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On Roter type warped products with 1-dimensional fibres

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ABSTRACT

We investigate warped products satisfying the Roter type equation. Important examples of such manifolds are certain generalized Robertson–Walker spacetimes, and a number of spherically symmetric static spacetimes like, for example, Reissner–Nordström–de Sitter spacetime. We mainly focus on warped products with 1-dimensional fibre.

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1. Introduction

Let R, S, C and κ be the curvature tensor, the Ricci tensor, the Weyl conformal curvature tensor and the scalar curvature of a semi-Riemannian manifold (M, g), $n = \dim M \geq 4$, respectively. Further, let $\mathcal{U} \subset M$ be a set of all points at which for any $\alpha \in \mathbb{R}$, the Ricci tensor S satisfies $\operatorname{rank}(S - \alpha g) > 1$ and the Weyl tensor C is non-zero. The manifold (M, g), $n \geq 4$, is said to be a *Roter type manifold* (cf. [1]) if

$$R = \frac{\phi}{2} S \wedge S + \mu g \wedge S + \eta G \tag{1}$$

holds on $\mathcal{U} \subset M$, where ϕ, μ and η are some functions on this set and $G = \frac{1}{2} g \wedge g$. We note that $\mathcal{U} \subset U_C \cap U_S$, where $U_C = \{x \in M \mid C \neq 0 \text{ at } x\}$ and $U_S = \{x \in M \mid S - \frac{\kappa}{n} g \neq 0 \text{ at } x\}$. For precise definitions of the symbols used here, we refer to Section 2 of this paper (and also to Sections 2 and 3 of [2]). Obviously, we consider Roter type manifolds with $\mathcal{U}$ non-empty. We refer to [1,3,4] for a survey of results on Roter type manifolds. In particular Roter type manifolds $\overline{M} \times_F \widetilde{N}$ with $\dim \overline{M} = p \geq 2$ and $\dim \widetilde{N} = n - p \geq 2$, respectively with $\dim \overline{M} = 1$ and $\dim \widetilde{N} = n - 1 \geq 3$, were investigated in [2,5], respectively, in [6]. Semi-Riemannian manifolds fulfilling (1), with some additional assumptions imposed on the functions ϕ, μ and η, were investigated in [7,8].

We recall that if $n \geq 4$, $p = 1$, and the fibre manifold $(\widetilde{N}, \widetilde{g})$ is a Riemannian manifold, then $\overline{M} \times_F \widetilde{N}$ with $(\overline{M}, \overline{g}) = (I, -dt^2)$, I an interval, will be called a *generalized Robertson–Walker spacetime* (see [9] and references therein). Clearly,

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the class of generalized Robertson–Walker spacetimes is a natural extension of the class of Robertson–Walker spacetimes (e.g. see [10]).

Warped products $\overline{M}\times_F\widetilde{N}$ with the Riemannian base $(\overline{M},\overline{g})$ of dimension $p=n-1\geq 3$ and fibre $(\widetilde{M},\widetilde{g})=(I,-dt^2)$ are important examples of static spacetimes, called standard static [11]. As it was stated in [2] spherically symmetric static spacetimes, with the metric tensor of the form (14), satisfy (1) (see Remark 2.3(ii)). This together with the fact that certain generalized Robertson–Walker spacetimes also satisfy (1) (e.g. see [6]) leads to our aim to investigate curvature properties of warped products $\overline{M}\times_F\widetilde{N}$ with $\dim\overline{M}=n-1\geq 3$, satisfying (1). Among other things we will show that if they are Roter type then under certain additional assumptions the base manifolds $(\overline{M},\overline{g})$ satisfy a special form of (1).

In Section 2 basic definitions are presented. Section 3 contains preliminary results on Roter type warped products $\overline{M}\times_F\widetilde{N}$ with $\dim\overline{M}=n-1\geq 3$ and $\dim\widetilde{N}=1$. In Section 4 we continue investigations of those warped products. Finally, we use the obtained results for the construction of suitable examples of dimension $n\geq 4$ and $n=4$, respectively (Examples 4.1 and 4.2).

2. Preliminaries

Throughout this paper all manifolds are assumed to be connected paracompact C^∞-manifolds. Let (M,g) be an n-dimensional, $n\geq 3$, semi-Riemannian manifold, ∇ its Levi-Civita connection and $\Xi(M)$ the Lie algebra of vector fields on M. We define the endomorphisms $X\wedge_A Y$ and $\mathcal{R}(X,Y)$ of $\Xi(M)$ by

$$(X\wedge_A Y)Z=A(Y,Z)X-A(X,Z)Y,$$
$$\mathcal{R}(X,Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z,$$

respectively, where A is a symmetric $(0,2)$-tensor on M and $X,Y,Z\in\Xi(M)$. The Ricci tensor S, the Ricci operator $\mathcal{S}$ and the scalar curvature κ of (M,g) are defined by

$$S(X,Y)=\operatorname{tr}\{Z\to\mathcal{R}(Z,X)Y\},\qquad g(\mathcal{S}X,Y)=S(X,Y),\qquad \kappa=\operatorname{tr}\mathcal{S},$$

respectively. The endomorphism $\mathcal{C}(X,Y)$ is defined by

$$\mathcal{C}(X,Y)Z=\mathcal{R}(X,Y)Z-\frac{1}{n-2}\left(X\wedge_g\mathcal{S}Y+\mathcal{S}X\wedge_g Y-\frac{\kappa}{n-1}X\wedge_g Y\right)Z.$$

