Journal of Geometry and Physics 62 (2012) 1677–1696

Contents lists available at SciVerse ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Rational representations of the Yangian $Y(\mathfrak{gl}_n)$

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ARTICLE INFO

Article history:
Received 28 October 2011
Received in revised form 4 March 2012
Accepted 5 March 2012
Available online 13 March 2012

Keywords:
Yangians
Representation theory
Mickelsson algebras

ABSTRACT

We construct a series of rational representations of $Y(\mathfrak{gl}_n)$ and intertwining operators between them. We find explicit expressions for the images of highest-weight vectors under intertwining operators. Finally, we state a conjecture that all irreducible finite-dimensional rational $Y(\mathfrak{gl}_n)$-modules can be realized as images of the constructed intertwining operators.

0. Introduction

Let $Y(\mathfrak{gl}_n)$ be the Yangian of the general linear Lie algebra $\mathfrak{gl}_n$. We call two finite-dimensional representations of the algebra $Y(\mathfrak{gl}_n)$ similar if they differ by an automorphism of the form (1.10). Up to similarity the irreducible finite-dimensional $Y(\mathfrak{gl}_n)$-modules were classified in [1]. Due to this classification every irreducible finite-dimensional $Y(\mathfrak{gl}_n)$-module is described by a set of Drinfeld polynomials. Later on, analogous results on the representations of shifted Yangians and W-algebras were obtained in [2].

Consider the subalgebra of $Y(\mathfrak{gl}_n)$ consisting of all elements which are invariant under every automorphism of the form (1.10). This subalgebra is called *special Yangian* and is isomorphic to the Yangian $Y(\mathfrak{sl}_n)$ of the special linear Lie algebra $\mathfrak{sl}_n \subset \mathfrak{gl}_n$ considered in [3,1]. Therefore, two $Y(\mathfrak{gl}_n)$-modules are similar if and only if their restrictions to the special Yangian are isomorphic. Thus, the sets of Drinfeld polynomials correspond to the finite-dimensional irreducible representations of the Yangian $Y(\mathfrak{sl}_n)$.

In the works [4,5], a certain functor $\mathcal{E}_m$ from the category of $\mathfrak{gl}_m$-modules to the category of $Y(\mathfrak{gl}_n)$-modules was investigated. This functor arose as a composition of Drinfeld (see [3]) and Cherednik (see [6,7]) functors. The construction is closely related to the $(\mathrm{GL}_m, \mathfrak{gl}_n)$ Howe duality (see [8,9]) and can be regarded as a reformulation of Olshanski centralizer construction (see [10,11]). An application of the functor $\mathcal{E}_m$ to the Verma modules of the algebra $\mathfrak{gl}_m$ produced a series of *standard* representations of the Yangian $Y(\mathfrak{gl}_n)$. Then intertwining operators between the standard $Y(\mathfrak{gl}_n)$-modules were constructed using the theory of Zhelobenko operators for Mickelsson algebras (see [12–15]). Finally, in the works [16,17] it was shown that all irreducible finite-dimensional $Y(\mathfrak{gl}_n)$-modules considered up to similarity can be realized as the images of the intertwining operators constructed in [4,5]. The approach introduced by Khoroshkin and Nazarov in [4,5] led to the more explicit realization of $Y(\mathfrak{gl}_n)$-modules. An analogous result for representations of quantum affine algebras appeared earlier in [18].

Let us call a representation of the Yangian $Y(\mathfrak{gl}_n)$ *polynomial* if it is a subquotient of a tensor product of vector representations of $Y(\mathfrak{gl}_n)$. Then, all $Y(\mathfrak{gl}_n)$-modules constructed in [4,5] are polynomial. Moreover, any polynomial $Y(\mathfrak{gl}_n)$-module appears that way. In this paper, we consider a modification $\mathcal{E}_{p,q}$ of the functor $\mathcal{E}_m$ with the use of $(\mathrm{U}_{p,q}, \mathfrak{gl}_n)$ Howe duality (see [19–21]). This modification leads to a bigger class of $Y(\mathfrak{gl}_n)$-modules called the *rational* modules. We

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doi:10.1016/j.geomphys.2012.03.002

say that a representation of the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ is *rational* if it is a subquotient of a tensor product of vector and dual vector representations of $\mathrm{Y}(\mathfrak{gl}_n)$. Irreducible rational representations of $\mathrm{Y}(\mathfrak{gl}_n)$ associated with skew Young diagrams were investigated by Nazarov in [22]. Irreducible rational modules may be similar but not isomorphic. Using parabolic induction we decompose the images of Verma $\mathfrak{gl}_m$-modules under $\mathcal{E}_{p,q}$ into a tensor product of evaluation modules of the Yangian. With the help of technique developed for twisted Yangians $\mathrm{Y}(\mathfrak{so}_{2n})$, $\mathrm{Y}(\mathfrak{sp}_{2n})$ in [23,24] we construct intertwining operators between obtained tensor products and compute the images of highest-weight vectors under intertwining operators. Using the results of [16] we observe that under some conditions on the parameters of the modules the image of the certain intertwining operator is an irreducible rational $\mathrm{Y}(\mathfrak{gl}_n)$-module. Finally, we state as a conjecture that all irreducible rational $\mathrm{Y}(\mathfrak{gl}_n)$-modules can be obtained by this construction. We return to this problem in the forthcoming publication.

1. Basics

1.1. Yangian $\mathrm{Y}(\mathfrak{gl}_n)$

The Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ is a deformation of the universal enveloping algebra of the polynomial current Lie algebra $\mathfrak{gl}_n[u]$ in the class of Hopf algebras, see for instance [25]. The unital associative algebra $\mathrm{Y}(\mathfrak{gl}_n)$ has a family of generators

$$T_{ij}^{(1)}, T_{ij}^{(2)}, \dots \quad \text{where } i,j = 1,\dots,n.$$

Consider the generating functions

$$T_{ij}(u) = \delta_{ij} + T_{ij}^{(1)}u^{-1} + T_{ij}^{(2)}u^{-2} + \cdots \in \mathrm{Y}(\mathfrak{gl}_n)[\![u^{-1}]\!] \tag{1.1}$$

with formal parameter u. The defining relations in the associative algebra $\mathrm{Y}(\mathfrak{gl}_n)$ can be written as

$$(u-v)\cdot[T_{ij}(u), T_{kl}(v)] = T_{kj}(u)T_{il}(v) - T_{kj}(v)T_{il}(u) \tag{1.2}$$

where $i,j,k,l = 1,\dots,n$.

If $n = 1$, the algebra $\mathrm{Y}(\mathfrak{gl}_n)$ is commutative. The relations (1.2) imply that for any $z \in \mathbb{C}$ assignments

$$\tau_z\colon T_{ij}(u) \mapsto T_{ij}(u-z) \quad \text{for } i,j = 1,\dots,n \tag{1.3}$$

define an automorphism τ_z of the algebra $\mathrm{Y}(\mathfrak{gl}_n)$. Here each of the formal series $T_{ij}(u-z)$ in $(u-z)^{-1}$ should be re-expanded in u^{-1}, and the assignment (1.3) is a correspondence between the respective coefficients of series in u^{-1}.

Now let $E_{ij} \in \mathfrak{gl}_n$ with $i,j = 1,\dots,n$ be the standard matrix units. Sometimes E_{ij} will also denote elements of the algebra $\mathrm{End}(\mathbb{C}^n)$ but this should not cause any confusion. The Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ contains the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_n)$ as a subalgebra, the embedding $\mathrm{U}(\mathfrak{gl}_n) \to \mathrm{Y}(\mathfrak{gl}_n)$ can be defined by the assignments

$$E_{ij} \mapsto T_{ij}^{(1)} \quad \text{for } i,j = 1,\dots,n.$$

Moreover, there is a homomorphism $\pi_n\colon \mathrm{Y}(\mathfrak{gl}_n) \to \mathrm{U}(\mathfrak{gl}_n)$ which is identical on the subalgebra $\mathrm{U}(\mathfrak{gl}_n) \subset \mathrm{Y}(\mathfrak{gl}_n)$ and is given by

$$\pi_n\colon T_{ij}^{(2)}, T_{ij}^{(3)}, \dots \mapsto 0 \quad \text{for } i,j = 1,\dots,n. \tag{1.4}$$

Let $T(u)$ be an $n \times n$ matrix whose i,j entry is the series $T_{ij}(u)$. The relations (1.2) can be rewritten by means of the *Yang R-matrix*

$$R(u) = 1 \otimes 1 - \sum_{i,j=1}^{n} \frac{E_{ij} \otimes E_{ji}}{u} \tag{1.5}$$

where the tensor factors E_{ij} and E_{ji} are regarded as $n \times n$ matrices. Note that

$$R(u)\,R(-u) = 1 - \frac{1}{u^2}. \tag{1.6}$$

Consider two $n^2 \times n^2$ matrices whose entries are series with coefficients in the algebra $\mathrm{Y}(\mathfrak{gl}_n)$,

$$T_1(u) = T(u) \otimes 1 \quad \text{and} \quad T_2(v) = 1 \otimes T(v).$$

Then a collection of relations (1.2) for all possible indices i,j,k,l is equivalent to

$$R(u-v)\,T_1(u)\,T_2(v) = T_2(v)\,T_1(u)\,R(u-v). \tag{1.7}$$

Further, the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ is a Hopf algebra over the field $\mathbb{C}$. We define the comultiplication $\Delta : \mathrm{Y}(\mathfrak{gl}_n) \to \mathrm{Y}(\mathfrak{gl}_n) \otimes \mathrm{Y}(\mathfrak{gl}_n)$ by the assignment

$$\Delta : T_{ij}(u) \mapsto \sum_{k=1}^{n} T_{ik}(u) \otimes T_{kj}(u). \tag{1.8}$$

When taking tensor products of modules over $\mathrm{Y}(\mathfrak{gl}_n)$ we use the comultiplication (1.8). The counit homomorphism ε : $\mathrm{Y}(\mathfrak{gl}_n) \to \mathbb{C}$ is defined by

$$\varepsilon\colon T_{ij}(u) \mapsto \delta_{ij} \cdot 1.$$

The antipode S on $\mathrm{Y}(\mathfrak{gl}_n)$ is given by

$$\mathrm{S}\colon T(u) \mapsto T(u)^{-1}$$

and defines an anti-automorphism of the associative algebra $\mathrm{Y}(\mathfrak{gl}_n)$.

Let $T'(u)$ be the transpose to the matrix $T(u)$. Then the i, j entry of the matrix $T'(u)$ is $T_{ji}(u)$. Consider $n^2 \times n^2$ matrices

$$T_1'(u) = T'(u) \otimes 1 \quad \text{and} \quad T_2'(v) = 1 \otimes T'(v).$$

Note that the Yang R-matrix (1.5) is invariant under applying the transposition to both tensor factors. Hence the relation (1.7) implies

$$T_1'(u)\, T_2'(v)\, R(u-v) = R(u-v)\, T_2'(v)\, T_1'(u),$$

$$R(u-v)\, T_1'(-u)\, T_2'(-v) = T_2'(-v)\, T_1'(-u)\, R(u-v). \tag{1.9}$$

To obtain the latter relation we used (1.6). By comparing the relations (1.7) and (1.9), an involutive automorphism of the algebra $\mathrm{Y}(\mathfrak{gl}_n)$ can be defined by the assignment

$$\omega\colon T(u) \mapsto T'(-u),$$

understood as a correspondence between the respective matrix entries. For further details on the algebra $\mathrm{Y}(\mathfrak{gl}_n)$ see [26, Chapter 1].

1.2. *Representations of Yangian* $\mathrm{Y}(\mathfrak{gl}_n)$

Let Φ be an irreducible finite-dimensional $\mathrm{Y}(\mathfrak{gl}_n)$-module. A non-zero vector $\varphi \in \Phi$ is said to be of *highest weight* if it is annihilated by all the coefficients of the series $T_{ij}(u)$ with $1 \leqslant i < j \leqslant n$ and is an eigenvector for all the coefficients of the series $T_{ii}(u)$ for $1 \leqslant i \leqslant n$. In that case φ is unique up to a scalar multiplier and for $i = 1, \ldots, n-1$ holds

$$T_{ii}(u) T_{i+1,i+1}(u)^{-1} \varphi = P_i\left(u + \frac{1}{2}\right) P_i\left(u - \frac{1}{2}\right)^{-1} \varphi$$

where $P_i(u)$ is a monic polynomial in u with coefficients in $\mathbb{C}$. Then $P_1(u), \ldots, P_{n-1}(u)$ are called the *Drinfeld polynomials* of Φ. Any sequence of $n-1$ monic polynomials with complex coefficients arises this way. An irreducible finite-dimensional $\mathrm{Y}(\mathfrak{gl}_n)$-module is defined by the set of eigenfunctions $\Lambda_{ii}(u)$ such that

$$T_{ii}(u)\varphi = \Lambda_{ii}(u)\varphi$$

for $i = 1, \ldots, n$. Thus, an irreducible finite-dimensional $\mathrm{Y}(\mathfrak{gl}_n)$-module is defined by a set of polynomials $P_1(u), \ldots, P_{n-1}(u)$ and some normalizing factor, for example $\Lambda_{nn}(u)$.

Relations (1.2) show that for any formal power series $g(u)$ in u^{-1} with coefficients from $\mathbb{C}$ and leading term 1, the assignments

$$T_{ij}(u) \mapsto g(u) T_{ij}(u) \tag{1.10}$$

define an automorphism of the algebra $\mathrm{Y}(\mathfrak{gl}_n)$. The subalgebra in $\mathrm{Y}(\mathfrak{gl}_n)$ consisting of all elements which are invariant under every automorphism of the form (1.10), is called the *special Yangian* of $\mathfrak{gl}_n$. The special Yangian of $\mathfrak{gl}_n$ is a Hopf subalgebra of $\mathrm{Y}(\mathfrak{gl}_n)$ and is isomorphic to the Yangian $\mathrm{Y}(\mathfrak{sl}_n)$ of the special linear Lie algebra $\mathfrak{sl}_n \subset \mathfrak{gl}_n$ considered in [3,1]. For the proofs of the latter two assertions see [27, Subsection 1.8].

Two irreducible finite-dimensional $\mathrm{Y}(\mathfrak{gl}_n)$-modules are called *similar* if and only if their restrictions to the special Yangian are isomorphic. Therefore, irreducible finite-dimensional $\mathrm{Y}(\mathfrak{sl}_n)$-modules are defined by a set of Drinfeld polynomials, while irreducible finite-dimensional $\mathrm{Y}(\mathfrak{gl}_n)$-modules are parameterized by their Drinfeld polynomials only up to similarity. For further details on representations of $\mathrm{Y}(\mathfrak{gl}_n)$ see [1,28].

Now, let us define the fundamental representation V_z and the dual fundamental representation V_z' of $\mathrm{Y}(\mathfrak{gl}_n)$. As a vector space $V_z = V_z' = \mathbb{C}^n$, the corresponding actions are given by evaluation and dual evaluation homomorphisms

$$\pi_z = \pi_n \circ \tau_{-z}\colon \mathrm{Y}(\mathfrak{gl}_n) \longrightarrow \mathrm{U}(\mathfrak{gl}_n), \qquad T_{ij}(u) \mapsto \delta_{ij} + \frac{E_{ij}}{u+z},$$

$$\pi_z' = \pi_n \circ \tau_z \circ \omega\colon \mathrm{Y}(\mathfrak{gl}_n) \longrightarrow \mathrm{U}(\mathfrak{gl}_n), \qquad T_{ij}(u) \mapsto \delta_{ij} - \frac{E_{ji}}{u+z}.$$

Let also Ω_z and Ω_z' denote one-dimensional representations of $\mathrm{Y}(\mathfrak{gl}_n)$ defined as evaluation and dual evaluation homomorphisms in $\Lambda^n(\mathbb{C}^n)$ with standard action of $\mathfrak{gl}_n$. Thus,

$$T_{ij}(u) \mapsto \delta_{ij} \cdot \frac{u+z+1}{u+z} \quad \text{and} \quad T_{ij}(u) \mapsto \delta_{ij} \cdot \frac{u+z-1}{u+z}$$

on Ω_z and Ω_z' correspondingly.

