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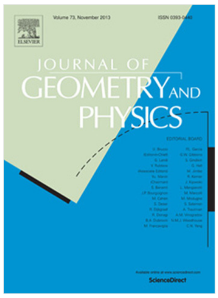

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Generalized derivation extensions of 3-Lie algebras and corresponding Nambu-Poisson structures *

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Abstract

In this paper, we introduce the notion of a generalized derivation on a 3-Lie algebra. We construct a new 3-Lie algebra using a generalized derivation and call it the generalized derivation extension. We show that the corresponding Leibniz algebra on the space of fundamental objects is the double of a matched pair of Leibniz algebras. We also determine the corresponding Nambu-Poisson structures under some conditions.

1 Introduction

The notion of an n-Lie algebra, or a Filippov algebra, was introduced in [10] and have many applications in mathematical physics. See the review article [8] for more details. Ternary Lie algebras are related to Nambu mechanics and Nambu-Poisson structures [22], generalizing Hamiltonian mechanics by using more than one hamiltonian. The algebraic formulation of this theory is due to Takhtajan [25], see also [11]. Moreover, 3-Lie algebras appeared in String Theory. See [4, 16] for classifications of 3-Lie algebras and n-Lie algebras. Deformations of 3-Lie algebras and n-Lie algebras are studied in [9, 26], see [21] for a review. It is very useful to construct new 3-Lie and n-Lie algebras. 3-Lie algebras were constructed using Nambu-Poisson structures in [14, 22, 25]; More generally, one can construct 3-Lie algebras, which are called Jacobian algebras, using a commutative associative algebra with some derivations [10, 23]; 3-Lie algebras can also be constructed from Dirac γ-matrix [13] and quadratic Lie algebras that related to integrable systems [15]; Moreover, R. Bai and her collaborators gave some construction of 3-Lie algebras using Lie algebras and linear functions [3, 6]; Construction of $(n+1)$-Lie algebras from n-Lie algebras are studied in [2, 5]. Furthermore, abelian extensions of 3-Lie algebras are studied in [17] using generalized representations, which is a generalization of the usual representation introduced in [20].

The notion of a Leibniz algebra was introduced by Loday [18, 19], which is a noncommutative generalization of a Lie algebra. The notion of a matched pair of Leibniz algebras was introduced in [1], which is an approach to construct new Leibniz algebras. 3-Lie algebras and Leibniz algebras are closely related. Through fundamental objects one may represent a 3-Lie algebra and more generally

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an n-Lie algebra by a Leibniz algebra [7], and one can study n-Lie algebras by the corresponding Leibniz algebras.

The purpose of this paper is to give an approach to construct new 3-Lie algebras that generalize the method of derivation extension for Lie algebras. Recall that given a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ and a derivation D, we can construct a new Lie algebra structure $[\cdot,\cdot]_D$ on the direct sum $\mathfrak{g} \oplus \mathbb{K}D$, which is called the derivation extension, by

$$[x + k_1 D, y + k_2 D]_D = [x, y]_{\mathfrak{g}} + k_1 D(y) - k_2 D(x), \quad \forall x, y \in \mathfrak{g}, k_1, k_2 \in \mathbb{K}.$$

In [24], the author studied classification of 4-dimensional Lie algebras and Lie-Poisson structures using the method of derivation extension of Lie algebras. However, for 3-Lie algebras, since there are three variables, it is impossible to construct new brackets using the usual derivations on 3-Lie algebras. To solve this difficulty, we introduce the notion of a generalized derivation on a 3-Lie algebra, by which we can construct new 3-Lie algebras, which we call generalized derivation extensions of 3-Lie algebras. We study the corresponding Leibniz algebra on the space of fundamental objects, and show that the Leibniz algebra associated to a generalized derivation extension is a matched pair of Leibniz algebras. Finally, we study the relation with Nambu-Poisson structures. Under some conditions, we give the explicit formulas of the Nambu-Poisson structure corresponding to a generalized derivation extension.

The paper is organized as follows. In Section 2, we give a review on matched pairs of Leibniz algebras, 3-Lie algebras and Nambu-Poisson structures. In Section 3, we introduce the notion of a generalized derivation on a 3-Lie algebra, and show that one can construct new 3-Lie algebras using generalized derivations. In Section 4, we study the corresponding Leibniz algebra on the space of fundamental objects of a generalized derivation extension of 3-Lie algebras. In Section 5, we study the corresponding Nambu-Poisson structure of a generalized derivation extension of 3-Lie algebras.

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2 Preliminaries

In this paper, we work over an algebraically closed field $\mathbb{K}$ of characteristic 0 and all the vector spaces are over $\mathbb{K}$.

2.1 Matched pair of Leibniz algebras

A **Leibniz algebra** is a vector space $\mathfrak{L}$ endowed with a linear map $[\cdot,\cdot]_{\mathfrak{L}} : \mathfrak{L} \otimes \mathfrak{L} \longrightarrow \mathfrak{L}$ satisfying

$$[x, [y, z]_{\mathfrak{L}}]_{\mathfrak{L}} = [[x, y]_{\mathfrak{L}}, z]_{\mathfrak{L}} + [y, [x, z]_{\mathfrak{L}}]_{\mathfrak{L}}, \quad \forall\ x, y, z \in \mathfrak{L}. \tag{1}$$

This is in fact a left Leibniz algebra. In this paper, we only consider left Leibniz algebras. A linear map $D : \mathfrak{L} \longrightarrow \mathfrak{L}$ is called a left derivation on the Leibniz algebra $(\mathfrak{L}, [\cdot,\cdot]_{\mathfrak{L}})$ if

$$D([x, y]_{\mathfrak{L}}) = [D(x), y]_{\mathfrak{L}} + [x, D(y)]_{\mathfrak{L}}, \quad \forall x, y \in \mathfrak{L}.$$

A **representation** of a Leibniz algebra $(\mathfrak{L}, [\cdot,\cdot]_{\mathfrak{L}})$ [19] is a triple (V, ρ^L, ρ^R), where V is a vector space equipped with two linear maps $\rho^L, \rho^R : \mathfrak{L} \longrightarrow \mathfrak{gl}(V)$ such that the following equalities hold for $x, y \in \mathfrak{L}$:

$$\begin{aligned}
\rho^L([x,y]_{\mathfrak{L}}) &= [\rho^L(x), \rho^L(y)], &(2)\\
\rho^R([x,y]_{\mathfrak{L}}) &= [\rho^L(x), \rho^R(y)], &(3)\\
\rho^R(y)\circ\rho^L(x) &= -\rho^R(y)\circ\rho^R(x). &(4)
\end{aligned}$$

Definition 2.1. ([1]) *A pair $(\mathfrak{G}, \mathfrak{H})$ of two Leibniz algebras is called a* **matched pair** *if there exist a representation (ρ_1^L, ρ_1^R) of $\mathfrak{G}$ on $\mathfrak{H}$ and a representation of (ρ_2^L, ρ_2^R) of $\mathfrak{H}$ on $\mathfrak{G}$ such that the identities*

