



Noncommutative geometric spaces with boundary: Spectral action

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ARTICLE INFO

Article history:

Received 10 September 2010

Accepted 1 October 2010

Available online 8 October 2010

Keywords:

Noncommutative spaces with boundary

Spectral action

Standard model

ABSTRACT

We study spectral action for Riemannian manifolds with boundary, and then generalize this to noncommutative spaces which are products of a Riemannian manifold times a finite space. We determine the boundary conditions consistent with the hermiticity of the Dirac operator. We then define spectral triples of noncommutative spaces with boundary. In particular we evaluate the spectral action corresponding to the noncommutative space of the standard model and show that the Einstein–Hilbert action gets modified by the addition of the extrinsic curvature terms with the right sign and coefficient necessary for consistency of the Hamiltonian. We also include effects due to the addition of a dilaton field.

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1. Introduction

Boundaries of manifolds play an important role in many physical theories, such as anomalies, Chern–Simons theories, topological theories, conformal theories and gravity. Riemannian geometry of manifolds with boundaries is well understood. This is not the case in noncommutative geometry where the spectral triple associated with the boundary of noncommutative space has not been defined. The spectral action principle in noncommutative geometry states that the physical action depends only on the spectrum. In particular, the simple assumption that space–time is a product of a continuous manifold times a finite space of KO -dimension 6 results uniquely, in the symplectic-unitary irreducible case, in the noncommutative space of the standard model, predicting the number of fermions to be 16 per generation and determines the correct representations of these fermions with respect to the gauge symmetry group $SU(3) \times SU(2) \times U(1)$ [1–3]. The spectral action is defined as the trace of an arbitrary function of the Dirac operator for the bosonic part and a Dirac type action for the fermionic part including all their interactions. The action is then uniquely defined and the only arbitrariness one encounters is in the first few moments of the function which enter in the spectral expansion with the higher coefficients suppressed by the high-energy scale. One essential point in the analysis is that the formulation is defined in terms of operators of compact resolvent, and thus the space considered is Euclidean. Therefore the model thus defined will correspond to Euclidean quantum gravity and will need Wick rotation to go to spaces with Lorentzian signature. Space–time is then assumed to have the topology of $\Sigma \times R$ where Σ is a three dimensional space. In studying the dynamics of the gravitational field by performing the $3 + 1$ splitting, one discovers that the Hamiltonian obtained from the Einstein–Hilbert action, contains an additional unwanted surface term that could be eliminated exactly by adding a surface term equal to twice the extrinsic curvature of the boundary three space. Alternatively, the variation of the Einstein–Hilbert action is inconsistent for manifolds with boundary without the addition of the extrinsic curvature term. The question we have to face is whether the spectral action for manifolds with boundary gives the correct boundary terms. This will be a severe test on the spectral action principle,

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because the boundary terms are completely fixed and there is essentially no freedom allowed to change any of these terms. The plan of this paper is as follows. In Section 2 we summarize properties of Riemannian manifolds with boundary. In Section 3 we evaluate the spectral action for a Dirac operator on a Riemannian manifold with boundary. In Section 4 we define the spectral triple associated with a noncommutative space with boundary. In Section 5 we evaluate the spectral action for the noncommutative space of the standard model taken to be with boundary. In Section 6 we include the effects of a dilaton. Section 7 is the conclusion. Appendix A is our calibrating example where the manifold is taken to be the disk, and this can be used to check the sign conventions. Appendix B is a summary of the variation of the Einstein–Hilbert action in the presence of the boundary term. The results in this paper were announced in [4].

2. Riemannian manifolds with boundary

We shall first give some definitions concerning embedding of hypersurfaces in a manifold that will enable us to perform the computations in a covariant way. Let us denote the coordinates of the manifold M by $\{x^\mu\}$ and of the hypersurface by $\{y^a\}$ and define the unit *inward* normal to the hypersurface by n^μ such that $g_{\mu\nu}n^\mu n^\nu = 1$ where $g_{\mu\nu}$ is the metric on M which is assumed to be Euclidean. Define the functions $e^\mu(y^a)$ as the embedding of the hypersurface in M and let [5]

$$e_a^\mu = \frac{\partial x^\mu}{\partial y^a} \quad (1)$$

then the metric $g_{\mu\nu}$ on M induces a metric h_{ab} on the hypersurface such that

$$h_{ab} = g_{\mu\nu} e_a^\mu e_b^\nu \quad (2)$$

and where the inward normal n^μ is orthogonal to e_a^μ

$$g_{\mu\nu} n^\mu e_a^\nu = 0. \quad (3)$$

It is convenient to define $n_\mu = g_{\mu\nu} n^\nu$ so that $n_\mu e_a^\mu = 0$. We now define the inverse functions e_μ^a by

$$e_a^\mu e_\mu^b = \delta_a^b \quad (4)$$

which satisfies the two conditions

$$e_a^\mu e_\mu^a = \delta_\nu^\nu - n^\mu n_\mu, \quad n_\mu e_a^\mu = 0. \quad (5)$$

We therefore can write

$$g_{\mu\nu} = h_{ab} e_\mu^a e_\nu^b + n_\mu n_\nu. \quad (6)$$

The inverse to the metric h_{ab} is given by

$$h^{ab} = g^{\mu\nu} e_\mu^a e_\nu^b \quad (7)$$

and fulfills the relation

$$g^{\mu\nu} = h^{ab} e_\mu^a e_\nu^b + n^\mu n^\nu \quad (8)$$

where $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$. This shows that any tensor can be projected into the hypersurface using the completeness relations for the basis $\{e_\mu^a, n_\mu\}$. We now define the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}, \quad \mu, \nu = 1, \dots, \dim M \quad (9)$$

and project these to define

$$\gamma^n = \gamma^\mu n_\mu, \quad \gamma^a = \gamma^\mu e_\mu^a \quad (10)$$

which satisfy the properties

$$\gamma^n \gamma^n = -1, \quad \{\gamma^a, \gamma^b\} = -2h^{ab}, \quad \{\gamma^a, \gamma^n\} = 0 \quad (11)$$

which follow from the relation

$$\gamma^\mu = e_\mu^a \gamma^a + n^\mu \gamma^n. \quad (12)$$

We will specialize to manifolds of dimension 4 so that a local coordinate system on ∂M will be denoted by $\{y^a\} = \{y^1, y^2, y^3\}$ and for M denoted by $\{x^\mu\} = \{x^1, x^2, x^3, x^4\}$. We then define on ∂M

$$\chi = -\frac{\sqrt{h}}{3!} \epsilon^{abc} \gamma_a \gamma_b \gamma_c, \quad \gamma_5 = \chi \gamma_n \quad (13)$$

which satisfy

$$\chi^2 = 1, \quad \chi \gamma_a = \gamma_a \chi, \quad \chi \gamma_n = -\gamma_n \chi \quad (14)$$

$$\gamma_5^2 = 1, \quad \chi \gamma_5 = -\gamma_5 \chi. \quad (15)$$

The normal vector n^μ satisfies the properties ([6] Chapter 3)

$$n_{\mu;\nu} = -K_{ab} e_\mu^a e_\nu^b \quad (16)$$

where the covariant derivative ; ν is the space–time covariant derivative and K_{ab} is the extrinsic curvature whose symmetry follows from the relation $e_{a;b}^\mu = e_{b;a}^\mu$. The Gauss–Weingarten equation is [6]

$$e_{a;b}^\mu = K_{ab} n^\mu + {}^{(3)}\Gamma_{ab}^c e_c^\mu \quad (17)$$

where ${}^{(3)}\Gamma_{ab}^c$ is the three dimensional affine connection and is given by

$${}^{(3)}\Gamma_{ab}^c = e_\mu^c e_{a;\nu}^\mu e_b^\nu. \quad (18)$$

3. Spectral action for noncommutative spaces with boundary

To compute the spectral action including boundary terms, for noncommutative spaces, we will utilize the known results which list the Seeley–deWitt coefficients for elliptic operators which are the square of the Dirac operator. An important ingredient in the calculation is to specify the boundary conditions that must be imposed on the Dirac operator [7,8]. We start with the observation that the Dirac operator must satisfy the hermiticity condition

$$\langle \Psi, D\Psi \rangle = \langle D\Psi, \Psi \rangle. \quad (19)$$

This condition is satisfied provided that the following Dirichlet boundary condition is imposed ([9] (3.30) p. 297)

$$\Pi_- \Psi|_{\partial M} = 0 \quad (20)$$

where the projector Π_- is given by

$$\Pi_- = \frac{1}{2}(1 - \chi). \quad (21)$$

We first write the square of the Dirac operator in the form [10,11]

$$P = D^2 = -(g^{\mu\nu} \partial_\mu \partial_\nu + \mathbb{A}^\mu + \mathbb{B}) \quad (22)$$

$$= -(g^{\mu\nu} \nabla'_\mu \nabla'_\nu + E) \quad (23)$$

where

$$\nabla'_\mu = \partial_\mu + \omega'_\mu \quad (24)$$

and

$$E = \mathbb{B} - g^{\mu\nu} (\partial_\mu \omega'_\nu + \omega'_\mu \omega'_\nu - \Gamma_{\mu\nu}^\rho \omega'_\rho) \quad (25)$$

$$\omega'_\mu = \frac{1}{2} g_{\mu\nu} (\mathbb{A}^\nu + g^{\rho\sigma} \Gamma_{\rho\sigma}^\nu(g)) \quad (26)$$

$$\Omega_{\mu\nu} = \partial_\mu \omega'_\nu - \partial_\nu \omega'_\mu + \omega'_\mu \omega'_\nu - \omega'_\nu \omega'_\mu. \quad (27)$$