The $(0,4)$-tensor G, the Riemann–Christoffel curvature tensor R and the Weyl conformal curvature tensor C of (M,g) are defined by

$$G(X_1,X_2,X_3,X_4)=g((X_1\wedge_g X_2)X_3,X_4),$$
$$R(X_1,X_2,X_3,X_4)=g(\mathcal{R}(X_1,X_2)X_3,X_4),$$
$$C(X_1,X_2,X_3,X_4)=g(\mathcal{C}(X_1,X_2)X_3,X_4),$$

where $X_1,\dots,X_4\in\Xi(M)$. Let $\mathcal{B}(X,Y)$ be a skew-symmetric endomorphism of $\Xi(M)$ and let B be a $(0,4)$-tensor associated with $\mathcal{B}(X,Y)$ by

$$B(X_1,X_2,X_3,X_4)=g(\mathcal{B}(X_1,X_2)X_3,X_4).\tag{2}$$

The tensor B is said to be a *generalized curvature tensor* if

$$B(X_1,X_2,X_3,X_4)+B(X_2,X_3,X_1,X_4)+B(X_3,X_1,X_2,X_4)=0,$$
$$B(X_1,X_2,X_3,X_4)=B(X_3,X_4,X_1,X_2).$$

Let $\mathcal{B}(X,Y)$ be a skew-symmetric endomorphism of $\Xi(M)$ and let B be the tensor defined by (2). We extend $\mathcal{B}(X,Y)$ to the derivation $\mathcal{B}(X,Y)\cdot$ of the algebra of tensor fields on M, assuming that it commutes with contractions and $\mathcal{B}(X,Y)\cdot f=0$ for every smooth function f on M. Now, for a $(0,k)$-tensor field T, $k\geq 1$, we define the $(0,k+2)$-tensor $B\cdot T$ by

$$\begin{aligned}(B\cdot T)(X_1,\dots,X_k;X,Y)&=(\mathcal{B}(X,Y)\cdot T)(X_1,\dots,X_k)\\&=-T(\mathcal{B}(X,Y)X_1,X_2,\dots,X_k)-\cdots-T(X_1,\dots,X_{k-1},\mathcal{B}(X,Y)X_k).\end{aligned}$$

In addition, if A is a symmetric $(0,2)$-tensor, we define the $(0,k+2)$-tensor $Q(A,T)$ by

$$\begin{aligned}Q(A,T)(X_1,\dots,X_k;X,Y)&=(X\wedge_A Y\cdot T)(X_1,\dots,X_k)\\&=-T((X\wedge_A Y)X_1,X_2,\dots,X_k)-\cdots-T(X_1,\dots,X_{k-1},(X\wedge_A Y)X_k).\end{aligned}$$

In this manner we obtain the $(0,6)$-tensors $B\cdot B$ and $Q(A,B)$. Setting in the above formulae $\mathcal{B}=\mathcal{R}$ or $\mathcal{B}=\mathcal{C}$, $T=R$ or $T=C$ or $T=S$, $A=g$ or $A=S$, we get the tensors $R\cdot R$, $R\cdot S$, $C\cdot C$, $Q(g,R)$, $Q(g,S)$, $Q(g,C)$, $Q(S,R)$ and $Q(S,G)$. For symmetric $(0,2)$-tensors E and F we define their *Kulkarni–Nomizu product* $E\wedge F$ by

$$(E\wedge F)(X_1,X_2,X_3,X_4)=E(X_1,X_4)F(X_2,X_3)+E(X_2,X_3)F(X_1,X_4)-E(X_1,X_3)F(X_2,X_4)-E(X_2,X_4)F(X_1,X_3).$$

Clearly, the tensors R, C, G and $E \wedge F$ are generalized curvature tensors. For the Weyl tensor we can thus write

$$C = R - \frac{1}{n-2}\, g \wedge S + \frac{\kappa}{(n-2)(n-1)}\, G. \tag{3}$$

We also have the following identity (see e.g. [12, Section 3]):

$$Q(E, E \wedge F) = -Q(F, \overline{E}). \tag{4}$$

A semi-Riemannian manifold (M, g), $n \geq 3$, is said to be *pseudosymmetric* (see e.g. [13,14]) if the $(0, 6)$-tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent at every point of M. This is equivalent to

$$R \cdot R = L_R\, Q(g, R) \tag{5}$$

on $U_R = \{x \in M \mid R - \frac{\kappa}{(n-1)n}\, G \neq 0 \text{ at } x\}$, where L_R is some function on U_R. It is easy to check that L_R is uniquely determined on U_R. According to [15], if L_R is a constant on U_R, then the pseudosymmetric manifold (M, g) is called of *constant type*. It is obvious that every *semisymmetric manifold*, i.e. $R \cdot R = 0$ holds on M [16], is pseudosymmetric. The converse is not true (see [13]). A geometric interpretation of the notion of pseudosymmetry is given in [17] (see also [18]). According to [17], pseudosymmetric manifolds are called *pseudosymmetric in the sense of Deszcz* (see also [19]).

A manifold (M, g), $n \geq 3$, is said to be *Ricci-pseudosymmetric* (see e.g. [13,14]) if at every point of M the tensors $R \cdot S$ and $Q(g, S)$ are linearly dependent. This is equivalent to

$$R \cdot S = L_S\, Q(g, S) \tag{6}$$

on U_S, where L_S is some function on U_S. The class of Ricci-pseudosymmetric manifolds is an extension of the class of *Ricci-semisymmetric* manifolds, i.e. $R \cdot S = 0$ holds on M, as well as of the class of pseudosymmetric manifolds. A geometric interpretation of the notion of Ricci-pseudosymmetry is given in [19]. According to [3], if L_S is constant on U_S then a Ricci-pseudosymmetric manifold is of *constant type*.

In [19] Ricci-pseudosymmetric manifolds were named *Ricci pseudosymmetric in the sense of Deszcz*. The class of Ricci-pseudosymmetric manifolds is an extension of the class of pseudosymmetric manifolds (e.g. see [13]).

According to [1], the Eq. (1) is called a *Roter type equation* and a semi-Riemannian manifold (M, g), $n \geq 4$, satisfying (1) on $\mathcal{U} \subset M$ is called a Roter type manifold. We mention that the decomposition of R on $\mathcal{U}$ in terms of $S \wedge S$, $g \wedge S$ and G is unique [20, Lemma 3.2]. It is easy to check that (1) implies (5) on $\mathcal{U}$ with

$$L_R = \phi^{-1}\left((n-2)(\mu^2 - \phi\eta) - \mu\right). \tag{7}$$

Further, we note that (1) can be presented in the form

$$R + \phi^{-1}(\mu^2 - \phi\eta)\, G = \frac{\phi}{2}\,(S + \phi^{-1}\mu\, g) \wedge (S + \phi^{-1}\mu\, g).$$

The Eq. (1) also implies (see e.g. [1,4,21])

$$R \cdot R - Q(S, R) = L\, Q(g, C), \tag{8}$$

$$C \cdot C = L_C\, Q(g, C), \tag{9}$$

with

$$L = L_R + \phi^{-1}\mu = (n-2)\phi^{-1}(\mu^2 - \phi\eta),$$

$$L_C = L_R - \frac{\kappa}{n-1} + \phi^{-1}\left(\frac{1}{n-2} - \mu\right).$$

It is obvious that (9) is invariant under the conformal changes of the metric g.