(i) $\rho_1^R(x)[u,v]_{\mathfrak{H}} = [u, \rho_1^R(x)v]_{\mathfrak{H}} - [v, \rho_1^R(x)u]_{\mathfrak{H}} + \rho_1^R(\rho_2^L(v)x)u - \rho_1^R(\rho_2^L(u)x)v$;

(ii) $\rho_1^L(x)[u,v]_{\mathfrak{H}} = [\rho_1^L(x)u, v]_{\mathfrak{H}} + [u, \rho_1^L(x)v]_{\mathfrak{H}} + \rho_1^L(\rho_2^R(u)x)v + \rho_1^R(\rho_2^R(v)x)u$;

(iii) $[\rho_1^L(x)u, v]_{\mathfrak{H}} + \rho_1^L(\rho_2^R(u)x)v + [\rho_1^R(x)u, v]_{\mathfrak{H}} + \rho_1^L(\rho_2^L(u)x)v = 0$;

(iv) $\rho_2^R(u)[x,y]_{\mathfrak{G}} = [x, \rho_2^R(u)y]_{\mathfrak{G}} - [y, \rho_2^R(u)x]_{\mathfrak{G}} + \rho_2^R(\rho_1^L(y)u)x - \rho_2^R(\rho_1^L(x)u)y$;

(v) $\rho_2^L(u)[x,y]_{\mathfrak{G}} = [\rho_2^L(u)x, y]_{\mathfrak{G}} + [x, \rho_2^L(u)y]_{\mathfrak{G}} + \rho_2^L(\rho_1^R(x)u)y + \rho_2^R(\rho_1^R(y)u)x$;

(vi) $[\rho_2^L(u)x, y]_{\mathfrak{G}} + \rho_2^L(\rho_1^R(x)u)y + [\rho_2^R(u)x, y]_{\mathfrak{G}} + \rho_2^L(\rho_1^L(x)u)y = 0$,

hold for all $x, y \in \mathfrak{G}$ and $u, v \in \mathfrak{H}$.

Lemma 2.2. ([1]) *Given a matched pair $(\mathfrak{G}, \mathfrak{H})$ of Leibniz algebras, there is a Leibniz algebra structure $\mathfrak{G} \bowtie \mathfrak{H}$ on the direct sum vector space $\mathfrak{G} \oplus \mathfrak{H}$ with bracket*

$$[x+u, y+v]_{\mathfrak{G}\bowtie\mathfrak{H}} = [x,y]_{\mathfrak{G}} + \rho_2^R(v)x + \rho_2^L(u)y + [u,v]_{\mathfrak{H}} + \rho_1^L(x)v + \rho_1^R(y)u.$$

Conversely, if $\mathfrak{G} \oplus \mathfrak{H}$ has a Leibniz algebra structure for which $\mathfrak{G}$ and $\mathfrak{H}$ are Leibniz subalgebras, then the representations defined by

$$[x,u]_{\mathfrak{G}\oplus\mathfrak{H}} = \rho_2^R(u)x + \rho_1^L(x)u, \qquad [u,x]_{\mathfrak{G}\oplus\mathfrak{H}} = \rho_2^L(u)x + \rho_1^R(x)u,$$

endow the couple $(\mathfrak{G}, \mathfrak{H})$ with a structure of a matched pair.

Definition 2.3. ([10]) *A* **3-Lie algebra** *is a vector space $\mathfrak{g}$ together with a skew-symmetric linear map $[\cdot,\cdot,\cdot]_{\mathfrak{g}} : \otimes^3\mathfrak{g} \to \mathfrak{g}$ such that the following* **Fundamental Identity (FI)** *holds:*

$$\begin{aligned}
& F_{x_1,x_2,x_3,x_4,x_5}\\
\triangleq\ & [x_1,x_2,[x_3,x_4,x_5]_{\mathfrak{g}}]_{\mathfrak{g}} - [[x_1,x_2,x_3]_{\mathfrak{g}},x_4,x_5]_{\mathfrak{g}} - [x_3,[x_1,x_2,x_4]_{\mathfrak{g}},x_5]_{\mathfrak{g}} - [x_3,x_4,[x_1,x_2,x_5]_{\mathfrak{g}}]_{\mathfrak{g}}\\
=\ & 0. \qquad (5)
\end{aligned}$$

A linear map $D : \mathfrak{g} \longrightarrow \mathfrak{g}$ is called a derivation on the 3-Lie algebra $(\mathfrak{g}, [\cdot,\cdot,\cdot]_{\mathfrak{g}})$ if the following equality holds:

$$D([x,y,z]_{\mathfrak{g}}) = [D(x),y,z]_{\mathfrak{g}} + [x,D(y),z]_{\mathfrak{g}} + [x,y,D(z)]_{\mathfrak{g}}, \quad \forall x,y,z \in \mathfrak{g}.$$

For all $x, y \in \mathfrak{g}$, define $\mathsf{ad}_{x,y} : \mathfrak{g} \longrightarrow \mathfrak{g}$ by

$$\mathsf{ad}_{x,y}z = [x,y,z]_{\mathfrak{g}}.$$

Then $\mathsf{ad}_{x,y}$ is a derivation on $\mathfrak{g}$, which is called an inner derivation.

Elements in $\wedge^2\mathfrak{g}$ are called **fundamental objects** of the 3-Lie algebra $(\mathfrak{g}, [\cdot,\cdot,\cdot]_{\mathfrak{g}})$. There is a bilinear operation $[\cdot,\cdot]_{\mathrm{F}}$ on $\wedge^2\mathfrak{g}$, which is given by

$$[\mathfrak{X},\mathfrak{Y}]_{\mathrm{F}} = [x_1,x_2,y_1]_{\mathfrak{g}}\wedge y_2 + y_1\wedge[x_1,x_2,y_2]_{\mathfrak{g}}, \quad \forall\mathfrak{X} = x_1\wedge x_2,\ \mathfrak{Y} = y_1\wedge y_2. \tag{6}$$

It is well-known that $(\wedge^2\mathfrak{g}, [\cdot,\cdot]_{\mathrm{F}})$ is a Leibniz algebra [7], which plays an important role in the theory of 3-Lie algebras.

Proposition 2.4. ([4]) *There is a unique non-trivial* 3*-dimensional complex* 3*-Lie algebra. It has a basis* $\{e_1,e_2,e_3\}$ *with respect to which the non-zero product is given by* $[e_1,e_2,e_3] = e_1$.

