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Polynomial separable indefinite natural systems

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ABSTRACT

We review the conditions for separability of 2-dimensional indefinite natural Hamiltonian systems. We examine the possibility that the separability condition is satisfied on a given energy hypersurface only (weak integrability) and derive the additional requirement necessary to have separability at arbitrary values of the Hamiltonian (strong integrability). We give a list of separable polynomial potentials and discuss the kind of separable structures they admit.

1. Introduction

The method of separation of variables stays at the foundations of analytical mechanics. The application to classical and quantum Hamiltonian systems leads to a thorough understanding of the conditions for separability and of the form of separated solutions. Even restricting to natural systems implies an enormous bibliography, among which we just mention the classical contributions of Jacobi [1] and Eisenhart [2] and the more recent systematic work by Benenti [3]. It is therefore surprising to observe that in the case of natural systems with indefinite kinetic energy, or *indefinite systems* for short, a systematic investigation of their separation has been performed only very recently [4,5]. Maybe, the reasons for this neglect stay in the prevalent application of indefinite natural systems in celestial mechanics [6], where the main focus is on non-integrability and perturbation methods. In this framework, the role of indefinite quadratic Hamiltonians is quite clear in the context of normal form theory [7] even if we are just starting to understand the structure of indefinite resonant normal forms [8]. On the other hand, an exhaustive investigation of separable potentials of natural indefinite systems is still lacking even for the simplest case of two degrees of freedom.

The study of separation of variables by means of a general approach based on conformal coordinate transformations was introduced in [9] and applied in [10,11]. It has been generalized in [4] to get a complete classification of separating coordinate systems of the *indefinite* Hamilton–Jacobi equation with the corresponding formal separated potentials and second integral of motion. Separability of free motion on the flat hyperbolic plane has already been investigated [12,13]. However, in [4,5] a more general picture is provided, showing how different kinds of separation structures appear. In particular, it is in general necessary to use different separating variables, even for the integration of a single orbit [14]. Associating as usual the existence of a 2nd-rank Killing tensor to that of a system of separating coordinates [2], the picture can be illustrated as

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follows: for (1 + 1)-dimensional systems there are three possible types of conformal Killing tensors, and therefore, three distinct separability structures in contrast to the single standard (*Liouville*) type separation of the positive definite case [10]. One of the new separability structures is the *complex*-Liouville/harmonic type which is characterized by complex separation variables and the potential is a harmonic function. The other new type is the *linear/null* separation which occurs when the conformal Killing tensor has a null eigenvector so that it has the structure of a Jordan block and the potential depends only linearly on one of the separation variables.

Aim of this paper is to rework the presentation of standard quadratic separability in a more 'mechanical' (i.e. less geometric) fashion, adopting the usual 'direct' approach [15,16]. However, we will find useful in several places to exploit also geometrical properties. Several examples are detailed and, in this case, a comparison with the standard positive definite counterpart is made. Since they are useful toy models for mechanical applications, a special emphasis is accorded to classify indefinite separable *polynomial* potentials: it turns out that the set of separable polynomial families is quite larger than in the positive definite case.

The layout of the paper is as follows: in Section 2 we introduce the coordinate transformations which preserve the Hamiltonians in 'null' form; in Section 3 we get the systems admitting an integral of the motion quadratic in the momenta; Section 4 accounts for the systems with polynomial potentials; Section 5 contains concluding remarks.

2. Separation of 2-dimensional and of (1 + 1)-dimensional systems

We consider the general 2-degrees of freedom natural system described by the Hamiltonian

$$H = \tfrac{1}{2}(p_x^2 + \sigma p_y^2) + V(x, y) \tag{1}$$

where $\sigma = +1$ for positive definite 2-dimensional systems and $\sigma = -1$ for indefinite, (1 + 1)-dimensional systems. We may then look for a phase–space function $I = I(p_x, p_y, x, y)$ preserved along the flow given by Eq. (1), namely such that

$$\{I, H\} = 0. \tag{2}$$

Following the approach already used in the positive definite case [10,17–19], we consistently define, for any given energy E of the system, the *null* Hamiltonian

$$\mathscr{H} \equiv H - E = \tfrac{1}{2}(p_x^2 + \sigma p_y^2) - G(x, y; E), \tag{3}$$

where

$$G \equiv E - V. \tag{4}$$

In the positive definite case it turns out to be very helpful to work with complex variables [10,20]. In the indefinite case, it is useful instead [4] to work with coordinates which are null (*lightlike*) with respect to the pseudo-Euclidean metric

$$ds^2 = dx^2 - dy^2. \tag{5}$$

In both cases, such variables are naturally adapted to the action of the conformal group which plays an essential role for 2-dimensional systems. Introducing the label

$$\varepsilon = i^{\frac{1+\sigma}{2}} = \begin{cases} i & (\sigma = +1), \\ 1 & (\sigma = -1), \end{cases} \tag{6}$$

we perform the canonical point transformation given by

$$z = x + \varepsilon y, \qquad p = \frac{1}{2}\left(p_x + \frac{1}{\varepsilon} p_y\right), \tag{7}$$

$$\hat{z} = x - \varepsilon y, \qquad \hat{p} = \frac{1}{2}\left(p_x - \frac{1}{\varepsilon} p_y\right), \tag{8}$$

so that the null Hamiltonian can be written in the form

$$\mathscr{H} = 2p\hat{p} - G(z, \hat{z}; E). \tag{9}$$

The forms of the null Hamiltonian and of its conformal transforms are the same in both cases $\sigma = \pm 1$ and we can follow a unified approach based on the direct search of polynomial constants of the motion. To come back to the explicit forms in the two cases, in the positive definite case the variable z is the standard complex coordinate on the plane so that the hat can be interpreted as the bar of complex conjugation,

$$z = x + iy, \tag{10}$$

$$\hat{z} \equiv \bar{z} = x - iy. \tag{11}$$

In the indefinite case, the variables are real,

$$z = x + y, \tag{12}$$

$$\hat{z} = x - y, \tag{13}$$

and give a null coordinate system on the Minkowski plane $\mathbb{M}^2$.