Definition 1.1. (a) Representation of Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ is called polynomial if it is isomorphic to a subquotient of tensor product of fundamental representations V_z with arbitrary values of z.
(b) Representation of Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ is called rational if it is isomorphic to a subquotient of tensor product of fundamental and dual fundamental representations V_z and V'_z with arbitrary values of z.

Note that representations V_z and Ω_z are polynomial while representations V'_z and Ω'_z are rational. We would also like to point out that the modules Ω_z and Ω'_z are central, i.e. in tensor products of $\mathrm{Y}(\mathfrak{gl}_n)$-modules one can permute them with other modules. More precisely, the form of the comultiplication map Δ implies that for any $\mathrm{Y}(\mathfrak{gl}_n)$-module M there is a pair of canonical isomorphism

$$M \otimes \Omega_z \cong \Omega_z \otimes M \quad \text{and} \quad M \otimes \Omega'_z \cong \Omega'_z \otimes M$$

sending $m \otimes a \mapsto a \otimes m$ where $m \in M$ and $a \in \Omega_z$ or $a \in \Omega'_z$.

1.3. Functor

Let E be an $m \times m$ matrix whose a, b entry is the generator $E_{ab} \in \mathfrak{gl}_m$. Let E' be the transposed matrix. Consider a matrix

$$X(u) = \left(u + \theta E'\right)^{-1} \quad \text{with } \theta = \pm 1 \tag{1.11}$$

whose a, b entry is a formal power series in u^{-1}

$$X_{ab}(u) = u^{-1}\left(\delta_{ab} + \sum_{s=0}^{\infty} X_{ab}^{(s)} u^{-s-1}\right). \tag{1.12}$$

For $a, b = 1, \dots, m$ elements $X_{ab}^{(s)} \in \mathrm{U}(\mathfrak{gl}_m)$, moreover

$$X_{ab}^{(0)} = -\theta E_{ba} \quad \text{and} \quad X_{ab}^{(s)} = \sum_{c_1,\dots,c_s=1}^{m} (-\theta)^{s+1} E_{c_1 a} E_{c_2 c_1} \cdots E_{c_s c_{s-1}} E_{b c_s} \quad \text{for } s \geqslant 1.$$

Consider the ring $\mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^n)$ of polynomial functions on $\mathbb{C}^m \otimes \mathbb{C}^n$ with coordinate functions x_{ai} where $a = 1, \dots, m$ and $i = 1, \dots, n$. Let $\mathcal{PD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ be the ring of differential operators with polynomial coefficients on $\mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^n)$, and let ∂_{ai} be the partial derivation corresponding to x_{ai}.

Consider also the Grassmann algebra $\mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^n)$ of the vector space $\mathbb{C}^m \otimes \mathbb{C}^n$. It is generated by the elements x_{ai} subject to the anticommutation relations $x_{ai}x_{bj} = -x_{bj}x_{ai}$ for all indices $a, b = 1, \dots, m$ and $i, j = 1, \dots, n$. Let ∂_{ai} be the operator of left derivation on $\mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^n)$ corresponding to the variable x_{ai}. Let $\mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ denote the ring of $\mathbb{C}$-endomorphisms of $\mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^n)$ generated by all operators of left multiplication x_{ai} and by all operators ∂_{ai}.

Let us define

$$\mathcal{H}\left(\mathbb{C}^m \otimes \mathbb{C}^n\right) = \mathcal{P}\left(\mathbb{C}^m \otimes \mathbb{C}^n\right) \quad \text{and} \quad \mathcal{HD}\left(\mathbb{C}^m \otimes \mathbb{C}^n\right) = \mathcal{PD}\left(\mathbb{C}^m \otimes \mathbb{C}^n\right) \quad \text{if } \theta = 1,$$
$$\mathcal{H}\left(\mathbb{C}^m \otimes \mathbb{C}^n\right) = \mathcal{G}\left(\mathbb{C}^m \otimes \mathbb{C}^n\right) \quad \text{and} \quad \mathcal{HD}\left(\mathbb{C}^m \otimes \mathbb{C}^n\right) = \mathcal{GD}\left(\mathbb{C}^m \otimes \mathbb{C}^n\right) \quad \text{if } \theta = -1.$$

Therefore, algebra $\mathcal{H}(\mathbb{C}^m \otimes \mathbb{C}^n)$ is generated by the elements x_{ai}, $a = 1, \dots, m$, $i = 1, \dots, n$ subject to relations

$$x_{ai}x_{bj} - \theta x_{bj}x_{ai} = 0.$$

Algebra $\mathcal{HD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ is generated by the elements x_{ai} and ∂_{bj}, $a, b = 1, \dots, m$, $i, j = 1, \dots, n$ subject to relations

$$\begin{aligned} &x_{ai}x_{bj} - \theta x_{bj}x_{ai} = 0, \\ &\partial_{ai}\partial_{bj} - \theta \partial_{bj}\partial_{ai} = 0, \\ &\partial_{ai}x_{bj} - \theta x_{bj}\partial_{ai} = \delta_{ab}\delta_{ij}. \end{aligned} \tag{1.13}$$

Note that $\theta = 1$ corresponds to the case of commuting variables, while $\theta = -1$ corresponds to the case of anticommuting variables.

From now on and till the end of the paper we assume $m = p+q$, where p and q are non-negative integers. Let us introduce new coordinates

$$p_{ci} = \begin{cases} -\theta x_{ci}, & \text{for } c = 1, \dots, p \\ \partial_{ci}, & \text{for } c = p+1, \dots, m \end{cases} \qquad q_{ci} = \begin{cases} \partial_{ci}, & \text{for } c = 1, \dots, p \\ x_{ci}, & \text{for } c = p+1, \dots, m. \end{cases} \tag{1.14}$$

Now, relations (1.13) can be rewritten in the following form

$$\begin{aligned} &q_{ai}q_{bj} - \theta q_{bj}q_{ai} = 0, \\ &p_{ai}p_{bj} - \theta p_{bj}p_{ai} = 0, \\ &p_{ai}q_{bj} - \theta q_{bj}p_{ai} = \delta_{ab}\delta_{ij}. \end{aligned}$$

Define the elements $\hat{E}_{ai,bj} \in \mathcal{HD}\left(\mathbb{C}^m \otimes \mathbb{C}^n\right)$ as

$$\hat{E}_{ai,bj} = q_{ai} p_{bj}. \tag{1.15}$$

Elements $\hat{E}_{ai,bj}$ satisfy relations

$$\left[\hat{E}_{ai,bj}, \hat{E}_{ck,dl}\right] = \delta_{bc}\delta_{jk}\hat{E}_{ai,dl} - \delta_{ad}\delta_{il}\hat{E}_{ck,bj}, \tag{1.16}$$

$$\hat{E}_{ai,bj}\hat{E}_{ck,dl} - \theta\hat{E}_{ck,bj}\hat{E}_{ai,dl} = \delta_{bc}\delta_{jk}\hat{E}_{ai,dl} - \theta\delta_{ab}\delta_{ij}\hat{E}_{ck,dl} \tag{1.17}$$

which also imply

$$\hat{E}_{ck,dl}\hat{E}_{ai,bj} - \theta\hat{E}_{ck,bj}\hat{E}_{ai,dl} = \delta_{ad}\delta_{il}\hat{E}_{ck,bj} - \theta\delta_{ab}\delta_{ij}\hat{E}_{ck,dl}. \tag{1.18}$$

There is an action of the algebra $\mathfrak{gl}_m$ on the space $\mathcal{H}\left(\mathbb{C}^m \otimes \mathbb{C}^n\right)$ which is defined by homomorphism $\zeta_n \colon \mathrm{U}(\mathfrak{gl}_m) \mapsto \mathcal{HD}\left(\mathbb{C}^m \otimes \mathbb{C}^n\right)$:

$$\zeta_n\left(E_{ab}\right) = \theta\delta_{ab}\frac{n}{2} + \sum_{k=1}^{n} \hat{E}_{ak,bk}. \tag{1.19}$$

The homomorphism property can be verified using the relation (1.16). Hence, there exists an embedding $\mathrm{U}(\mathfrak{gl}_m) \hookrightarrow \mathrm{U}(\mathfrak{gl}_m) \otimes \mathcal{HD}\left(\mathbb{C}^m \otimes \mathbb{C}^n\right)$ defined for $a, b = 1, \ldots, m$ by the mappings

$$E_{ab} \mapsto E_{ab} \otimes 1 + 1 \otimes \zeta_n(E_{ab}). \tag{1.20}$$

Proposition 1.2. (i) *One can define a homomorphism* $\alpha_m \colon \mathrm{Y}(\mathfrak{gl}_n) \to \mathrm{U}(\mathfrak{gl}_m) \otimes \mathcal{HD}\left(\mathbb{C}^m \otimes \mathbb{C}^n\right)$ *by mapping*

$$\alpha_m \colon T_{ij}(u) \mapsto \delta_{ij} + \sum_{a,b=1}^{m} X_{ab}(u) \otimes \hat{E}_{ai,bj}. \tag{1.21}$$

(ii) *The image of* $\mathrm{Y}(\mathfrak{gl}_n)$ *under the homomorphism* (1.21) *commutes with the image of* $\mathrm{U}(\mathfrak{gl}_m)$ *under the embedding* (1.20).

Consider an automorphism of the algebra $\mathcal{HD}\left(\mathbb{C}^m \otimes \mathbb{C}^n\right)$ such that for all $a = 1, \ldots, m$ and $i = 1, \ldots, n$

$$x_{ai} \mapsto q_{ai} \quad \text{and} \quad \partial_{ai} \mapsto p_{ai}. \tag{1.22}$$

Now, proof of Proposition 1.2 can be obtained by applying the automorphism (1.22) to the results of [4, Proposition 1.3] if $\theta = 1$ and to the results of [5, Proposition 1.3] if $\theta = -1$. In the Appendix we give an explicit proof of the Proposition 1.2.

Finally, let V be an arbitrary $\mathfrak{gl}_m$-module. Then we can define a $\mathrm{U}(\mathfrak{gl}_m) \otimes \mathcal{HD}\left(\mathbb{C}^m \otimes \mathbb{C}^n\right)$-module

$$\mathcal{E}_{p,q}(V) = V \otimes \mathcal{H}\left(\mathbb{C}^m \otimes \mathbb{C}^n\right). \tag{1.23}$$

The results of Proposition 1.2 turns $\mathcal{E}_{p,q}$ into a functor from the category of $\mathfrak{gl}_m$-modules to the category of $\mathfrak{gl}_m$ and $\mathrm{Y}(\mathfrak{gl}_n)$ bimodules, where the actions of the algebras $\mathfrak{gl}_m$ and $\mathrm{Y}(\mathfrak{gl}_n)$ are defined by homomorphisms (1.20) and (1.21) respectively.

2. Reduction to standard modules

2.1. Parabolic induction

For any positive integer l let U be a module over the Lie algebra $\mathfrak{gl}_l$. Let $l = l_1 + l_2$, then $\mathcal{E}_{l_1,l_2}(U)$ is a $\mathrm{Y}(\mathfrak{gl}_n)$-module. For any $z \in \mathbb{C}$ denote by $\mathcal{E}^z_{l_1,l_2}(U)$ the $\mathrm{Y}(\mathfrak{gl}_n)$-module obtained from $\mathcal{E}_{l_1,l_2}(U)$ via pull-back through the automorphism $\tau_{-\theta z}$ of $\mathrm{Y}(\mathfrak{gl}_n)$, in other words, the underlying vector space of $\mathcal{E}^z_{l_1,l_2}(U)$ is the same as of $\mathcal{E}_{l_1,l_2}(U)$, but the action of $T_{ij}(u)$ on $\mathcal{E}^z_{l_1,l_2}(U)$ is given by the same formula as the action of $T_{ij}(u+\theta z)$ on $\mathcal{E}_{l_1,l_2}(U)$. Note that as a $\mathfrak{gl}_l$-module $\mathcal{E}^z_{l_1,l_2}(U)$ coincides with $\mathcal{E}_{l_1,l_2}(U)$.

The decomposition $\mathbb{C}^{m+l} = \mathbb{C}^m \oplus \mathbb{C}^l$ determines an embedding of the direct sum $\mathfrak{gl}_m \oplus \mathfrak{gl}_l$ of Lie algebras into $\mathfrak{gl}_{m+l}$. As a subalgebra of $\mathfrak{gl}_{m+l}$, the direct summand $\mathfrak{gl}_m$ is spanned by the matrix units $E_{ab} \in \mathfrak{gl}_{m+l}$ where $a, b = 1, \ldots, m$. The direct summand $\mathfrak{gl}_l$ is spanned by the matrix units E_{ab} where $a, b = m+1, \ldots, m+l$. Let $\mathfrak{q}$ and $\mathfrak{q}'$ be the Abelian subalgebras of $\mathfrak{gl}_{m+l}$ spanned respectively by matrix units E_{ba} and E_{ab} for all $a = 1, \ldots, m$ and $b = m+1, \ldots, m+l$. Put $\mathfrak{p} = \mathfrak{gl}_m \oplus \mathfrak{gl}_l \oplus \mathfrak{q}'$. Then $\mathfrak{p}$ is a maximal parabolic subalgebra of the reductive Lie algebra $\mathfrak{gl}_{m+l}$, and moreover

$$\mathfrak{gl}_{m+l} = \mathfrak{q} \oplus \mathfrak{p}.$$

Denote by $V \boxtimes U$ the $\mathfrak{gl}_{m+l}$-module *parabolically induced* from the $\mathfrak{gl}_m \oplus \mathfrak{gl}_l$-module $V \otimes U$. To define $V \boxtimes U$, one first extends the action of the Lie algebra $\mathfrak{gl}_m \oplus \mathfrak{gl}_l$ on $V \otimes U$ to the Lie algebra $\mathfrak{p}$, so that any element of the subalgebra $\mathfrak{q}' \subset \mathfrak{p}$ acts on $V \otimes U$ as zero. By definition, $V \boxtimes U$ is the $\mathfrak{gl}_{m+l}$-module induced from the $\mathfrak{p}$-module $V \otimes U$.

Now consider bimodules $\mathcal{E}_{p,q+r}(V \boxtimes U)$ and $\mathcal{E}_{p+r,q}(V \boxtimes U)$ over $\mathfrak{gl}_{m+r}$ and $\mathrm{Y}(\mathfrak{gl}_n)$, which are parabolically induced from the $\mathfrak{gl}_p \oplus \mathfrak{gl}_{q+r}$ or $\mathfrak{gl}_{p+r} \oplus \mathfrak{gl}_q$-module $V \otimes U$. The action of $\mathrm{Y}(\mathfrak{gl}_n)$ commutes with the action of the Lie algebra $\mathfrak{gl}_{m+r}$, and hence with the action of the subalgebra $\mathfrak{q} \subset \mathfrak{gl}_{m+r}$. For any $\mathfrak{gl}_m$-module W denote by $W_{\mathfrak{q}}$ the vector space $W/\mathfrak{q} \cdot W$ of the coinvariants of the action of the subalgebra $\mathfrak{q} \subset \mathfrak{gl}_m$ on W. Then vector spaces $\mathcal{E}_{p,q+r}(V \boxtimes U)_{\mathfrak{q}}$ and $\mathcal{E}_{p+r,q}(V \boxtimes U)_{\mathfrak{q}}$ are quotients of the $\mathrm{Y}(\mathfrak{gl}_n)$-modules $\mathcal{E}_{p,q+r}(V \boxtimes U)$ and $\mathcal{E}_{p+r,q}(V \boxtimes U)$. Note that the subalgebras $\mathfrak{gl}_p \oplus \mathfrak{gl}_{q+r}$ and $\mathfrak{gl}_{p+r} \oplus \mathfrak{gl}_q$ also act on these quotient spaces.