Proposition 2.5. ([4]) *Let* A *be a non-trivial* 4*-dimensional complex* 3*-Lie algebra. Then* A *has a basis* $\{e_1,e_2,e_3,e_4\}$ *with respect to which the product of the* 3*-Lie algebra is given by one of the following:*

(a) $[e_1,e_2,e_3] = e_4, \quad [e_2,e_3,e_4] = e_1, \quad [e_1,e_3,e_4] = e_2, \quad [e_1,e_2,e_4] = e_3;$

(b) $[e_1,e_2,e_3] = e_1;$

(c) $[e_2,e_3,e_4] = e_1;$

(d) $[e_2,e_3,e_4] = e_1, \quad [e_1,e_3,e_4] = e_2;$

(e) $[e_2,e_3,e_4] = e_2, \quad [e_1,e_3,e_4] = e_1;$

(f) $[e_2,e_3,e_4] = \alpha e_1 + e_2,\ \alpha\neq 0, \quad [e_1,e_3,e_4] = e_2;$

(g) $[e_2,e_3,e_4] = e_1, \quad [e_1,e_3,e_4] = e_2, \quad [e_1,e_2,e_4] = e_3.$

2.2 Nambu-Poisson structures

Nambu-Poisson structures were introduced in [25] by Takhtajan in order to find an axiomatic formalism for Nambu-mechanics which is a generalization of Hamiltonian mechanics.

Definition 2.6. ([25]) *A* **Nambu-Poisson structure** *of order* $n-1$ *on* M *is an* n*-linear map* $\{\cdot,\cdots,\cdot\} : C^\infty(M)\times\cdots\times C^\infty(M)\longrightarrow C^\infty(M)$ *satisfying the following properties:*

(1) *skew-symmetry, i.e. for all* $f_1,\cdots,f_n\in C^\infty(M)$ *and* $\sigma\in\mathrm{Sym}(n)$,

$$\{f_1,\cdots,f_n\} = (-1)^{|\sigma|}\{f_{\sigma(1)},\cdots,f_{\sigma(n)}\};$$

(2) *the Leibniz rule, i.e. for all* $g\in C^\infty(M)$*, we have*

$$\{f_1 g, f_2,\cdots,f_n\} = f_1\{g,f_2,\cdots,f_n\} + g\{f_1,f_2,\cdots,f_n\};$$

(3) *integrability condition, i.e. for all* $f_1,\cdots,f_{n-1},g_1,\cdots,g_n\in C^\infty(M)$,

$$\{f_1,\cdots,f_{n-1},\{g_1,\cdots,g_n\}\} = \sum_{i=1}^{n}\{g_1,\cdots,\{f_1,\cdots,f_{n-1},g_i\},\cdots,g_n\}.$$

In particular, a Nambu-Poisson structure of order 1 is exactly a usual Poisson structure. Similar to the fact that a Poisson structure is determined by a bivector field, a Nambu-Poisson structure of order $n-1$ is determined by an n-vector field $\pi \in \mathfrak{X}^n(M)$ such that

$$\{f_1, \cdots, f_n\} = \pi(df_1, \cdots, df_n).$$

An n-vector field $\pi \in \mathfrak{X}^n(M)$ is a Nambu-Poisson structure if and only if for all $f_1, \cdots, f_{n-1} \in C^\infty(M)$, we have

$$L_{\pi^\sharp(df_1 \wedge \cdots \wedge df_{n-1})}\pi = 0,$$

where $\pi^\sharp : \wedge^{n-1}T^*M \longrightarrow TM$ is defined by

$$\langle \pi^\sharp(\xi_1 \wedge \cdots \wedge \xi_{n-1}), \xi_n \rangle = \pi(\xi_1 \wedge \cdots \wedge \xi_{n-1} \wedge \xi_n), \quad \forall \xi_1, \cdots, \xi_n \in \Omega^1(M).$$

3 Generalized derivation extensions of 3-Lie algebras

In this section, we introduce the notion of a generalized derivation on a 3-Lie algebra, by which we can construct a new 3-Lie algebra, called the generalized derivation extension of 3-Lie algebras.

Definition 3.1. *Let $(\mathfrak{g}, [\cdot,\cdot,\cdot]_\mathfrak{g})$ be a 3-Lie algebra. A linear map $D : \wedge^2\mathfrak{g} \to \mathfrak{g}$ is called a* **generalized derivation***, if for all $x, y, z, u \in \mathfrak{g}$, the following conditions are satisfied:*

(a) $D(x, [y, z, u]_\mathfrak{g}) = [D(x, y), z, u]_\mathfrak{g} + [y, D(x, z), u]_\mathfrak{g} + [y, z, D(x, u)]_\mathfrak{g}$;

(b) $D([x, y, z]_\mathfrak{g}, u) + D(z, [x, y, u]_\mathfrak{g}) = [x, y, D(z, u)]_\mathfrak{g} - [z, u, D(x, y)]_\mathfrak{g}$;

(c) $D(x, D(y, z)) + D(y, D(z, x)) + D(z, D(x, y)) = 0$.

We analyze the three conditions in the above definition. First we have

Lemma 3.2. *Let $D : \wedge^2\mathfrak{g} \to \mathfrak{g}$ be a generalized derivation on a 3-Lie algebra $(\mathfrak{g}, [\cdot,\cdot,\cdot]_\mathfrak{g})$. Then D defines a Lie algebra structure on the vector space $\mathfrak{g}$.*

Proof. It follows from Condition (c) in Definition 3.1. ■

For a linear map $D : \wedge^2\mathfrak{g} \longrightarrow \mathfrak{g}$, define $D^\sharp : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ by

$$D^\sharp(x)(y) = D(x, y), \quad \forall x, y \in \mathfrak{g}.$$

Lemma 3.3. *Let $D : \wedge^2\mathfrak{g} \to \mathfrak{g}$ be a generalized derivation on a 3-Lie algebra $(\mathfrak{g}, [\cdot,\cdot,\cdot]_\mathfrak{g})$. Then for all $x \in \mathfrak{g}$, $D^\sharp(x)$ is a derivation on the 3-Lie algebra $(\mathfrak{g}, [\cdot,\cdot,\cdot]_\mathfrak{g})$.*

Proof. It follows from Condition (a) in Definition 3.1. ■

Consider the Leibniz algebra $(\wedge^2\mathfrak{g}, [\cdot,\cdot]_\mathrm{F})$. Define $\rho^L : \wedge^2\mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ by

$$\rho^L(x \wedge y)(z) = [x, y, z]_\mathfrak{g}, \quad \forall x \wedge y \in \wedge^2\mathfrak{g}, z \in \mathfrak{g}.$$

Lemma 3.4. *Let $(\mathfrak{g}, [\cdot,\cdot,\cdot]_\mathfrak{g})$ be a 3-Lie algebra. Then $(\rho^L, \rho^R = -\rho^L)$ is a representation of the Leibniz algebra $(\wedge^2\mathfrak{g}, [\cdot,\cdot]_\mathrm{F})$ on $\mathfrak{g}$. Consequently, we have a semidirect product Leibniz algebra $(\wedge^2\mathfrak{g} \oplus \mathfrak{g}, [\cdot,\cdot]_s)$, where the Leibniz bracket is given by*

$$[x_1 \wedge y_1 + z_1, x_2 \wedge y_2 + z_2]_s = [x_1 \wedge y_1, x_2 \wedge y_2]_\mathrm{F} + [x_1, y_1, z_2]_\mathfrak{g} - [x_2, y_2, z_1]_\mathfrak{g}, \tag{7}$$

for all $x_1, y_1, z_1, x_2, y_2, z_2 \in \mathfrak{g}$.