In the direct approach an ansatz is made for the 'candidate' integral

$$I = I(p, \hat{p}, z, \hat{z}), \tag{14}$$

and one tries to solve Eq. (2). The usual ansatz is the polynomial one [16]. Separability is associated with integrals linear or quadratic in the momenta. We will find it useful to exploit a conformal transformation to standardize the frame and the coordinate representation of the integral of motion [9]. To that end we may introduce new coordinates $w, \hat{w}$ by means of the analytic functions

$$z = F(w), \tag{15}$$

$$\hat{z} = \hat{F}(\hat{w}). \tag{16}$$

The conformal transformation

$$z \to w \equiv X + \varepsilon Y, \qquad \hat{z} \to \hat{w} \equiv X - \varepsilon Y \tag{17}$$

is given by (15), (16) and determines the canonical point transformation

$$w = F^{-1}(z), \qquad P = F'p, \tag{18}$$

$$\hat{w} = \hat{F}^{-1}(\hat{z}), \qquad \hat{P} = \hat{F}'\hat{p}, \tag{19}$$

where the apex denotes the derivative and $P, \hat{P}$ are the momenta conjugate to $w, \hat{w}$. The transformation induces a corresponding canonical transformation from the old physical coordinates p_x, p_y, x, y to the new ones P_X, P_Y, X, Y where X, Y are introduced in (17) and, in analogy with (7), (8),

$$P = \frac{1}{2}\left(P_X + \frac{1}{\varepsilon}P_Y\right), \tag{20}$$

$$\hat{P} = \frac{1}{2}\left(P_X - \frac{1}{\varepsilon}P_Y\right). \tag{21}$$

Under the transformation (18), (19), the Hamiltonian (9) transforms into the new null Hamiltonian

$$\widetilde{\mathscr{H}} = \frac{2P\hat{P} - \widetilde{G}(w, \hat{w}; E)}{F'\hat{F}'}, \tag{22}$$

where

$$\widetilde{G} = F'\hat{F}'G \tag{23}$$

is the new 'potential'. On the 'shell' $\widetilde{\mathscr{H}} = 0$ we can work with the 'standard' null Hamiltonian

$$\widetilde{\mathscr{H}_S} = 2P\hat{P} - \widetilde{G}(w, \hat{w}; E) = 0. \tag{24}$$

A conserved quantity stays conserved if transformed between the two gauges (22) and (24). In terms of real variables, (24) is given by

$$\widetilde{\mathscr{H}_S} = \frac{1}{2}(P_X^2 + \sigma P_Y^2) - \widetilde{G}(X, Y; E) = 0. \tag{25}$$

In the following, we will refer to X and Y as *separating* variables, because, as shown in [5,4], separation of the Hamilton–Jacobi equation

$$\left(\frac{\partial \mathscr{W}}{\partial X}\right)^2 + \sigma\left(\frac{\partial \mathscr{W}}{\partial Y}\right)^2 - 2\,\widetilde{G}(X, Y; E) = 0 \tag{26}$$

occurs in general, even if in a nonstandard fashion. The procedure common to all cases is based on a conformal transformation that allows us to get a unique form of the linear differential equation for the integrability condition of the potential, the same for every system of coordinates. Specific separating coordinates are determined by 'generating' functions that at the same time define the leading order term of the integral. For weak integrability (namely integrability at a fixed value of the energy, [21]) those functions are arbitrary, whereas strong integrability ('standard' integrability at arbitrary values of the energy) requires that they must be polynomials [20].

In the positive definite case, only the region $V < E$ is relevant implying $G > 0$, whereas this is in general not necessary in the indefinite case. However, the additional very important difference is that $z, \hat{z}$ and $w, \hat{w}$ are complex conjugates in the positive definite case, while they are real variables in the indefinite case. This difference turns out to be crucial and leads to a much richer structure than in the positive definite case.

3. Separability and quadratic constants of the motion

The present paper is devoted to the indefinite case. However, for the sake of completeness and to enlighten the similarity between the two settings, we present also the positive definite case in a unified approach.

3.1. The quadratic integral

We look for systems admitting a second integral of motion which is a quadratic function in the momenta. The ansatz is

$$\mathscr{I}_2 = Sp^2 + \hat{S}\hat{p}^2 + \frac{1}{2}K, \tag{27}$$

where the two coefficients S, $\hat{S}$ and K are assumed to be functions of the coordinates $z, \hat{z}$. We remark that (27) is indeed the most general quadratic polynomial, since mixed terms of the form $p\hat{p}$ are absorbed in K via the Hamiltonian constraint (9) and linear terms are absent due to the reversibility of the system. We apply the direct method [22,15,23,16], so the system of equations ensuing from the conservation condition (2) is the following [20]:

$$S_{\hat{z}} = 0, \tag{28}$$

$$\hat{S}_z = 0, \tag{29}$$

$$K_z + 2\hat{S}G_{\hat{z}} + \hat{S}_{\hat{z}}G = 0, \tag{30}$$

$$K_{\hat{z}} + 2SG_z + S_zG = 0 \tag{31}$$

where, from hereinafter, with the subscript we denote the partial derivative with respect to the corresponding variable. Looking at this system, we see that Eqs. (28), (29) are readily solved:

$$S = S(z), \qquad \hat{S} = \hat{S}(\hat{z}); \tag{32}$$

that is S and $\hat{S}$ are arbitrary functions of a single variable. Concerning Eqs. (30)–(31) we now have the following 'Darboux' integrability condition

$$2G_{zz}S(z) - 2G_{\hat{z}\hat{z}}\hat{S}(\hat{z}) + 3G_zS'(z) - 3G_{\hat{z}}\hat{S}'(\hat{z}) + G[S''(z) - \hat{S}''(\hat{z})] = 0 \tag{33}$$

where (32) is already exploited. As discussed above we can simplify Eq. (33) by using a conformal transformation like in (17). The solution of the integrability condition is determined by the structure of the conformal part of the integral [4,5]. If the product $S\hat{S}$ is strictly positive the procedure is the same in the definite and indefinite cases ('Liouville' separability), if $S\hat{S}$ vanishes and/or changes sign we get respectively the 'null/linear' and the 'complex/harmonic' separability. These three possibilities affect the form of the generalized Darboux equation (33) in the new variables. In fact, recalling the conformal potential (23), the coordinate transformations leading to the standard form of the integral, provide the three different equations

$$\widetilde{G}_{ww} - s\,\widetilde{G}_{\hat{w}\hat{w}} = 0, \quad s = +1, -1, 0. \tag{34}$$

We now examine each of these cases in turn.