Theorem 2.1. (i) *The bimodule $\mathcal{E}_{p,q+r}(V \boxtimes U)_{\mathfrak{q}}$ over the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ and the direct sum $\mathfrak{gl}_p \oplus \mathfrak{gl}_{q+r}$ is isomorphic to the tensor product $\mathcal{E}_{p,q}(V) \otimes \mathcal{E}^m_{0,r}(U)$.*
(ii) *The bimodule $\mathcal{E}_{p+r,q}(V \boxtimes U)_{\mathfrak{q}}$ over the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ and the direct sum $\mathfrak{gl}_{p+r} \oplus \mathfrak{gl}_q$ is isomorphic to the tensor product $\mathcal{E}_{r,0}(V) \otimes \mathcal{E}^r_{p,q}(U)$.*

The Theorem 2.1 is equivalent to [4, Theorem 2.1] under the action of automorphism (1.22) if $\theta = 1$ and to [5, Theorem 2.1] under the action of automorphism (1.22) if $\theta = -1$. In both cases Theorem 2.1 was proved by establishing a linear map

$$\chi : \mathcal{E}_m(V) \otimes \mathcal{E}^m_l(U) \to \mathcal{E}_{m+l}(V \boxtimes U)_{\mathfrak{q}}.$$

Both the source and the target are bimodules over the algebras $\mathfrak{gl}_m \oplus \mathfrak{gl}_l$ and $\mathrm{Y}(\mathfrak{gl}_n)$, while χ is a bijective map intertwining actions of algebras. One can easily show that the map χ commutes with the automorphism (1.22), hence the intertwining property follows in our case. The proof that the map χ is bijective can be almost word by word taken from [4] or [5] except for automorphism (1.22) alters the filtration described in the papers just mentioned. Thus, we should take descending filtrations

$$\bigoplus_{N=K}^{\infty} \mathcal{P}(\mathbb{C}^m) \otimes \mathcal{P}^N(\mathbb{C}^r) \quad \text{and} \quad \bigoplus_{N=K}^{\infty} \mathcal{P}^N(\mathbb{C}^m) \otimes \mathcal{P}(\mathbb{C}^r) \quad \text{if } \theta = 1,$$
$$\bigoplus_{N=K}^{r} \mathcal{G}(\mathbb{C}^m) \otimes \mathcal{G}^N(\mathbb{C}^r) \quad \text{and} \quad \bigoplus_{N=K}^{m} \mathcal{G}^N(\mathbb{C}^m) \otimes \mathcal{G}(\mathbb{C}^r) \quad \text{if } \theta = -1$$

for cases (i) and (ii) of the Theorem 2.1 respectively. Therefore, the proof of the theorem follows in our case.

Let us consider the *triangular decomposition* of the Lie algebra $\mathfrak{gl}_m$,

$$\mathfrak{gl}_m = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}'. \tag{2.1}$$

Here $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{gl}_m$ with the basis vectors $E_{11}, \dots, E_{mm}$. Further, $\mathfrak{n}$ and $\mathfrak{n}'$ are the nilpotent subalgebras spanned respectively by the elements E_{ba} and E_{ab} for all $a, b = 1, \dots, m$ such that $a < b$. Denote by $V_{\mathfrak{n}}$ the vector space $V/\mathfrak{n} \cdot V$ of the coinvariants of the action of the subalgebra $\mathfrak{n} \subset \mathfrak{gl}_m$ on V. Note that the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{gl}_m$ acts on the vector space $V_{\mathfrak{n}}$. Now consider the bimodule $\mathcal{E}_{p,q}(V)$. The action of $\mathrm{Y}(\mathfrak{gl}_n)$ on this bimodule commutes with the action of the Lie algebra $\mathfrak{gl}_m$, and hence with the action of the subalgebra $\mathfrak{n} \subset \mathfrak{gl}_m$. Therefore, the space $\mathcal{E}_{p,q}(V)_{\mathfrak{n}}$ of coinvariants of the action of $\mathfrak{n}$ is a quotient of the $\mathrm{Y}(\mathfrak{gl}_n)$-module $\mathcal{E}_{p,q}(V)$. Thus, we get a functor from the category of all $\mathfrak{gl}_m$-modules to the category of bimodules over $\mathfrak{h}$ and $\mathrm{Y}(\mathfrak{gl}_n)$

$$V \mapsto \mathcal{E}_{p,q}(V)_{\mathfrak{n}} = \left(V \otimes \mathcal{H}\left(\mathbb{C}^m \otimes \mathbb{C}^n\right)\right)_{\mathfrak{n}}. \tag{2.2}$$

By the transitivity of induction, Theorem 2.1 can be extended from the maximal to all parabolic subalgebras of the Lie algebra $\mathfrak{gl}_m$. Consider the case of the Borel subalgebra $\mathfrak{h} \oplus \mathfrak{n}'$ of $\mathfrak{gl}_m$. Apply the functor (2.2) to the $\mathfrak{gl}_m$-module $V = M_\mu$, where M_μ is the Verma module of weight $\mu \in \mathfrak{h}^*$. We obtain the $\mathrm{Y}(\mathfrak{gl}_n)$ and $\mathfrak{h}$-bimodule

$$\mathcal{E}_{p,q}(M_\mu)_{\mathfrak{n}} = \left(M_\mu \otimes \mathcal{H}\left(\mathbb{C}^m \otimes \mathbb{C}^n\right)\right)_{\mathfrak{n}}.$$

Using the basis $E_{11}, \dots, E_{mm}$ we identify $\mathfrak{h}$ with the direct sum of m copies of the Lie algebra $\mathfrak{gl}_1$. Consider the Verma modules $M_{\mu_1}, \dots, M_{\mu_m}$ over $\mathfrak{gl}_1$. By applying Theorem 2.1 repeatedly we get the next result.

Corollary 2.2. *The bimodule $\mathcal{E}_{p,q}(M_\mu)_{\mathfrak{n}}$ of $\mathfrak{h}$ and $\mathrm{Y}(\mathfrak{gl}_n)$ is isomorphic to the tensor product*

$$\mathcal{E}_{1,0}(M_{\mu_1}) \otimes \mathcal{E}^1_{1,0}(M_{\mu_2}) \otimes \cdots \otimes \mathcal{E}^{p-1}_{1,0}(M_{\mu_p}) \otimes \mathcal{E}^p_{0,1}(M_{\mu_{p+1}}) \otimes \cdots \otimes \mathcal{E}^{m-1}_{0,1}(M_{\mu_m}).$$

2.2. Standard modules

Let us now describe the bimodules $\mathcal{E}^z_{1,0}(M_t)$ and $\mathcal{E}^z_{0,1}(M_t)$ over $\mathfrak{gl}_1$ and $\mathrm{Y}(\mathfrak{gl}_n)$ for arbitrary $t, z \in \mathbb{C}$. The Verma module M_t over $\mathfrak{gl}_1$ is one-dimensional, and the element $E_{11} \in \mathfrak{gl}_1$ acts on M_t by multiplication by t. The vector space of bimodules $\mathcal{E}_{1,0}(M_t)$ and $\mathcal{E}_{0,1}(M_t)$ is the algebra $\mathcal{H}(\mathbb{C}^1 \otimes \mathbb{C}^n) = \mathcal{H}(\mathbb{C}^n)$. Then E_{11} acts on the bimodules $\mathcal{E}_{1,0}(M_t)$ and $\mathcal{E}_{0,1}(M_t)$ as differential operators

$$t + \theta \frac{n}{2} - \theta \sum_{k=1}^{n} \partial_{1k} x_{1k} \quad \text{and} \quad t + \theta \frac{n}{2} + \sum_{k=1}^{n} x_{1k} \partial_{1k}$$

respectively. The action of E_{11} on $\mathcal{E}^z_{p,q}(M_t)$ is the same as on $\mathcal{E}_{p,q}(M_t)$. The action of $\mathrm{Y}(\mathfrak{gl}_n)$ on $\mathcal{E}^z_{1,0}(M_t)$ and $\mathcal{E}^z_{0,1}(M_t)$ is given by

$$T_{ij}(u) \mapsto \delta_{ij} - \theta\frac{\partial_{1i}x_{1j}}{u+\theta(t-z)} \quad \text{and} \quad T_{ij}(u) \mapsto \delta_{ij} + \frac{x_{1i}\partial_{1j}}{u+\theta(t-z)} \tag{2.3}$$

respectively, this is what Proposition 1.2 states in the case $m = 1$. Note that both operators $x_{1i}\partial_{1j}$ and $-\theta\partial_{1i}x_{1j}$ describe actions of the element $E_{ij} \in \mathfrak{gl}_n$ on $\mathcal{H}(\mathbb{C}^1 \otimes \mathbb{C}^n) = \mathcal{H}(\mathbb{C}^n)$. When speaking about these actions we will omit the first indices and write x_i and ∂_j instead of x_{1i} and ∂_{1j}. Hence, actions of the algebra $\mathrm{Y}(\mathfrak{gl}_n)$ on $\mathcal{E}^z_{1,0}(M_t)$ and $\mathcal{E}^z_{0,1}(M_t)$ can be obtained from the actions of $\mathfrak{gl}_n$ on $\mathcal{H}(\mathbb{C}^n)$ by pulling back through the evaluation homomorphism $\pi_{\theta(t-z)}$.

Now, consider $\mathrm{Y}(\mathfrak{gl}_n)$-modules $\widetilde{\Phi}_z$ and Φ_z with the underlying vector space $\mathcal{H}(\mathbb{C}^n)$ and Yangian actions defined by

$$T_{ij}(u) \mapsto \delta_{ij} - \theta\frac{\partial_i x_j}{u+\theta z} \quad \text{and} \quad T_{ij}(u) \mapsto \delta_{ij} + \frac{x_i\partial_j}{u+\theta z}$$

correspondingly. Therefore, the bimodules $\mathcal{E}^z_{1,0}(M_t)$ and $\mathcal{E}^z_{0,1}(M_t)$ are respectively isomorphic to $\widetilde{\Phi}_{t-z}$ and Φ_{t-z} as the $\mathrm{Y}(\mathfrak{gl}_n)$-modules. Moreover, Corollary 2.2 implies that the bimodule $\mathcal{E}_{p,q}(M_\mu)_\mathfrak{n}$ of $\mathfrak{h} \in \mathfrak{gl}_m$ and $\mathrm{Y}(\mathfrak{gl}_n)$ is isomorphic as a $\mathrm{Y}(\mathfrak{gl}_n)$-module to the tensor product

$$\widetilde{\Phi}_{\mu_1+\rho_1} \otimes \cdots \otimes \widetilde{\Phi}_{\mu_p+\rho_p} \otimes \Phi_{\mu_{p+1}+\rho_{p+1}} \otimes \cdots \otimes \Phi_{\mu_m+\rho_m} \tag{2.4}$$

where $\rho_i = 1 - i$.

Let us also define a $\mathrm{Y}(\mathfrak{gl}_n)$-module Φ'_z with the underlying vector space $\mathcal{H}(\mathbb{C}^n)$ and the $\mathrm{Y}(\mathfrak{gl}_n)$-action is given by

$$T_{ij}(u) \mapsto \delta_{ij} - \frac{x_j\partial_i}{u+\theta(z-1)}.$$

Using commutation relation $\partial_i x_j - \theta x_j\partial_i = \delta_{ij}$ one can verify that

$$\delta_{ij} - \theta\frac{\partial_i x_j}{u+\theta z} = \frac{u+\theta(z-1)}{u+\theta z}\left(\delta_{ij} - \frac{x_j\partial_i}{u+\theta(z-1)}\right)$$

which implies the isomorphism of $\mathrm{Y}(\mathfrak{gl}_n)$-modules

$$\widetilde{\Phi}_z \cong \Omega'_z \otimes \Phi'_z \quad \text{if } \theta = 1,$$
$$\widetilde{\Phi}_z \cong \Omega_{-z} \otimes \Phi'_z \quad \text{if } \theta = -1.$$

Hence, the bimodule $\mathcal{E}_{p,q}(M_\mu)_\mathfrak{n}$ of $\mathfrak{h} \in \mathfrak{gl}_m$ and $\mathrm{Y}(\mathfrak{gl}_n)$ is isomorphic as the $\mathrm{Y}(\mathfrak{gl}_n)$-module to the tensor product

$$\bigotimes_{i=1}^{p} \Omega^*_{\theta(\mu_i+\rho_i)} \otimes \bigotimes_{i=1}^{p} \Phi'_{\mu_i+\rho_i} \otimes \bigotimes_{i=p+1}^{m} \Phi_{\mu_i+\rho_i} \tag{2.5}$$

where

$$\Omega^*_z = \Omega'_z \quad \text{if } \theta = 1 \quad \text{and} \quad \Omega^*_z = \Omega_z \quad \text{if } \theta = -1.$$

Note that the $\mathrm{Y}(\mathfrak{gl}_n)$-modules Φ_z and Φ'_z can be also realized as evaluation and dual evaluation homomorphisms to $\mathcal{P}(\mathbb{C}^n)$ or $\mathcal{G}(\mathbb{C}^n)$. Moreover, modules Φ_z and Φ'_z are rational, and hence so are their subquotients. We call an $\mathrm{Y}(\mathfrak{gl}_n)$-module a *standard rational module* if it is a tensor product of modules Φ_z and Φ'_z with arbitrary values of z.

3. Zhelobenko operators

3.1. Definition

Consider $\mathfrak{S}_m$ as the Weyl group of the reductive Lie algebra $\mathfrak{gl}_m$. Let $E^*_{11}, \ldots, E^*_{mm}$ be the basis of $\mathfrak{h}^*$ dual to the basis $E_{11}, \ldots, E_{mm}$ of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{gl}_m$. The group $\mathfrak{S}_m$ acts on the space $\mathfrak{h}^*$ so that for any $\sigma \in \mathfrak{S}_m$ and $a = 1, \ldots, m$

$$\sigma\colon E^*_{aa} \mapsto E^*_{\sigma(a)\sigma(a)}.$$

If we identify each weight $\mu \in \mathfrak{h}^*$ with the sequence $(\mu_1, \ldots, \mu_m)$ of its labels, then

$$\sigma\colon (\mu_1, \ldots, \mu_m) \mapsto (\mu_{\sigma^{-1}(1)}, \ldots, \mu_{\sigma^{-1}(m)}).$$

Let $\rho \in \mathfrak{h}^*$ be the weight with sequence of labels $(0, -1, \ldots, 1-m)$. The *shifted* action of any element $\sigma \in \mathfrak{S}_m$ on $\mathfrak{h}^*$ is defined by the assignment

$$\mu \mapsto \sigma \circ \mu = \sigma(\mu + \rho) - \rho.$$

The Weyl group also acts on the vector space $\mathfrak{gl}_m$ so that for any $\sigma \in \mathfrak{S}_m$ and $a, b = 1, \ldots, m$

$$\sigma : E_{ab} \mapsto E_{\sigma(a)\sigma(b)}.$$

The latter action extends to an action of the group $\mathfrak{S}_m$ by automorphisms of the associative algebra $\mathrm{U}(\mathfrak{gl}_m)$. The group $\mathfrak{S}_m$ also acts by automorphisms of the space $\mathcal{HD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ so that element $\sigma \in \mathfrak{S}_m$ maps

$$p_{ai} \mapsto p_{\sigma(a)i} \quad \text{and} \quad q_{ai} \mapsto q_{\sigma(a)i}.$$

Note that homomorphisms (1.20) and (1.21) of the algebras $\mathfrak{gl}_m$ and $\mathrm{Y}(\mathfrak{gl}_n)$ into the algebra $\mathrm{U}(\mathfrak{gl}_m) \otimes \mathcal{HD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ are $\mathfrak{S}_m$-equivariant.

Let A be the associative algebra generated by the algebras $\mathrm{U}(\mathfrak{gl}_m)$ and $\mathcal{HD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ with the cross relations

$$[X, Y] = [\zeta_n(X), Y] \tag{3.1}$$

for any $X \in \mathfrak{gl}_m$ and $Y \in \mathcal{HD}(\mathbb{C}^m \otimes \mathbb{C}^n)$. The brackets at the left hand side of the relation (3.1) denote the commutator in A, while the brackets at the right hand side denote the commutator in the algebra $\mathcal{HD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ embedded into A. In particular, we will regard $\mathrm{U}(\mathfrak{gl}_m)$ as a subalgebra of A. An isomorphism of the algebra A with the tensor product $\mathrm{U}(\mathfrak{gl}_m) \otimes \mathcal{HD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ can be defined by mapping the elements $X \in \mathfrak{gl}_m$ and $Y \in \mathcal{HD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ in A respectively to the elements

$$X \otimes 1 + 1 \otimes \zeta_n(X) \quad \text{and} \quad 1 \otimes Y$$

in $\mathrm{U}(\mathfrak{gl}_m) \otimes \mathcal{HD}(\mathbb{C}^m \otimes \mathbb{C}^n)$. The action of the group $\mathfrak{S}_m$ on A is defined via the isomorphism of A with the tensor product $\mathrm{U}(\mathfrak{gl}_m) \otimes \mathcal{HD}(\mathbb{C}^m \otimes \mathbb{C}^n)$. Since the homomorphism ζ_n is $\mathfrak{S}_m$-equivariant the same action of $\mathfrak{S}_m$ is obtained by extending the actions of $\mathfrak{S}_m$ from the subalgebras $\mathrm{U}(\mathfrak{gl}_m)$ and $\mathcal{HD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ to A.