Remark 2.1 (*See [6], Remark 2.1*)**.**

(i) In [22] (see Theorem 4.1) it was shown that every warped product $\overline{M} \times_F \widetilde{N}$ with $\dim \overline{M} = 1$ and $\dim \widetilde{N} = 3$ satisfies (8) with some function L. In particular, every 4-dimensional generalized Robertson–Walker spacetime has this property.

(ii) From Theorem 6.1 of [22] it follows that the warped product $\overline{M} \times_F \widetilde{N}$ of an $(n-1)$-dimensional space of constant curvature $(\overline{M}, \overline{g})$, $n \geq 4$, and a line or a circle $(\widetilde{N}, \widetilde{g})$ satisfies (8) with

$$L = -\frac{\overline{\kappa}}{n-1}, \tag{10}$$

where $\overline{\kappa}$ is the scalar curvature of $(\overline{M}, \overline{g})$.

Remark 2.2. (i) First of all we note that $U_C \subset U_R \subset M$ for any semi-Riemannian manifold (M, g), $n \geq 4$. In fact, if we had $x \in U_C \cap (M \setminus U_R)$, then – in a standard way – we would obtain $C = 0$ from $R = \frac{\kappa}{(n-1)n}\, G$, a contradiction.

(ii) Let (M, g), $n \geq 4$, be a semi-Riemannian manifold such that (5) and (8) hold on $U_C \subset M$, where L_R and L are some functions on U_C. Applying (3)–(5) in (8) we get on U_C

$$(L_R - L)\, Q(g, R) = Q\left(S, R + \frac{L}{n-2}\, G\right).$$

This, by making use of the identity $Q(g, G) = 0$, turns into

$$Q\left(S - (L_R - L)\, g, R + \frac{L}{n-2}\, G\right) = 0. \tag{11}$$

The last relation, in view of Proposition 4.1 of [23], implies

$$R = \lambda\,(S - (L_R - L)\, g) \wedge (S - (L_R - L)\, g) - \frac{L}{n-2}\, G$$

on $\mathcal{U}$, where λ is some function on this set. Clearly, the last equation is of the form (1).

(iii) Pseudosymmetric conformally flat manifolds, as well as pseudosymmetric Einsteinian manifolds were investigated among others in [24,25], respectively.

(iv) Let (M, g), $n \geq 3$, be a *quasi-Einstein manifold*, that is a semi-Riemannian manifold with the Ricci tensor S given by

$$S = \alpha\, g + \beta\, w \otimes w \tag{12}$$

for every $x \in M$, where $w \in T_x^* M$, $\alpha,\ \beta \in \mathbb{R}$. Quasi-Einstein manifolds arose in the study of exact solutions of the Einstein field equations as well as in considerations of quasi-umbilical hypersurfaces of conformally flat spaces. We note that for every point of $U_S \subset M$ the condition (12) is equivalent to

$$\operatorname{rank}(S - \alpha\, g) = 1 \tag{13}$$

and this is equivalent to

$$(S - \alpha\, g) \wedge (S - \alpha\, g) = 0.$$

Let now (M, g) be a quasi-Einstein manifold of dimension ≥ 4. It is easy to verify that if (1) is satisfied on $U_S \subset M$ then $C = 0$ on this set. Also some converse statement is true [6, Theorem 2.1].

(v) First results on pseudosymmetric manifolds of constant type, as well on Ricci-pseudosymmetric manifolds of constant type, were published in [26]. Among others in that paper it was proved that the type number two hypersurfaces in semi-Riemannian spaces of constant curvature are pseudosymmetric manifolds of constant type. An example of a quasi-Einstein (respectively non-quasi-Einstein) type number two hypersurfaces in a semi-Riemannian space of constant curvature is given in [20, Example 5.1] (respectively [6, Section 4]). Non-quasi-Einstein type number two hypersurfaces in semi-Riemannian spaces of constant curvature satisfy (1) (e.g. see [6, Proposition 4.1]). We refer to [27] for recent results on quasi-Einstein hypersurfaces in semi-Riemannian spaces of constant curvature satisfying some curvature conditions. We also refer to Section 5 of [21] for a survey of results on quasi-Einstein manifolds.

(vi) An example of a generalized Robertson–Walker spacetime satisfying (5), (8) and (13) with $\alpha = \frac{\kappa}{n}$ is given in Example 4.1 of [28].

Remark 2.3. (i) Let $(M, g) = M_{s_1}^p(c_1) \times M_{s_2}^{n-p}(c_2)$ be the Cartesian product of semi-Riemannian spaces of constant curvature $M_{s_1}^p(c_1)$ and $M_{s_2}^{n-p}(c_2)$, $2 \leq p \leq n-2$, where $c_1 = \frac{\kappa_1}{(p-1)p}$, $c_2 = \frac{\kappa_2}{(n-p-1)(n-p)}$ and κ_1 and κ_2 are the scalar curvatures and $(s_1, p - s_1)$ and $(s_2, n - p - s_2)$ signatures of $M_{s_1}^p(c_1)$ and $M_{s_2}^{n-p}(c_2)$, respectively. It is well-known that (M, g) is a semisymmetric manifold, i.e. $R \cdot R = 0$ holds on M [16, Theorem 4.5]. We can check that if c_1 and c_2 satisfy

$$c_1 + c_2 \neq 0 \quad \text{and} \quad (p-1)\, c_1 - (n-p-1)\, c_2 \neq 0$$

then (M, g) is a non-conformally flat and non-Einstein manifold. Precisely, we have $\mathcal{U} = M$. Moreover, as it was stated in Section 3 of [3], (1) holds on M. Now we can easily deduce that every Clifford torus $S^p(r_1) \times S^{n-p}(r_2)$, where $S^p(r_1)$ and $S^{n-p}(r_2)$ are the standard spheres with $r_1 = \sqrt{\frac{p}{n}}$, $r_2 = \sqrt{\frac{n-p}{n}}$, $2 \leq p \leq n-2$ and $n \neq 2p$, is a hypersurface isometrically immersed in the sphere $S^{n+1}(1)$ satisfying (1) [3, Corollary 3.1].