3.2. Liouville separability

In the standard case $s = +1$, the separation variables are directly given by [9,24,5]

$$\frac{dz}{dw} = F'(w) \equiv \sqrt{S(z(w))}, \tag{35}$$

$$\frac{d\hat{z}}{d\hat{w}} = \hat{F}'(\hat{w}) \equiv \sqrt{\hat{S}(\hat{z}(\hat{w}))}, \tag{36}$$

or equivalently

$$w = \int \frac{dz}{\sqrt{S}}, \qquad \hat{w} = \int \frac{d\hat{z}}{\sqrt{\hat{S}}}. \tag{37}$$

In the new variables $(w, \hat{w})$ the generalized Darboux equation becomes

$$\widetilde{G}_{ww} - \widetilde{G}_{\hat{w}\hat{w}} = \widetilde{G}_{XY} = 0 \tag{38}$$

the solution of which, like in the positive definite case, is

$$\widetilde{G} = B_1(X) + B_2(Y), \tag{39}$$

with B_1 and B_2 arbitrary functions of their argument. The null Hamiltonian then takes the explicitly separated form

$$\widetilde{\mathscr{H}_S} = 2P\hat{P} - \widetilde{G}(w, \hat{w}; E) = \frac{1}{2}(P_X{}^2 - P_Y{}^2) + B_1(X) + B_2(Y). \tag{40}$$

The equations for the trace (30)–(31) become

$$K_X = -2\widetilde{G}_X, \qquad K_Y = 2\widetilde{G}_Y. \tag{41}$$

Using (39), the solution is

$$K = 2\,[B_2(Y) - B_1(X)] \tag{42}$$

and the second integral of motion (27) can be written as

$$\mathscr{I}_2 = P^2 + \hat{P}^2 + \frac{1}{2}K = \frac{1}{2}(P_X^2 + p_Y^2) + B_2(Y) - B_1(X). \tag{43}$$

3.3. Complex/harmonic separability

Let us now consider the case $S\hat{S} < 0,\; s = -1$, which has no counterpart for positive definite systems. The separation variables ((18)–(19)) are now given by

$$w = \int \frac{dz}{\sqrt{S}}, \qquad \hat{w} = -\int \frac{d\hat{z}}{\sqrt{\hat{S}}} \tag{44}$$

and the form of the generalized Darboux equation written in the separation variables then changes from the wave equation to the Laplace equation

$$\widetilde{G}_{ww} + \widetilde{G}_{\hat{w}\hat{w}} = \tfrac{1}{2}(\widetilde{G}_{XX} + \widetilde{G}_{YY}) = 0. \tag{45}$$

It follows that the general solution is a harmonic function given by

$$\widetilde{G} = \Re\{Q(Z)\}, \tag{46}$$

where $Q(Z)$ is an arbitrary holomorphic function of $Z = X + iY$. This means that the system separates if it is written in the complex variables Z and $\bar{Z} = X - iY$, since the null Hamiltonian can be written as

$$\mathscr{H} = p_Z^2 + p_{\bar{Z}}^2 + \Re\{Q(Z)\}. \tag{47}$$

We therefore refer to this case as *harmonic* or *complex separation* in contrast to the additive Hamilton–Jacobi separation. The equations for the trace ((30)–(31)) are

$$K_Z + K_{\bar{Z}} = 2i(\widetilde{G}_{\bar{Z}} - \widetilde{G}_Z), \qquad K_Z - K_{\bar{Z}} = -2i(\widetilde{G}_{\bar{Z}} + \widetilde{G}_Z). \tag{48}$$

Using (46), we find that the solution is given by

$$K = i\left(Q(Z) - \hat{Q}(\bar{Z})\right). \tag{49}$$

The second integral then takes the form

$$\mathscr{I}_2 = P_X P_Y + \frac{1}{2}K = i(P_Z^2 - p_{\bar{Z}}^2) - \Im\{Q(Z)\}. \tag{50}$$

3.4. Linear/null separation

We finally consider the third type of separation which occurs when $s = 0$ and for which there is again no counterpart in the positive definite case. We may then assume $S \neq 0$ and $\hat{S} = 0$ (or vice versa). The generalized Darboux equation becomes

$$\widetilde{G}_{ww} = 0. \tag{51}$$

One separation variable is $w = A(z)$ where $A(z)$ satisfies $[A'(z)]^{-2} = |S(z)|$ as in the previous cases. There is no restriction on the other variable which can therefore be any function independent of w. It follows that the general solution must have the form

$$\widetilde{G} = C(\hat{w})w + D(\hat{w}), \tag{52}$$

where $C(\hat{w})$ and $D(\hat{w})$ are arbitrary functions. This case is referred to as *linear* or *null separation*, but is mentioned as *Lie* case in Ref. [24]. The terminology 'null separation' stems from the fact that the conformal Killing tensor has a double null (or lightlike) eigenvector [5]. The integrability by quadrature can be checked by following the recipe provided in [4, Section III.E] and [5, Section 2.2.2].

Table 1
The possible conformal transformation functions for (1 + 1)-dimensional integrable Hamiltonians with a second degree invariant.