It is convenient to write the Dirac operator in the form

$$D = \gamma^\mu \nabla_\mu - \Phi \quad (28)$$

where $\nabla_\mu = \partial_\mu + \omega_\mu$ with

$$\omega_\mu = \frac{1}{4} \omega_\mu^{\alpha\beta} \gamma_{\alpha\beta} \quad (29)$$

is the spin connection determined by the vanishing of the vierbein covariant derivative

$$\partial_\mu e_\nu^\alpha - \omega_\mu^{\alpha\beta} e_{\nu\beta} - \Gamma_{\mu\nu}^\rho(g) e_\rho^\alpha = 0 \quad (30)$$

where

$$\Gamma_{\mu\nu}^\rho(g) = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \quad (31)$$

is the Christoffel connection of $g_{\mu\nu} = e_{\mu}^{\alpha} e_{\nu\alpha}$. Note that e_{μ}^a should not be confused with e_{μ}^{α} as the index α refers to the tangent space $T(M)$ and is four dimensional and has the flat metric $\delta_{\alpha\beta}$. The covariant derivative ∇'_n is along the normal direction and is defined by

$$n^{\mu} \nabla'_{\mu} \quad (32)$$

and the index n always refers to the projection of the vector index along the normal direction. The boundary conditions for D^2 are then equivalent to [7,8]

$$\mathcal{B}_{\chi} \Psi = \Pi_{-}(\Psi)|_{\partial M} \oplus \Pi_{+}(\nabla'_n + S)\Pi_{+}(\Psi)|_{\partial M} = 0. \quad (33)$$

Here $\Pi_{+} = 1 - \Pi_{-}$, and the operator S is

$$S = \Pi_{+} \left(\gamma_n \Phi - \frac{1}{2} \gamma_n \gamma^a \nabla'_a \chi \right) \Pi_{+} \quad (34)$$

with

$$\nabla'_a \chi = \partial_a \chi + [\omega'_a, \chi] = K_{ab} \chi \gamma^n \gamma^b + [\theta_a, \chi] \quad (35)$$

where

$$\theta_a = \omega'_a - \omega_a. \quad (36)$$

To prove the above relation we write

$$\Pi_{-}(\gamma^n \nabla'_n + \gamma^a \nabla'_a - \Phi) \Psi|_{\partial M} = \gamma^n (\nabla'_n + \gamma_n \Phi) \Pi_{+} \Psi|_{\partial M} + [\Pi_{-}, \gamma^a \nabla'_a] \Psi|_{\partial M} \quad (37)$$

where we have used $\Pi_{-} \Psi|_{\partial M} = 0$ and $\gamma^a \nabla'_a (\Pi_{-} \Psi|_{\partial M}) = 0$. We then have

$$\begin{aligned} [\Pi_{-}, \gamma^a \nabla'_a] \Psi|_{\partial M} &= \frac{1}{2} \gamma^a \nabla'_a \chi (\Pi_{-} \Psi + \Pi_{+} \Psi)|_{\partial M} \\ &= \Pi_{-} \left(\frac{1}{2} \gamma^a \nabla'_a \chi \right) \Pi_{+} \Psi|_{\partial M} \\ &= \gamma^n \Pi_{+} \left(\frac{1}{2} \gamma_n \gamma^a \nabla'_a \chi \right) \Pi_{+} \Psi|_{\partial M}. \end{aligned}$$

We also have the relations

$$E = \gamma^{\mu} \nabla_{\mu} \Phi - \Phi^2 - \frac{1}{2} \gamma^{\mu\nu} \Omega_{\mu\nu}, \quad (38)$$

$$\Omega_{\mu\nu} = \partial_{\mu} \omega'_{\nu} - \partial_{\nu} \omega'_{\mu} + \omega'_{\mu} \omega'_{\nu} - \omega'_{\nu} \omega'_{\mu}. \quad (39)$$

The Seeley–deWitt coefficients for second order operators on manifolds with boundary were calculated by Branson and Gilkey [7,8] and are given by

$$a_0(P, \chi) = \frac{1}{16\pi^2} \int_M d^4x \sqrt{g} \text{Tr}(1) \quad (40)$$

$$a_1(P, \chi) = 0 \quad (41)$$

$$a_2(P, \chi) = \frac{1}{96\pi^2} \left(\int_M d^4x \sqrt{g} \text{Tr}(6E + R) + \int_{\partial M} d^3x \sqrt{h} \text{Tr}(2K + 12S) \right) \quad (42)$$

$$a_3(P, \chi) = \frac{1}{384(4\pi)^{\frac{3}{2}}} \int_{\partial M} d^3x \sqrt{h} \text{Tr}(96\chi E + 3K^2 + 6K_{ab}K^{ab} + 96SK + 192S^2 - 12\nabla'_a \chi \nabla'^a \chi) \quad (43)$$

$$\begin{aligned} a_4(P, \chi) &= \frac{1}{360} \frac{1}{16\pi^2} \left\{ \int_M d^4x \sqrt{g} \text{Tr} \left(60RE + 180E^2 + 30\Omega_{\mu\nu} \Omega^{\mu\nu} + 12(R + 5E)_{;\mu}^{\mu} \right. \right. \\ &\quad \left. \left. + 5R^2 - 2R_{\mu\nu} R^{\mu\nu} + 2R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right) + \int_{\partial M} d^3x \sqrt{h} \text{Tr} \left(180\chi \nabla'_n E + 120EK + 20RK \right. \right. \\ &\quad \left. \left. + 4R_{nan}^a K - 12R_{nbn}^a K_a^b + 4R_{acb}^c K^{ab} + \frac{1}{21} (160K^3 - 48KK_{ab}K^{ab} + 272K_b^a K_c^b K_a^c) \right. \right. \\ &\quad \left. \left. + 720SE + 120SR + 144SK^2 + 48SK_{ab}K^{ab} + 480S^2K + 480S^3 \right. \right. \\ &\quad \left. \left. + 60\chi \nabla'^a \chi \Omega_{an} - 12\nabla'_a \chi \nabla'^a \chi (K + 10S) - 24\nabla'_a \chi \nabla'_b \chi K^{ab} \right) \right\}. \quad (44) \end{aligned}$$

The Riemann tensor is defined by

$$R_{\mu\nu\rho\sigma} = g_{\sigma\tau}(\partial_\mu \Gamma_{\nu\rho}^\tau - \partial_\nu \Gamma_{\mu\rho}^\tau + \Gamma_{\mu\kappa}^\tau \Gamma_{\nu\rho}^\kappa - \Gamma_{\nu\kappa}^\tau \Gamma_{\mu\rho}^\kappa) \quad (45)$$

and its contractions are

$$R_{\mu\nu} = g^{\rho\sigma} R_{\mu\rho\sigma\nu}, \quad R = g^{\mu\nu} R_{\mu\nu}. \quad (46)$$

We are using the conventions of Gilkey. They are related to the ones used by Misner–Thorn–Wheeler by [12]

$$R_{\mu\nu\rho\sigma}^G = -R_{\mu\nu\rho\sigma}^{\text{MTW}}, \quad R_{\mu\nu}^G = R_{\mu\nu}^{\text{MTW}}, \quad R^G = R^{\text{MTW}}. \quad (47)$$

The curvature defined by the spin connection is

$$R_{\mu\nu}^{\alpha\beta}(\omega) = \partial_\mu \omega_\nu^{\alpha\beta} - \partial_\nu \omega_\mu^{\alpha\beta} - \omega_\mu^{\alpha\gamma} \omega_\nu^\beta + \omega_\nu^{\alpha\gamma} \omega_\mu^\beta \quad (48)$$

and is related to the curvature of the Christoffel connection by

$$R_{\mu\nu}^{\alpha\beta}(\omega) e_{\rho\alpha} e_{\sigma\beta} = R_{\mu\nu\rho\sigma}^G, \quad R_{\mu\nu}^{\alpha\beta}(\omega) e_\alpha^\mu e_\beta^\nu = -R^G. \quad (49)$$

We note that the R we used in [13] has the opposite sign to R^G where the curvature is positive for spheres.