(ii) Let $\overline{M} = \{(t, r) \in \mathbb{R}^2 : r > 0\}$ be an open connected non-empty subset of $\mathbb{R}^2$ and let on $\overline{M}$ the metric tensor $\overline{g}$ be defined by $\overline{g}_{11} = -H$, $\overline{g}_{12} = \overline{g}_{21} = 0$ and $\overline{g}_{22} = H^{-1}$, where $H = H(r)$ is a smooth positive (or negative) function on $\overline{M}$. Let $S^{n-2}(1)$, $n \geq 4$, be the $(n-2)$-dimensional unit sphere. The warped product $\overline{M} \times_F S^{n-2}(1)$ of $(\overline{M}, \overline{g})$ and $S^{n-2}(1)$, with the warping function $F = F(r) = r^2$, is a pseudosymmetric manifold satisfying (1) on the set $\mathcal{U} \subset \overline{M} \times_F S^{n-2}(1)$ [2, Theorem 4.1], [25, Example 1]. In particular (1) holds on the set $\mathcal{U} \subset \overline{M} \times_F S^2(1)$. Clearly the warped product $\overline{M} \times_F S^2(1)$ is a spherically symmetric static spacetime. The metric g of that spacetime can be presented in the form

$$g = -H dt^2 + H^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2. \tag{14}$$

In addition when $H(r) = 1 - \frac{2m}{r} + \frac{e^2}{r^2} - \frac{\Lambda r^2}{3}$, where $m = \text{const.}$, $e = \text{const.}$ and $\Lambda = \text{const.}$, the warped product $\overline{M} \times_F S^2(1)$ is called the *Reissner–Nordström–de Sitter spacetime* [29]. The coefficients ϕ, μ and η of the decomposition (1) of the curvature tensor R of that spacetime were found in [2] (Example 5.3). We recall that if $e = 0$ and $\Lambda = 0$, respectively, $e = 0$ and $\Lambda \neq 0$ and $e \neq 0$ and $\Lambda = 0$, then the spacetime $\overline{M} \times_F S^2(1)$ is called the *Schwarzschild spacetime*, respectively the *Kottler* and the *Reissner–Nordström spacetime* [30, Section 13]. It is well-known that the Schwarzschild spacetime is Ricci flat and the Kottler spacetime is a non-Ricci flat Einstein manifold.

(iii) In [31, Theorem 2] it was stated that (9) is satisfied on every 4-dimensional warped product with 2-dimensional base. Thus in particular, (9) holds on every spherically symmetric static spacetime. It is also known that the product of two semi-Riemannian spaces of constant curvature of dimensions ≥ 2 is a semisymmetric manifold satisfying (9) (e.g. see [14, Section 10.4]).

(iv) Examples of warped products with p-dimensional base, $2 \leq p \leq n-2$, satisfying

$$R = \frac{\phi}{2}\, S \wedge S, \tag{15}$$

i.e. the special form of (1), are given in [3] (Example 3.1) and [8] (Examples 4.5–4.7). Moreover, in [8] (Example 4.4) a warped product with $(n-1)$-dimensional base, $n \geq 3$, satisfying (15) was constructed. It is easy to check that manifolds satisfying (15) are semisymmetric.

3. Warped products satisfying (5) and (8)

Let $(\overline{M}, \overline{g})$ and $(\widetilde{N}, \widetilde{g})$ with $\dim \overline{M} = p$ and $\dim \widetilde{N} = n-p$, $1 \leq p < n$, be semi-Riemannian manifolds covered by systems of charts $\{U; x^a\}$ and $\{V; y^\alpha\}$, respectively. Further, let $F : \overline{M} \to \mathbb{R}^+$ be a positive smooth function on $\overline{M}$. The *warped product* $\overline{M} \times_F \widetilde{N}$ is the product manifold $\overline{M} \times \widetilde{N}$ with the metric $g = \overline{g} \times_F \widetilde{g} = \pi_1^* \overline{g} + (F \circ \pi_1)\, \pi_2^* \widetilde{g}$, where π_1 and π_2 are the natural projections. Let $\{\overline{U} \times \widetilde{V} : x^1, \ldots, x^p, x^{p+1} = y^1, \ldots, x^n = y^{n-p}\}$ be a product chart for $\overline{M} \times \widetilde{N}$. The local components of the metric $g = \overline{g} \times_F \widetilde{g}$ with respect to this chart read: $g_{hk} = \overline{g}_{ab}$ if $h = a$ and $k = b$, $g_{hk} = F\, \widetilde{g}_{\alpha\beta}$ if $h = \alpha$ and $k = \beta$, and $g_{hk} = 0$ otherwise, where $a, b, c, \ldots, \in \{1, \ldots, p\}$, $\alpha, \beta, \gamma, \ldots \in \{p+1, \ldots, n\}$ and $h, i, j, k \ldots \in \{1, 2, \ldots, n\}$. We will mark by bars (respectively, tildes) tensors formed from $\overline{g}$ (respectively, $\widetilde{g}$). The local components R_{hijk} of the curvature tensor R and the local components S_{hk} of the Ricci tensor S of $\overline{M} \times_F \widetilde{N}$ which generally do not vanish identically are the following (see e.g. [2,6,32]):

$$R_{abcd} = \overline{R}_{abcd}, \tag{16}$$

$$R_{\alpha bc\beta} = -\frac{1}{2}\, T_{bc}\, \widetilde{g}_{\alpha\beta},$$

$$R_{\alpha\beta\gamma\delta} = F\left(\widetilde{R}_{\alpha\beta\gamma\delta} - \frac{\Delta_1 F}{4F}\, \widetilde{G}_{\alpha\beta\gamma\delta}\right), \tag{17}$$

$$S_{ab} = \overline{S}_{ab} - \frac{n-p}{2F}\, T_{ab},$$

$$S_{\alpha\beta} = \widetilde{S}_{\alpha\beta} - \left(\frac{\operatorname{tr} T}{2} + (n-p-1)\, \frac{\Delta_1 F}{4F}\right) \widetilde{g}_{\alpha\beta}, \tag{18}$$

$$T_{ab} = \overline{\nabla}_b F_a - \frac{1}{2F}\, F_a F_b, \qquad \Delta_1 F = \Delta_{1\overline{g}} F = \overline{g}^{ab} F_a F_b,$$

where T denotes the $(0,2)$-tensor with local components T_{ab} and $\operatorname{tr} T = \operatorname{tr}_{\overline{g}} T = \overline{g}^{ab} T_{ab}$. The scalar curvature κ of $\overline{M} \times_F \widetilde{N}$ satisfies

$$\kappa = \overline{\kappa} + \frac{\widetilde{\kappa}}{F} - \frac{n-p}{F}\left(\operatorname{tr} T + (n-p-1)\, \frac{\Delta_1 F}{4F}\right). \tag{19}$$