For any $a, b = 1, \ldots, m$ put $\eta_{ab} = E^*_{aa} - E^*_{bb} \in \mathfrak{h}^*$ and $\eta_c = \eta_{c,c+1}$ with $c = 1, \ldots, m-1$. Put also

$$E_c = E_{c,c+1}, \qquad F_c = E_{c+1,c} \quad \text{and} \quad H_c = E_{cc} - E_{c+1,c+1}. \tag{3.2}$$

For any $c = 1, \ldots, m-1$ these three elements form an $\mathfrak{sl}_2$-triple.

Let $\overline{\mathrm{U}(\mathfrak{h})}$ be the ring of fractions of the commutative algebra $\mathrm{U}(\mathfrak{h})$ relative to the set of denominators

$$\{E_{aa} - E_{bb} + z \mid 1 \leqslant a, b \leqslant m;\; a \neq b;\; z \in \mathbb{Z}\}. \tag{3.3}$$

The elements of this ring can also be regarded as rational functions on the vector space $\mathfrak{h}^*$. The elements of $\mathrm{U}(\mathfrak{h}) \subset \overline{\mathrm{U}(\mathfrak{h})}$ are then regarded as polynomial functions on $\mathfrak{h}^*$. Denote by $\bar{\mathrm{A}}$ the ring of fractions of A relative to the set of denominators (3.3), regarded as elements of A using the embedding of $\mathfrak{h} \subset \mathfrak{gl}_m$ into A. The ring $\bar{\mathrm{A}}$ is defined due to the following relations in the algebras $\mathrm{U}(\mathfrak{gl}_m)$ and A: for $a, b = 1, \ldots, m$ and $H \in \mathfrak{h}$

$$[H, E_{ab}] = \eta_{ab}(H)E_{ab}, \qquad [H, p_{ak}] = -\theta E^*_{aa}(H)p_{ak}, \qquad [H, q_{bk}] = E^*_{bb}(H)q_{bk}, \tag{3.4}$$

where p_{ak} and q_{bk} are given by (1.14). Therefore, the ring A satisfies the Ore condition relative to its subset (3.3). Using left multiplication by elements of $\overline{\mathrm{U}(\mathfrak{h})}$, the ring of fractions $\bar{\mathrm{A}}$ becomes a module over $\overline{\mathrm{U}(\mathfrak{h})}$.

The ring $\bar{\mathrm{A}}$ is also an associative algebra over the field $\mathbb{C}$. The action of the group $\mathfrak{S}_m$ on A preserves the set of denominators (3.3) so that $\mathfrak{S}_m$ also acts by automorphisms of the algebra $\bar{\mathrm{A}}$. For each $c = 1, \ldots, m-1$ define a linear map $\xi_c : \mathrm{A} \to \bar{\mathrm{A}}$ by setting

$$\xi_c(Y) = Y + \sum_{s=1}^{\infty} (s!H_c^{(s)})^{-1} E_c^s \widehat{F}_c^s(Y) \tag{3.5}$$

for $Y \in \mathrm{A}$. Here

$$H_c^{(s)} = H_c(H_c - 1) \cdots (H_c - s + 1)$$

and $\widehat{F}_c$ is the operator of adjoint action corresponding to the element $F_c \in \mathrm{A}$, so that

$$\widehat{F}_c(Y) = [F_c, Y].$$

For any given element $Y \in \mathrm{A}$ only finitely many terms of the sum (3.5) differ from zero, hence the map ξ_c is well defined.

Let J and $\bar{\mathrm{J}}$ be the right ideals of the algebras A and $\bar{\mathrm{A}}$ respectively, generated by all elements of the subalgebra $\mathfrak{n} \subset \mathfrak{gl}_m$. Let J' be the left ideal of the algebra A, generated by the elements $X - \zeta_n(X)$ or equivalently by the elements

$$X \otimes 1 \in \mathrm{U}(\mathfrak{gl}_m) \otimes 1 \subset \mathrm{U}(\mathfrak{gl}_m) \otimes \mathcal{HD}\left(\mathbb{C}^m \otimes \mathbb{C}^n\right), \tag{3.6}$$

where $X \in \mathfrak{n}'$. Denote $\bar{\mathrm{J}}' = \overline{\mathrm{U}(\mathfrak{h})}\mathrm{J}'$, then $\bar{\mathrm{J}}'$ is a left ideal of the algebra $\bar{\mathrm{A}}$.

Now we give a short observation of some results proved in [4]. For any elements $X \in \mathfrak{h}$ and $Y \in \mathrm{A}$ we have

$$\xi_a(XY) \in (X + \eta_a(X))\xi_a(Y) + \bar{\mathrm{J}}. \tag{3.7}$$

The property (3.7) allows us to define a linear map $\bar{\xi}_a : \bar{\mathrm{A}} \to \bar{\mathrm{J}} \setminus \bar{\mathrm{A}}$ by setting

$$\bar{\xi}_a(XY) = Z\xi_a(Y) + \bar{\mathrm{J}} \quad \text{for } X \in \overline{\mathrm{U}(\mathfrak{h})} \text{ and } Y \in \mathrm{A} \tag{3.8}$$

where the element $Z \in \overline{\mathrm{U}(\mathfrak{h})}$ is defined by the equality

$$Z(\mu) = X(\mu + \eta_a) \quad \text{for } \mu \in \mathfrak{h}^* \tag{3.9}$$

and both X and Z are regarded as rational functions on $\mathfrak{h}^*$. The backslash in $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}$ indicates that the quotient is taken relative to a *right* ideal of $\bar{\mathrm{A}}$.

The action of the group $\mathfrak{S}_m$ on the algebra $\mathrm{U}(\mathfrak{gl}_m)$ extends to an action on $\overline{\mathrm{U}(\mathfrak{h})}$ so that for any $\sigma \in \mathfrak{S}_m$

$$(\sigma(X))(\mu) = X(\sigma^{-1}(\mu)),$$

when the element $X \in \overline{\mathrm{U}(\mathfrak{h})}$ is regarded as a rational function on $\mathfrak{h}^*$. The action of $\mathfrak{S}_m$ by automorphisms of the algebra A then extends to an action by automorphisms of $\bar{\mathrm{A}}$. For any $c = 1, \dots, m-1$ let $\sigma_c \in \mathfrak{S}_m$ be the transposition of c and $c+1$. Then, by [4, Proposition 3.5] we have

$$\bar{\xi}_c(\sigma_c(\bar{\mathrm{J}}')) \subset \bar{\mathrm{J}}' + \bar{\mathrm{J}}. \tag{3.10}$$

Now consider the image $\sigma_c(\bar{\mathrm{J}})$, that is again a right ideal of $\bar{\mathrm{A}}$. By [4, Proposition 3.2] we also have

$$\sigma_c(\bar{\mathrm{J}}) \subset \ker \bar{\xi}_c.$$

This allows us to define for any $c = 1, \dots, m-1$ a linear map

$$\check{\xi}_c : \bar{\mathrm{J}} \setminus \bar{\mathrm{A}} \to \bar{\mathrm{J}} \setminus \bar{\mathrm{A}} \tag{3.11}$$

as the composition $\bar{\xi}_c \sigma_c$ applied to the elements of $\bar{\mathrm{A}}$ taken modulo $\bar{\mathrm{J}}$. The operators $\check{\xi}_1, \dots, \check{\xi}_{m-1}$ on the vector space $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}$ are called *the Zhelobenko operators*. By [4, Proposition 3.3] the Zhelobenko operators $\check{\xi}_1, \dots, \check{\xi}_{m-1}$ on $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}$ satisfy the braid relations

$$\check{\xi}_c\, \check{\xi}_{c+1}\, \check{\xi}_c = \check{\xi}_{c+1}\, \check{\xi}_c\, \check{\xi}_{c+1} \quad \text{for } c < m-1,$$
$$\check{\xi}_b\, \check{\xi}_c = \check{\xi}_c\, \check{\xi}_b \quad \text{for } b < c-1.$$

Therefore, for any reduced decomposition $\sigma = \sigma_{c_1} \cdots \sigma_{c_K}$ in $\mathfrak{S}_m$ the composition $\check{\xi}_{c_1} \cdots \check{\xi}_{c_K}$ of operators on $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}$ does not depend on the choice of decomposition of σ. Finally, for any $\sigma \in \mathfrak{S}_m$, $X \in \overline{\mathrm{U}(\mathfrak{h})}$, and $Y \in \bar{\mathrm{J}} \setminus \bar{\mathrm{A}}$ we have relations

$$\check{\xi}_\sigma(XY) = (\sigma \circ X)\check{\xi}_\sigma(Y) \tag{3.12}$$

which follows from [4, Proposition 3.1].

3.2. Intertwining properties

Let $\delta = (\delta_1, \dots, \delta_m)$ be a sequence of m elements from the set $\{1, -1\}$. Let the symmetric group $\mathfrak{S}_m$ act on the set of sequences $\{\delta\}$ by

$$\sigma(\delta) = \overline{\sigma \cdot \overline{\delta}}$$

where

$$\overline{\delta} = (\delta_1, \dots, \delta_p, -\delta_{p+1}, \dots, -\delta_{p+q}) \quad \text{and} \quad \sigma \cdot \delta = (\delta_{\sigma^{-1}(1)}, \dots, \delta_{\sigma^{-1}(m)}).$$

Denote

$$\delta^+ = (1, \dots, 1) \quad \text{and} \quad \delta' = \overline{\delta^+} = (\underbrace{1, \dots, 1}_{p}, \underbrace{-1, \dots, -1}_{q}).$$

For a given sequence δ let ϖ denote a composition of automorphisms of the ring $\mathcal{HD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ such that

$$x_{ak} \mapsto -\theta\partial_{ak} \quad \text{and} \quad \partial_{ak} \mapsto x_{ak} \quad \text{whenever } \delta_a = -1.$$

For any $\mathfrak{gl}_m$-module V we define a bimodule $\mathcal{E}_\delta(V)$ over $\mathfrak{gl}_m$ and $\mathrm{Y}(\mathfrak{gl}_n)$. Its underlying vector space is $\mathcal{H}(\mathbb{C}^m \otimes \mathbb{C}^n)$ for every δ. The action of the algebra $\mathfrak{gl}_m$ on the module $\mathcal{E}_\delta(V)$ is defined by pushing the homomorphism ξ_n forward through the automorphism ϖ

$$E_{ab} \mapsto E_{ab} \otimes 1 + 1 \otimes \varpi(\xi_n(E_{ab})).$$

The action of the algebra $\mathrm{Y}(\mathfrak{gl}_n)$ on the module $\mathcal{E}_\delta(V)$ is defined by pushing the homomorphism α_n forward through the automorphism ϖ, applied to the second tensor factor $\mathcal{HD}\,(\mathbb{C}^m \otimes \mathbb{C}^n)$ of the target of α_n

$$T_{ij}(u) \mapsto \delta_{ij} + \sum_{a,b=1}^{m} X_{ab}(u) \otimes \varpi\,(\hat{E}_{ai,bj}).$$

For instance, we have $\mathcal{E}_{p,q}(V) = \mathcal{E}_{\delta^+}(V)$.

Let $\mu \in \mathfrak{h}^*$ be a generic weight of $\mathfrak{gl}_m$, which means that

$$\mu_a - \mu_b \notin \mathbb{Z} \quad \text{for all } a, b = 1, \ldots, m. \tag{3.13}$$

In the remaining of this section we show that the Zhelobenko operator $\check{\xi}_\sigma$ determines an intertwining operator

$$\mathcal{E}_{p,q}(M_\mu)_{\mathfrak{n}} \to \mathcal{E}_\delta(M_{\sigma\circ\mu})_{\mathfrak{n}} \quad \text{where } \delta = \sigma(\delta^+). \tag{3.14}$$

Let I_δ be the left ideal of the algebra A generated by the elements $x_{ak},\ k = 1, \ldots, n$, for $\delta_a = -1$ and by the elements $\partial_{ak},\ k = 1, \ldots, n$, for $\delta_a = 1$. For instance, ideal I_{δ^+} is generated by the elements ∂_{ak} with $a = 1, \ldots, m$ and $k = 1, \ldots, n$. Let $\bar{\mathrm{I}}_\delta$ be the left ideal of the algebra $\bar{\mathrm{A}}$ generated by the same elements as the ideal I_δ in A. Occasionally, $\bar{\mathrm{I}}_\delta$ will denote the image of the ideal $\bar{\mathrm{I}}_\delta$ in the quotient space $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}$.

Proposition 3.1. *For any $\sigma \in \mathfrak{S}_m$ the operator $\check{\xi}_\sigma$ maps the subspace $\bar{\mathrm{I}}_{\delta^+}$ to $\bar{\mathrm{I}}_{\sigma(\delta^+)}$.*

Proof. For all $a = 1, \ldots, m-1$ consider the operator $\widehat{F_a}$. Due to (3.1), (3.2) and (1.19) we have

$$\widehat{F_a}(Y) = \sum_{k=1}^{n} [p_{ak}q_{a+1\,k}, Y]$$

for any $Y \in \mathcal{HD}\,(\mathbb{C}^m \otimes \mathbb{C}^n)$. The above description of the action of $\widehat{F_a}$ with $a = 1, \ldots, m-1$ on the vector space $\mathcal{HD}\,(\mathbb{C}^m \otimes \mathbb{C}^n)$ shows that this action preserves each of the two $2n$ dimensional subspaces spanned by the vectors

$$q_{ai} \quad \text{and} \quad q_{a+1\,i} \quad \text{where } i = 1, \ldots, n; \tag{3.15}$$

$$p_{ai} \quad \text{and} \quad p_{a+1\,i} \quad \text{where } i = 1, \ldots, n. \tag{3.16}$$

This action also maps to zero the $2n$ dimensional subspace spanned by

$$p_{ai} \quad \text{and} \quad q_{a+1\,i} \quad \text{where } i = 1, \ldots, n. \tag{3.17}$$

Therefore, for any δ the operator $\bar{\xi}_a$ maps the ideal $\bar{\mathrm{I}}_\delta$ of $\bar{\mathrm{A}}$ to the image of $\bar{\mathrm{I}}_\delta$ in $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}$ unless $\delta_a = \delta'_a$ and $\delta_{a+1} = -\delta'_{a+1}$ for $a = 1, \ldots, m-1$. Hence, the operator $\check{\xi}_a = \bar{\xi}_a\sigma_a$ maps the subspace $\bar{\mathrm{I}}_\delta$ to the image of $\bar{\mathrm{I}}_{\sigma(\delta)}$ unless $\delta_a = -\delta'_a$ and $\delta_{a+1} = \delta'_{a+1}$.