The formulas expressing the projections of the Riemann tensor on the boundary in terms of the three curvature and the extrinsic curvature are

$$R_{abcd} = {}^{(3)}R_{abcd} + (K_{ac}K_{bd} - K_{ad}K_{bc})$$

$$R_{nan}^a = g^{\mu\rho} n^\nu (n_{\mu;\nu\rho} - n_{\mu;\rho\nu}) = K^2 - K_{ab}K^{ab} + \text{cov.div.}$$

In particular, we can apply these results to the square of the Dirac operator. We shall start with the simplest example of the Dirac operator of a pure gravitational fields, and later generalize the results to the general case of the standard model.

4. Spectral action for Riemannian manifolds with boundary

In this case we have

$$D = \gamma^\mu (\partial_\mu + \omega_\mu). \quad (50)$$

To use the above formulas we have

$$\omega'_\mu = \omega_\mu, \quad \Phi = 0 \quad (51)$$

and

$$S = \Pi_+ \left(-\frac{1}{2} \gamma_n \gamma^a \chi K_{ab} \gamma^n \gamma^b \right) = -\frac{1}{2} K \Pi_+ \quad (52)$$

where we used $\gamma_n \gamma^a \chi \gamma^n = -\chi \gamma_n \gamma^a \gamma^n = -\chi \gamma^a$. We also have

$$E = -\frac{1}{4} R, \quad \nabla'_a \chi = K_{ab} \chi \gamma^n \gamma^b. \quad (53)$$

Substituting $\text{Tr}(1) = 4$ and $\text{Tr}(S) = -K$ we have

$$a_0(P, \chi) = \frac{1}{4\pi^2} \int_M d^4x \sqrt{g}. \quad (54)$$

Next we calculate

$$a_2(P, \chi) = \frac{1}{96\pi^2} \left(\int_M d^4x \sqrt{g} \text{Tr}(6E + R) + \int_{\partial M} d^3x \sqrt{h} \text{Tr}(2K + 12S) \right) \quad (55)$$

we use

$$\text{Tr}(6E + R) = -\frac{R}{2} \text{Tr}(1)$$

$$\text{Tr}(2K + 12S) = -K \text{Tr}(1) \quad (56)$$

because $\text{Tr}(\Pi_+) = \frac{1}{2}$. In this case $\text{Tr}(1) = 4$ (trace over Dirac matrices). Substituting into the formula for a_2 gives

$$a_2(P, \chi) = \frac{1}{24\pi^2} \left(\int_M d^4x \sqrt{g} \left(-\frac{1}{2} R \right) + \int_{\partial M} d^3x \sqrt{h} (-K) \right). \quad (57)$$

The important point in the above result is the emergence of the combination

$$- \int_M d^4x \sqrt{g} R - 2 \int_{\partial M} d^3x \sqrt{h} K \quad (58)$$

as the lowest term of the gravitational action which is known to be the required correction to the Einstein action including the surface term which makes the Hamiltonian formalism consistent. The consistency of the variation of this action is summarized in appendix 2. This is remarkable because both the sign and the coefficients are correct. The only assumption we made is that the boundary conditions are taken to satisfy the hermiticity of the Dirac operator. This is yet another miracle concerning the correct signs obtained in the spectral action of the Dirac operator. We also notice that the relative coefficient between R and K depends on the nature of the Laplacian. The desired answer is obtained naturally for the Dirac operator, but not for a general Laplacian.

We continue to compute

$$a_3(P, \chi) = \frac{1}{384(4\pi)^{\frac{3}{2}}} \int_{\partial M} d^3x \sqrt{h} \text{Tr}(96\chi E + 3K^2 + 6K_{ab}K^{ab} + 96SK + 192S^2 - 12\nabla'_a \chi \nabla'^a \chi). \quad (59)$$

We first note that $\text{Tr}(96\chi E) = 0$ and

$$\text{Tr}(3K^2 + 6K_{ab}K^{ab} + 96SK + 192S^2) = \text{Tr}(1)(3K^2 + 6K_{ab}K^{ab}) \quad (60)$$

while

$$\text{Tr}(-12\nabla'_a \chi \nabla'^a \chi) = \text{Tr}(-12K_{ab}\chi \gamma^n \gamma^b K_{ac}\chi \gamma^n \gamma^c) = -12K_{ab}K^{ab}\text{Tr}(1) \quad (61)$$

where we have used $\chi \gamma^n \gamma^b \chi \gamma^n \gamma^c = -\gamma^n \gamma^b \chi \gamma^n \gamma^c = -\gamma^b \gamma^c$. Collecting the above terms gives

$$a_3(P, \chi) = \frac{1}{32(4\pi)^{\frac{3}{2}}} \int_{\partial M} d^3x \sqrt{h} (K^2 - 2K_{ab}K^{ab}). \quad (62)$$

Finally we turn our attention to the computation of a_4 which is rather complicated. First we evaluate

$$\begin{aligned} & \text{Tr}(60RE + 180E^2 + 30\Omega_{\mu\nu}\Omega^{\mu\nu} + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 3R_{;\mu}^{\mu}) \\ &= \text{Tr}(1) \frac{1}{4} (5R^2 - 8R_{\mu\nu}R^{\mu\nu} - 7R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 12R_{;\mu}^{\mu}). \end{aligned} \quad (63)$$

We then use the identities

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} = R^*R^* - R^2 \quad (64)$$

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} = C_{\mu\nu\rho\sigma}^2 - \frac{1}{3}R^2 \quad (65)$$

where $R^*R^* = \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\delta}R_{\mu\nu}^{\alpha\beta}R_{\rho\sigma}^{\gamma\delta}$. These identities are solved to give

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = 2C_{\mu\nu\rho\sigma}^2 + \frac{1}{3}R^2 - R^*R^* \quad (66)$$

$$R_{\mu\nu}R^{\mu\nu} = \frac{1}{2}C_{\mu\nu\rho\sigma}^2 + \frac{1}{3}R^2 - \frac{1}{2}R^*R^* \quad (67)$$

and can be combined to show that

$$5R^2 - 8R_{\mu\nu}^2 - 7R_{\mu\nu\rho\sigma}^2 = 11R^*R^* - 18C_{\mu\nu\rho\sigma}^2. \quad (68)$$

We continue by evaluating

$$\begin{aligned} & \text{Tr}(180\chi \nabla'_n E + 120EK + 20RK + 4R_{nan}^a K - 12R_{nbn}^a K_a^b + 4R_{acb}^c K^{ab}) \\ &= \text{Tr}(1)(-10RK + 4R_{nan}^a K - 12R_{nbn}^a K_a^b + 4R_{acb}^c K^{ab}). \end{aligned}$$

In addition the expression

$$\frac{1}{21} \text{Tr}(160K^3 - 48KK_{ab}K^{ab} + 272K_b^a K_c^b K_a^c)$$

cannot be simplified. Next we have

$$\begin{aligned} & \text{Tr}(720SE + 120SR + 144SK^2 + 48SK_{ab}K^{ab} + 480S^2K + 480S^3 \\ &+ 60\chi \nabla'^a \chi \Omega_{an} - 12\nabla'_a \chi \nabla'^a \chi (K + 10S) - 24\nabla'_a \chi \nabla'_b \chi K^{ab}) \\ &= \text{Tr} \left(720 \left(-\frac{K}{4} \right) \left(-\frac{R}{4} \right) + 120 \left(-\frac{K}{4} R \right) + 144 \left(-\frac{K}{4} K^2 \right) + 48 \left(-\frac{K}{4} K_{ab} K^{ab} \right) \right. \\ &+ 480 \frac{K^2}{4} \frac{1}{2} K + 480 \left(-\frac{K^3}{8} \frac{1}{2} \right) + 60K_{ab}\chi^2 \gamma^n \gamma^b \frac{1}{4} R_{an}^{\alpha\beta} \gamma_{\alpha\beta} \\ &\left. - 12 \left(K - \frac{10}{4} K \right) K_{ac} \chi \gamma^n \gamma^c K_{ad} \chi \gamma^n \gamma^d - 24K_{ac} \chi \gamma^n \gamma^c K_{bd} \chi \gamma^n \gamma^d K^{ab} \right). \end{aligned}$$

The factors of $\frac{1}{2}$ appearing above are due to the presence of $\Pi_+ = \frac{1}{2}(1 + \chi)$. Evaluating the traces, the above expression simplifies to

$$\text{Tr}(1)(15KR - 6K^3 + 30R_{an}^{bn}K^{ab} + 6KK_{ab}K^{ab} - 24K_a^bK_b^cK_c^a).$$

Combining all the above terms gives

$$\text{Tr}(1) \left(5RK + 4R_{nan}^aK + 18R_{nbn}^aK_a^b + 4R_{acb}^cK^{ab} + \frac{2}{21}(17K^3 + 39KK_{ab}K^{ab} - 116K_a^bK_b^cK_c^a) \right). \quad (69)$$