In particular, if $p = n-1$ then (17)–(19) reduce to

$$R_{nbcn} = -\frac{1}{2}\, T_{bc}\, \widetilde{g}_{nn}, \tag{20}$$

$$S_{ab} = \overline{S}_{ab} - \frac{1}{2F}\, T_{ab},$$

$$S_{nn} = -\frac{1}{2} \operatorname{tr} T\, \widetilde{g}_{nn}, \tag{21}$$

$$\kappa = \overline{\kappa} - \frac{\operatorname{tr} T}{F},$$

respectively. Further, the local components of the tensors $(R\cdot R)_{hijklm}$, $(Q(S,R))_{hijklm}$ and $(Q(g,C))_{hijklm}$ of $\overline{M}\times_F\widetilde{N}$ with $\dim\overline{M}=n-1$ and $\dim\widetilde{N}=1$ which generally do not vanish identically are the following (see e.g. [22,33,34]):

$$
\begin{aligned}
&(R\cdot R)_{abcdef}=(\overline{R}\cdot\overline{R})_{abcdef},\\
&(R\cdot R)_{nabcdn}=\frac{1}{2}\left(T_{de}\overline{g}^{ef}\overline{R}_{fabc}-\frac{1}{2F}(T_{ac}T_{bd}-T_{ab}T_{cd})\right)\widetilde{g}_{nn},\\
&(R\cdot R)_{nabncd}=-\frac{1}{2}\,(T_{ae}\overline{g}^{ef}\overline{R}_{fbcd}+T_{be}\overline{g}^{ef}\overline{R}_{facd})\widetilde{g}_{nn}.
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
&(Q(S,R))_{abcdef}=(Q(\overline{S},\overline{R}))_{abcdef}-\frac{1}{2F}\,(Q(T,\overline{R}))_{abcdef},\\
&(Q(S,R))_{nabcdn}=-\frac{1}{2}(\operatorname{tr}T\,\overline{R}_{dabc}+T_{ab}S_{cd}-T_{ac}S_{bd})\widetilde{g}_{nn},\\
&(Q(S,R))_{nabncd}=-\frac{1}{2}\,(Q(S,T))_{abcd}\,\widetilde{g}_{nn}.
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
&(Q(g,C))_{abcdef}=(Q(\overline{g},\overline{R}))_{abcdef}+\frac{1}{n-2}\,(Q(\overline{S},\overline{G}))_{abcdef},\ -\frac{1}{2(n-2)F}\,(Q(T,\overline{G}))_{abcdef},\\
&(Q(g,C))_{nabcdn}=F\left(\frac{1}{n-2}(\overline{g}_{bd}A_{ac}-\overline{g}_{cd}A_{ab})-C_{dabc}\right)\widetilde{g}_{nn},\\
&(Q(g,C))_{nabncd}=-\frac{F}{n-2}\,(Q(\overline{g},A))_{abcd}\,\widetilde{g}_{nn},\\
&A_{ab}=\overline{S}_{ab}+\frac{n-3}{2F}\,T_{ab}-\frac{1}{n-1}\left(\overline{\kappa}+\frac{n-3}{2F}\operatorname{tr}T\right)\overline{g}_{ab}.
\end{aligned}
\tag{24}
$$

If $\overline{M}\times_F\widetilde{N}$ is a pseudosymmetric manifold, i.e. if (5) holds on $U_R\subset\overline{M}\times_F\widetilde{N}$, then we define on U_R the $(0,2)$-tensor H by (e.g. see [31, Lemma 4])

$$
H=\frac{1}{2}T+FL_R\overline{g}.
\tag{25}
$$

We have on U_R (Eq. (18) of [31])

$$
H_{de}\overline{g}^{ef}\overline{R}_{fabc}=\frac{1}{2F}(T_{ac}H_{bd}-T_{ab}H_{cd}),
\tag{26}
$$

which is equivalent to

$$
\frac{1}{2}T_{de}\overline{g}^{ef}\overline{R}_{fabc}=-FL_R\,\overline{R}_{dabc}+\frac{1}{4F}\,(T_{ac}T_{bd}-T_{ab}T_{cd})+\frac{L_R}{2}\,(\overline{g}_{bd}T_{ac}-\overline{g}_{cd}T_{ab}).
\tag{27}
$$

Proposition 3.1. *Let $\overline{M}\times_F\widetilde{N}$ be the warped product of $(\overline{M},\overline{g})$, $\dim\overline{M}=n-1\geq 3$, and a line or a circle $(\widetilde{M},\widetilde{g})$. If (5) and (8) hold on $U_R\cap U_C\subset\overline{M}\times_F\widetilde{N}$ then at every $x\in U_R\cap U_C$ we have:* (i) $\overline{S}-\frac{\overline{\kappa}}{n-1}\overline{g}=0$ *or* (ii) $\overline{S}-\frac{\overline{\kappa}}{n-1}\overline{g}\neq 0$ *and*

$$
T-\frac{\operatorname{tr}T}{n-1}\overline{g}=\lambda\left(\overline{S}-\frac{\overline{\kappa}}{n-1}\overline{g}\right),\quad \lambda\in\mathbb{R}.
\tag{28}
$$

on some neighbourhood $V\subset U_R\cap U_C$ of x.