From now on we will denote the image of the ideal $\bar{\mathrm{I}}_\delta$ in the quotient space $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}$ by the same symbol $\bar{\mathrm{I}}_\delta$. Put

$$\widehat{\delta} = \sum_{a=1}^{p} \delta_a E^*_{aa} - \sum_{a=p+1}^{p+q} \delta_a E^*_{aa}.$$

Then for every $\sigma \in \mathfrak{S}_m$ we have the equality $\widehat{\sigma(\delta)} = \sigma(\widehat{\delta})$ where at the right hand side we use the action of the group $\mathfrak{S}_m$ on $\mathfrak{h}^*$. Let $(\,,\,)$ be the standard bilinear form on $\mathfrak{h}^*$ so that the basis of weights E^*_{aa} with $a = 1, \ldots, m$ is orthonormal. The above remarks on the action of the Zhelobenko operators on $\bar{\mathrm{I}}_\delta$ can now be rewritten as

$$\check{\xi}_a(\bar{\mathrm{I}}_\delta) \subset \bar{\mathrm{I}}_{\sigma_a(\delta)} \quad \text{if } (\widehat{\delta}, \eta_a) \geqslant 0. \tag{3.18}$$

We will prove the proposition by induction on the length of the reduced decomposition of σ. Recall that the length $\ell(\sigma)$ of a reduced decomposition of σ is the total number of the factors $\sigma_1, \ldots, \sigma_{m-1}$ in that decomposition. This number is independent of the choice of decomposition and is equal to the number of elements in the set

$$\Delta_\sigma = \left\{\eta \in \Delta^+ \mid \sigma(\eta) \notin \Delta^+\right\}$$

where Δ^+ denotes the set of all positive roots of the Lie algebra $\mathfrak{gl}_m$.

If σ is the identity element of $\mathfrak{S}_m$ then the statement of the proposition is trivial. Suppose that for some $\sigma \in \mathfrak{S}_m$

$$\check{\xi}_\sigma\left(\bar{\mathrm{I}}_{\delta^+}\right) \subset \bar{\mathrm{I}}_{\sigma(\delta^+)}.$$

Take σ_a such that

$$\ell(\sigma_a\sigma) = \ell(\sigma) + 1. \tag{3.19}$$

Now it is only left to prove that

$$\check{\xi}_a\left(\bar{\mathrm{I}}_{\sigma(\delta^+)}\right) \subset \bar{\mathrm{I}}_{\sigma_a\sigma(\delta)}.$$

Due to (3.18), the desired property will take place if

$$(\widehat{\sigma(\delta^+)}, \eta_a) = (\sigma(\widehat{\delta^+}), \eta_a) \geqslant 0.$$

Note that η_a is a simple root of the algebra $\mathfrak{gl}_m$ and hence, $\sigma_a(\eta) \in \Delta^+$ for any $\eta \in \Delta^+$ such that $\eta \neq \eta_a$. Since $\ell(\sigma)$ and $\ell(\sigma_a\sigma)$ are the numbers of elements in Δ_σ and $\Delta_{\sigma_a\sigma}$ respectively, condition (3.19) implies that $\eta_a \in \sigma(\Delta^+)$. Therefore, $\eta_a = \sigma(E^*_{bb} - E^*_{cc})$ for some $1 \leqslant b < c \leqslant m$. Thus

$$\left(\sigma(\widehat{\delta^+}), \eta_a\right) = \left(\sigma(\widehat{\delta^+}), \sigma(E^*_{bb} - E^*_{cc})\right) = \left(\sum_{a=1}^{m} \delta'_a E^*_{aa}, E^*_{bb} - E^*_{cc}\right) \geqslant 0. \quad \square$$

Corollary 3.2. *For any $\sigma \in \mathfrak{S}_m$ the operator $\check{\xi}_\sigma$ on $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}$ maps*

$$\bar{\mathrm{J}} \setminus (\bar{\mathrm{J}}' + \bar{\mathrm{I}}_{\delta^+} + \bar{\mathrm{J}}) \to \bar{\mathrm{J}} \setminus (\bar{\mathrm{J}}' + \bar{\mathrm{I}}_{\sigma(\delta^+)} + \bar{\mathrm{J}}).$$

Proof. For the proof of the Corollary 3.2 see [23, Corollary 5.2]. $\square$

For generic μ let $\mathrm{I}_{\mu,\delta}$ be the left ideal of the algebra A generated by $\mathrm{I}_\delta + \mathrm{J}'$ and by the elements

$$E_{aa} - \zeta_n(E_{aa}) - \mu_a \quad \text{where } a = 1, \dots, m.$$

Recall that under the isomorphism of the algebra A with $\mathrm{U}(\mathfrak{gl}_m) \otimes \mathcal{HD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ element $X - \zeta_n(X) \in \mathrm{A}$ maps to the element (3.6) for every $X \in \mathfrak{gl}_m$. Let $\bar{\mathrm{I}}_{\mu,\delta}$ denote the subspace $\overline{\mathrm{U}(\mathfrak{h})}\, \mathrm{I}_{\mu,\delta}$ of $\bar{\mathrm{A}}$. Note that $\bar{\mathrm{I}}_{\mu,\delta}$ is a left ideal of the algebra $\bar{\mathrm{A}}$.

Theorem 3.3. *For any element $\sigma \in \mathfrak{S}_m$ the operator $\check{\xi}_\sigma$ on $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}$ maps*

$$\bar{\mathrm{J}} \setminus (\bar{\mathrm{I}}_{\mu,\delta^+} + \bar{\mathrm{J}}) \to \bar{\mathrm{J}} \setminus (\bar{\mathrm{I}}_{\sigma\circ\mu,\sigma(\delta^+)} + \bar{\mathrm{J}}).$$

Proof. Let κ be a weight of $\mathfrak{gl}_m$ with the sequence of labels $(\kappa_1, \dots, \kappa_m)$. Suppose that the weight κ satisfies the conditions (3.13) instead of μ. Let $\tilde{\mathrm{I}}_{\kappa,\delta}$ denote the left ideal of $\bar{\mathrm{A}}$ generated by $\mathrm{I}_\delta + \mathrm{J}'$ and by the elements

$$E_{aa} - \kappa_a \quad \text{where } a = 1, \dots, m.$$

Relation (3.10) and Corollary 3.2 imply that the operator $\check{\xi}_\sigma$ on $\bar{\mathrm{J}} \setminus \mathrm{A}$ maps

$$\bar{\mathrm{J}} \setminus (\tilde{\mathrm{I}}_{\kappa,\delta^+} + \bar{\mathrm{J}}) \to \bar{\mathrm{J}} \setminus (\tilde{\mathrm{I}}_{\sigma\circ\kappa,\sigma(\delta^+)} + \bar{\mathrm{J}}).$$

Now choose

$$\kappa = \mu - \theta\frac{n}{2}\delta' \tag{3.20}$$

where the sequence δ' is regarded as a weight of $\mathfrak{gl}_m$ by identifying the weights with their sequence of labels. Then the conditions on κ stated in the beginning of this proof are satisfied. For every $\sigma \in \mathfrak{S}_m$ we shall prove the equality of left ideals of $\bar{\mathrm{A}}$,

$$\tilde{\mathrm{I}}_{\sigma\circ\kappa,\sigma(\delta^+)} = \bar{\mathrm{I}}_{\sigma\circ\mu,\sigma(\delta^+)}. \tag{3.21}$$

Theorem 3.3 will then follow. Denote $\delta = \sigma \cdot \delta'$. Then by our choice of κ, we have

$$\sigma \circ \kappa = \sigma \circ \mu - \theta\frac{n}{2}\delta. \tag{3.22}$$

Let the index a run through $1, \dots, m-1$, then

$$\zeta_n(E_{aa}) + \theta\frac{n}{2} = \theta\sum_{k=1}^{n} p_{ak}q_{ak} \in \mathrm{I}_{\sigma(\delta^+)} \quad \text{if } \delta_a = 1,$$

$$\zeta_n(E_{aa}) - \theta\frac{n}{2} = \sum_{k=1}^{n} q_{ak}p_{ak} \in \mathrm{I}_{\sigma(\delta^+)} \quad \text{if } \delta_a = -1.$$

Hence, the relation (3.22) implies the equality (3.21). $\square$

Consider the quotient vector space $\mathrm{A}/\mathrm{I}_{\mu,\delta}$ for any sequence δ. The algebra $\mathrm{U}(\mathfrak{gl}_m)$ acts on this quotient via left multiplication, being regarded as a subalgebra of A. The algebra $\mathrm{Y}(\mathfrak{gl}_n)$ also acts on this quotient via left multiplication, using the homomorphism $\alpha_m : \mathrm{Y}(\mathfrak{gl}_n) \to \mathrm{A}$. Recall that in Section 1 the target algebra of the homomorphism α_m was defined as the tensor product $\mathrm{U}(\mathfrak{gl}_m) \otimes \mathcal{HD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ isomorphic to the algebra A by means of the cross relations (3.1). Part (ii) of the Proposition 1.2 implies that the image of α_m in A commutes with the subalgebra $\mathrm{U}(\mathfrak{gl}_m) \subset \mathrm{A}$. Thus, the vector space $\mathrm{A}/\mathrm{I}_{\mu,\delta}$ becomes a bimodule over $\mathfrak{gl}_m$ and $\mathrm{Y}(\mathfrak{gl}_n)$.

Consider the bimodule $\mathcal{E}_\delta(M_\mu)$ over $\mathfrak{gl}_m$ and $\mathrm{Y}(\mathfrak{gl}_n)$ defined in the beginning of this section. This bimodule is isomorphic to $\mathrm{A}/\mathrm{I}_{\mu,\delta}$. Indeed, let Z run through $\mathcal{H}(\mathbb{C}^m \otimes \mathbb{C}^n)$. Then a bijective linear map

$$\mathcal{E}_\delta(M_\mu) \to \mathrm{A}/\mathrm{I}_{\mu,\delta},$$

intertwining the actions of $\mathfrak{gl}_m$ and $\mathrm{Y}(\mathfrak{gl}_n)$, can be determined by mapping the element

$$1_\mu \otimes Z \in M_\mu \otimes \mathcal{H}\left(\mathbb{C}^m \otimes \mathbb{C}^n\right)$$

to the image of

$$\varpi^{-1}(Z) \in \mathcal{HD}\left(\mathbb{C}^m \otimes \mathbb{C}^n\right) \subset \mathrm{A}$$

in the quotient $\mathrm{A}/\mathrm{I}_{\mu,\delta}$. The intertwining property here follows from the definitions of $\mathcal{E}_\delta(M_\mu)$ and $\mathrm{I}_{\mu,\delta}$. The same mapping determines a bijective linear map

$$\mathcal{E}_\delta(M_\mu) \to \bar{\mathrm{A}}/\bar{\mathrm{I}}_{\mu,\delta}. \tag{3.23}$$

In particular, the space $\mathcal{E}_\delta(M_\mu)_{\mathfrak{n}}$ of $\mathfrak{n}$-coinvariants of $\mathcal{E}_\delta(M_\mu)$ is isomorphic to the quotient $\bar{\mathrm{J}} \backslash \bar{\mathrm{A}}/\bar{\mathrm{I}}_{\mu,\delta}$ as a bimodule over the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{gl}_m$ and over $\mathrm{Y}(\mathfrak{gl}_n)$. Theorem 3.3 implies that the operator $\check{\xi}_\sigma$ on $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}$ determines a linear operator

$$\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}/\bar{\mathrm{I}}_{\mu,\delta^+} \to \bar{\mathrm{J}} \setminus \bar{\mathrm{A}}/\bar{\mathrm{I}}_{\sigma\circ\mu,\sigma(\delta^+)}. \tag{3.24}$$

The definition (3.5) and the fact that the image of $\mathrm{Y}(\mathfrak{gl}_n)$ in A under α_m commutes with the subalgebra $\mathrm{U}(\mathfrak{gl}_m) \subset \mathrm{A}$ imply that the latter operator intertwines the actions of $\mathrm{Y}(\mathfrak{gl}_n)$ on the source and the target vector spaces. We also use the invariance of the image of $\mathrm{Y}(\mathfrak{gl}_n)$ in A under the action of $\mathfrak{S}_m$. Recall that $\mathcal{E}_{p,q}(V) = \mathcal{E}_{\delta^+}(V)$. Hence, by using the equivalences (3.23) for the sequences $\delta = \delta^+$ and $\delta = \sigma(\delta^+)$, the operator (3.24) becomes the desired $\mathrm{Y}(\mathfrak{gl}_n)$-intertwining operator (3.14).

As usual, for any $\mathfrak{gl}_m$-module V and any element $\lambda \in \mathfrak{h}^*$ let $V^\lambda \subset V$ be the subspace of vectors *of weight* λ so that any $X \in \mathfrak{h}$ acts on V^λ via multiplication by $\lambda(X) \in \mathbb{C}$. It now follows from the property (3.12) of $\check{\xi}_\sigma$ that the restriction of our operator (3.14) to the subspace of weight λ is an $\mathrm{Y}(\mathfrak{gl}_n)$-intertwining operator

$$\mathcal{E}_{p,q}(M_\mu)^\lambda_{\mathfrak{n}} \to \mathcal{E}_\delta(M_{\sigma\circ\mu})^{\sigma\circ\lambda}_{\mathfrak{n}} \quad \text{where } \delta = \sigma(\delta^+). \tag{3.25}$$

Consider $\mathrm{Y}(\mathfrak{gl}_n)$-modules Φ_z and $\widetilde{\Phi}_z$ described at the end of the Section 2. The underlying vector space of each of these modules coincides with the algebra $\mathcal{H}(\mathbb{C}^n)$. Note that the action of $\mathrm{Y}(\mathfrak{gl}_n)$ on each of these modules preserves the polynomial degree. Now for any $N = 1, 2, \ldots$ denote respectively by Ψ_z^N and Ψ_z^{-N} the submodules in Φ_z and $\widetilde{\Phi}_z$ consisting of the polynomial functions of degree N. It will also be convenient to denote by Ψ_z^0 the vector space $\mathbb{C}$ with the trivial action of $\mathrm{Y}(\mathfrak{gl}_n)$.

The element $E_{11} \in \mathfrak{gl}_1$ acts on $\mathcal{E}_{1,0}(M_t)$ and $\mathcal{E}_{0,1}(M_t)$ by

$$t - \theta\frac{n}{2} - \deg \quad \text{and} \quad t + \theta\frac{n}{2} + \deg$$

respectively where deg is the degree operator. Therefore, Corollary 2.2 yields the isomorphism between the source $\mathrm{Y}(\mathfrak{gl}_n)$-module in (3.25) and the tensor product

$$\Psi^{-\nu_1}_{\mu_1+\rho_1} \otimes \cdots \otimes \Psi^{-\nu_p}_{\mu_p+\rho_p} \otimes \Psi^{\nu_{p+1}}_{\mu_{p+1}+\rho_{p+1}} \otimes \cdots \otimes \Psi^{\nu_m}_{\mu_m+\rho_m} \tag{3.26}$$

where

$$\begin{aligned} \nu_a &= -\lambda_a + \mu_a - \theta\frac{n}{2} \quad \text{for } a = 1, \ldots, p, \\ \nu_a &= \lambda_a - \mu_a - \theta\frac{n}{2} \quad \text{for } a = p+1, \ldots, m. \end{aligned} \tag{3.27}$$

Let us now consider the target $\mathrm{Y}(\mathfrak{gl}_n)$-module in (3.25). For each $a = 1, \ldots, m$ denote

$$\widetilde{\mu}_a = \mu_{\sigma^{-1}(a)}, \qquad \widetilde{\nu}_a = \nu_{\sigma^{-1}(a)}, \qquad \widetilde{\rho}_a = \rho_{\sigma^{-1}(a)}.$$

The above description of the source $\mathrm{Y}(\mathfrak{gl}_n)$-module in (3.25) can now be generalized to the target $\mathrm{Y}(\mathfrak{gl}_n)$-module which depends on an arbitrary element $\sigma \in \mathfrak{S}_m$.