Thus the final expression for a_4 is given by

$$\begin{aligned} a_4(P, \chi) = & \frac{1}{360} \frac{1}{16\pi^2} \left\{ \int_M d^4x \sqrt{g} (5R^2 - 8R_{\mu\nu}^2 - 7R_{\mu\nu\rho\sigma}^2 - 12R_{;\mu}^\mu) \right. \\ & + 4 \int_{\partial M} d^3x \sqrt{h} \left(\frac{2}{21} (17K^3 + 39KK_{ab}K^{ab} - 116K_a^bK_b^cK_c^a) \right) \\ & \left. + (5RK + 4KR_{nan}^a + 4K_{ab}R_{acb}^c + 18R_{anbn}K^{ab}) \right\}. \quad (70) \end{aligned}$$

Collecting terms, the spectral action is then given by [11]

$$I = 2(f_4\Lambda^4a_0 + f_2\Lambda^2a_2 + f_1\Lambda a_3 + f_0a_4) + O\left(\frac{1}{\Lambda^2}\right). \quad (71)$$

5. Spectral action for the noncommutative space of the standard model with boundary

It was shown recently [2,1] by making the basic assumption at some high-energy scale that space–time is described by a noncommutative space which is a product of a continuous four-dimensional Riemannian manifold times a finite space, it is possible to almost uniquely determine the algebra and Hilbert space of the finite space. The main constraints come from the axioms of noncommutative geometry, as well as from the physical requirement that there is a mixing between the fermions and their conjugates, which turns out to imply that the neutrinos get a Majorana mass through the see-saw mechanism. Under these conditions, the algebra is given by $\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_F$ where the algebra \mathcal{A}_F is finite dimensional, $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, and $\mathbb{H} \subset M_2(\mathbb{C})$ is the algebra of quaternions. The important point to emphasize is that the number of fermions is predicted to be $4^2 = 16$, and the representations of the fermions follow from the decomposition of the representation $(4, 4)$ with respect to the subalgebra $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ of $\mathbb{H} \oplus \mathbb{H} \oplus M_4(\mathbb{C})$. The spectral geometry of \mathcal{A} is given by the product rule

$$\mathcal{H} = L^2(M, S) \otimes \mathcal{H}_F, \quad D = D_M \otimes 1 + \gamma_5 \otimes D_F$$

where $L^2(M, S)$ is the Hilbert space of L^2 spinors, and D_M is the Dirac operator of the Levi-Civita spin connection on M . The operator D_F anticommutes with the chirality operator γ_F on \mathcal{H}_F . The spectral geometry does not change if one replaces D by the equivalent operator

$$D = D_M \otimes \gamma_F + 1 \otimes D_F$$

but this equivalence fails when M has a boundary and it is only the latter choice which has conceptual meaning since γ_5 no longer anticommutes with D_M when $\partial M \neq \emptyset$. The noncommutative space defined by a spectral triple has to satisfy the basic axioms of noncommutative geometry. The charge conjugation operator J for the product geometry is then given by

$$J = J_M \gamma_5 \otimes J_F$$

which commutes with the operator D since in even dimension J_M commutes with D_M while in dimension 6 modulo 8, J_F anticommutes with γ_F . The KO -dimension of the noncommutative space must be taken to be equal to 6 to insure that the fermions and their conjugates are not independent, and thus avoiding the fermion doubling problem.

Our main interest now is to derive again the spectral action of the standard model, including boundary contributions. The computations are very complicated, and because of this it is important to devise a way to make this calculation tractable. The starting point is the observation that the inner fluctuations under the action of the unitary transformations of the algebra, forces the Dirac operator to be modified to

$$D \rightarrow D_A = D + A + JAJ^{-1}, \quad A = \sum a[D, b]. \quad (72)$$

The Dirac operator acts on the 96 dimensional space of the three families of 16 dimensional spinors and their conjugates, and splits into a leptonic sector and a quark sector. It turns out that we can get a handle on this calculation by considering first the much simpler problem of a Dirac operator of the type:

$$D_A = \begin{pmatrix} \gamma^\mu ((\partial_\mu + \omega_\mu)1_N + B_\mu) \otimes \gamma_F & H \\ H^\dagger & \gamma^\mu ((\partial_\mu + \omega_\mu)1_M + B_\mu) \otimes \gamma_F \end{pmatrix} \quad (73)$$

where B_μ is an $N \times N$ matrix valued gauge field. We shall then define substitutions which will enable us to find the answer for the general case without much difficulty.

Having defined $D = \gamma^\mu \nabla_\mu - \Phi$ we can easily deduce that

$$\Phi = - \begin{pmatrix} \gamma^\mu B_\mu \otimes \gamma_F & H \\ H^\dagger & \gamma^\mu B_\mu \otimes \gamma_F \end{pmatrix}. \quad (74)$$

We then evaluate D^2 and put it in canonical form to find that

$$\mathbb{A}^\mu = (2g^{\mu\nu} \omega_\nu - g^{\rho\sigma} \Gamma_{\rho\sigma}^\mu) 1_{N+2} + 2g^{\mu\nu} B_\nu 1_2 \quad (75)$$

$$\mathbb{B} = (\partial^\mu \omega_\mu + \omega^\mu \omega_\mu - \Gamma^\mu \omega_\mu - R) 1_{N+2} + 2\omega_\mu g^{\mu\nu} B_\nu 1_2 + \begin{pmatrix} X \otimes \gamma_F & \gamma^\mu \nabla_\mu H \\ -\gamma^\mu \nabla_\mu H^\dagger & X \otimes \gamma_F \end{pmatrix} \quad (76)$$

where

$$X = (\partial^\mu + \omega^\mu - \Gamma^\mu) B_\mu - \frac{1}{2} \gamma^{\mu\nu} F_{\mu\nu} + B^\mu B_\mu \quad (77)$$

$$\nabla_\mu H = \partial_\mu H + [B_\mu, H] \quad (78)$$

$$F_{\mu\nu}(B) = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]. \quad (79)$$

From these we can construct ω'_μ , E and $\Omega_{\mu\nu}$:

$$\omega'_\mu = (\omega_\mu) 1_{N+M} + B_\mu 1_2 \quad (80)$$

$$\Omega_{\mu\nu} = \frac{1}{4} (R_{\mu\nu}^{\alpha\beta} \gamma_{\alpha\beta}) 1_{N+2} + F_{\mu\nu}(B) 1_2 \quad (81)$$

$$E = \left(-\frac{1}{4} R \right) 1_{N+2} + \begin{pmatrix} -\frac{1}{2} \gamma^{\mu\nu} F_{\mu\nu}(B) - HH^\dagger & \gamma^\mu \nabla_\mu H \otimes \gamma_F \\ -\gamma^\mu \nabla_\mu H^\dagger \otimes \gamma_F & -\frac{1}{2} \gamma^{\mu\nu} F_{\mu\nu}(B) - H^\dagger H \end{pmatrix} \quad (82)$$

$$\theta_\mu = \omega'_\mu - \omega_\mu = B_\mu 1_2. \quad (83)$$

From these relations, and assuming the boundary condition that the normal components of the vectors vanish on the boundary

$$B_n|_{\partial M} = 0$$

we deduce that:

$$S = \Pi_+ \left(-\frac{1}{2} K 1_{N+2} \right). \quad (84)$$

The reason for the vanishing of all Higgs and vector terms from S is the relation $\Pi_+ \gamma_n \Pi_+ = 0$ and $\Pi_+ \gamma_n \gamma_a \Pi_+ = 0$.