1.	$F_1(w) = w$	$A_1(z) = z$	$S_1(z) = 1$	$k = b = 0,\ c \neq 0$
2.	$F_2(w) = w^2$	$A_2(z) = \sqrt{z}$	$S_2(z) = 4z$	$k = c = 0,\ b \neq 0$
3.	$F_3(w) = e^w$	$A_3(z) = \ln z$	$S_3(z) = z^2$	$k \neq 0,\ D = 0$
4.	$F_4(w) = \Delta \cosh w$	$A_4(z) = \mathrm{acosh}(z/\Delta)$	$S_4(z) = z^2 - \Delta^2$	$k \neq 0,\ D > 0$
5.	$F_5(w) = \Delta \sinh w$	$A_5(z) = \mathrm{asinh}(z/\Delta)$	$S_5(z) = z^2 + \Delta^2$	$k \neq 0,\ D < 0$

3.5. Strong separability

We now focus our attention on *strong* separability which, considering definition (4), means that the system is separable for arbitrary values of E. This is the ordinary separation property which has as a particular case the free (geodesic) motion on the 'flat' ($G = 1$) hyperbolic plane [12]. When the system is separable (or integrable) only at certain fixed values of E we speak of *weak* separability (or integrability in the general case of non-quadratic integrals [18]). A discussion of this topic for positive definite systems is given in [20].

Strong separation is obtained by imposing that integrability condition (33) should not depend on E leading to

$$S''(z) = \hat{S}''(\hat{z}). \tag{53}$$

The solutions of (53) are

$$S(z) = kz^2 + bz + c, \qquad \hat{S}(\hat{z}) = k\hat{z}^2 + \hat{b}\hat{z} + \hat{c} \tag{54}$$

where all arbitrary constants are real. In the positive definite case, the corresponding solution is $S(z) = az^2 + \beta z + \gamma$ [10], where a is a real constant whereas β and γ are complex. In both cases the total number of free constants is five and the leading order coefficients can be assumed to take the values 1 or 0.

In the positive definite case, there arises four distinct cases when evaluating $w(z)$ to obtain the separating variables $(w \pm \bar{w})$. They correspond precisely to the four classical cases of separability [10]: Cartesian, parabolic, polar and elliptical. In the indefinite case, the equations to be integrated are (37), so that the functions $w(z)$ and $\hat{w}(\hat{z})$ may assume five distinct forms. The forms are enumerated in Table 1, where

$$A \doteq F^{-1}$$

and, for the cases with $k = 0$, the standard forms with either $c = 1, b = 0$ or $c = 0, b = 4$ are chosen and, for the cases with $k \neq 0$,

$$D = b^2 - 4kc, \qquad \Delta = \tfrac{1}{2}\sqrt{|D/k|}. \tag{55}$$

The corresponding "hatted" quantities $\hat{A}(\hat{z})$, $\hat{F}(\hat{z})$, $\hat{D}$ and $\hat{\Delta}$ are defined in an analogous way in terms of $\hat{b}$ and $\hat{c}$. There appears a fifth class of transformations since, in the case of real variables, the choice of the hyperbolic sine or cosine provides two independent coordinate systems. Rather, in the positive definite case, the analogous transformation corresponds to a hyperbolic function of a *complex* variable which generates elliptic–hyperbolic coordinates: the choice of the hyperbolic sine rather than the cosine simply gives a $\pi/2$ rotation of the foci of the confocal families of coordinate lines.

When combining the five cases, we have to keep only combinations with the same value of the leading-order constant k, since k appears both in S and $\hat{S}$. Since there are no other restrictions, this condition gives four classes with $k = 0$ and nine with $k \neq 0$, thirteen classes in total. However, it is reasonable not to distinguish systems that can be transformed into each other by the inversion $(z, \hat{z}) \to (\hat{z}, z)$ or equivalently $x \to -x$. This reduces the number of inequivalent classes to three for $k = 0$ and six for $k \neq 0$, nine in total. Using the numbers 1–5 appearing in the first column of the table and the corresponding "hatted" figures $\hat{1}$–$\hat{5}$ the set of possible independent separating coordinates is given, in an obvious notation, by the combinations

$$\begin{array}{llllll} k = 0: & 1\hat{1} & 1\hat{2} & 2\hat{2} & & \\ k \neq 0: & 3\hat{3} & 3\hat{4} & 3\hat{5} & 4\hat{4} & 4\hat{5} \quad 5\hat{5}. \end{array} \tag{56}$$

For each combination of S and $\hat{S}$, we need to check the possible values of s (± 1 or 0): the negative sign can only be present when S_1, S_2 and S_4 are involved, whereas $S\hat{S}$ may only vanish when S_1 and S_2 are involved.

In each separable system, we have solutions with the metric factor given by $\widetilde{G} = \Psi G$, where $\Psi(X, Y) \doteq F'\hat{F}'$. We recall from (4) that the physical potential is given by

$$V = G + E = \frac{\widetilde{G}}{\Psi} + E. \tag{57}$$

For the 'strong' Liouville separability, it turns out that also the conformal factor associated with the corresponding coordinate transformation separates so that

$$\Psi(X,Y) = \Psi_1(X) + \Psi_2(Y). \tag{58}$$

Actually, it is straight-forward to check [25] that a necessary and sufficient condition for

$$\Psi_{XY} = 0$$

is provided just by (53). Using (57), we can then write the physical potential in the form

$$V = \frac{f_1(X) + f_2(Y)}{\Psi_1(X) + \Psi_2(Y)}, \tag{59}$$

where f_1 and f_2 are arbitrary functions. Referring to (40) we have

$$\begin{aligned} B_1(X) &= f_1(X) - E\Psi_1(X), \\ B_2(Y) &= f_2(Y) - E\Psi_2(Y), \end{aligned} \tag{60}$$

so that the physical Hamiltonian (1) in separating coordinates can be written as

$$H = \frac{1}{\Psi_1(X) + \Psi_2(Y)} \left[\frac{1}{2}({P_X}^2 - {P_Y}^2) + f_1(X) + f_2(Y) \right] \tag{61}$$

and the second integral of motion is

$$I_2 = \frac{1}{\Psi_1(X) + \Psi_2(Y)} \left[\Psi_2(Y)(P_X^2 - 2f_1(X)) + \Psi_1(X)(P_Y^2 + 2f_2(Y)) \right]. \tag{62}$$