Proposition 3.4. *For $\delta = \sigma(\delta^+)$ the* $\mathrm{Y}(\mathfrak{gl}_n)$*-module* $\mathcal{E}_\delta(M_{\sigma\circ\mu})_{\mathfrak{n}}^{\sigma\circ\lambda}$ *is isomorphic to the tensor product*

$$\Psi^{-\delta_1\widetilde{\nu}_1}_{\widetilde{\mu}_1+\widetilde{\rho}_1} \otimes \cdots \otimes \Psi^{-\delta_p\widetilde{\nu}_p}_{\widetilde{\mu}_p+\widetilde{\rho}_p} \otimes \Psi^{\delta_{p+1}\widetilde{\nu}_{p+1}}_{\widetilde{\mu}_{p+1}+\widetilde{\rho}_{p+1}} \otimes \cdots \otimes \Psi^{\delta_m\widetilde{\nu}_m}_{\widetilde{\mu}_m+\widetilde{\rho}_m}. \tag{3.28}$$

Proof. Consider the bimodule $\mathcal{E}_{p,q}(M_{\sigma\circ\mu})_{\mathfrak{n}}^{\sigma\circ\lambda}$ over $\mathfrak{h}$ and $\mathrm{Y}(\mathfrak{gl}_n)$. By Corollary 2.2 and the arguments just above this proposition, this bimodule is isomorphic to the tensor product

$$\Psi^{\widetilde{\nu}_1}_{\widetilde{\mu}_1+\widetilde{\rho}_1} \otimes \cdots \otimes \Psi^{\widetilde{\nu}_p}_{\widetilde{\mu}_p+\widetilde{\rho}_p} \otimes \Psi^{\widetilde{\nu}_{p+1}}_{\widetilde{\mu}_{p+1}+\widetilde{\rho}_{p+1}} \otimes \cdots \otimes \Psi^{\widetilde{\nu}_m}_{\widetilde{\mu}_m+\widetilde{\rho}_m} \tag{3.29}$$

as a $\mathrm{Y}(\mathfrak{gl}_m)$-module. The bimodule $\mathcal{E}_\delta(M_{\sigma\circ\mu})_{\mathfrak{n}}$ can be obtained by pushing forward the actions of $\mathfrak{h}$ and $\mathrm{Y}(\mathfrak{gl}_n)$ on $\mathcal{E}_{p,q}(M_{\sigma\circ\mu})_{\mathfrak{n}}$ through the composition of automorphisms

$$x_a \to -\theta\partial_a \quad \text{and} \quad \partial_a \to x_a \tag{3.30}$$

for every tensor factor with number a such that $\delta_a = -1$. These automorphisms exchange $\mathrm{Y}(\mathfrak{gl}_n)$-modules Φ_{z_a} and $\widetilde{\Phi}_{z_a}$ but leave invariant the degree of polynomials. Therefore, they interchange $\Psi^N_{z_a}$ and $\Psi^{-N}_{z_a}$ which implies the resulting $\mathrm{Y}(\mathfrak{gl}_n)$-module to be as in (3.28). $\square$

Thus, for any non-negative integers $\nu_1, \ldots, \nu_m$ we have shown that the Zhelobenko operator $\check{\xi}_\sigma$ on $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}$ defines the intertwining operator between the $\mathrm{Y}(\mathfrak{gl}_n)$-modules

$$\Psi^{-\nu_1}_{\mu_1+\rho_1} \otimes \cdots \otimes \Psi^{-\nu_p}_{\mu_p+\rho_p} \otimes \Psi^{\nu_{p+1}}_{\mu_{p+1}+\rho_{p+1}} \otimes \cdots \otimes \Psi^{\nu_m}_{\mu_m+\rho_m}$$

and

$$\Psi^{-\delta_1\widetilde{\nu}_1}_{\widetilde{\mu}_1+\widetilde{\rho}_1} \otimes \cdots \otimes \Psi^{-\delta_p\widetilde{\nu}_p}_{\widetilde{\mu}_p+\widetilde{\rho}_p} \otimes \Psi^{\delta_{p+1}\widetilde{\nu}_{p+1}}_{\widetilde{\mu}_{p+1}+\widetilde{\rho}_{p+1}} \otimes \cdots \otimes \Psi^{\delta_m\widetilde{\nu}_m}_{\widetilde{\mu}_m+\widetilde{\rho}_m}.$$

Moreover, the operator $\check{\xi}_\sigma$ permutes tensor factors of the module (3.26), therefore modules $\mathcal{E}_{p,q}(M_\mu)_{\mathfrak{n}}^\lambda$ and $\mathcal{E}_\delta(M_{\sigma\circ\mu})_{\mathfrak{n}}^{\sigma\circ\lambda}$, written in the form (2.5), contain similar tensor factors Ω_{z_a} or Ω'_{z_a}. Now recall that modules Ω_{z_a} and Ω'_{z_a} are central and that the Zhelobenko operator $\check{\xi}_\sigma$ acts on them as identity operator. These two observations allow us to exclude Ω_{z_a} and Ω'_{z_a} from both source and target modules of the mapping (3.25) and to define an intertwining operator ξ'_σ between tensor products of modules Φ_z and Φ'_z. For any integer N denote respectively by Φ^N_z and Φ^{-N}_z the submodules of Φ_z and Φ'_z consisting of polynomial functions of degree N. Note that Φ^N_z coincides with Ψ^N_z for positive N but differs for negative N. Hence, the following theorem holds.

Theorem 3.5. *Given a generic weight μ and non-negative integers $\nu_1, \ldots, \nu_m$, the map ξ'_σ intertwines rational* $\mathrm{Y}(\mathfrak{gl}_n)$*-modules*

$$\Phi^{-\nu_1}_{\mu_1+\rho_1} \otimes \cdots \otimes \Phi^{-\nu_p}_{\mu_p+\rho_p} \otimes \Phi^{\nu_{p+1}}_{\mu_{p+1}+\rho_{p+1}} \otimes \cdots \otimes \Phi^{\nu_m}_{\mu_m+\rho_m} \tag{3.31}$$

and

$$\Phi^{-\delta_1\widetilde{\nu}_1}_{\widetilde{\mu}_1+\widetilde{\rho}_1} \otimes \cdots \otimes \Phi^{-\delta_p\widetilde{\nu}_p}_{\widetilde{\mu}_p+\widetilde{\rho}_p} \otimes \Phi^{\delta_{p+1}\widetilde{\nu}_{p+1}}_{\widetilde{\mu}_{p+1}+\widetilde{\rho}_{p+1}} \otimes \cdots \otimes \Phi^{\delta_m\widetilde{\nu}_m}_{\widetilde{\mu}_m+\widetilde{\rho}_m}.$$

4. Highest weight vectors

4.1. Symmetric case

In this subsection we consider only the case of commuting variables, hence from now on and till the end of the subsection we assume $\theta = 1$. Proposition 4.4 determines the image of the highest weight vector v^λ_μ of the $\mathrm{Y}(\mathfrak{gl}_n)$-module $\mathcal{E}_{p,q}(M_\mu)^\lambda_{\mathfrak{n}}$ under the action of the operator $\check{\xi}_\sigma$. The proof of the Proposition 4.4 is based on the following three lemmas. Proofs of the first two of them are similar to the proof of Lemma 5.6 in [23]. Proof of the last one is similar to the proof of Lemma 5.7 in [23]. Let $s, t = 0, 1, 2, \ldots$ and $k, \ell = 1, \ldots, n$.

Lemma 4.1. *For any $a = 1, \ldots, m-1$ the operator $\check{\xi}_a$ on $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}$ maps the image in $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}$ of $p^s_{ak}p^t_{a+1\,k} \in \bar{\mathrm{A}}$ to the image in $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}$ of the product*

$$p^t_{ak}p^s_{a+1\,k} \cdot \prod_{r=1}^{s} \frac{H_a + r + 1}{H_a + r - t}$$

plus the images in $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}$ of elements of the left ideal in $\bar{\mathrm{A}}$ generated by $\bar{\mathrm{J}}'$ and (3.15).

Lemma 4.2. *For any $a = 1, \ldots, m-1$ the operator $\check{\xi}_a$ on $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}$ maps the image in $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}$ of $q^s_{ak}q^t_{a+1\,k} \in \bar{\mathrm{A}}$ to the image in $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}$ of the product*

$$q^t_{ak} q^s_{a+1\,k} \cdot \prod_{r=1}^{t} \frac{H_a + r + 1}{H_a + r - s}$$

plus the images in $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}$ *of elements of the left ideal in* $\bar{\mathrm{A}}$ *generated by* $\bar{\mathrm{J}}'$ *and* (3.16).

Lemma 4.3. *For any* $a = 1, \ldots, m-1$ *the operator* $\check{\xi}_a$ *on* $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}$ *maps the image in* $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}$ *of* $p^s_{ak} q^t_{a+1\,\ell} \in \bar{\mathrm{A}}$ *to the image in* $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}$ *of the product*

$$q^t_{a\ell} p^s_{a+1\,k} \cdot \begin{cases} \prod_{r=1}^{s} \dfrac{H_a + r}{H_a + r + t} & \text{if } n = 1 \text{ and } k = \ell = 1, \\ 1 & \text{if } n > 1 \text{ and } k \neq \ell \end{cases}$$

plus the images in $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}$ *of elements of the left ideal in* $\bar{\mathrm{A}}$ *generated by* $\bar{\mathrm{J}}'$ *and* (3.17).

We keep assuming that the weight μ satisfies the condition (3.13). Let $(\mu^*_1, \ldots, \mu^*_m)$ be the sequence of labels of weight $\mu + \rho - \frac{n}{2}\delta'$. Suppose that for all $a = 1, \ldots, m$ the number ν_a defined by (3.27) is a non-negative integer. For each positive root $\eta = E^*_{bb} - E^*_{cc} \in \Delta^+$ with $1 \leqslant b < c \leqslant m$ define a number $z_\eta \in \mathbb{C}$ by

$$z_\eta = \begin{cases} \prod_{r=1}^{\nu_b} \dfrac{\mu^*_b - \mu^*_c - r}{\lambda^*_b - \lambda^*_c + r} & \text{if } b, c = 1, \ldots, p, \\ \prod_{r=1}^{\nu_c} \dfrac{\mu^*_b - \mu^*_c - r}{\lambda^*_b - \lambda^*_c + r} & \text{if } b, c = 1, \ldots, p+1, \\ \prod_{r=1}^{\upsilon} \dfrac{\mu^*_b - \mu^*_c - r + 1}{\lambda^*_b - \lambda^*_c + r} & \text{if } \begin{matrix} b = 1, \ldots, p, \\ c = p+1, \ldots, m, \end{matrix} \text{ and } n = 1, \\ 1, & \text{if } \begin{matrix} b = 1, \ldots, p, \\ c = p+1, \ldots, m, \end{matrix} \text{ and } n > 1. \end{cases}$$

Here $\upsilon = \min(\nu_b, \nu_c)$. Let v^λ_μ denote the image of the vector

$$\prod_{a=1}^{p} p^{\nu_a}_{an} \cdot \prod_{a=p+1}^{m} q^{\nu_a}_{a1} \in \bar{\mathrm{A}}$$

in the quotient space $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}} / \bar{\mathrm{I}}_{\mu,\delta^+}$.

Proposition 4.4. (i) *The vector* v^λ_μ *does not belong to the zero coset in* $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}} / \bar{\mathrm{I}}_{\mu,\delta^+}$.

(ii) *The vector* v^λ_μ *is of weight* λ *under the action of* $\mathfrak{h}$ *on* $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}} / \bar{\mathrm{I}}_{\mu,\delta^+}$ *and is of highest weight with respect to the action of* $\mathrm{Y}(\mathfrak{gl}_n)$ *on the same quotient space.*

(iii) *For any* $\sigma \in \mathfrak{S}_m$ *the operator* (3.24) *determined by* $\check{\xi}_\sigma$ *maps the vector* v^λ_μ *to the image in* $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}} / \bar{\mathrm{I}}_{\sigma\circ\mu,\sigma(\delta^+)}$ *of* $\sigma(v^\lambda_\mu) \in \bar{\mathrm{A}}$ *multiplied by* $\prod_{\eta \in \Delta_\sigma} z_\eta$.

Proof. Part (i) follows directly from the definition of the ideal $\bar{\mathrm{I}}_{\mu,\delta^+}$. Let us now prove Part (ii). Subalgebra $\mathfrak{h}$ acts on the quotient space $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}} / \bar{\mathrm{I}}_{\mu,\delta^+}$ via left multiplication on $\bar{\mathrm{A}}$. Let symbol $\equiv$ denote the equalities in $\bar{\mathrm{A}}$ modulo the left ideal $\bar{\mathrm{I}}_{\mu,\delta^+}$. Then we have

$$\begin{aligned} E_{aa} v^\lambda_\mu &= v^\lambda_\mu E_{aa} + \left[\zeta_n(E_{aa}), v^\lambda_\mu\right] = v^\lambda_\mu (E_{aa} \mp \nu_a) \equiv v^\lambda_\mu \left(\zeta_n(E_{aa}) + \mu_a \mp \nu_a\right) \\ &\equiv v^\lambda_\mu \left(\mp \frac{n}{2} + \mu_a \mp \nu_a\right) = \lambda_a v^\lambda_\mu \end{aligned}$$

where one should choose the upper sign for $a = 1, \ldots, p$ and the lower sign for $a = p+1, \ldots, m$. Next, $\mathrm{Y}(\mathfrak{gl}_n)$-modules $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}} / \bar{\mathrm{I}}_{\mu,\delta^+}$ and $\mathcal{E}_{p,q}(M_\mu)_\mathfrak{n}$ are isomorphic. Recall that $T_{ij}(u)$ acts on $\mathcal{E}_{p,q}(M_\mu)_\mathfrak{n}$ with the help of the comultiplication Δ. Now Corollary 2.2 and formulas (2.3) imply that $T_{ij}(u) v^\lambda_\mu = 0$ for all $1 \leqslant i < j \leqslant n$. Moreover, one can check that

$$T_{ii}(u) v^\lambda_\mu = v^\lambda_\mu \cdot \begin{cases} \prod_{a=p+1}^{m} \dfrac{u + \mu_a + \rho_a + \nu_a}{u + \mu_a + \rho_a}, & i = 1 \\ \prod_{a=1}^{p} \dfrac{u + \mu_a + \rho_a - \nu_a - 1}{u + \mu_a + \rho_a}, & i = n \\ 1, & \text{otherwise.} \end{cases}$$

Therefore, v^λ_μ is a highest weight vector with respect to the action of $\mathrm{Y}(\mathfrak{gl}_n)$ on the quotient space $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}} / \bar{\mathrm{I}}_{\mu,\delta^+}$.

We will prove Part (iii) by induction on the length of a reduced decomposition of σ. If σ is the identity element of $\mathfrak{S}_m$, then the required statement is tautological. Now suppose that for some $\sigma \in \mathfrak{S}_m$ the statement of (iii) is true. Take any simple reflection $\sigma_a \in \mathfrak{S}_m$ with $1 \leqslant a \leqslant m-1$ such that $\sigma_a\sigma$ has a longer reduced decomposition in terms of $\sigma_1, \ldots, \sigma_m$ than σ.

Consider the simple root η_a, corresponding to the reflection σ_a. Let $\eta = \sigma^{-1}(\eta_a)$ then

$$\sigma_a\sigma(\eta) = \sigma_a(\eta_a) = -\eta_a \notin \Delta^+.$$

Thus, $\Delta_{\sigma_a\sigma} = \Delta_\sigma \sqcup \{\eta\}$. Let $\kappa \in \mathfrak{h}^*$ be the weight with the labels determined by (3.20). Using the proof of Theorem 3.3, we get the equality of two left ideals of the algebra A,

$$\bar{\mathrm{I}}_{(\sigma_a\sigma)\circ\mu,(\sigma_a\sigma)(\delta^+)} = \tilde{\mathrm{I}}_{(\sigma_a\sigma)\circ\kappa,(\sigma_a\sigma)(\delta^+)}.$$

Modulo the second of these two ideals the element H_a equals

$$\begin{aligned}((\sigma_a\sigma)\circ\kappa)(H_a) &= (\sigma_a\sigma(\kappa+\rho)-\rho)(H_a) = (\kappa+\rho)(\sigma^{-1}\sigma_a(H_a)) - \rho(H_a)\\ &= -(\kappa+\rho)(\sigma^{-1}(H_a)) - 1 = -(\kappa+\rho)(H_\eta) - 1 = -\mu_b^* + \mu_c^* - 1.\end{aligned}$$

Here $H_\eta = \sigma^{-1}(H_a)$ is the coroot corresponding to the root η, and we use the standard bilinear form on $\mathfrak{h}^*$.

Let us use the statement of (iii) as the induction assumption. Denote $\delta = \sigma(\delta^+)$. Consider three cases.