Proof. It is straightforward by the Fundamental Identity. ■

Lemma 3.5. *Let $D : \wedge^2\mathfrak{g} \to \mathfrak{g}$ be a generalized derivation on a 3-Lie algebra $(\mathfrak{g}, [\cdot,\cdot,\cdot]_{\mathfrak{g}})$. Then $\begin{pmatrix} 0 & 0 \\ D & 0 \end{pmatrix}$ is a left derivation on the Leibniz algebra $(\wedge^2\mathfrak{g} \oplus \mathfrak{g}, [\cdot,\cdot]_s)$.*

Proof. By Condition (b) in Definition 3.1, we have

$$\begin{array}{rl} & \begin{pmatrix} 0 & 0 \\ D & 0 \end{pmatrix}[x_1 \wedge y_1 + z_1, x_2 \wedge y_2 + z_2]_s - [\begin{pmatrix} 0 & 0 \\ D & 0 \end{pmatrix}(x_1 \wedge y_1 + z_1), x_2 \wedge y_2 + z_2]_s \\ & -[x_1 \wedge y_1 + z_1, \begin{pmatrix} 0 & 0 \\ D & 0 \end{pmatrix}(x_2 \wedge y_2 + z_2)]_s \\ = & D[x_1 \wedge y_1, x_2 \wedge y_2]_{\mathrm{F}} - [D(x_1, y_1), x_2 \wedge y_2 + z_2]_s - [x_1 \wedge y_1 + z_1, D(x_2, y_2)]_s \\ = & D[x_1 \wedge y_1, x_2 \wedge y_2]_{\mathrm{F}} + [D(x_1, y_1), x_2, y_2]_{\mathfrak{g}} - [x_1, y_1, D(x_2, y_2)]_{\mathfrak{g}} \\ = & 0, \end{array}$$

which implies the conclusion. ■

For all $v \in \mathfrak{g}$, define $\mathfrak{ad} : \mathfrak{g} \longrightarrow \mathsf{Hom}(\wedge^2\mathfrak{g}, \mathfrak{g})$ by

$$\mathfrak{ad}_v(x, y) = [v, x, y]_{\mathfrak{g}}. \tag{8}$$

Then we have

Lemma 3.6. *For all $v \in \mathfrak{g}$, $\mathfrak{ad}_v$ is a generalized derivation on the 3-Lie algebra $(\mathfrak{g}, [\cdot,\cdot,\cdot]_{\mathfrak{g}})$, which is called an inner generalized derivation.*

Proof. First for all $x \in \mathfrak{g}$, we have

$$(\mathfrak{ad}_v)^\sharp(x) = \mathsf{ad}_{v,x},$$

which implies that Condition (a) in Definition 3.1 holds.

For all $x, y, z, u \in \mathfrak{g}$, by the Fundamental Identity, we have

$$\begin{array}{rl} & \mathfrak{ad}_v([x, y, z]_{\mathfrak{g}}, u) + \mathfrak{ad}_v(z, [x, y, u]_{\mathfrak{g}}) - [x, y, \mathfrak{ad}_v(z, u)]_{\mathfrak{g}} + [z, u, \mathfrak{ad}_v(x, y)]_{\mathfrak{g}} \\ = & [v, [x, y, z]_{\mathfrak{g}}, u]_{\mathfrak{g}} + [v, z, [x, y, u]_{\mathfrak{g}}]_{\mathfrak{g}} - [x, y, [v, z, u]_{\mathfrak{g}}]_{\mathfrak{g}} + [z, u, [v, x, y]_{\mathfrak{g}}]_{\mathfrak{g}} \\ = & 0, \end{array}$$

which implies that Condition (b) in Definition 3.1 holds.

Finally, by the Fundamental Identity, we can deduce that $\mathfrak{ad}_v$ defines a Lie algebra structure. The proof is finished. ■

Remark 3.7. *A derivation, no matter on a Lie algebra or a 3-Lie algebra can be viewed as a 1-cocycle with the coefficient in the adjoint representation. It is interesting to investigate whether one can realize a generalized derivation as a 1-cocycle, and an inner generalized derivation as a 1-coboundary, associated to some cohomological complex.*

For any linear map $D : \wedge^2\mathfrak{g} \to \mathfrak{g}$, denote by $\mathbb{K}D$ the 1-dimensional vector space generated by D. On the direct sum $\mathfrak{g} \oplus \mathbb{K}D$, define a totally skew-symmetric linear map $[\cdot,\cdot,\cdot]_D : \wedge^3(\mathfrak{g} \oplus \mathbb{K}D) \longrightarrow \mathfrak{g} \oplus \mathbb{K}D$ by

$$[x + k_1 D, y + k_2 D, z + k_3 D]_D = [x, y, z]_{\mathfrak{g}} + k_1 D(y, z) + k_2 D(z, x) + k_3 D(x, y), \tag{9}$$

for all $x, y, z \in \mathfrak{g}$, $k_1, k_2, k_3 \in \mathbb{K}$.

Theorem 3.8. *Let $(\mathfrak{g}, [\cdot,\cdot,\cdot]_{\mathfrak{g}})$ be a 3-Lie algebra and $D : \wedge^2\mathfrak{g} \to \mathfrak{g}$ a linear map. Then $(\mathfrak{g} \oplus \mathbb{K}D, [\cdot,\cdot,\cdot]_D)$ is a 3-Lie algebra if and only if D is a generalized derivation on $\mathfrak{g}$.*

Proof. For all $x, y, z, u, v \in \mathfrak{g}$ and $k_i \in \mathbb{K}$, $i = 1, \cdots, 5$, by direct computation, we have

$$\begin{aligned}
& [x + k_1D, y + k_2D, [z + k_3D, u + k_4D, v + k_5D]_D]_D \\
= \;& [x + k_1D, y + k_2D, [z, u, v]_{\mathfrak{g}} + k_3D(u, v) + k_4D(v, z) + k_5D(z, u)]_D \\
= \;& [x, y, [z, u, v]_{\mathfrak{g}} + [x, y, k_3D(u, v)]_{\mathfrak{g}} + [x, y, k_4D(v, z)]_{\mathfrak{g}} + [x, y, k_5D(z, u)]_{\mathfrak{g}} \\
& + k_1D(y, [z, u, v]_{\mathfrak{g}}) + k_1D(y, k_3D(u, v)) + k_1D(y, k_4D(v, z)) + k_1D(y, k_5D(z, u)) \\
& + k_2D([z, u, v]_{\mathfrak{g}}, x) + k_2D(k_3D(u, v), x) + k_2D(k_4D(v, z), x) + k_2D(k_5D(z, u), x),
\end{aligned}$$