In the case of complex/harmonic separation, the conformal factor always separates in the form

$$\Psi = \tfrac{1}{2}\big(\psi(Z) + \bar{\psi}(\bar{Z})\big) = \Re\{\psi(Z)\}, \tag{63}$$

so that, proceeding as in the Liouville case, we can write the physical potential in the form

$$V = \frac{Q_1(Z) + \bar{Q}_1(\bar{Z})}{\psi(Z) + \bar{\psi}(\bar{Z})} = \frac{\Re\{Q_1(Z)\}}{\Re\{\psi(Z)\}}, \tag{64}$$

where $Q_1(Z)$ is an arbitrary function. Referring now to (47) we can write

$$Q(Z) = Q_1(Z) - E\psi(Z) \tag{65}$$

which leads to

$$H = \frac{\Re\{2P_Z^2 + Q_1(Z)\}}{\Re\{\psi(Z)\}}, \tag{66}$$

while the second integral of motion becomes

$$I_2 = \frac{\Re\{-i\bar{\psi}(2P_Z^2 + Q_1(Z))\}}{\Re\{\psi(Z)\}}. \tag{67}$$

Finally, considering linear/null separation, the simplest case is that in which $S = S_1 = 1, \hat{S} = 0$ (Cartesian-zero case). The Hamiltonian and the second integral are then given by

$$\mathscr{H} = -2P_w P_{\hat{w}} + C(\hat{w})w + D(\hat{w}) \tag{68}$$

and the second invariant is

$$\mathscr{I}_2 = P_w^2 + \frac{1}{2}K = P_w^2 - \int C(\hat{w})d\hat{w} \tag{69}$$

and are therefore already in the desired separated form. For the other non-trivial cases, we refer to [4].

4. Polynomial separable systems

Here we list the cases of potentials which are polynomial in the 'physical' coordinates which separate in one of the classes given above. This list is compiled in analogy with that appearing in the seminal review by Hietarinta [16], where in the analysis of quadratic integrable systems also the complex cases were included. Under suitable transformations, some of these cases correspond to indefinite systems: however, the analysis performed in [16] is incomplete.

4.1. The Cartesian–Cartesian $(1\hat{1})$ class

The system separates in the physical coordinates coinciding with the separating coordinates. Actually, for the sake of simplicity, we do not consider dilations and rotations by assuming $c \neq \hat{c} \neq 1$ in (54). The conformal factor and the coordinate transformations have the trivial forms

$$
\begin{aligned}
&S_1(z) = 1, \qquad \hat{S}_1(\hat{z}) = 1 \\
&F_1(w) = w = z = A_1(z) \\
&\hat{F}_1(\hat{w}) = \hat{w} = s\hat{z} = \hat{A}_1(\hat{z}), \quad s = \begin{cases} +1 & \text{Liouville separation} \\ -1 & \text{complex/harmonic separation} \\ 0 & \text{linear/null separation} \end{cases} \\
&X = x, \qquad Y = y, \qquad s = 1 \\
&X = y, \qquad Y = x, \qquad s = -1.
\end{aligned}
\tag{70}
$$

In the linear/null case, only one separating variable is actually fixed, but it is convenient to use $\hat{w}(=\hat{z})$ as the second variable.

4.1.1. Sub-class $(1\hat{1})_+$

The Hamiltonian and the second integral are given by

$$
H = \tfrac{1}{2}({p_x}^2 - {p_y}^2) + x^m \pm y^n, \quad m, n \in \mathbb{N}, \qquad \tilde{I}_2 = \tfrac{1}{2}{p_x}^2 + x^m. \tag{71}
$$

4.1.2. Sub-class $(1\hat{1})_-$

Defining the complex variable $z = x + iy$, for a polynomial potential of degree m, the Hamiltonian and the second integral are given by

$$
\begin{aligned}
&H = \tfrac{1}{2}({p_x}^2 - {p_y}^2) + \Re\{z^m\} = {p_z}^2 + {p_{\bar{z}}}^2 + \tfrac{1}{2}(z^m + \bar{z}^m), \\
&I_2 = p_x p_y - \Im\{z^m\}, \qquad \tilde{I}_2 = {p_z}^2 + \tfrac{1}{2}z^m.
\end{aligned}
\tag{72}
$$

4.1.3. Sub-class $(1\hat{1})_0$

Since there is just one separating variable which we take as $w = z = x + y$, it is convenient to use the complementary null variable $\hat{w} = \hat{z} = x - y$ as the other independent variable. The Hamiltonian and the second integral are then given by the expressions

$$
\begin{aligned}
&H = \tfrac{1}{2}({p_x}^2 - {p_y}^2) + (x + y)(x - y)^{m-1} + (x - y)^m, \\
&I_2 = \tfrac{1}{4}(p_x + p_y)^2 - \frac{1}{m}(x - y)^m.
\end{aligned}
\tag{73}
$$

4.2. The Cartesian–parabolic $(1\hat{2})$ class

This is the first nontrivial class, since it combines rotated Cartesian and parabolic coordinates. As in the Cartesian–Cartesian class we do not include dilations and rotations. The conformal coordinate transformation is given by

$$
\begin{aligned}
&S_1(z) = 1, \qquad \hat{S}_2(\hat{z}) = 4\hat{z} \\
&w = A_1(z) = z, \qquad F_1(w) = w \\
&\hat{w} = \hat{A}_2(\hat{z}) = \sqrt{s\hat{z}}, \qquad \hat{F}_2(\hat{w}) = s\hat{w}^2, \quad s = \begin{cases} +1 & \text{Liouville separation} \\ -1 & \text{complex/harmonic separation} \\ 0 & \text{linear/null separation} \end{cases} \\
&X = \tfrac{1}{2}(x + y + \sqrt{x - y}), \qquad Y = \tfrac{1}{2}(x + y - \sqrt{x - y}), \qquad s = +1, \quad x > y \\
&X = \tfrac{1}{2}(x + y + \sqrt{y - x}), \qquad Y = \tfrac{1}{2}(x + y - \sqrt{y - x}), \qquad s = -1, \quad x < y.
\end{aligned}
\tag{74}
$$

Again, in the linear/null case, only one separating variable is actually fixed, but it is convenient to use $\hat{w}(=\hat{z})$ as the second variable.