We now derive the Seeley–de Witt coefficients a_n . Starting with a_0 we have:

$$a_0(P, \chi) = \frac{\text{Tr}(1_{N+2})}{16\pi^2} \int_M d^4x \sqrt{g}. \quad (85)$$

(To facilitate going to the standard model at a later stage and to make use of the results there, we will include all numerical factors in the $\text{Tr}(\dots)$, thus in what follows we will not take out the factor 4 coming from tracing over Dirac matrices.) Next we find a_2 by evaluating the various parts

$$\text{Tr}(6E + R) = -\frac{R}{2} \text{Tr}(1_{N+2}) - 12 \text{Tr} H^\dagger H \quad (86)$$

$$\text{Tr}(2K + 12S) = -K \text{Tr}(1) \quad (87)$$

because $\text{Tr}(\Pi_+) = \frac{1}{2}$. Thus

$$a_2(P, \chi) = -\frac{1}{96\pi^2} \left(\int_M d^4x \sqrt{g} \left(\frac{1}{2} R \text{Tr}(1_{N+2}) + 12 \text{Tr}(H^\dagger H) \right) + \int_{\partial M} d^3x \sqrt{h} \text{Tr}(1_{N+2}) K \right). \quad (88)$$

Next we find a_3 by computing its parts:

$$\text{Tr}(96\chi E) = 0 \quad (89)$$

$$\text{Tr}(3K^2 + 6K_{ab} K^{ab} + 96SK + 192S^2) = 3\text{Tr}(1_{N+2})(K^2 + 2K_{ab} K^{ab}) \quad (90)$$

$$\text{Tr}(-12\nabla'_a \chi \nabla'^a \chi) = -12K_{ab} K^{ab} \text{Tr}(1_{N+2}). \quad (91)$$

Therefore after substituting, a_3 simplifies to:

$$a_3(P, \chi) = \frac{1}{128(4\pi)^{\frac{3}{2}}} \int_{\partial M} d^3x \sqrt{h} ((K^2 - 2K_{ab}K^{ab}) \text{Tr}(1_{N+2})). \quad (92)$$

Finally, we turn our attention to a_4 and concentrate on the terms which were not present in the pure Riemannian case. First we simplify the combination:

$$\begin{aligned} & \text{Tr}(60RE + 180E^2 + 30\Omega_{\mu\nu}\Omega^{\mu\nu} + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 12(R + 5E)_{;\mu}^{\mu}) \\ &= \text{Tr} \left(\left(-15 + \frac{45}{4} + 5 \right) R^2 + \left(2 - \frac{15}{4} \right) R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} - 3(R)_{;\mu}^{\mu} - 120(H^\dagger H)_{;\mu}^{\mu} \right) \\ & \quad + (180 - 120)RH^\dagger H - (180 - 60)F_{\mu\nu}^2 + 360(H^\dagger H)^2 + 360\nabla_\mu H^\dagger \nabla_\mu H \\ &= \text{Tr}(1_{N+2}) \frac{1}{4} (-18C_{\mu\nu\rho\sigma}^2 + 11R^*R^* - 12(R)_{;\mu}^{\mu}) \\ & \quad + 360 \left(\text{Tr}(H^\dagger H)^2 + \text{Tr}\nabla_\mu H^\dagger \nabla_\mu H + \frac{1}{6}R\text{Tr}H^\dagger H - \frac{1}{3}\text{Tr}F_{\mu\nu}^2 \right) - 120\text{Tr}(H^\dagger H)_{;\mu}^{\mu}. \end{aligned}$$

Next we consider:

$$\begin{aligned} & \text{Tr} (180\chi \nabla_n' E + 120EK + 20RK + 4R_{nan}^a K - 12R_{nbn}^a K_a^b + 4R_{acb}^c K^{ab}) \\ &= \text{Tr}(1_{N+2}) \left(0 + 120 \left(-\frac{R}{4} - 2H^\dagger H \right) K + 20RK + 4R_{nan}^a K - 12R_{nbn}^a K_a^b + 4R_{acb}^c K^{ab} \right) \\ &= \text{Tr}(1_{N+2}) (-10RK + 4R_{nan}^a K - 12R_{nbn}^a K_a^b + 4R_{acb}^c K^{ab}) - 240K \text{Tr}H^\dagger H \end{aligned}$$

where the only change from the purely gravitational case is the addition of $-240K \text{Tr}H^\dagger H$. The combination

$$\frac{1}{21} \text{Tr}(160K^3 - 48KK_{ab}K^{ab} + 272K_b^a K_c^b K_a^c)$$

does not simplify. Next we consider

$$\begin{aligned} & \text{Tr} (720SE + 120SR + 144SK^2 + 48SK_{ab}K^{ab} + 480S^2K + 480S^3 \\ & \quad + 60\chi \nabla'^a \chi \Omega_{an} - 12\nabla'_a \chi \nabla'^a \chi (K + 10S) - 24\nabla'_a \chi \nabla'_b \chi K^{ab}) \end{aligned}$$

and note that the only change from the purely gravitational case is that $\text{Tr}(720SE)$ will give the extra term $360K \text{Tr}H^\dagger H$. The next three contributions

$$\begin{aligned} \text{Tr}(60\chi \nabla'^a \chi \Omega_{an}) &= \text{Tr} \left(60\chi K_{ab} \chi \gamma^n \gamma^b \left(\frac{1}{4} R_{an}^{\alpha\beta} \gamma_{\alpha\beta} + F_{an} \right) \right) \\ &= \text{Tr}(1_{N+2}) (30R_{nbn}^a K_a^b) \\ \text{Tr}(-12\nabla'_a \chi \nabla'^a \chi (K + 10S)) &= \text{Tr}(1_{N+2}) (18KK_{ab}K^{ab}) \\ \text{Tr}(-24\nabla'_a \chi \nabla'_b \chi K^{ab}) &= \text{Tr}(1_{N+2}) (-24K_b^a K_c^b K_a^c) \end{aligned}$$

remain unchanged. Therefore the total change for the boundary term from the purely gravitational case is the addition of

$$(-240 + 360)K \text{Tr}H^\dagger H = 120K \text{Tr}H^\dagger H. \quad (93)$$

Combining all the above terms gives

$$\begin{aligned} a_4(P, \chi) &= \frac{1}{16\pi^2} \left\{ \int_M d^4x \sqrt{g} \left\{ \frac{1}{360} \left(\frac{11}{4} R^*R^* - \frac{9}{2} C_{\mu\nu\rho\sigma}^2 - 3(R)_{;\mu}^{\mu} \right) \text{Tr}1_{N+2} \right. \right. \\ & \quad + \text{Tr} \left(|\nabla_\mu H|^2 + (H^\dagger H)^2 + \frac{1}{6}RH^\dagger H - \frac{1}{3}F_{\mu\nu}^2 \right) - \frac{1}{3}\text{Tr}(H^\dagger H)_{;\mu}^{\mu} \Big\} \\ & \quad + \frac{1}{360} \int_{\partial M} d^3x \sqrt{h} \left\{ \frac{2}{21} (17K^3 + 39KK_{ab}K^{ab} - 116K_a^b K_b^c K_c^a) \text{Tr}1_{N+2} \right. \\ & \quad \left. \left. + (5RK + 4KR_{nan}^a + 4K_{ab}R_{acb}^c + 18R_{anbn}K^{ab}) \text{Tr}1_{N+2} + 120K \text{Tr}H^\dagger H \right\} \right\}. \quad (94) \end{aligned}$$

We can now apply these formulas to the Dirac operator of the Standard model by making the following substitutions

$$\text{Tr}1_{N+2} \rightarrow (96 + 288) = 384. \quad (95)$$

For the trace in the leptonic sector we have

$$\text{Tr}(1) = 4 \cdot 3 \cdot 4 \cdot 2 = 96$$

where the first 4 is from the trace of gamma matrices, the 3 is for the number of generations, the 4 is the dimension of the basis for leptons, and 2 is for summing over fermions and conjugate fermions. In the quarks sector we have

$$\text{Tr}(1) = 4 \cdot 3 \cdot 4 \cdot 2 \cdot 3 = 288$$

where the last factor of 3 is for color. Next for the Higgs field we make the substitution

$$\text{Tr}(H^\dagger H) \rightarrow 8 \left(a|\varphi|^2 + \frac{1}{2}c \right) \quad (96)$$

where

$$a = \text{tr}(3|k^u|^2 + 3|k^d|^2 + |k^e|^2 + |k^v|^2) \quad \text{and} \quad c = \text{tr}(|k^{v_R}|^2). \quad (97)$$

and the factor $8 = 4 \cdot 2$ with the 4 coming from the trace of the Dirac gamma matrices, and 2 because of summing over fermionic and conjugate fermions. The term $4c$ appears because of the mass mixing term between the fermions and their conjugates. Next we have

$$-\frac{1}{3}\text{Tr}(F_{\mu\nu}^2) \rightarrow 8 \left[g_3^2 (G_{\mu\nu}^i)^2 + g_2^2 (F_{\mu\nu}^\alpha)^2 + \frac{5}{3} g_1^2 (B_{\mu\nu})^2 \right] \quad (98)$$

where $G_{\mu\nu}^i$, $F_{\mu\nu}^\alpha$ and $B_{\mu\nu}$ are the $SU(3)$, $SU(2)$ and $U(1)$ gauge curvatures. The contributions from the leptonic sector are

$$-\frac{1}{16\pi^2} \frac{1}{6} \text{Tr} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{16\pi^2} \frac{24}{6} \left(2 \left(-i \frac{g_2}{2} \right)^2 F_{\mu\nu}^\alpha F^{\mu\nu\alpha} + 2 \left(i \frac{g_1}{2} \right)^2 B_{\mu\nu} B^{\mu\nu} + (ig_1)^2 B_{\mu\nu} B^{\mu\nu} \right)$$

and from the quarks sector

$$\begin{aligned} -\frac{1}{16\pi^2} \frac{1}{6} \text{Tr} F_{\mu\nu} F^{\mu\nu} = & -\frac{24}{6} \frac{1}{16\pi^2} \left(2 \cdot 3 \left(-i \frac{g_2}{2} \right)^2 F_{\mu\nu}^\alpha F^{\mu\nu\alpha} + 2 \cdot 3 \left(-i \frac{g_1}{6} \right)^2 B_{\mu\nu} B^{\mu\nu} \right. \\ & \left. + 3 \cdot \left(\frac{i}{3} g_1 \right)^2 B_{\mu\nu} B^{\mu\nu} + 3 \cdot \left(-\frac{2i}{3} g_1 \right)^2 B_{\mu\nu} B^{\mu\nu} + 2 \cdot 4 \cdot \left(-\frac{i}{2} g_3 \right)^2 G_{\mu\nu}^i G^{\mu\nu i} \right) \end{aligned}$$

where $24 = 4 \cdot 3 \cdot 2$, the 4 is due to the trace on the gamma matrices, 3 from generations and 2 from fermions and their conjugates. Next, $\frac{180}{360} \text{Tr}(E^2)$ gives the extra contribution of

$$\frac{1}{2} \left(2e|\varphi|^2 + \frac{1}{2}d \right) \quad (99)$$

where

$$d = \text{tr}(|k^{v_R}|^4) \quad \text{and} \quad e = \text{tr}(|k^{v_R}|^2 |k^v|^2). \quad (100)$$