$$\begin{aligned}
& [[x + k_1D, y + k_2D, z + k_3D]_D, u + k_4D, v + k_5D]_D \\
= \;& [[x, y, z]_{\mathfrak{g}} + k_1D(y, z) + k_2D(z, x) + k_3D(x, y), u + k_4D, v + k_5D]_D \\
= \;& [[x, y, z]_{\mathfrak{g}}, u, v] + [k_1D(y, z), u, v]_{\mathfrak{g}} + [k_2D(z, x), u, v]_{\mathfrak{g}} + [k_3D(x, y), u, v]_{\mathfrak{g}} \\
& + k_4D(v, [x, y, z]_{\mathfrak{g}}) + k_4D(v, k_1D(y, z)) + k_4D(v, k_2D(z, x)) + k_4D(v, k_3D(x, y)) \\
& + k_5D([x, y, z]_{\mathfrak{g}}, u) + k_5D(k_1D(y, z), u) + k_5D(k_2D(z, x), u) + k_5D(k_3D(x, y), u), \\
& [z + k_3D, [x + k_1D, y + k_2D, u + k_4D]_D, v + k_5D]_D \\
= \;& [z + k_3D, [x, y, u]_{\mathfrak{g}} + k_1D(y, u) + k_2D(u, x) + k_4D(x, y), v + k_5D]_D \\
= \;& [z, [x, y, u]_{\mathfrak{g}}, v]_{\mathfrak{g}} + [z, k_1D(y, u), v]_{\mathfrak{g}} + [z, k_2D(u, x), v]_{\mathfrak{g}} + [z, k_4D(x, y), v]_{\mathfrak{g}} \\
& + k_3D([x, y, u]_{\mathfrak{g}}, v) + k_3D(k_1D(y, u), v) + k_3D(k_2D(u, x), v) + k_3D(k_4D(x, y), v) \\
& + k_5D(z, [x, y, u]_{\mathfrak{g}}) + k_5D(z, k_1D(y, u)) + k_5D(z, k_2D(u, x)) + k_5D(z, k_4D(x, y)),
\end{aligned}$$

and

$$\begin{aligned}
& [z + k_3D, u + k_4D, [x + k_1D, y + k_2D, v + k_5D]_D]_D \\
= \;& [z + k_3D, u + k_4D, [x, y, v]_{\mathfrak{g}} + k_1D(y, v) + k_2D(v, x) + k_5D(x, y)]_D \\
= \;& [z, u, [x, y, v]_{\mathfrak{g}} + [z, u, k_1D(y, v)]_{\mathfrak{g}} + [z, u, k_2D(v, x)]_{\mathfrak{g}} + [z, u, k_5D(x, y)]_{\mathfrak{g}} \\
& + k_3D(u, [x, y, v]_{\mathfrak{g}}) + k_3D(u, k_1D(y, v)) + k_3D(u, k_2D(v, x)) + k_3D(u, k_5D(x, y)) \\
& + k_4D([x, y, v]_{\mathfrak{g}}, z) + k_4D(k_1D(y, v), z) + k_4D(k_2D(v, x), z) + k_4D(k_5D(x, y), z).
\end{aligned}$$

Define $\Delta, \Theta : \otimes^4\mathfrak{g} \longrightarrow \mathfrak{g}$ and $\Lambda : \otimes^3\mathfrak{g} \longrightarrow \mathfrak{g}$ by

$$\begin{aligned}
\Delta(x, y, z, u) &= D(x, [y, z, u]_{\mathfrak{g}}) - [D(x, y), z, u]_{\mathfrak{g}} - [y, D(x, z), u]_{\mathfrak{g}} - [y, z, D(x, u)]_{\mathfrak{g}}, \\
\Theta(x, y, z, u) &= D([x, y, z]_{\mathfrak{g}}, u) + D(z, [x, y, u]_{\mathfrak{g}}) - [x, y, D(z, u)]_{\mathfrak{g}} + [z, u, D(x, y)]_{\mathfrak{g}}, \\
\Lambda(x, y, z) &= D(x, D(y, z)) + D(y, D(z, x)) + D(z, D(x, y)).
\end{aligned}$$

Then $(\mathfrak{g} \oplus \mathbb{R}D, [\cdot,\cdot,\cdot]_D)$ is a 3-Lie algebra if and only if the following equalities hold:

$$\begin{array}{rclrclrcl}
k_1\Delta(y, z, u, v) &=& 0, & k_2\Delta(x, z, u, v) &=& 0, & & & \\
k_3\Theta(x, y, u, v) &=& 0, & k_4\Theta(x, y, z, v) &=& 0, & k_5\Theta(x, y, z, u) &=& 0, \\
k_1k_3\Lambda(y, u, v) &=& 0, & k_1k_4\Lambda(y, z, v) &=& 0, & k_1k_5\Lambda(y, z, u) &=& 0, \\
k_2k_3\Lambda(x, u, v) &=& 0, & k_2k_4\Lambda(x, z, v) &=& 0, & k_2k_5\Lambda(x, z, u) &=& 0.
\end{array}$$

Therefore, $(\mathfrak{g} \oplus \mathbb{K}D, [\cdot,\cdot,\cdot]_D)$ is a 3-Lie algebra if and only if D is generalized derivation. The proof is finished. ■

Remark 3.9. *In [12, Example 1], the author give a way to construct a 3-Lie algebra from a Lie algebra. Since a generalized derivation gives a Lie algebra structure naturally, the generalized derivation extension can be viewed as a generalization of the approach given in [12] in some sense.*

As expected, if two generalized derivation are differed by an inner generalized derivation, then the corresponding generalized extension are isomorphic.

Proposition 3.10. *Let D and D' be two generalized derivation on a 3-Lie algebra $(\mathfrak{g}, [\cdot,\cdot,\cdot]_{\mathfrak{g}})$. If there exists some $v \in \mathfrak{g}$ such that $D = D' + \mathfrak{ad}_v$, then the corresponding generalized derivation extensions $(\mathfrak{g} \oplus \mathbb{K}D, [\cdot,\cdot,\cdot]_D)$ and $(\mathfrak{g} \oplus \mathbb{K}D', [\cdot,\cdot,\cdot]_{D'})$ are isomorphic.*