4.2.1. Sub-class $(1\hat{2})_+$

The conformal factor is $\Psi = 2\hat{w} = 2(X - Y) = 2\sqrt{x - y}$, so that

$$
\Psi_1 = 2X, \qquad \Psi_2 = -2Y. \tag{75}
$$

This leads to the physical potential

$$
V = \frac{f_1(X) + f_2(Y)}{X - Y}. \tag{76}
$$

The Hamiltonian and the second integral are given by

$$H = \tfrac{1}{2}({p_x}^2 - {p_y}^2) + \frac{f_1\big(x+y+\sqrt{x-y}\big) + f_2\big(x+y-\sqrt{x-y}\big)}{2\sqrt{x-y}} \tag{77}$$

and

$$I_2 = \tfrac{1}{4}(p_x+p_y)^2 - 2(p_x-p_y)(yp_x+xp_y) + \frac{\big(x+y-\sqrt{x-y}\big)f_1(X) + \big(x+y+\sqrt{x-y}\big)f_2(Y)}{\sqrt{x-y}}. \tag{78}$$

With the choice

$$f_1 = X^m, \qquad f_2 = -Y^m, \quad m \in \mathbb{N}, \tag{79}$$

we get the sequence of polynomial potentials

$$V = x - y + 3(x+y)^2, \qquad m = 3, \tag{80}$$

$$V = x^2 - y^2 + (x+y)^3, \qquad m = 4, \tag{81}$$

$$V = (x-y)^2 + 10(x-y)(x+y)^2 + 5(x+y)^4, \qquad m = 5 \tag{82}$$

and so forth. These potentials can be freely superposed.

4.2.2. Sub-class $(1\hat{2})_-$

With the complex variable $Z = X + iY$, the separating variables are

$$Z = \tfrac{1}{2}\Big[(1+i)(x+y) - (1-i)\sqrt{y-x}\Big], \qquad \bar{Z} = \tfrac{1}{2}\Big[(1-i)(x+y) - (1+i)\sqrt{y-x}\Big] \tag{83}$$

so that the conformal factor is

$$\Psi = 2(Y - X) = (1+i)Z + (1-i)\bar{Z}. \tag{84}$$

This leads to a physical potential of the form

$$V = \frac{\Re\Big\{Q_1\Big(\tfrac{1}{2}\big[(1+i)(x+y) - (1-i)\sqrt{y-x}\,\big]\Big)\Big\}}{2\sqrt{y-x}}. \tag{85}$$

The Hamiltonian and the second integral are given by

$$H = \tfrac{1}{2}({p_x}^2 - {p_y}^2) + \frac{\Re\{Q_1(Z)\}}{2\sqrt{y-x}} \tag{86}$$

and

$$I_2 = \tfrac{1}{4}(p_x+p_y)^2 - 2(p_x-p_y)(yp_x+xp_y) - \frac{\Re\big\{\big(x+y-i\sqrt{y-x}\big)Q_1(Z)\big\}}{\sqrt{y-x}}. \tag{87}$$

The sequence of polynomial potentials corresponding to those given above is provided by the choice

$$Q_1 = (1-i)^{m-2} Z^m. \tag{88}$$

4.2.3. Sub-class $(1\hat{2})_0$

In view of the choice of the second separating variable, this coincides for all expressions with the sub-class $(1\hat{1})_0$ above.

4.3. The parabolic–parabolic $(2\hat{2})$ *class*

In this purely parabolic case, the coordinate transformation is given by

$$\begin{aligned}
&S_2(z) = 4z, \qquad \hat{S}_2(\hat{z}) = 4\hat{z}\\
&w = A_2(z) = \sqrt{z}, \qquad F_2(w) = w^2\\
&\hat{w} = \hat{A}_2(\hat{z}) = \sqrt{s\hat{z}}, \qquad \hat{F}_2(\hat{w}) = s\hat{w}^2, \quad s = \begin{cases} +1 & \text{Liouville separation} \\ -1 & \text{complex/harmonic separation} \\ 0 & \text{linear/null separation} \end{cases}\\
&X = \tfrac{1}{2}\big(\sqrt{x+y} + \sqrt{x-y}\big), \qquad Y = \tfrac{1}{2}\big(\sqrt{x+y} - \sqrt{x-y}\big), \quad s = +1, \qquad x^2 - y^2 > 0\\
&X = \tfrac{1}{2}\big(\sqrt{x+y} + \sqrt{y-x}\big), \qquad Y = \tfrac{1}{2}\big(\sqrt{x+y} - \sqrt{y-x}\big), \qquad s = -1, \qquad x^2 - y^2 < 0.
\end{aligned} \tag{89}$$

In the linear/null case we assume $w = A_1(z) = \sqrt{z}$ for the first variable and keep $\hat{z}(=\hat{w})$ as the second variable.

4.3.1. Sub-class $(2\hat{2})_+$

The conformal factor is

$$\Psi = 4w\hat{w} = 4(X^2 - Y^2) = 4\sqrt{x^2 - y^2} \tag{90}$$

leading to

$$\Psi_1 = 4X^2, \qquad \Psi_2 = -4Y^2. \tag{91}$$

The physical potential then takes the form

$$V = \frac{f_1(X) + f_2(Y)}{X^2 - Y^2}. \tag{92}$$

The Hamiltonian and the second invariant are given by

$$H = \tfrac{1}{2}({p_x}^2 - {p_y}^2) + \frac{f_1\big(\sqrt{x+y} + \sqrt{x-y}\big) + f_2\big(\sqrt{x+y} - \sqrt{x-y}\big)}{\sqrt{x^2 - y^2}} \tag{93}$$

and

$$I_2 = p_y(xp_y + yp_x) - \frac{\big(x - \sqrt{x^2 - y^2}\,\big)f_1(X) + \big(x + \sqrt{x^2 - y^2}\,\big)f_2(Y)}{\sqrt{x^2 - y^2}}. \tag{94}$$

With the choice

$$f_1 = X^{2m}, \qquad f_2 = -Y^{2m}, \quad m \in \mathbb{N}, \tag{95}$$

we get the sequence

$$V = 4x^2 - y^2, \qquad m = 3, \tag{96}$$

$$V = 2x^3 - xy^2, \qquad m = 4, \tag{97}$$

$$V = 16x^4 - 12x^2y^2 + y^4, \qquad m = 5 \tag{98}$$

and so forth.