Proof. We consider the equation

$$
((R\cdot R))_{nabcdn}=(Q(S,R))_{nabcdn}+L\,(Q(g,C))_{nabcdn}.
$$

This, by (21)–(24) and (27), reads

$$
\begin{aligned}
&\left(\frac{\operatorname{tr}T}{2}+FL_R\right)\overline{R}_{dabc}+\frac{L_R}{2}\,(\overline{g}_{cd}T_{ab}-\overline{g}_{bd}T_{ac})+\frac{1}{2}\,(\overline{S}_{cd}T_{ab}-\overline{S}_{bd}T_{ac})-\frac{1}{4}\,(T_{ab}T_{cd}-T_{ac}T_{bd})\\
&\quad=\frac{FL}{n-2}\,(\overline{g}_{cd}\overline{S}_{ab}-\overline{g}_{bd}\overline{S}_{ac})+\frac{(n-3)L}{2(n-2)}\,(\overline{g}_{cd}T_{ab}-\overline{g}_{bd}T_{ac})+FL\,C_{dabc}.
\end{aligned}
\tag{29}
$$

Symmetrizing (29) in d, a we find

$$
(Q(\overline{S},T))_{adbc}+\left(L_R-\frac{(n-3)L}{n-2}\right)(Q(\overline{g},T))_{adbc}=\frac{2FL}{n-2}\,(Q(\overline{g},\overline{S}))_{adbc}.
\tag{30}
$$

Now (30), in view of Lemma 2.4(ii) of [35], yields our assertion. □

4. Warped products satisfying (1)

Let $\overline{M}\times_F\widetilde{N}$ with $\dim\overline{M}=n-1\geq 3$ and $\dim\widetilde{N}=1$ be a Roter type manifold. Thus (5) holds on $\mathcal{U}\subset\overline{M}\times_F\widetilde{N}$. In the local components (1) reads

$$R_{hijk}=\phi\,(S_{hk}S_{ij}-S_{hj}S_{ik})+\mu\,(g\wedge S)_{hijk}+\eta\,G_{hijk}, \tag{31}$$

where R_{hijk}, G_{hijk}, S_{hk} and g_{hk} are the local components of the tensors R, G, S and g, respectively. Setting in (31) $h=a$, $i=b, j=c, k=d$ and $h=k=n, i=a, j=b$ we get

$$\begin{aligned}R_{abcd}&=\phi\,(S_{ad}S_{bc}-S_{ac}S_{bd})+\mu\,(g\wedge S)_{abcd}+\eta\,G_{abcd},\\ R_{nabn}&=\phi\,S_{nn}S_{ab}+\mu\,(g_{ab}S_{nn}+g_{nn}S_{ab})+\eta\,g_{ab}g_{nn},\end{aligned} \tag{32}$$

respectively. The last relation, by making use of (20) and (21), turns into

$$T_{ab}=(\phi\,\mathrm{tr}\,T-2\mu F)\,S_{ab}-(\mu\,\mathrm{tr}\,T-2\eta F)\,g_{ab}. \tag{33}$$

Proposition 4.1. *Let $V\subset\mathcal{U}\subset\overline{M}\times_F\widetilde{N}$ be a coordinate neighbourhood of $x\in\mathcal{U}$ in the warped product $\overline{M}\times_F\widetilde{N}$ with $\dim\overline{M}=n-1\geq 3$ and $\dim\widetilde{N}=1$.*

(i) *If (5) holds on V then on this set we have*

$$H_{ae}\overline{g}^{ef}S_{fb}=\left(nL_R-\frac{\mathrm{tr}\,H}{F}\right)H_{ab}. \tag{34}$$

(ii) *If (1) holds on V then on this set we have*

$$H^2_{ab}=\left(\rho\left(L_R-\frac{\mathrm{tr}\,H}{2F}\right)+\rho_0\right)H_{ab}, \tag{35}$$

where $H^2_{ab}=H_{ae}\overline{g}^{ef}H_{fb}$, $\mathrm{tr}\,H=\frac{\mathrm{tr}\,T}{2}+(n-1)FL_R$ and

$$\rho=\frac{\phi\,\mathrm{tr}\,T}{2}-\mu F,\qquad \rho_0=FL_R+\frac{\mu\,\mathrm{tr}T}{2F}-\eta F. \tag{36}$$

(iii) *Let (1) be satisfied on V. If the tensor $\overline{S}-\frac{\overline{\kappa}}{n-1}\overline{g}$ is non-zero at x then on some non-empty open set $W\subset V$ we have*

$$\overline{R}_{abcd}=\overline{\phi}\,(\overline{S}_{ad}\overline{S}_{bc}-\overline{S}_{ac}\overline{S}_{bd})+\overline{\mu}\,(\overline{g}\wedge\overline{S})_{abcd}+\overline{\eta}\,\overline{G}_{abcd}, \tag{37}$$

where

$$\overline{\phi}=\alpha^2\phi,\qquad \overline{\mu}=\alpha(\mu-\alpha\beta\phi),\qquad \overline{\eta}=\alpha^2\beta^2\phi-2\alpha\beta\mu+\eta, \tag{38}$$

$$\alpha=\left(1-\mu+\frac{\phi\mathrm{tr}\,T}{2F}\right)^{-1},\qquad \beta=-\frac{\mu\mathrm{tr}\,T}{2F}+\eta. \tag{39}$$

Moreover, on W we have

$$L_R=L_{\overline{R}}, \tag{40}$$

where L_R is defined by (7) and

$$L_{\overline{R}}=\overline{\phi}^{-1}\,((n-3)(\overline{\mu}^2-\overline{\phi}\overline{\eta})-\overline{\mu}).$$

Proof. First of all we note that on V the Eq. (26) by (25) is equivalent to

$$H_{de}\overline{g}^{ef}\overline{R}_{fabc}=\frac{1}{F}(H_{ac}H_{bd}-H_{ab}H_{cd})+L_R(g_{ab}H_{cd}-g_{ac}H_{bd}). \tag{41}$$

This by contraction with $\overline{g}^{ab}$ yields

$$H_{de}\overline{g}^{ef}\overline{S}_{fc}=\frac{1}{F}H^2_{cd}+\left((n-1)L_R-\frac{\mathrm{tr}\,H}{F}\right)H_{cd},$$

hence, by an application of (21), we get

$$H_{de}\overline{g}^{ef}S_{fc}+\frac{1}{2F}H_{de}\overline{g}^{ef}T_{fc}=\frac{1}{F}H^2_{cd}+\left((n-1)L_R-\frac{\mathrm{tr}\,H}{F}\right)H_{cd}.$$

Similarly, from the last equation by making use of (25) we obtain

$$H_{de}\overline{g}^{ef}S_{fc}+\frac{1}{F}H_{dc}^2=\frac{1}{F}H_{cd}^2+\left(nL_R-\frac{\operatorname{tr}H}{F}\right)H_{cd},$$

i.e., the equality (34).

(ii) We note that (5) holds on V. Further, (33) by making use of (25) and (36) turns into

$$H_{ab}=\rho\,S_{ab}+\rho_0\,\overline{g}_{ab}, \tag{42}$$

hence by transvection with $\overline{g}^{bc}H_{cd}$ we obtain

$$\rho\,H_{dc}\overline{g}^{cb}S_{ba}=H_{da}^2-\rho_0\,H_{da}.$$

Applying in this (34) we get (35), which completes the proof of (ii).