Proof. Define $\overline{v} : \mathbb{K}D \longrightarrow \mathfrak{g}$ by

$$\overline{v}(kD) = kv, \quad \forall k \in \mathbb{K}.$$

Then $\begin{pmatrix} \mathrm{Id}_{\mathfrak{g}} & \overline{v} \\ 0 & 1 \end{pmatrix}$ is an isomorphism between 3-Lie algebras $(\mathfrak{g} \oplus \mathbb{K}D, [\cdot,\cdot,\cdot]_D)$ and $(\mathfrak{g} \oplus \mathbb{K}D', [\cdot,\cdot,\cdot]_{D'})$. In fact, for all $x, y, z \in \mathfrak{g}$ and $k_1, k_2, k_3 \in \mathbb{K}$, we have

$$\begin{pmatrix} \mathrm{Id}_{\mathfrak{g}} & \overline{v} \\ 0 & 1 \end{pmatrix} [x + k_1 D, y + k_2 D, z + k_3 D]_D = [x, y, z]_{\mathfrak{g}} + k_1 D(y, z) + k_2 D(z, x) + k_3 D(x, y),$$

and

$$\begin{aligned} & [\begin{pmatrix} \mathrm{Id}_{\mathfrak{g}} & \overline{v} \\ 0 & 1 \end{pmatrix}(x + k_1 D), \begin{pmatrix} \mathrm{Id}_{\mathfrak{g}} & \overline{v} \\ 0 & 1 \end{pmatrix}(y + k_2 D), \begin{pmatrix} \mathrm{Id}_{\mathfrak{g}} & \overline{v} \\ 0 & 1 \end{pmatrix}(z + k_3 D)]_{D'} \\ = \; & [x + k_1 v + k_1 D', y + k_2 v + k_2 D', z + k_3 v + k_3 D']_{D'} \\ = \; & [x + k_1 v, y + k_2 v, z + k_3 v]_{\mathfrak{g}} + k_1 D'(y + k_2 v, z + k_3 v) + k_2 D'(z + k_3 v, x + k_1 v) \\ & + k_3 D'(x + k_1 v, y + k_2 v) \\ = \; & [x, y, z]_{\mathfrak{g}} + k_1 [v, y, z]_{\mathfrak{g}} + k_2 [v, z, x]_{\mathfrak{g}} + k_3 [v, x, y]_{\mathfrak{g}} + k_1 D'(y, z) + k_2 D'(z, x) + k_3 D'(x, y). \end{aligned}$$

Thus by $D = D' + \mathfrak{ad}_v$, we deduce that $\begin{pmatrix} \mathrm{Id}_{\mathfrak{g}} & \overline{v} \\ 0 & 1 \end{pmatrix}$ is an isomorphism between 3-Lie algebras. ■

Example 3.11. Let $\mathfrak{g}$ be a 2-dimensional trivial 3-Lie algebra with a basis $\{e_1, e_2\}$. Then $D : \wedge^2 \mathfrak{g} \longrightarrow \mathfrak{g}$ defined by

$$D(e_1 \wedge e_2) = e_1 \tag{10}$$

is a generalized derivation on the 2-dimensional trivial 3-Lie algebra $\mathfrak{g}$. Acturally, any non-trivial 2-dimensional Lie algebra is isomorphic to the one given by (10). The 3-Lie algebra obtained by generalized derivation is exactly the one given in Proposition 2.4. Therefore, any 3-dimensional 3-Lie algebra can be realized as a generalized derivation extension.

Example 3.12. Let $\mathfrak{g}$ be a 3-dimensional abelian 3-Lie algebra with a basis $\{e_1, e_2, e_3\}$.

(i) the map $D : \wedge^2 \mathfrak{g} \longrightarrow \mathfrak{g}$ defined by

$$D(e_2 \wedge e_3) = e_1, \quad D(e_1 \wedge e_3) = 0, \quad D(e_1 \wedge e_2) = 0 \tag{11}$$

is a generalized derivation on the 3-dimensional abelian 3-Lie algebra $\mathfrak{g}$. The 3-Lie algebra obtained by generalized derivation is exactly the one given by (c) in Proposition 2.5.

(ii) the map $D : \wedge^2\mathfrak{g} \longrightarrow \mathfrak{g}$ defined by

$$D(e_2 \wedge e_3) = e_1, \quad D(e_1 \wedge e_3) = e_2, \quad D(e_1 \wedge e_2) = 0 \tag{12}$$

is a generalized derivation on the 3-dimensional abelian 3-Lie algebra $\mathfrak{g}$. The 3-Lie algebra obtained by generalized derivation is exactly the one given by (d) in Proposition 2.5.

(iii) the map $D : \wedge^2\mathfrak{g} \longrightarrow \mathfrak{g}$ defined by

$$D(e_2 \wedge e_3) = e_2, \quad D(e_1 \wedge e_3) = e_1, \quad D(e_1 \wedge e_2) = 0 \tag{13}$$

is a generalized derivation on the 3-dimensional abelian 3-Lie algebra $\mathfrak{g}$. The 3-Lie algebra obtained by generalized derivation is exactly the one given by (e) in Proposition 2.5.

(iv) the map $D : \wedge^2\mathfrak{g} \longrightarrow \mathfrak{g}$ defined by

$$D(e_2 \wedge e_3) = \alpha e_1 + e_2, \quad D(e_1 \wedge e_3) = e_2, \quad D(e_1 \wedge e_2) = 0 \tag{14}$$

is a generalized derivation on the 3-dimensional abelian 3-Lie algebra $\mathfrak{g}$. The 3-Lie algebra obtained by generalized derivation is exactly the one given by (f) in Proposition 2.5.

(v) the map $D : \wedge^2\mathfrak{g} \longrightarrow \mathfrak{g}$ defined by

$$D(e_2 \wedge e_3) = e_1, \quad D(e_1 \wedge e_3) = e_2, \quad D(e_1 \wedge e_2) = e_3 \tag{15}$$

is a generalized derivation on the 3-dimensional abelian 3-Lie algebra $\mathfrak{g}$. The 3-Lie algebra obtained by generalized derivation is exactly the one given by (g) in Proposition 2.5.

4 Matched pairs of Leibniz algebras

In this section, we always assume that $(\mathfrak{g}, [\cdot,\cdot,\cdot]_\mathfrak{g})$ is a 3-Lie algebra and $D : \wedge^2\mathfrak{g} \longrightarrow \mathfrak{g}$ is a generalized derivation on $\mathfrak{g}$. In the last section, we have obtain a 3-Lie algebra structure on $\mathfrak{g} \oplus \mathbb{K}D$. In this section, we analyze the corresponding Leibniz algebra structure on the space of fundamental objects. Note that $\wedge^2(\mathfrak{g} \oplus \mathbb{K}D) \cong (\wedge^2\mathfrak{g}) \oplus (\mathfrak{g} \otimes \mathbb{K}D)$ naturally.

