cosh s} \mathbb{S}^n,$$

where $\mathbb{S}^n$ means the Riemannian unit sphere. We observe that there are lots of possible choices for the unit timelike vector $a \in \mathbb{L}^{n+2}$ and, hence, a lot of ways to describe $\mathbb{S}_1^{n+1}$ as such a RW (cf. [14], Section 4). We notice that in this model, the equator of $\mathbb{S}_1^{n+1}$ is the slice $\{0\} \times \mathbb{S}^n$ and, consequently, $\phi'(s) = \sinh s$ vanishes only in this slice.

In this setting, from Theorem 3.2, we obtain the following:

Corollary 3.3. *Let $x : M^n \rightarrow \mathbb{S}_1^{n+1}$ be a closed strongly r -stable spacelike hypersurface. Suppose that the set of points in which M^n intersects the equator of $\mathbb{S}_1^{n+1}$ has empty interior. If $H_r \geq \max\{H_{r+1}, 0\}$, then M^n is either r -maximal or an umbilical round sphere.*

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