(iii) From (33), by an application of (21) and (39), we get

$$\overline{S}_{ab}=\left(1-\mu+\frac{\phi\operatorname{tr}T}{2F}\right)S_{ab}+\beta\overline{g}_{ab}. \tag{43}$$

Since the tensor $\overline{S}-\frac{\overline{\kappa}}{n-1}\overline{g}$ is non-zero at x, then also the function $(1-\mu+\frac{\phi\operatorname{tr}T}{2F})$ must be non-zero at every point of some non-empty open set $W\subset V$. Therefore (43) and (39) implies on W

$$S_{ab}=\alpha(\overline{S}_{ab}-\beta\overline{g}_{ab}). \tag{44}$$

Applying this and (38) into (32) we obtain (37). Finally, using (7) and (38) we can check that (40) holds on W, which completes the proof of our proposition. □

Proposition 4.2. *Let $\overline{M}\times_F\widetilde{N}$ be the warped product of a space of constant curvature $(\overline{M},\overline{g})$, $\dim\overline{M}=n-1\geq 3$, and a line or a circle $(\widetilde{N},\widetilde{g})$.*

(i) *If (5) holds on $\mathcal{U}\subset\overline{M}\times_F\widetilde{N}$ then on this set we have:*

$$\operatorname{rank}H=1, \tag{45}$$

$$L_R=\frac{\overline{\kappa}}{(n-2)(n-1)}. \tag{46}$$

(ii) *If (5) and (46) hold on $\mathcal{U}\subset\overline{M}\times_F\widetilde{N}$ then on this set we have*

$$Q\left(S-\frac{\overline{\kappa}}{n-2}\,g,R-\frac{\overline{\kappa}}{(n-1)(n-2)}\,G\right)=0. \tag{47}$$

Moreover, if at $x\in\mathcal{U}$ we have $\operatorname{rank}(S-\frac{\overline{\kappa}}{n-2}\,g)>1$ then on some non-empty open set $W\subset\mathcal{U}$ we have

$$R-\frac{\overline{\kappa}}{(n-2)(n-1)}\,G=\frac{\phi}{2}\left(S-\frac{\overline{\kappa}}{n-2}\,g\right)\wedge\left(S-\frac{\overline{\kappa}}{n-2}\,g\right). \tag{48}$$

Proof. (i) Let $V\subset\mathcal{U}$ be a coordinate neighbourhood of a point $x\in\mathcal{U}$. In view of Lemma 4 of [31] we can state that (5) is equivalent to (26) on V. Further, in the proof of Proposition 4.1 it was noted that on V the relations (26) and (41) are equivalent. Since $(\overline{M},\overline{g})$ is a space of constant curvature, (41) turns into

$$H_{ac}H_{bd}-H_{ab}H_{cd}=F\left(\frac{\overline{\kappa}}{(n-1)(n-2)}-L_R\right)(g_{ab}H_{cd}-g_{ac}H_{bd}),$$

which gives

$$2(H_{ac}H_{bd}-H_{ab}H_{cd})=F\left(\frac{\overline{\kappa}}{(n-1)(n-2)}-L_R\right)(g\wedge H)_{adcb}.$$

If $H=0$ at x then, by making use of the formulae (12)–(16) of [31], we can check that the Weyl tensor C of the considered warped product vanishes at x, a contradiction. Thus H is non-zero at x. Now, in view of Lemma 3.1 of [7], $\operatorname{rank}H=1$ and (46) hold on V.

(ii) It is clear that (11), by making use of (10) and (46), yields (47). But this, in view of Proposition 4.1 of [23], implies

$$R-\frac{\overline{\kappa}}{(n-2)(n-1)}\,G=\frac{\lambda}{2}\left(S-\frac{\overline{\kappa}}{n-2}\,g\right)\wedge\left(S-\frac{\overline{\kappa}}{n-2}\,g\right),$$

where λ is some function on $\mathcal{U}$. Since the decomposition of R is unique on $\mathcal{U}$, we have $\lambda=\phi$ on $\mathcal{U}$. The last remark completes the proof. □

Example 4.1. Let $\overline{M}\times_F\widetilde{N}$ be the warped product of a space of constant curvature $(\overline{M},\overline{g})$, $\dim\overline{M}=n-1\geq 3$, and a line or a circle $(\widetilde{M},\widetilde{g})$ satisfying (5) on $\mathcal{U}\subset\overline{M}\times_F\widetilde{N}$.

(i) In view of Proposition 3.1(i), on some neighbourhood $V\subset\mathcal{U}$ of a point $x\in\mathcal{U}$ (45) reads

$$\frac{1}{2}\,T_{ab}+FL_R\,\overline{g}_{ab}=\varepsilon\,u_au_b,\quad \varepsilon=\pm1, \tag{49}$$

where u_a are the local components of a covector u at x. Applying (46) and (49) in (21) we obtain

$$S_{ab}-\frac{\overline{\kappa}}{n-2}\,g_{ab}=-\frac{\varepsilon}{F}\,u_au_b,$$
$$S_{nn}-\frac{\overline{\kappa}}{n-2}\,g_{nn}=-\frac{\varepsilon}{F}\overline{g}^{ef}u_eu_f\,g_{nn}.$$

From Proposition 3.1(ii) it follows that (48) holds for all points of V at which $g^{ef}u_eu_f\neq 0$.

(ii) (cf. [33]) (Example 5.2) and [12] (Example 4.1). Let $\overline{M}$ be a non-empty open connected subset of $\mathbb{R}^{n-1}$, $n\geq 4$, equipped with the standard metric $\overline{g}$, $\overline{g}_{ab}=\varepsilon_a\,\delta_{ab}$, $\varepsilon_a=\pm1$. We set $F=F(x^1,\ldots,x^p)=\exp(\xi_ax^a)$, where $\xi_1,\ldots,\xi_{n-1}$ are constants such that $\xi_1^2+\cdots+\xi_{n-1}^2>0$ and $\overline{g}^{ab}\xi_a\xi_b>0$. We consider the warped product $\overline{M}\times_F\tilde{N}$. Now (16), (20) and (21) turn into

$$R_{abcd}=0,\qquad R_{nabn}=-\frac{1}{4}\,\xi_a\xi_b\,g_{nn},$$
$$T_{ab}=\frac{F}{2}\,\xi_a\xi_b,\qquad \operatorname{tr}T=\overline{g}^{ab}\xi_a\xi_b>0,$$
$$S_{ab}=-\frac{1}{4}\,\xi_a\xi_b,\qquad S_{nn}=-\frac{\operatorname{tr}T}{2F}\,g_{nn}, \tag{50}$$

respectively. Since $\overline{\kappa}=0$, (48) yields

$$R_{nabn}=\phi\,S_{nn}S_{ab},$$

which by (50) implies $\phi=-\frac{2F}{\operatorname{tr}T}$.