First we introduce a representation of the Leibniz algebra $(\wedge^2\mathfrak{g}, [\cdot,\cdot]_\mathrm{F})$ on $\mathfrak{g} \otimes \mathbb{K}D$. Define $\rho_1^L, \rho_1^R : \wedge^2\mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g} \otimes \mathbb{K}D)$ by

$$\rho_1^L(x \wedge y)(u \otimes D) = [x, y, u]_\mathfrak{g} \otimes D, \tag{16}$$
$$\rho_1^R(x \wedge y)(u \otimes D) = 0, \tag{17}$$

for all $x, y, u \in \mathfrak{g}$. Then we have

Lemma 4.1. *With the above notations, (ρ_1^L, ρ_1^R) is a representation of the Leibniz algebra $(\wedge^2\mathfrak{g}, [\cdot,\cdot]_\mathrm{F})$ on $\mathfrak{g} \otimes \mathbb{K}D$.*

Proof. For all $x, y, u, v, w \in \mathfrak{g}$, by direct computation, we have

$$\begin{aligned}
\rho_1^L([u \wedge v, w \wedge x]_\mathrm{F})(y \otimes D) &= \rho_1^L([u, v, w]_\mathfrak{g} \wedge x + w \wedge [u, v, x]_\mathfrak{g})(y \otimes D) \\
&= [[u, v, w]_\mathfrak{g}, x, y]_\mathfrak{g} \otimes D + [w, [u, v, x]_\mathfrak{g}, y]_\mathfrak{g} \otimes D, \\
[\rho_1^L(u \wedge v), \rho_1^L(w \wedge x)](y \otimes D) &= \rho_1^L(u \wedge v)\rho_1^L(w \wedge x)(y \otimes D) - \rho_1^L(w \wedge x)\rho_1^L(u \wedge v)(y \otimes D) \\
&= [u, v, [w, x, y]_\mathfrak{g}]_\mathfrak{g} \otimes D - [w, x, [u, v, y]_\mathfrak{g}]_\mathfrak{g} \otimes D.
\end{aligned}$$

By the Fundamental Identity, we deduce that

$$\rho_1^L([u \wedge v, w \wedge x]_{\mathrm{F}}) = [\rho_1^L(u \wedge v), \rho_1^L(w \wedge x)].$$

Since $\rho_1^R = 0$, it is obvious that (ρ_1^L, ρ_1^R) is a representation of the Leibniz algebra $\wedge^2\mathfrak{g}$ on $\mathfrak{g} \otimes \mathbb{K}D$. The proof is finished. ■

On the tensor space $\mathfrak{g} \otimes \mathbb{K}D$, define a skewsymmetric linear map $\{\cdot,\cdot\} : \wedge^2(\mathfrak{g} \otimes \mathbb{K}D) \longrightarrow \mathfrak{g} \otimes \mathbb{K}D$ by

$$\{u \otimes D, v \otimes D\} = -D(u, v) \otimes D. \tag{18}$$

Then we have

Proposition 4.2. *With the above notations,* $(\mathfrak{g} \otimes \mathbb{K}D, \{\cdot,\cdot\})$ *is a Lie algebra.*

Proof. For all $u, v, w \in \mathfrak{g}$, by Condition (c) in Definition 3.1, we have

$$\begin{aligned}
&\{u \otimes D, \{v \otimes D, w \otimes D\}\} + \{v \otimes D, \{w \otimes D, u \otimes D\}\} + \{w \otimes D, \{u \otimes D, v \otimes D\}\} \\
&= \{u \otimes D, D(w, v) \otimes D\} + \{v \otimes D, D(u, w) \otimes D\} + \{w \otimes D, D(v, u) \otimes D\} \\
&= \big(D(D(w, v), u) + D(D(u, w), v) + D(D(v, u), w)\big) \otimes D \\
&= 0.
\end{aligned}$$

Thus, $(\mathfrak{g} \otimes \mathbb{K}D, \{\cdot,\cdot\})$ is a Lie algebra. ■

Now we view $\mathfrak{g} \otimes \mathbb{K}D$ as a Leibniz algebra and define $\rho_2^L, \rho_2^R : \mathfrak{g} \otimes \mathbb{K}D \longrightarrow \mathfrak{gl}(\wedge^2\mathfrak{g})$ by

$$\begin{aligned}
\rho_2^L(u \otimes D)(x \wedge y) &= D(x, u) \wedge y + x \wedge D(y, u), \\
\rho_2^R(u \otimes D)(x \wedge y) &= u \wedge D(x, y).
\end{aligned}$$

Lemma 4.3. *With the above notations,* (ρ_2^L, ρ_2^R) *is a representation of the Leibniz algebra* $(\mathfrak{g} \otimes \mathbb{K}D, \{\cdot,\cdot\})$ *on* $\wedge^2\mathfrak{g}$.

Proof. For all $u, v, x, y \in \mathfrak{g}$, we have

$$\begin{aligned}
\rho_2^L(\{u \otimes D, v \otimes D\})(x \wedge y) &= \rho_2^L(-D(u, v) \otimes D)(x \wedge y) \\
&= -D(x, D(u, v)) \wedge y - x \wedge D(y, D(u, v)),
\end{aligned}$$

and

$$\begin{aligned}
&[\rho_2^L(u \otimes D), \rho_2^L(v \otimes D)](x \wedge y) \\
=\ & \rho_2^L(u \otimes D)\rho_2^L(v \otimes D)(x \wedge y) - \rho_2^L(v \otimes D)\rho_2^L(u \otimes D)(x \wedge y) \\
=\ & \rho_2^L(u \otimes D)(D(x, v) \wedge y + x \wedge D(y, v)) - \rho_2^L(v \otimes D)(D(x, u) \wedge y + x \wedge D(y, u)) \\
=\ & -(D(D(x, u), v) - D(D(x, v), u)) \wedge y - x \wedge (D(D(y, u), v) - D(D(y, v), u)).
\end{aligned}$$

By Condition (c) in Definition 3.1, we have

$$\rho_2^L(\{u \otimes D, v \otimes D\}) = [\rho_2^L(u \otimes D), \rho_2^L(v \otimes D)].$$

Obviously, we have

$$\rho_2^R(\{u \otimes D, v \otimes D\})(x \wedge y) = \rho_2^R(-D(u, v) \otimes D)(x \wedge y) = -D(u, v) \wedge D(x, y).$$

By Condition (c) in Definition 3.1, we have

$$\begin{aligned}
& [\rho_2^L(u\otimes D),\rho_2^R(v\otimes D)](x\wedge y)\\
= & \rho_2^L(u\otimes D)(v\wedge D(x,y))-\rho_2^R(v\otimes D)(D(x,u)\wedge y+x\wedge D(y,u))\\
= & D(v,u)\wedge D(x,y)+v\wedge D(D(x,y),u)-v\wedge D(D(x,u),y)-v\wedge D(x,D(y,u))\\
= & \rho_2^R(\{u\otimes D,v\otimes D\})(x\wedge y).
\end{aligned}$$

Finally, we have

$$\begin{aligned}
& (\rho_2^R(v\otimes D)\circ\rho_2^L(u\otimes D)+\rho_2^R(v\otimes D)\circ\rho_2^R(u\otimes D))(x\wedge y)\\
= & \rho_2^R(v\otimes D)\circ(D(x,u)\wedge y+x\wedge D(y,u))+\rho_2^R(v\otimes D)\circ(u\wedge D(x,y))\\
= & v\wedge D(D(x,u),y)+v\wedge D(x,D(y,u))+v\wedge D(u,D(x,y))\\
= & 0.
\end{aligned}$$

Therefore, (ρ_2^L,ρ_2^R) is a representation of $\mathfrak{g}\otimes\mathbb{K}D$ on $\wedge^2\mathfrak{g}$. ■

Now we are ready to give the main result in this section.