Finally

$$\text{Tr}(H^\dagger H)^2 \rightarrow b|\varphi|^4 \quad (101)$$

where

$$b = \text{tr}(3|k^u|^4 + 3|k^d|^4 + |k^e|^4 + |k^v|^4). \quad (102)$$

Summarizing, we have

$$a_0 = \frac{384}{16\pi^2} \int_M d^4x \sqrt{g} \quad (103)$$

$$a_2 = \frac{4}{\pi^2} \left(\int_M d^4x \sqrt{g} \left(-\frac{1}{2}R - \frac{1}{4} \left(a|\varphi|^2 + \frac{1}{2}c \right) \right) - \int_{\partial M} d^3x \sqrt{h} K \right) \quad (104)$$

$$a_3 = \frac{1}{(4\pi)^{\frac{3}{2}}} \int_{\partial M} d^3x \sqrt{h} (3(K^2 - 2K_{ab}K^{ab})) \quad (105)$$

$$\begin{aligned}
a_4 &= \frac{1}{16\pi^2} \left\{ \int_M d^4x \sqrt{g} \left(\frac{384}{360} \frac{1}{4} \left(-18C_{\mu\nu\rho\sigma}^2 + 11R^*R^* - 12(R)_{;\mu}^\mu \right) \right. \right. \\
&\quad + 8 \left(a|D_\mu\varphi|^2 + \frac{1}{6}R \left(a|\varphi|^2 + \frac{1}{2}c \right) + b|\varphi|^4 + \frac{1}{2}d - \frac{1}{3}a(|\varphi|^2)_{;\mu}^\mu \right) \\
&\quad + 8 \left(g_3^2 (G_{\mu\nu}^i)^2 + g_2^2 (F_{\mu\nu}^\alpha)^2 + \frac{5}{3}g_1^2 (B_{\mu\nu})^2 \right) \left. + \int_{\partial M} d^3x \sqrt{h} \left(\frac{1}{3}K \left(a|\varphi|^2 + \frac{1}{2}c \right) \right. \right. \\
&\quad + \frac{384}{360} (5RK + 4KR_{nan}^a + 4K_{ab}R_{acb}^c + 18R_{anbn}K^{ab}) + \frac{384}{360} \frac{2}{21} (17K^3 + 39KK_{ab}K^{ab} - 116K_a^b K_b^c K_c^a) \left. \right) \left. \right\} \\
&= \frac{1}{2\pi^2} \left\{ \int_M d^4x \sqrt{g} \left(\left(-\frac{3}{5}C_{\mu\nu\rho\sigma}^2 + \frac{11}{30}R^*R^* - \frac{2}{5}(R)_{;\mu}^\mu \right) \right. \right. \\
&\quad + a|D_\mu\varphi|^2 + \frac{1}{6}R \left(a|\varphi|^2 + \frac{1}{2}c \right) + b|\varphi|^4 + 2e|\varphi|^2 + \frac{1}{2}d - \frac{1}{3}a(|\varphi|^2)_{;\mu}^\mu \left. \right) \\
&\quad + \int_{\partial M} d^3x \sqrt{h} \left(\frac{1}{3}K \left(a|\varphi|^2 + \frac{1}{2}c \right) + \frac{2}{15} (5RK + 4KR_{nan}^a + 4K_{ab}R_{acb}^c + 18R_{anbn}K^{ab}) \right. \\
&\quad \left. \left. + \frac{4}{315} (17K^3 + 39KK_{ab}K^{ab} - 116K_a^b K_b^c K_c^a) \right) \right\}. \tag{106}
\end{aligned}$$

Thus we reach the final result that the spectral action for the standard model including all boundary terms is given by

$$\begin{aligned}
I &= \frac{48\Lambda^4}{\pi^2} f_4 \int_M d^4x \sqrt{g} + \frac{8\Lambda^2}{\pi^2} f_2 \left\{ \int_M d^4x \sqrt{g} \left(-\frac{1}{2}R - \frac{1}{4} \left(a|\varphi|^2 + \frac{1}{2}c \right) \right) - \int_{\partial M} d^3x \sqrt{h} K \right\} \\
&\quad + \frac{2\Lambda}{(4\pi)^{\frac{3}{2}}} f_1 \int_{\partial M} d^3x \sqrt{h} (3(K^2 - 2K_{ab}K^{ab})) + \frac{f_0}{2\pi^2} \left\{ \int_M d^4x \sqrt{g} \left(-\frac{3}{5}C_{\mu\nu\rho\sigma}^2 + \frac{11}{30}R^*R^* - (2/5)R_{;\mu}^\mu \right. \right. \\
&\quad + a|D_\mu\varphi|^2 + \frac{1}{6}R \left(a|\varphi|^2 + \frac{1}{2}c \right) + g_3^2 (G_{\mu\nu}^i)^2 + g_2^2 (F_{\mu\nu}^\alpha)^2 + \frac{5}{3}g_1^2 (B_{\mu\nu})^2 \left. \right) + b|\varphi|^4 + 2e|\varphi|^2 + \frac{1}{2}d - \frac{1}{3}a(|\varphi|^2)_{;\mu}^\mu \left. \right\} \\
&\quad + \frac{f_0}{2\pi^2} \left\{ \int_{\partial M} d^3x \sqrt{h} \left(\frac{1}{3}K \left(a|\varphi|^2 + \frac{1}{2}c \right) + \frac{2}{15} (5RK + 4KR_{nan}^a + 4K_{ab}R_{acb}^c + 18R_{anbn}K^{ab}) \right) \right. \\
&\quad \left. + \frac{4}{315} (17K^3 + 39KK_{ab}K^{ab} - 116K_a^b K_b^c K_c^a) \right\}, \tag{107}
\end{aligned}$$

where

$$f_n = \int_0^\infty v^{n-1} f(v) dv. \tag{108}$$

There are two things to be noted about the form of the boundary terms. First, the Higgs fields do contribute through the combination

$$\frac{1}{3}K \left(a|\varphi|^2 + \frac{1}{2}c \right).$$

This is dictated by the presence of the term

$$\frac{1}{6}R \left(a|\varphi|^2 + \frac{1}{2}c \right)$$

and therefore, they again appear together, with the same sign and relative factor of 2. This is remarkable and means that the spectral action takes care of its own consistency. The second thing, is the absence of the contributions of the gauge fields to boundary terms. It is known that both in the Hamiltonian formulation, or Lagrangian path integrals, a boundary term is added to make the definition of conjugate momenta possible and to enforce the Gauss constraint on the divergence of the electric field. It was, however, shown by Vassilevich [9] (section 3.4) and [14] that the Yang–Mills action

$$\frac{1}{4} \int_M d^4x \sqrt{g} (F_{\mu\nu}^\alpha F^{\mu\nu\alpha}) \tag{109}$$

where

$$F_{\mu\nu}^\alpha = \partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha + f_{\beta\gamma}^\alpha A_\mu^\beta A_\nu^\gamma \tag{110}$$

can be put into the form

$$\frac{1}{2} \int_M d^4x \sqrt{g} A^{\rho\alpha} ((-g_{\rho\sigma} g^{\mu\nu} \nabla_\mu \nabla_\nu + \nabla_\rho \nabla_\sigma + R_{\rho\sigma}) g_{\alpha\beta} + 2F_{\rho\sigma}^\gamma (B) f_{\alpha\beta}^\gamma) A^{\sigma\beta} + \frac{1}{2} \int_{\partial M} d^3x \sqrt{h} A^{\nu\alpha} (\nabla_n A_{\nu\alpha} - \nabla_\nu A_{n\alpha})$$

where the field A_μ^α is expanded around a background B_μ^α . By imposing the gauge condition

$$\nabla^\mu A_\mu^\alpha = 0 \quad (111)$$

and one of the two boundary conditions

$$A_n|_{\partial M} = 0, \quad (\nabla_n \delta_{ab} - K_{ab})A_b|_{\partial M} = 0 \quad (112)$$

or

$$(\nabla_n - K)A_n|_{\partial M} = 0, \quad A_a|_{\partial M} = 0 \quad (113)$$

the boundary term will vanish in both cases. We have noted that we have taken the first condition to avoid any appearance of gauge fields in the boundary. In other words the spectral action needs only part of the first two conditions. It also seems to imply that the second part of the boundary conditions arises as an integrability condition derived from the boundary conditions of the Dirac operator.