4.3.2. Sub-class $(2\hat{2})_-$

Using the complex variable $Z = X + iY$, the separating coordinates are

$$Z = \tfrac{1}{2}(1+i)\big(\sqrt{x+y} + i\sqrt{y-x}\big), \qquad \bar{Z} = \tfrac{1}{2}(1-i)\big(\sqrt{x+y} - i\sqrt{y-x}\big) \tag{99}$$

corresponding to the conformal factor

$$\Psi = 4\sqrt{y^2 - x^2} = 2(Z^2 + \bar{Z}^2). \tag{100}$$

This leads the physical potential

$$V = \frac{\Re\big\{Q_1\big(\tfrac{1}{2}(1+i)\big[\sqrt{x+y} + i\sqrt{y-x}\,\big]\big)\big\}}{\sqrt{y^2 - x^2}} \tag{101}$$

where Q_1 is an arbitrary function. The Hamiltonian and the second invariant are given by

$$H = \tfrac{1}{2}({p_x}^2 - {p_y}^2) + \frac{\Re\{Q_1(Z)\}}{\sqrt{y^2 - x^2}} \tag{102}$$

and

$$I_2 = p_y(xp_y + yp_x) + \frac{\Re\big\{\big(u - i\sqrt{y^2 - x^2}\,\big)Q_1(Z)\big\}}{\sqrt{y^2 - x^2}}. \tag{103}$$

The sequence of polynomial potentials corresponding to those of the previous sub-class is provided by the choice

$$Q_1 = i^{m+1}Z^{2m}. \tag{104}$$

4.3.3. Sub-class $(2\hat{2})_0$

In this case there is just one separating variable which we take as $w = \sqrt{z}$. It is convenient to use the complementary null variable $\hat{w} = \hat{z} = x - y$ as the other independent variable. The Hamiltonian and the second invariant are then given by

$$\begin{aligned} H &= \tfrac{1}{2}({p_x}^2 - {p_y}^2) + \sqrt{x+y}C'(x-y) + D(x-y) \\ I_2 &= (p_x + p_y)(xp_x + yp_y) - C(x-y) - \frac{x-y}{\sqrt{x+y}}D(x-y) - (x-y)C'(x-y) \end{aligned} \tag{105}$$

where Y and D are arbitrary functions.

4.4. The polar–elliptical class of the first kind $(3\hat{4})$

The cases $(3\hat{4})_s$ combine polar and elliptical coordinates and admit standard and harmonic separations. The conformal coordinate transformation is (with $\hat{\Delta} = 1$)

$$\begin{aligned} &S_3(z) = z^2, \qquad \hat{S}_4(\hat{z}) = \hat{z}^2 - 1, \qquad w = A_3(z) = \ln|z|, \qquad F_3(w) = e^w \\ &\hat{w} = \hat{A}_4(\hat{z}) = \int \frac{d\hat{z}}{\sqrt{s(\hat{z}^2-1)}} = \begin{cases} \operatorname{arccosh}\hat{z}, & \hat{F}_4(\hat{w}) = \cosh\hat{w}, \quad s = +1, \quad \text{Liouville separation} \\ \arcsin\hat{z}, & \hat{F}_4(\hat{w}) = \sin\hat{w}, \quad s = -1, \quad \text{harmonic separation} \end{cases} \end{aligned} \tag{106}$$

or in non-null coordinates

$$\left.\begin{aligned} X &= \tfrac{1}{2}\left[\ln(x+y) + \ln\left(x - y + \sqrt{(x-y)^2-1}\right)\right] \\ Y &= \tfrac{1}{2}\left[\ln(x+y) - \ln\left(x - y + \sqrt{(x-y)^2-1}\right)\right] \end{aligned}\right\} \quad s = +1, \qquad |x-y| > 1 \tag{107}$$

and

$$\left.\begin{aligned} X &= \tfrac{1}{2}\Big[\ln(x+y) + \arcsin(x-y)\Big] \\ Y &= \tfrac{1}{2}\Big[\ln(x+y) - \arcsin(x-y)\Big] \end{aligned}\right\} \quad s = -1, \qquad |x-y| < 1. \tag{108}$$

4.4.1. Sub-class $(3\hat{4})_+$

The conformal factor in this standard Liouville case is given by

$$\Psi = e^w \sinh\hat{w} = (x+y)\sqrt{(x-y)^2-1} \tag{109}$$

so that

$$\Psi_1 = \tfrac{1}{2}e^{2X}, \qquad \Psi_2 = -\tfrac{1}{2}e^{2Y}. \tag{110}$$

The physical potential then takes the form

$$V = \frac{f_1(X) + f_2(Y)}{e^{2X} - e^{2Y}}. \tag{111}$$

The Hamiltonian and the second invariant are given by

$$H = \tfrac{1}{2}({p_x}^2 - {p_y}^2) + \frac{f_1(X) + f_2(Y)}{(x+y)\sqrt{(x-y)^2-1}} \tag{112}$$

and

$$I_2 = (xp_y + yp_y)^2 - \tfrac{1}{4}(p_x - p_y)^2 - \left[\frac{x-y}{\sqrt{(x-y)^2-1}} - 1\right]f_1(X) - \left[\frac{x-y}{\sqrt{(x-y)^2-1}} + 1\right]f_2(Y). \tag{113}$$

With the choice

$$f_1 = e^{2mX}, \qquad f_2 = -e^{2mY}, \quad m \in \mathbb{N}, \tag{114}$$

we get the sequence

$$V = 4(x-y)(x+y), \qquad m = 2, \tag{115}$$

$$V = 2(-1 + 4(x-y)^2)(x+y)^2, \qquad m = 3, \tag{116}$$

$$V = 8(-1 + 2(x-y)^2)(x-y)(x+y)^3, \qquad m = 4 \tag{117}$$

and so forth.