I. Let $b, c = 1, \ldots, p$, then $\delta_a = \delta'_a$ and $\delta_{a+1} = \delta'_{a+1}$. Hence,

$$\sigma\left(v_\mu^\lambda\right) = p_{an}^{\nu_b} p_{a+1\,n}^{\nu_c} Y$$

where Y is an element of the subalgebra of $\mathcal{PD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ generated by all x_{dk} and ∂_{dk} with $d \neq a, a+1$. Now we apply the Lemma 4.1 with $s = \nu_b$ and $t = \nu_c$. After substitution $-\mu_b^* + \mu_c^* - 1$ for H_a, the fraction in the lemma turns into

$$\prod_{r=1}^{\nu_b} \frac{-\mu_b^* + \mu_c^* + r}{-\mu_b^* + \mu_c^* + \nu_b - \nu_c - r} = \prod_{r=1}^{\nu_b} \frac{\mu_b^* - \mu_c^* - r}{\lambda_b^* - \lambda_c^* + r}.$$

II. Let $b, c = p+1, \ldots, m$, then $\delta_a = -\delta'_a$ and $\delta_{a+1} = -\delta'_{a+1}$. Hence,

$$\sigma\left(v_\mu^\lambda\right) = q_{a1}^{\nu_b} q_{a+1\,1}^{\nu_c} Y$$

where Y is an element of the subalgebra of $\mathcal{PD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ generated by all x_{dk} and ∂_{dk} with $d \neq a, a+1$. Now we apply the Lemma 4.2 with $s = \nu_b$ and $t = \nu_c$. After substitution $-\mu_b^* + \mu_c^* - 1$ for H_a, the fraction in the lemma turns into

$$\prod_{r=1}^{\nu_c} \frac{-\mu_b^* + \mu_c^* + r}{-\mu_b^* + \mu_c^* + \nu_c - \nu_b - r} = \prod_{r=1}^{\nu_c} \frac{\mu_b^* - \mu_c^* - r}{\lambda_b^* - \lambda_c^* + r}.$$

III. Let $b = 1, \ldots, p$, $c = p+1, \ldots, m$, then $\delta_a = \delta'_a$ and $\delta_{a+1} = -\delta'_{a+1}$. Hence,

$$\sigma\left(v_\mu^\lambda\right) = p_{an}^{\nu_b} q_{a+1\,1}^{\nu_c} Y$$

where Y is an element of the subalgebra of $\mathcal{PD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ generated by all x_{dk} and ∂_{dk} with $d \neq a, a+1$. Now we apply the Lemma 4.3 with $s = \nu_b$ and $t = \nu_c$. When $n = 1$, after substitution $-\mu_b^* + \mu_c^* - 1$ for H_a, the fraction in the lemma turns into

$$\prod_{r=1}^{\upsilon} \frac{-\mu_b^* + \mu_c^* + r - 1}{-\mu_b^* + \mu_c^* + \nu_b + \nu_c - r} = \prod_{r=1}^{\upsilon} \frac{\mu_b^* - \mu_c^* - r + 1}{\lambda_b^* - \lambda_c^* + r}.$$

Thus, in the three cases considered above the operator

$$\check{\xi}_{\sigma_a\sigma} : \bar{\mathrm{J}} \setminus \bar{\mathrm{A}}/\bar{\mathrm{I}}_{\mu,\delta^+} \to \bar{\mathrm{I}}_{(\sigma_a\sigma)\circ\mu,(\sigma_a\sigma)(\delta^+)}$$

maps the vector v_μ^λ to the image of

$$\sigma_a\sigma(v_\mu^\lambda) \cdot \prod_{\eta\in\Delta_{\sigma_a\sigma}} z_\eta \in \bar{\mathrm{A}}$$

in the vector space $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}/\bar{\mathrm{I}}_{(\sigma_a\sigma)\circ\mu,(\sigma_a\sigma)(\delta^+)}$. This observation makes the induction step. $\square$

4.2. Skew-symmetric case

In this subsection we consider only the case of anticommuting variables, hence from now on and till the end of this subsection we assume $\theta = -1$. Proposition 4.8 determines the image of the highest weight vector v_μ^λ of the $\mathrm{Y}(\mathfrak{gl}_n)$-module $\mathcal{E}_{p,q}(M_\mu)_{\mathrm{n}}^\lambda$ under the action of the operator $\check{\xi}_\sigma$. The proof of the Proposition 4.8 is based on the following three lemmas. Proofs of the first two of them are similar to the proof of Lemma 5.6 in [24]. Proof of the last one is similar to the proof of Lemma 5.7 in [24]. Let $s, t = 1, \ldots, n$, define

$$f_{as} = p_{a\,n-s+1} \cdots p_{an} \quad \text{and} \quad g_{as} = q_{a1} \cdots q_{as}.$$

Lemma 4.5. *For any $a = 1, \ldots, m-1$ the operator $\check{\xi}_a$ on $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}$ maps the image in $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}$ of $f_{as} f_{a+1\,t} \in \bar{\mathrm{A}}$ to the image in $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}$ of the product*

$$f_{a+1\,s} f_{at} \cdot \begin{cases} \dfrac{H_a + s - t + 1}{H_a + 1} & \text{if } s > t, \\ 1 & \text{if } s \leqslant t \end{cases}$$

plus the images in $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}$ of elements of the left ideal in $\bar{\mathrm{A}}$ generated by $\bar{\mathrm{J}}'$ and (3.15).

Lemma 4.6. *For any $a = 1, \ldots, m-1$ the operator $\check{\xi}_a$ on $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}$ maps the image in $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}$ of $g_{as} g_{a+1\,t} \in \bar{\mathrm{A}}$ to the image in $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}$ of the product*

$$g_{a+1\,s} g_{at} \cdot \begin{cases} \dfrac{H_a + t - s + 1}{H_a + 1} & \text{if } s < t, \\ 1 & \text{if } s \geqslant t \end{cases}$$

plus the images in $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}$ of elements of the left ideal in $\bar{\mathrm{A}}$ generated by $\bar{\mathrm{J}}'$ and (3.16).

Lemma 4.7. *For any $a = 1, \ldots, m-1$ the operator $\check{\xi}_a$ on $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}$ maps the image in $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}$ of $f_{as} g_{a+1\,t} \in \bar{\mathrm{A}}$ to the image in $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}$ of the product*

$$f_{a+1\,s} g_{at} \cdot \begin{cases} \dfrac{H_a + s + t - n + 1}{H_a + 1} & \text{if } s + t > n, \\ 1 & \text{if } s + t \leqslant n \end{cases}$$

plus the images in $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}$ of elements of the left ideal in $\bar{\mathrm{A}}$ generated by $\bar{\mathrm{J}}'$ and (3.17).

We keep assuming that the weight μ satisfies the condition (3.13). We also assume that $\nu_1, \ldots, \nu_m \in \{0, 1, \ldots, n\}$. Let $(\mu_1^*, \ldots, \mu_m^*)$ be the sequence of labels of weight $\mu + \rho + \frac{n}{2}\delta'$. For each positive root $\eta = E_{bb}^* - E_{cc}^* \in \Delta^+$ with $1 \leqslant b < c \leqslant m$ define a number $z_\eta \in \mathbb{C}$ by

$$z_\eta = \begin{cases} \dfrac{\lambda_b^* - \lambda_c^*}{\mu_b^* - \mu_c^*} & \text{if } b, c = 1, \ldots, p \text{ and } \nu_b > \nu_c, \\ \dfrac{\lambda_b^* - \lambda_c^*}{\mu_b^* - \mu_c^*} & \text{if } b, c = 1, \ldots, p+1 \text{ and } \nu_b < \nu_c, \\ \dfrac{\lambda_b^* - \lambda_c^* + n}{\mu_b^* - \mu_c^*} & \text{if } \begin{matrix} b = 1, \ldots, p, \\ c = p+1, \ldots, m, \end{matrix} \text{ and } \nu_b + \nu_c > n, \\ 1, & \text{otherwise.} \end{cases}$$

Let v_μ^λ denote the image of the vector

$$\prod_{a=1}^{p} f_{a\nu_a} \cdot \prod_{a=p+1}^{m} g_{a\nu_a} \in \bar{\mathrm{A}}$$

in the quotient space $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}} / \bar{\mathrm{I}}_{\mu,\delta^+}$.

Proposition 4.8. (i) *The vector v_μ^λ does not belong to the zero coset in $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}} / \bar{\mathrm{I}}_{\mu,\delta^+}$.*

(ii) *The vector v_μ^λ is of weight λ under the action of $\mathfrak{h}$ on $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}} / \bar{\mathrm{I}}_{\mu,\delta^+}$ and is of highest weight with respect to the action of $\mathrm{Y}(\mathfrak{gl}_n)$ on the same quotient space.*

(iii) *For any $\sigma \in \mathfrak{S}_m$ the operator (3.24) determined by $\check{\xi}_\sigma$ maps the vector v_μ^λ to the image in $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}} / \bar{\mathrm{I}}_{\sigma\circ\mu,\sigma(\delta^+)}$ of $\sigma(v_\mu^\lambda) \in \bar{\mathrm{A}}$ multiplied by $\prod_{\eta \in \Delta_\sigma} z_\eta$.*

Proof. Part (i) follows directly from the definition of the ideal $\bar{\mathrm{I}}_{\mu,\delta^+}$. Let us now prove Part (ii). Subalgebra $\mathfrak{h}$ acts on the quotient space $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}/\bar{\mathrm{I}}_{\mu,\delta^+}$ via left multiplication on $\bar{\mathrm{A}}$. Let symbol $\equiv$ denote the equalities in $\bar{\mathrm{A}}$ modulo the left ideal $\bar{\mathrm{I}}_{\mu,\delta^+}$. Then we have

$$\begin{aligned} E_{aa}v_\mu^\lambda &= v_\mu^\lambda E_{aa} + \left[\zeta_n(E_{aa}), v_\mu^\lambda\right] = v_\mu^\lambda(E_{aa} \mp \nu_a) \equiv v_\mu^\lambda\left(\zeta_n(E_{aa}) + \mu_a \mp \nu_a\right) \\ &\equiv v_\mu^\lambda\left(\pm\frac{n}{2} + \mu_a \mp \nu_a\right) = \lambda_a v_\mu^\lambda \end{aligned}$$

where one should choose the upper sign for $a = 1, \ldots, p$ and the lower sign for $a = p+1, \ldots, m$. Next, $\mathrm{Y}(\mathfrak{gl}_n)$-modules $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}/\bar{\mathrm{I}}_{\mu,\delta^+}$ and $\mathcal{E}_{p,q}(M_\mu)_{\mathfrak{n}}$ are isomorphic. Recall that $T_{ij}(u)$ acts on $\mathcal{E}_{p,q}(M_\mu)_{\mathfrak{n}}$ with the help of the comultiplication Δ. Now, Corollary 2.2 and formulas (2.3) imply that $T_{ij}(u)v_\mu^\lambda = 0$ for all $1 \leqslant i < j \leqslant n$. Moreover, one can check that

$$T_{ii}(u)v_\mu^\lambda = v_\mu^\lambda \cdot \prod_{\substack{1\leqslant a\leqslant p,\\ \nu_a\leqslant n-i}} \frac{u-\mu_a-\rho_a+1}{u-\mu_a-\rho_a} \prod_{\substack{p+1\leqslant a\leqslant m,\\ \nu_a\geqslant i}} \frac{u-\mu_a-\rho_a+1}{u-\mu_a-\rho_a}.$$

Therefore, v_μ^λ is a highest weight vector with respect to the action of $\mathrm{Y}(\mathfrak{gl}_n)$ on the quotient space $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}/\bar{\mathrm{I}}_{\mu,\delta^+}$.

We will prove Part (iii) by induction on the length of a reduced decomposition of σ. If σ is the identity element of $\mathfrak{S}_m$, then the required statement is tautological. Now suppose that for some $\sigma \in \mathfrak{S}_m$ the statement of (iii) is true. Take any simple reflection $\sigma_a \in \mathfrak{S}_m$ with $1 \leqslant a \leqslant m-1$ such that $\sigma_a\sigma$ has a longer reduced decomposition in terms of $\sigma_1, \ldots, \sigma_m$ than σ.

Consider the simple root η_a, corresponding to the reflection σ_a. Let $\eta = \sigma^{-1}(\eta_a)$ then

$$\sigma_a\sigma(\eta) = \sigma_a(\eta_a) = -\eta_a \notin \Delta^+.$$

Thus $\Delta_{\sigma_a\sigma} = \Delta_\sigma \sqcup \{\eta\}$. Let $\kappa \in \mathfrak{h}^*$ be the weight with the labels determined by (3.20) with $\theta = 1$. Using the proof of Theorem 3.3, we get the equality of two left ideals of the algebra A,

$$\bar{\mathrm{I}}_{(\sigma_a\sigma)\circ\mu,(\sigma_a\sigma)(\delta^+)} = \tilde{\mathrm{I}}_{(\sigma_a\sigma)\circ\kappa,(\sigma_a\sigma)(\delta^+)}.$$

Modulo the second of these two ideals the element H_a equals

$$\begin{aligned} ((\sigma_a\sigma)\circ\kappa)(H_a) &= (\sigma_a\sigma(\kappa+\rho)-\rho)(H_a) = (\kappa+\rho)(\sigma^{-1}\sigma_a(H_a)) - \rho(H_a) \\ &= -(\kappa+\rho)(\sigma^{-1}(H_a)) - 1 = -(\kappa+\rho)(H_\eta) - 1 = -\mu_b^* + \mu_c^* - 1. \end{aligned}$$

Here $H_\eta = \sigma^{-1}(H_a)$ is the coroot corresponding to the root η, and we use the standard bilinear form on $\mathfrak{h}^*$.

Let us use the statement of (iii) as the induction assumption. Denote $\delta = \sigma(\delta^+)$. Consider three cases.

I. Let $b, c = 1, \ldots, p$, then $\delta_a = \delta'_a$ and $\delta_{a+1} = \delta'_{a+1}$. Hence,

$$\sigma\left(v_\mu^\lambda\right) = f_{a\nu_b}f_{a+1\,\nu_c}Y$$

where Y is an element of the subalgebra of $\mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ generated by all x_{dk} and ∂_{dk} with $d \neq a, a+1$. Now we apply the Lemma 4.5 with $s = \nu_b$ and $t = \nu_c$. After substitution $-\mu_b^* + \mu_c^* - 1$ for H_a, the fraction in the lemma turns into

$$\frac{-\mu_b^*+\mu_c^*+\nu_b-\nu_c}{-\mu_b^*+\mu_c^*} = \frac{\lambda_b^*-\lambda_c^*}{\mu_b^*-\mu_c^*}.$$

II. Let $b, c = p+1, \ldots, m$, then $\delta_a = -\delta'_a$ and $\delta_{a+1} = -\delta'_{a+1}$. Hence,

$$\sigma\left(v_\mu^\lambda\right) = g_{a\nu_b}g_{a+1\,\nu_c}Y$$

where Y is an element of the subalgebra of $\mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ generated by all x_{dk} and ∂_{dk} with $d \neq a, a+1$. Now we apply the Lemma 4.6 with $s = \nu_b$ and $t = \nu_c$. After substitution $-\mu_b^* + \mu_c^* - 1$ for H_a, the fraction in the lemma turns into

$$\frac{-\mu_b^*+\mu_c^*+\nu_c-\nu_b}{-\mu_b^*+\mu_c^*} = \frac{\lambda_b^*-\lambda_c^*}{\mu_b^*-\mu_c^*}.$$

III. Let $b = 1, \ldots, p$, $c = p+1, \ldots, m$, then $\delta_a = \delta'_a$ and $\delta_{a+1} = -\delta'_{a+1}$. Hence,

$$\sigma\left(v_\mu^\lambda\right) = f_{a\nu_b}g_{a+1\,\nu_c}Y$$

where Y is an element of the subalgebra of $\mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ generated by all x_{dk} and ∂_{dk} with $d \neq a, a+1$. Now we apply the Lemma 4.7 with $s = \nu_b$ and $t = \nu_c$. When $n = 1$, after substitution $-\mu_b^* + \mu_c^* - 1$ for H_a, the fraction in the lemma turns into

$$\frac{-\mu_b^*+\mu_c^*+\nu_b+\nu_c-n}{-\mu_b^*+\mu_c^*} = \frac{\lambda_b^*-\lambda_c^*+n}{\mu_b^*-\mu_c^*}.$$

Thus, in the three cases considered above the operator

$$\check{\xi}_{\sigma_a\sigma} : \bar{\mathrm{J}} \setminus \bar{\mathrm{A}}/\bar{\mathrm{I}}_{\mu,\delta^+} \longrightarrow \bar{\mathrm{I}}_{(\sigma_a\sigma)\circ\mu,(\sigma_a\sigma)(\delta^+)}$$

maps the vector v^λ_μ to the image of

$$\sigma_a\sigma(v^\lambda_\mu) \cdot \prod_{\eta\in\Delta_{\sigma_a\sigma}} z_\eta \in \bar{\mathrm{A}}$$

in the vector space $\bar{\mathrm{J}} \setminus \bar{\mathrm{A}}/\bar{\mathrm{I}}_{(\sigma_a\sigma)\circ\mu,(\sigma_a\sigma)(\delta^+)}$. This observation makes the induction step. $\square$

5. Conjecture

The formula (3.5) yields that for any vector $v \in \bar{\mathrm{J}} \setminus \bar{\mathrm{A}}/\bar{\mathrm{I}}_{\mu,\delta^+}$ the image of v under the operator $\check{\xi}_\sigma$ is well defined unless $(H_a + 1 - r)v = 0$ for some integer r. As we have shown in the previous section H_a acts as $-\mu^*_b + \mu^*_c - 1$ for some $1 \leqslant b < c \leqslant m$ on the target module in the mapping (3.25). It follows from relations (3.4) that after commuting H_a with v, we will get $v(\lambda^*_b - \lambda^*_c + r)$. Therefore, the operators $\check{\xi}_\sigma$ are well defined unless

$$\lambda^*_b - \lambda^*_c = -1, -2, \dots \quad \text{for some } 1 \leqslant b < c \leqslant m. \tag{5.1}$$

In the latter case it can be shown that a factor of module (3.26) by the kernel of operator $\check{\xi}_{\sigma_0}$ is an irreducible (and non-zero under certain condition on the numbers $\nu_1, \dots, \nu_m$) $\mathrm{Y}(\mathfrak{gl}_n)$-module where σ_0 is the longest element of the Weyl group (see [16]).