From all these considerations we deduce that the simple requirement of having boundary conditions for the Dirac operator which are consistent with the self-adjointness of this operator, is enough to guarantee that the spectral action has all the correct boundary terms, including correct signs and coefficients.

6. Spectral action in the presence of a dilaton

We now deal with the question of what is the form of the action if a scaling is introduced through the operator

$$e^{-\phi} D^2 e^{-\phi} = - (G^{\mu\nu} \partial_\mu \partial_\nu + \mathcal{A}^\mu \partial_\mu + \mathcal{B}) \quad (114)$$

where

$$\begin{aligned} G^{\mu\nu} &= e^{-2\phi} g^{\mu\nu}, \\ \mathcal{A}^\mu &= e^{-2\phi} A^\mu - 2G^{\mu\nu} \partial_\nu \phi, \\ \mathcal{B} &= e^{-2\phi} B + G^{\mu\nu} (\partial_\mu \phi \partial_\nu \phi - \partial_\mu \partial_\nu \phi) - e^{-2\phi} A^\mu \partial_\mu \phi. \end{aligned}$$

We have shown in [15] that for this operator we have the identity

$$\mathcal{E} + \frac{1}{6} R(G) = e^{-2\phi} \left(E + \frac{1}{6} R(g) \right) \quad (115)$$

where

$$\begin{aligned} \mathcal{E} &= \mathcal{B} - G^{\mu\nu} (\partial_\mu \bar{\omega}'_\nu + \bar{\omega}'_\mu \bar{\omega}'_\nu - \Gamma_{\mu\nu}^\rho (G) \bar{\omega}'_\rho), \\ \bar{\omega}'_\mu &= \frac{1}{2} G_{\mu\nu} (\mathcal{A}^\nu + \Gamma^\nu (G)), \\ \Omega_{\mu\nu} &= \partial_\mu \bar{\omega}'_\nu - \partial_\nu \bar{\omega}'_\mu + [\bar{\omega}'_\mu, \bar{\omega}'_\nu]. \end{aligned}$$

It is then convenient to use these relations as well as

$$R(g) = e^{2\phi} (R(G) - 6 G^{\mu\nu} (-\nabla_\mu^G \nabla_\nu^G \phi + \partial_\mu \phi \partial_\nu \phi)), \quad (116)$$

to work out the spectral action for the scaled operator on manifolds with boundaries. A good starting point is the equality

$$\langle \Psi | D | \Psi \rangle = \langle \Psi' | D' | \Psi' \rangle' \quad (117)$$

where

$$|\Psi\rangle = e^{\frac{3}{2}\phi} |\Psi'\rangle \quad (118)$$

then the boundary conditions are taken to be

$$\Pi_- \Psi'|_{\partial M} = 0, \quad (119)$$

which implies the boundary condition for D^2

$$\Pi_- D' \Psi'|_{\partial M} = 0 \quad (120)$$

so that the function S is evaluated using the rescaled metric $G_{\mu\nu}$. To cut the story short, there are only a few places where we expect the dilaton to contribute. The terms in the bulk have already been evaluated, except for the total divergence $(5E + R)_{;\mu}^\mu$ which does receive a dilaton contribution equal to

$$\frac{5}{2} \int_M d^4x \sqrt{G} [G^{\kappa\lambda} (\partial_\kappa \phi \partial_\lambda \phi - \nabla_\kappa^G \nabla_\lambda^G \phi)]_{;\mu}^\mu \quad (121)$$

$$= \frac{5}{2} \int_M d^3y \sqrt{H} \partial_n [H^{ab} (\partial_a \phi \partial_b \phi - \nabla_a^H \nabla_b^H \phi) + (\partial_n \phi \partial_n \phi - \nabla_n \nabla_n \phi)]. \quad (122)$$

The remaining boundary terms could be simplified by observing that first modification occurs for $a_4(e^{-\phi}D^2e^{-\phi}, \chi)$ where we have the combination

$$720\text{Tr}\left(\varepsilon + \frac{1}{6}R(G)\right)\left(S + \frac{1}{6}K\right) \quad (123)$$

which, in the case of the standard model, is equal to

$$\frac{1}{16\pi^2} \frac{1}{360} f_0 \int_{\partial M} d^3y \sqrt{H} 720 \left(-\frac{1}{12}K\right) (384) \left(-\frac{1}{12}\right) (R(G) + 6H^{ab}(\nabla_a^H \nabla_b^H \phi - \partial_a \phi \partial_b \phi) + 6(\nabla_n \nabla_n \phi - \partial_n \phi \partial_n \phi))$$

which implies that the last term for a_4 gets modified by replacing $\frac{1}{8}(5RK + \dots)$ by

$$\frac{1}{3\pi^2} f_0 \int_{\partial M} d^3y \sqrt{H} K (R(G) + 6H^{ab}(\nabla_a^H \nabla_b^H \phi - \partial_a \phi \partial_b \phi) + 6(\nabla_n \nabla_n \phi - \partial_n \phi \partial_n \phi)) \quad (124)$$

and the boundary term, not being conformally invariant, gets a contribution dependent on the dilaton. Therefore the full action takes exactly the same form as before, but as a function of the metric $G_{\mu\nu}$ and the induced metric $H_{ab} = e^{2\phi} h_{ab}$ and the Higgs field $\phi' = e^{-\phi}\phi$ and the fermions $\Psi' = e^{-\frac{3}{2}\phi}\Psi$, plus the extra terms

$$\frac{12}{\pi^2} f_2 \int_M d^4x \sqrt{G} G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \quad (125)$$

$$+ \frac{2}{\pi^2} f_0 \int_{\partial M} d^3x \sqrt{H} (K + \partial_n) [H^{ab}(\nabla_a^H \nabla_b^H \phi - \nabla_a \phi \nabla_b \phi) + (\nabla_n \nabla_n \phi - \partial_n \phi \partial_n \phi)]. \quad (126)$$

Practical applications of these results will be dealt with in the future.

7. Appendix 1: the case of the disk

We take the case of the Dirac operator in the unit disk, in order to check the conventions for the extrinsic curvature. We take the Dirac operator in the form:

$$D = \begin{pmatrix} 0 & \partial_x + i\partial_y \\ -\partial_x + i\partial_y & 0 \end{pmatrix}$$

and we write it in polar coordinates (r, θ) using

$$\partial_x = \cos \theta \partial_r - \sin \theta \frac{1}{r} \partial_\theta, \quad \partial_y = \sin \theta \partial_r + \cos \theta \frac{1}{r} \partial_\theta$$

so that

$$D = i(\gamma_1(\theta) \frac{1}{r} \partial_\theta + \gamma_2(\theta) \partial_r)$$

where

$$\gamma_1(\theta) = \begin{pmatrix} 0 & e^{i\theta} \\ e^{-i\theta} & 0 \end{pmatrix}, \quad \gamma_2(\theta) = \begin{pmatrix} 0 & -ie^{i\theta} \\ ie^{-i\theta} & 0 \end{pmatrix}.$$

The $\gamma_j(\theta)$ are self-adjoint of square 1 and fulfill the Clifford relations

$$\gamma_1(\theta) \gamma_2(\theta) = -\gamma_2(\theta) \gamma_1(\theta) = i\gamma$$

where $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ gives the grading. Note also that

$$\partial_\theta(\gamma_1(\theta)) = -\gamma_2(\theta)$$

Lemma 1. The boundary condition for D is given by

$$\gamma_1(\theta) \xi = \xi$$

Proof. By definition the boundary condition is given as $\Pi_- \xi = 0$ on the boundary, where ([9] p. 297)

$$\Pi_- = \frac{1}{2}(1 - i\gamma_n \gamma)$$

with γ_n the Clifford multiplication by the normal, and γ the grading as above.

We have $D = i(\gamma_x \partial_x + \gamma_y \partial_y)$ where

$$\gamma_x = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and thus the inward normal corresponds to

$$\gamma_n = -\cos \theta \gamma_x - \sin \theta \gamma_y = -\gamma_2(\theta).$$

One has $i\gamma_n \gamma = -i\gamma_2(\theta) \gamma = \gamma_1(\theta)$ and thus $\Pi_- = \frac{1}{2}(1 - \gamma_1(\theta))$. \square

Lemma 2. The additional boundary condition for D^2 is given by

$$\left(\partial_n - \frac{1}{2}\right) \Pi_+ \xi = 0, \quad \Pi_+ = \frac{1}{2}(1 + \gamma_1(\theta))$$

where $\partial_n = -\partial_r$ is differentiation relative to the inward normal.