Proposition 4.3. *Let $V\subset\mathcal{U}\subset\overline{M}\times_F\widetilde{N}$ be a coordinate neighbourhood of $x\in\mathcal{U}$ in the warped product $\overline{M}\times_F\widetilde{N}$ with $\dim\overline{M}=n-1\geq 3$ and $\dim\widetilde{N}=1$ such that*

$$\overline{C}_{abcd}=0 \tag{51}$$

and (1) holds on V. If the tensor $\overline{S}-\frac{\overline{\kappa}}{n-1}\,\overline{g}$ is non-zero at x then on some non-empty open set $W\subset V$ we have

$$\overline{S}_{ab}=\left(\frac{1}{n-3}-\overline{\mu}\right)\overline{\phi}^{-1}\,\overline{g}_{ab}+\varepsilon\,u_au_b,\quad \varepsilon=\pm1, \tag{52}$$

$$\left(\frac{1}{n-3}-\overline{\mu}\right)^2=\overline{\phi}\left(\frac{\overline{\kappa}}{(n-3)(n-2)}+\overline{\eta}\right), \tag{53}$$

$$S_{ab}=\alpha\left(\left(\frac{1}{n-3}-\overline{\mu}\right)\overline{\phi}^{-1}-\beta\right)\overline{g}_{ab}+\varepsilon\alpha\,u_au_b, \tag{54}$$

$$H_{ab}=\left(FL_R+\frac{\mu\operatorname{tr}T}{2}-\eta F+\alpha\left(\frac{\phi\operatorname{tr}T}{2}-\mu F\right)\left(\left(\left(\frac{1}{n-3}-\overline{\mu}\right)\overline{\phi}^{-1}-\beta\right)\right)\right)\overline{g}_{ab}$$
$$+\varepsilon\alpha\left(\frac{\phi\operatorname{tr}T}{2}-\mu F\right)u_au_b, \tag{55}$$

where u_a are the local components of a covector on W and the functions $\overline{\phi}$, $\overline{\mu}$, $\overline{\eta}\alpha$, β, and $L_{\overline{R}}$, are defined by (38)–(40), respectively.

Proof. First of all we note that in view of Proposition 4.1(iii) (37) holds on some non-empty set $W\subset V$. Now from (37) and (51) it follows

$$\overline{S}_{ad}\overline{S}_{bc}-\overline{S}_{ac}\overline{S}_{bd}=\left(\frac{1}{n-3}-\overline{\mu}\right)\overline{\phi}^{-1}\,(\overline{g}\wedge\overline{S})_{abcd}-\left(\frac{\overline{\kappa}}{(n-3)(n-2)}+\overline{\eta}\right)\overline{\phi}^{-1}\,\overline{G}_{abcd}. \tag{56}$$

Applying Lemma 3.1 of [7] to (56) we obtain (52) and (53). Further, (54) is an immediate consequence of (44) and (52). Finally, (42) and (54) lead to (55), which completes the proof. □

Example 4.2. Consider a semi-Riemannian 4-manifold (M, g) the metric of which can be expressed as

$$g = dt^2 - e^{2\theta t}\left(dx^2 + \Omega(x)dy^2 + dz^2\right)$$

in local coordinates (t, x, y, z), where θ is a real constant, and Ω is some nowhere vanishing function. On the set where

$$\xi(x) = 2\Omega''(x)\Omega(x) - \Omega'(x)^2$$

is non-zero, define the functions

$$\mu(x) = 12e^{2\theta t}\theta^2\Omega(x)^2\xi(x)^{-1},$$
$$\phi(x) = \frac{1}{3\theta^2}\mu(x),$$
$$\eta(x) = \theta^2\left(1 + 3\mu(x)\right).$$

Then Eq. (1) is satisfied with the above coefficients. For the non-vanishing components of $Z = S - \Lambda g$ with $\Lambda \in \mathbb{R}$ one has

$$-Z_{tt} = Z_{zz} = 3\theta^2 + \Lambda\,,$$

and for the case $\Lambda = -3\theta^2$ we compute

$$Z_{xx} = Z_{yy} = \frac{1}{4}\Omega^{-2}\xi(x).$$

Thus, (M, g) is an Einstein space if and only if ξ vanishes everywhere. In the given coordinates, up to affine coordinate transformations, this is the case if and only if $\Omega(x) = \pm x^2$ for $x > 0$. Here, we even have a space of constant curvature, and for the Lorentzian case in particular, the de Sitter space.

Computing the components of the Weyl tensor, one finds that the metrics with $\xi = 0$ are also the only conformally flat ones in this class of metrics.

The semi-Riemannian manifold (M, g) can be viewed as a warped product $\overline{M} \times \widetilde{N}$ with $\dim \widetilde{N} = 1$, metric on $\overline{M}$

$$\overline{g} = dt^2 - e^{2\theta t}\left(dx^2 + \Omega(x)dy^2\right)$$

and warping function $F(t) = \exp(2\theta t)$. In view of Proposition 4.1, one has $\alpha = 1$ and $\beta = \theta^2$. Thus (if $\xi \neq 0$) the Roter type equation is also satisfied on $\overline{M}$ with coefficients

$$\overline{\phi} = \phi,$$
$$\overline{\mu} = -\theta^2\phi + \mu,$$
$$\overline{\eta} = \theta^4\phi - 2\theta^2\mu + \eta.$$

Since $\dim \overline{M} = 3$, we have $\overline{C} = 0$ and all requirements for Proposition 4.3 are satisfied; for example we have

$$\begin{aligned}\overline{S}_{ab} &= \left(1 - \theta^2\phi + \mu\right)\phi^{-1}g_{ab} + \epsilon u_a u_b \\ &= \left(2\theta^2 - \frac{1}{4}\xi(x)\Omega(x)^2e^{-2\theta t}\right)g_{ab} + \epsilon u_a u_b\end{aligned}$$

with some covector u.

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