4.4.2. Sub-class $(3\hat{4})_-$

In this harmonic case the separating variables are

$$Z = \tfrac{1}{2}(1+i)\big[\ln(x+y) + i\arcsin(y-x)\big], \qquad \bar{Z} = \tfrac{1}{2}(1-i)\big[\ln(x+y) - i\arcsin(y-x)\big] \tag{118}$$

so that the conformal factor

$$\Psi = e^{w}\cos\hat{w} = (x+y)\sqrt{1-(x-y)^2} = \tfrac{1}{2}\big[e^{(1-i)Z} + e^{(1+i)\bar{Z}}\big] \tag{119}$$

can be written in the form

$$\Psi = \tfrac{1}{2}\big[e^{(1-i)Z} + e^{(1+i)\bar{Z}}\big]. \tag{120}$$

This leads to a physical potential of the form

$$V = \frac{\Re\left\{Q_1\left(\frac{1}{2}(1+i)\big[\ln(x+y) + i\arcsin(y-x)\big]\right)\right\}}{(x+y)\sqrt{1-(x-y)^2}} \tag{121}$$

where Q_1 is an arbitrary function. The Hamiltonian and the second invariant are given by

$$H = \tfrac{1}{2}({p_x}^2 - {p_y}^2) + \frac{\Re\{Q_1(Z)\}}{(x+y)\sqrt{1-(x-y)^2}} \tag{122}$$

and

$$I_2 = (xp_y + yp_y)^2 - \tfrac{1}{4}(p_x - p_y)^2 + \frac{\Re\left\{\big[x-y+i\sqrt{1-(x-y)^2}\big]Q_1(Z)\right\}}{(x+y)\sqrt{1-(x-y)^2}}. \tag{123}$$

The sequence of polynomial potentials corresponding to those of the previous sub-class is provided by the choice

$$Q_1 = e^{(1-i)mZ}. \tag{124}$$

4.5. The polar–elliptical class of the second kind $(3\hat{5})$

The polar–elliptical class $(3\hat{5})$ admits only the standard Liouville separation. The conformal coordinate transformation is (with $\hat{\Delta} = 1$)

$$\begin{aligned} &S_3(z) = z^2, \qquad \hat{S}_5(\hat{z}) = \hat{z}^2 + 1 \\ &w = A_3(z) = \ln|z|, \qquad F_3(w) = e^{w} \\ &\hat{w} = \hat{A}_5(\hat{z}) = \operatorname{arcsinh}\hat{z}, \qquad \hat{F}_5(\hat{w}) = \sinh\hat{w} \end{aligned} \tag{125}$$

or in non-null coordinates

$$\begin{aligned} X &= \tfrac{1}{2}\left[\ln(x+y) + \ln\left(x-y+\sqrt{(x-y)^2+1}\right)\right] \\ Y &= \tfrac{1}{2}\left[\ln\left(x+y\right) - \ln(x-y+\sqrt{(x-y)^2+1}\,)\right]. \end{aligned} \tag{126}$$

The conformal factor is

$$\Psi = e^{w}\cosh\hat{w} = (x+y)\sqrt{(x-y)^2+1} \tag{127}$$

so that

$$\Psi_1 = \tfrac{1}{2}e^{2X}, \qquad \Psi_2 = \tfrac{1}{2}e^{2Y}. \tag{128}$$

This leads to a physical potential of the form

$$V = \frac{f_1(X) + f_2(Y)}{e^{2X} + e^{2Y}}. \tag{129}$$

The Hamiltonian and the second invariant are given by

$$H = \tfrac{1}{2}({p_x}^2 - {p_y}^2) + \frac{f_1(X) + f_2(Y)}{(x+y)\sqrt{(x-y)^2+1}} \tag{130}$$

and

$$I_2 = (xp_y + yp_x)^2 + \tfrac{1}{4}(p_x - p_y)^2 - \left[\frac{x-y}{\sqrt{(x-y)^2+1}} - 1\right] f_1(X) - \left[\frac{x-y}{\sqrt{(x-y)^2+1}} + 1\right] f_2(Y). \tag{131}$$

With the choice

$$f_1 = e^{2mX}, \qquad f_2 = (-1)^{m-1} e^{2mY}, \quad m \in \mathbb{N}, \tag{132}$$

we get the sequence

$$V = 4(x-y)(x+y), \qquad m = 2, \tag{133}$$

$$V = 2(1+4(x-y)^2)(x+y)^2, \qquad m = 3, \tag{134}$$

$$V = 8(1+2(x-y)^2)(x-y)(x+y)^3, \qquad m = 4 \tag{135}$$

and so forth.

5. Comments and conclusions

The generalization of the standard (Liouville) separability notion for $(1+1)$-dimensional natural systems to include also the complex/harmonic and the linear/null separation structures has interesting consequences. Several classes of potentials which admit separation in one of the classes with linear or quadratic conformal factor, may require coordinate patches with different separation structure, also implying different coordinates for even a single orbit. On the other hand, in the cases in which different separation structures live *together* on the same subset of the pseudo-plane, we have examples of multi-separable potentials. The simplest example is provided by the indefinite harmonic oscillator: choosing $m = 2$ in Section 4.1, we get the potential

$$V = \frac{1}{2}(x^2 - y^2)$$

with integrals

$$I_2 = \tfrac{1}{2}({p_x}^2 + {p_y}^2) + \frac{1}{2}(x^2 + y^2)$$

in the Liouville case and

$$\tilde{I}_2 = p_x p_y - xy$$

in the complex/harmonic case. We remark on the agreement of these results with those obtained in the investigation of superintegrable systems [24], by observing that the class of systems just mentioned coincides, after suitable linear combinations of the integrals, with the *Class* II_2 introduced in Ref. [24].

The cases of polynomial systems listed in this work provide a large set of examples to play with for applications to geometry [26,27], celestial mechanics (in particular we mention the triangular Lagrangian equilibrium in the restricted 3-body problem [28]), singularity theory of resonant normal forms [29], relativity [30,31] and quantum mechanics, with a particular emphasis on the theory of Dirac's equation [32,33]. Among other possibilities, the exploration of separable natural systems can be extended by looking for more general homogeneous potentials that, in analogy to the examples given in [16], can be constructed in the classes $3\hat{3}$, $4\hat{4}$, $4\hat{5}$, $5\hat{5}$.

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