Proof. The additional boundary condition is

$$\Pi_- D \xi = 0, \quad \Pi_- = \frac{1}{2}(1 - \gamma_1(\theta)).$$

Up to an overall factor this gives $N\xi = 0$ with

$$N = (1 - \gamma_1(\theta)) \left(\gamma_1(\theta) \frac{1}{r} \partial_\theta + \gamma_2(\theta) \partial_r \right).$$

One has

$$(1 - \gamma_1(\theta)) \gamma_2(\theta) \partial_r = \gamma_2(\theta) \partial_r (1 + \gamma_1(\theta)) = 2\gamma_2(\theta) \partial_r \Pi_+.$$

Next, using the first boundary condition, one gets $\partial_\theta(1 - \gamma_1(\theta))\xi = 0$ on the boundary circle. One has moreover

$$\partial_\theta \xi = \partial_\theta(\gamma_1(\theta)\xi) = \partial_\theta(\gamma_1(\theta))\xi + \gamma_1(\theta)\partial_\theta \xi = -\gamma_2(\theta)\xi + \gamma_1(\theta)\partial_\theta \xi.$$

Thus on the boundary one has

$$(1 - \gamma_1(\theta))\partial_\theta \xi = -\gamma_2(\theta)\xi.$$

This yields, on the boundary,

$$N = \gamma_2(\theta) \frac{1}{r} + 2\gamma_2(\theta) \partial_r \Pi_+$$

and hence

$$-\frac{1}{2} \gamma_2(\theta) N = \left(\partial_n - \frac{1}{2}\right) \Pi_+$$

since $\partial_n = -\partial_r$. \square

8. Appendix 2: sign of boundary term in Einstein action

We use the notations of [6]. We check that in Euclidean signature the correct combination which gives the Einstein equation is

$$-\int_M R \sqrt{g} d^4x - 2 \int_{\partial M} K \sqrt{h} d^3y$$

where R is positive for the sphere, and K is positive for the ball. This fits with Hawking [16] from which one can also check that the Euclidean action is as above for the overall sign.

8.1. Sign of R

The Ricci scalar is defined by

$$R = g^{\mu\nu} R_{\mu\nu}, \quad R_{\mu\nu} = R^\rho_{\mu\rho\nu} \quad (127)$$

where in a geodesic coordinate system

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}(g_{\mu\sigma,\nu\rho} - g_{\mu\rho,\nu\sigma} - g_{\nu\sigma,\mu\rho} + g_{\nu\rho,\mu\sigma}). \quad (128)$$

Thus for the sphere with $g_{\mu\nu} = (1 + \frac{\Omega}{4}\rho^2)^{-2}\delta_{\mu\nu}$ the value of R is $\frac{n(n-1)}{2}\Omega$ which is positive since $\Omega > 0$.

8.2. Stokes formula and outer normal

We start with a vector field $X = X^\mu \partial_\mu$ on a manifold with volume form ω . The divergence of X is given by

$$\operatorname{div} X = d i_X \omega$$

which is the Lie derivative $\partial_X \omega$ of the volume form since $\partial_X = di_X + i_X d$ on forms. The Stokes formula gives

$$\int_M \operatorname{div} X = \int_{\partial M} i_X \omega$$

where both M and ∂M are oriented so that

$$\int_M d\alpha = \int_{\partial M} \alpha.$$

Using a Riemannian metric g on M and the induced metric h on ∂M we get a formula of the form

$$\int_M X_{;\mu}^{\mu} \sqrt{g} d^n x = - \int_{\partial M} X^{\mu} n_{\mu} \sqrt{h} d^{n-1} y \quad (129)$$

and we need to determine the sign of the normal $n_{\mu} = g_{\mu\nu} n^{\nu}$. Note that in this formula the choice of orientation of M has disappeared. To get the sign of n^{ν} one can take the one dimensional case where $M = [a, b]$ with $a < b$. Thus the coordinate x increases from a to b . One lets $X = f(x) \partial_x$. The left hand side of (129) gives

$$\int_a^b \partial_x f(x) dx = f(b) - f(a)$$

which shows that n^{ν} is the inward normal. More generally if we let $k(x)$ be a convex function such as $k(x) = \sum (x^{\mu})^2$ in \mathbb{R}^n and take $M = \{x | k(x) \leq 1\}$, we can take for X the gradient of k . Then the left hand side of (129) is positive and thus the normal is again the inward normal.

8.3. Extrinsic curvature

We now recall the definition of the extrinsic curvature [6]: The extrinsic curvature K_{ab} is defined by

$$K_{ab} = -n_{\mu;v} e_a^{\mu} e_b^v$$

where n^{μ} is the inward normal.

Let us compute it explicitly in the case of the disk of radius R in the plane with coordinates x^{μ} and flat metric $g_{\mu\nu} = \delta_{\mu\nu}$. We take for y the angular parameter $y = \theta$ so that

$$x^1(y) = R \cos \theta, \quad x^2(y) = R \sin \theta.$$

There is only one index $a = 1$ and one has

$$e_1^1 = \partial_{\theta} R \cos \theta = -R \sin \theta, \quad e_1^2 = \partial_{\theta} R \sin \theta = R \cos \theta.$$

The coordinates of the inward normal are

$$n^{\mu} = -x^{\mu} / \sqrt{(x^1)^2 + (x^2)^2}.$$

One finds by direct computation that

$$-n_{\mu;v} e_1^{\mu} e_1^v = \sqrt{(x^1)^2 + (x^2)^2} = R.$$

One has $h_{11} = R^2$ and thus $h^{11} = R^{-2}$ which gives in this case

$$K = h^{ab} K_{ab} = \frac{1}{R}.$$

One defines $h^{\mu\nu}$ by

$$h^{\mu\nu} = h^{ab} e_a^{\mu} e_b^v \quad (130)$$

then

$$K = h^{ab} K_{ab} = -h^{ab} n_{\mu;v} e_a^{\mu} e_b^v = -h^{\mu\nu} n_{\mu;v}.$$

8.4. Variation of the Einstein action

This is well known, but to fix the notation, we give the main steps. The intermediate steps are given in [6]. Variation of the Einstein action is

$$\begin{aligned} \delta I_E &= \int_M \delta(g^{\mu\nu} R_{\mu\nu} \sqrt{g}) d^4 x \\ &= \int_M \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) \delta g^{\mu\nu} \sqrt{g} d^4 x + \int_M g^{\mu\nu} \delta R_{\mu\nu} \sqrt{g} d^4 x. \end{aligned}$$

We then use

$$g^{\mu\nu} \delta R_{\mu\nu} = X^\mu_{;\mu} \quad (131)$$

where

$$X^\mu = g^{\nu\rho} \delta \Gamma^\mu_{\nu\rho} - g^{\nu\mu} \delta \Gamma^\rho_{\alpha\rho}.$$

Using Stokes theorem

$$\int_M X^\mu_{;\mu} \sqrt{g} d^4x = - \int_{\partial M} n^\mu X_\mu \sqrt{h} d^3y \quad (132)$$

$$= - \int_{\partial M} h^{\mu\nu} (\delta g_{\rho\nu, \mu} - \delta g_{\mu\nu, \rho}) n^\rho \sqrt{h} d^3y \quad (133)$$

$$= \int_{\partial M} h^{\mu\nu} \delta g_{\mu\nu, \rho} n^\rho \sqrt{h} d^3y \quad (134)$$

where in the last step we used that the variation of $g_{\mu\nu}$ and the tangential derivative of $\delta g_{\mu\nu}$ is zero on ∂M so that $\delta g_{\mu\nu, \alpha} e^\alpha_a = 0$ and $h^{\alpha\beta} \delta g_{\mu\nu, \alpha} = 0$.

For the variation of the boundary term we have

$$\delta \int_{\partial M} 2K \sqrt{h} d^3y = \int_{\partial M} 2h^{\mu\nu} \delta \Gamma^\rho_{\mu\nu} n_\rho \sqrt{h} d^3y \quad (135)$$

$$= - \int_{\partial M} h^{\mu\nu} \delta g_{\mu\nu, \rho} n^\rho \sqrt{h} d^3y. \quad (136)$$

Thus

$$\delta \left(\int_M d^4x \sqrt{g} R + 2 \int_{\partial M} d^3x \sqrt{h} K \right) = \int_M \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) \delta g^{\mu\nu} \sqrt{g} d^4x \quad (137)$$

Acknowledgement

The research of A. H. C. is supported in part by the National Science Foundation under Grant No. Phys-0854779.

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