Conjecture 5.1. *Every irreducible finite-dimensional rational module of the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ may be obtained as the factor of the module (3.31) over the kernel of the intertwining operator ξ'_{σ_0} for some weights μ and λ, with λ satisfying condition (5.1).*

Acknowledgments

I want to express my gratitude to Sergey Khoroshkin for introducing me to this subject, constant interest to my progress, and his many invaluable remarks and corrections. I am also very grateful to Maxim Nazarov for many fruitful discussions. Finally, I would like to thank the reviewer for many useful comments. The author was supported by RFBR grant 11-01-00962, joint CNRS-RFBR grant 09-01-93106-NCNIL, Federal Agency for Science and Innovations of Russian Federation under contract 02.740.11.5194, and by Russian Federal Nuclear Energy Agency under contract H.4e.45.90.11.1059.

Appendix

Let us first prove some properties of the matrix $X(u)$ defined by (1.11).

Proposition A.1. (i) *The following relation holds:*

$$(u - v) \cdot X(u)X(v) = X(v) - X(u); \tag{A.1}$$

(ii) *The elements $X_{ab}(u)$ satisfy the Yangian relation:*

$$(u - v) \cdot [X_{ab}(u), X_{cd}(v)] = \theta(X_{cb}(u)X_{ad}(v) - X_{cb}(v)X_{ad}(u)). \tag{A.2}$$

Proof. The part (i) follows from equality

$$(u - v) = \left(u + \theta E'\right) - \left(v + \theta E'\right)$$

multiplied by $X(u)$ from the left and by $X(v)$ from the right. Let us start the proof of the part (ii) with equality

$$\left[\left(u + \theta E'\right)_{ef}, \left(v + \theta E'\right)_{gh}\right] = \delta_{eh}E_{fg} - \delta_{fg}E_{he}.$$

We multiply the above equality by $X_{ae}(u)$ from the left, by $X_{fb}(u)$ from the right, and take a sum over indices e, f:

$$\sum_{e,f=1}^{m} \left(X_{ae}(u)\left(u + \theta E'\right)_{ef}\left(v + \theta E'\right)_{gh} X_{fb}(u) - X_{ae}(u)\left(v + \theta E'\right)_{gh}\left(u + \theta E'\right)_{ef} X_{fb}(u)\right)$$

$$= \sum_{e,f=1}^{m} \left(X_{ae}(u)\delta_{eh}E_{fg}X_{fb}(u) - X_{ae}(u)\delta_{fg}E_{he}X_{fb}(u)\right).$$

Thus, we get

$$\left[\left(v+\theta E'\right)_{gh}, X_{ab}(u)\right] = \sum_{e,f=1}^{m} \left(X_{ah}(u)E_{fg}X_{fb}(u) - X_{ae}(u)E_{he}X_{gb}(u)\right).$$

Using that

$$\sum_{f=1}^{m} E_{fg}X_{fb}(u) = \theta\left(\delta_{bg} - uX_{gb}(u)\right) \quad \text{and} \quad \sum_{e=1}^{m} X_{ae}(u)E_{he} = \theta\left(\delta_{ah} - uX_{ah}(u)\right),$$

we obtain

$$\left[\left(v+\theta E'\right)_{gh}, X_{ab}(u)\right] = \theta\left(\delta_{bg}X_{ah}(u) - \delta_{ah}X_{gb}(u)\right).$$

Multiplying the above equation by $X_{cg}(v)$ from the left, by $X_{hd}(v)$ from the right, and taking a sum over indices g, h, we arrive to equality

$$[X_{ab}(u), X_{cd}(v)] = \theta \sum_{g,h=1}^{m} \left[X_{cb}(v)X_{ah}(u)X_{hd}(v) - X_{cg}(v)X_{gb}(u)X_{ad}(v)\right].$$

Now using the result of the part (i), we get

$$(u-v)\cdot[X_{ab}(u), X_{cd}(v)] = \theta\Big(X_{cb}(v)\big(X_{ad}(v) - X_{ad}(u)\big) - \big(X_{cb}(v) - X_{cb}(u)\big)X_{ad}(v)\Big)$$

and the statement of the part (ii) follows. □

Proof of Proposition 1.2. We prove the part (i) by direct calculation. During the proof we will write $T_{ij}(u)$ instead of its image under α_m in the algebra $\mathrm{U}(\mathfrak{gl}_m) \otimes \mathcal{HD}\,(\mathbb{C}^m \otimes \mathbb{C}^n)$. Using relations (1.17), (1.18), we get

$$\begin{aligned}
(u-v)\cdot\left[T_{ij}(u), T_{kl}(v)\right] = &\sum_{a,b,c,d=1}^{m} (u-v)\cdot\Big(X_{ab}(u)X_{cd}(v)\otimes(\theta\hat{E}_{ck,bj}\hat{E}_{ai,dl} + \delta_{bc}\delta_{jk}\hat{E}_{ai,dl} - \theta\delta_{ab}\delta_{ij}\hat{E}_{ck,dl}) \\
&- X_{cd}(v)X_{ab}(u)\otimes(\theta\hat{E}_{ck,bj}\hat{E}_{ai,dl} + \delta_{ad}\delta_{il}\hat{E}_{ck,bj} - \theta\delta_{ab}\delta_{ij}\hat{E}_{ck,dl})\Big) \\
= &\sum_{a,b,c,d=1}^{m} (u-v)\cdot\Big([X_{ab}(u), X_{cd}(v)]\otimes\theta(\hat{E}_{ck,bj}\hat{E}_{ai,dl} - \delta_{ab}\delta_{ij}\hat{E}_{ck,dl}) \\
&+ X_{ab}(u)X_{cd}(v)\otimes\delta_{bc}\delta_{jk}\hat{E}_{ai,dl} - X_{cd}(v)X_{ab}(u)\otimes\delta_{ad}\delta_{il}\hat{E}_{ck,bj}\Big).
\end{aligned}$$

Next, using relations (A.1) and (A.2), we obtain

$$\begin{aligned}
(u-v)\cdot\left[T_{ij}(u), T_{kl}(v)\right] = &\sum_{a,b,c,d=1}^{m} \bigg(\Big(X_{cb}(u)X_{ad}(v) - X_{cb}(v)X_{ad}(u)\Big)\otimes(\hat{E}_{ck,bj}\hat{E}_{ai,dl} - \delta_{ab}\delta_{ij}\hat{E}_{ck,dl}) \\
&+ (u-v)\cdot\Big(X_{ab}(u)X_{cd}(v)\otimes\delta_{bc}\delta_{jk}\hat{E}_{ai,dl} - X_{cd}(v)X_{ab}(u)\otimes\delta_{ad}\delta_{il}\hat{E}_{ck,bj}\Big)\bigg).
\end{aligned}$$

Now, using the definition of the homomorphism α_m, we get

$$\begin{aligned}
&\big(T_{kj}(u) - \delta_{jk}\big)\big(T_{il}(v) - \delta_{il}\big) - \big(T_{kj}(v) - \delta_{jk}\big)\big(T_{il}(u) - \delta_{il}\big) - \delta_{ij}\sum_{c,d=1}^{m}\Big(\big(X(u)X(v)\big)_{cd} - \big(X(v)X(u)\big)_{cd}\Big)\otimes\hat{E}_{ck,dl} \\
&+ \delta_{jk}\sum_{a,d=1}^{m}(u-v)\big((X(u)X(v))_{ad}\big)\otimes\hat{E}_{ai,dl} + \delta_{il}\sum_{b,c=1}^{m}(u-v)\big((X(v)X(u))_{cb}\big)\otimes\hat{E}_{ck,bj}.
\end{aligned}$$

It follows from relation (A.1) that $X(u)X(v) = X(v)X(u)$. Therefore, we finish the proof of the part (i) by the following calculations:

$$\begin{aligned}
(u-v)\cdot\left[T_{ij}(u), T_{kl}(v)\right] &= T_{kj}(u)T_{il}(v) - T_{kj}(v)T_{il}(u) - \delta_{jk}\big(T_{il}(v) - T_{il}(u)\big) - \delta_{il}\big(T_{kj}(u) - T_{kj}(v)\big) \\
&\quad + \delta_{jk}\big(T_{il}(v) - T_{il}(u)\big) + \delta_{il}\big(T_{kj}(u) - T_{kj}(v)\big) \\
&= T_{kj}(u)T_{il}(v) - T_{kj}(v)T_{il}(u).
\end{aligned}$$

The part (ii) is also proved by straightforward verification:

$$\Big(E_{cd}\otimes 1+1\otimes\zeta_n(E_{cd}),\,T_{ij}(u)\Big)$$
$$=\left[E_{cd}\otimes 1,\,\sum_{a,b=1}^{m}X_{ab}(u)\otimes\hat{E}_{ai,bj}\right]+\left[1\otimes\sum_{k=1}^{n}\hat{E}_{ck,dk},\,\sum_{a,b=1}^{m}X_{ab}(u)\otimes\hat{E}_{ai,bj}\right]$$
$$=\sum_{a,b=1}^{m}\Big(\delta_{bd}X_{ac}(u)-\delta_{ac}X_{db}(u)\Big)\otimes\hat{E}_{ai,bj}+\sum_{k=1}^{n}\sum_{a,b=1}^{m}\Big(X_{ab}(u)\otimes\Big(\delta_{ad}\delta_{ik}\hat{E}_{ck,bj}-\delta_{bc}\delta_{jk}\hat{E}_{ai,dk}\Big)\Big)$$
$$=\sum_{a=1}^{m}X_{ac}(u)\otimes\hat{E}_{ai,dj}-\sum_{b=1}^{m}X_{db}(u)\otimes\hat{E}_{ci,bj}+\sum_{b=1}^{m}X_{db}(u)\otimes\hat{E}_{ci,bj}-\sum_{a=1}^{m}X_{ac}(u)\otimes\hat{E}_{ai,dj}=0.\quad\square$$

References

[1] V. Drinfeld, A new realization of Yangians and quantum affine algebras, Sov. Math. Dokl. 36 (1988) 212–216.
[2] J. Brundan, A. Kleshchev, Representations of shifted Yangians and finite W-algebras, Mem. Amer. Math. Soc. 196 (2008) 1–107.
[3] V. Drinfeld, Degenerate affine Hecke algebras and Yangians, Funct. Anal. Appl. 20 (1986) 56–58.
[4] S. Khoroshkin, M. Nazarov, Yangians and Mickelsson algebras I, Transform. Groups 11 (2006) 625–658.
[5] S. Khoroshkin, M. Nazarov, Yangians and Mickelsson algebras II, Mosc. Math. J. 6 (2006) 477–504.
[6] I. Cherednik, Lectures on Knizhnik–Zamolodchikov equations and Hecke algebras, in: Quantum Many-body Problems and Representation Theory, in: MSJ Mem., vol. 1, Math. Soc. Japan, Tokyo, 1998, pp. 1–96. MR1724948 (2001g:20004).
[7] T. Arakawa, T. Suzuki, A. Tsuchiya, Degenerate double affine Hecke algebras and conformal field theory, Progr. Math. 160 (1998) 1–34.
[8] R. Howe, Remarks on classical invariant theory, Trans. Amer. Math. Soc. 313 (1989) 539–570.
[9] R. Howe, Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond, Israel Math. Conf. Proc. 8 (1995) 1–182.
[10] G. Olshanski, Extension of the algebra $U(g)$ for infinite-dimensional classical Lie algebras g, and the Yangians $Y(gl(m))$, Sov. Math. Dokl. 36 (1988) 569–573.
[11] G. Olshanski, Twisted Yangians and infinite-dimensional classical Lie algebras, Lecture Notes in Math. 1510 (1992) 103–120.
[12] D. Zhelobenko, Extremal cocycles on Weyl groups, Funct. Anal. Appl. 21 (1987) 183–192.
[13] D. Zhelobenko, Extremal projectors and generalized Mickelsson algebras over reductive Lie algebras, Math. USSR Izvestiya 33 (1989) 85–100.
[14] S. Khoroshkin, Extremal projector and dynamical twist, Theoret. Math. Phys. 139 (2004) 582–597.
[15] S. Khoroshkin, O. Ogievetsky, Mickelsson algebras and Zhelobenko operators, J. Algebra 319 (2008) 2113–2165.
[16] S. Khoroshkin, M. Nazarov, Mickelsson algebras and representations of Yangians, arXiv:math/0912.1101.
[17] S. Khoroshkin, M. Nazarov, P. Papi, Irreducible representations of Yangians, arXiv:math/1105.5777.
[18] T. Akasaka, M. Kashiwara, Finite-dimensional representations of quantum affine algebras, Publ. Res. Inst. Math. Sci. 33 (1997) 839–867.
[19] T. Enright, R. Howe, N. Wallach, A classification of unitary highest weight modules, Progr. Math. 40 (1983) 97–143.
[20] T. Enright, R. Parthasarathy, A proof of a conjecture of Kashiwara and Vergne, Lecture Notes in Math. 880 (1981) 74–90.
[21] M. Kashiwara, M. Vergne, On the Segal–Shale–Weil representation and harmonic polynomials, Invent. Math. 44 (1978) 1–47.
[22] M. Nazarov, Rational representations of Yangians associated with skew Young diagrams, Math. Z. 247 (2004) 21–63.
[23] S. Khoroshkin, M. Nazarov, Twisted Yangians and Mickelsson algebras I, Selecta Math. 13 (2007) 69–136. and 14 (2009), 321.
[24] S. Khoroshkin, M. Nazarov, Twisted Yangians and Mickelsson algebras II, St. Petersburg Math. J. 21 (2010) 111–161.
[25] V. Drinfeld, Hopf algebras and the quantum Yang–Baxter equation, Sov. Math. Dokl. 32 (1985) 254–258.
[26] A. Molev, M. Nazarov, G. Olshanski, Yangians and classical Lie algebras, Russian Math. Surveys 51 (1996) 205–282.
[27] A. Molev, Yangians and Classical Lie Algebras, Amer. Math. Soc., Providence, 2007.
[28] V. Chari, A. Pressley, Fundamental representations of Yangians and singularities of R-matrices, J. Reine Angew. Math. 417 (1991) 87–128.