Theorem 4.4. *Let $(\mathfrak{g},[\cdot,\cdot,\cdot]_{\mathfrak{g}})$ be a 3-Lie algebra and $D:\wedge^2\mathfrak{g}\longrightarrow\mathfrak{g}$ a generalized derivation. Then $(\wedge^2\mathfrak{g},\ \mathfrak{g}\otimes\mathbb{K}D)$ is a matched pair of Leibniz algebras, whose double is the Leibniz algebra on the space of fundamental objects associated to the generalized derivation extension $(\mathfrak{g}\oplus\mathbb{K}D,[\cdot,\cdot,\cdot]_D)$.*

Proof. One can show that conditions (i)-(vi) in Definition 2.1 hold directly. Thus, $(\wedge^2\mathfrak{g},\ \mathfrak{g}\otimes\mathbb{K}D)$ is a matched pair of Leibniz algebras. Here we use a different approach to prove this theorem. Using the isomorphism between $\wedge^2(\mathfrak{g}\oplus\mathbb{K}D)$ and $\wedge^2\mathfrak{g}\oplus(\mathfrak{g}\otimes\mathbb{K}D)$, the Leibniz algebra structure on $\wedge^2(\mathfrak{g}\oplus\mathbb{K}D)$ is given by

$$\begin{aligned}
& [x_1\wedge y_1+z_1\otimes D,x_2\wedge y_2+z_2\otimes D]_{\mathrm{F}}\\
= & [x_1\wedge y_1,x_2\wedge y_2]_{\mathrm{F}}+[x_1\wedge y_1,z_2\otimes D]_{\mathrm{F}}+[z_1\otimes D,x_2\wedge y_2]_{\mathrm{F}}+[z_1\otimes D,z_2\otimes D]_{\mathrm{F}}\\
= & [x_1\wedge y_1,x_2\wedge y_2]_{\mathrm{F}}+[x_1,y_1,z_2]_{\mathfrak{g}}\otimes D+z_2\otimes D(x_1,y_1)\\
& +D(x_2,z_1)\wedge y_2+x_2\wedge D(y_2,z_1)+D(z_2,z_1)\otimes D\\
= & [x_1\wedge y_1,x_2\wedge y_2]_{\mathrm{F}}+\{z_1\otimes D,z_2\otimes D\}+\rho_1^L(x_1,y_1)(z_2\otimes D)\\
& +\rho_2^L(z_1\otimes D)(x_2\wedge y_2)+\rho_2^R(z_2\otimes D)(x_1\wedge y_1).
\end{aligned}$$

Thus, by Lemma 2.2, we deduce that $(\wedge^2\mathfrak{g},\ \mathfrak{g}\otimes\mathbb{K}D)$ is a matched pair of Leibniz algebras. ■

Example 4.5. Consider the generalized derivation extension given in Example 3.11. The corresponding Leibniz algebra on the space of fundamental objects is a matched pair of $\wedge^2\mathfrak{g}$ and $\mathfrak{g}\otimes\mathbb{K}D$, where $\wedge^2\mathfrak{g}$ is an abelian 1-dimensional Leibniz algebra with the basis $e_1\wedge e_2$ and $\mathfrak{g}\otimes\mathbb{K}D$ is a 2-dimensional Lie algebra which is isomorphic to the one given by (10). Here $\rho_1^L=\rho_1^R=0$, and ρ_2^L, ρ_2^R are given by

$$\begin{aligned}
\rho_2^L(e_1\otimes D)(e_1\wedge e_2) &= D(e_1,e_1)\wedge e_2+e_1\wedge D(e_2,e_1)=-e_1\wedge e_1=0,\\
\rho_2^L(e_2\otimes D)(e_1\wedge e_2) &= D(e_1,e_2)\wedge e_2+e_1\wedge D(e_2,e_2)=e_1\wedge e_2,\\
\rho_2^R(e_1\otimes D)(e_1\wedge e_2) &= e_1\wedge D(e_1,e_2)=e_1\wedge e_1=0,\\
\rho_2^R(e_2\otimes D)(e_1\wedge e_2) &= e_2\wedge D(e_1,e_2)=e_2\wedge e_1.
\end{aligned}$$

Example 4.6. Consider the generalized derivation extension given in Example 3.12.

(i) The corresponding Leibniz algebra on the space of fundamental objects is a matched pair of $\wedge^2\mathfrak{g}$ and $\mathfrak{g}\otimes\mathbb{K}D$, where $\wedge^2\mathfrak{g}$ is an abelian 3-dimensional Leibniz algebra with the basis $\{e_1\wedge e_2, e_1\wedge e_3, e_2\wedge e_3\}$ and $\mathfrak{g}\otimes\mathbb{K}D$ is a 3-dimensional Lie algebra which is isomorphic to the one given by (11). Here $\rho_1^L=\rho_1^R=0$, and nontrivial ρ_2^L, ρ_2^R are given by

$$\begin{array}{rclrcl}
\rho_2^L(e_2\otimes D)(e_2\wedge e_3) &=& e_1\wedge e_2, & \rho_2^L(e_3\otimes D)(e_2\wedge e_3) &=& e_1\wedge e_3,\\
\rho_2^R(e_2\otimes D)(e_2\wedge e_3) &=& -e_1\wedge e_2, & \rho_2^R(e_3\otimes D)(e_2\wedge e_3) &=& -e_1\wedge e_3.
\end{array}$$

Thus, it is straightforward to see that the Leibniz algebra on the space of fundamental objects is a Lie algebra.

(ii) The corresponding Leibniz algebra on the space of fundamental objects is a matched pair of $\wedge^2\mathfrak{g}$ and $\mathfrak{g}\otimes\mathbb{K}D$, where $\wedge^2\mathfrak{g}$ is an abelian 3-dimensional Leibniz algebra with the basis $\{e_1\wedge e_2, e_1\wedge e_3, e_2\wedge e_3\}$ and $\mathfrak{g}\otimes\mathbb{K}D$ is a 3-dimensional Lie algebra which is isomorphic to the one given by (12). Here $\rho_1^L=\rho_1^R=0$, and nontrivial ρ_2^L, ρ_2^R are given by

$$\begin{array}{rclrcl}
\rho_2^L(e_2\otimes D)(e_2\wedge e_3) &=& e_1\wedge e_2, & \rho_2^L(e_3\otimes D)(e_2\wedge e_3) &=& e_1\wedge e_3,\\
\rho_2^L(e_1\otimes D)(e_1\wedge e_3) &=& -e_1\wedge e_2, & \rho_2^L(e_3\otimes D)(e_1\wedge e_3) &=& e_2\wedge e_3,\\
\rho_2^R(e_2\otimes D)(e_2\wedge e_3) &=& -e_1\wedge e_2, & \rho_2^R(e_3\otimes D)(e_2\wedge e_3) &=& -e_1\wedge e_3,\\
\rho_2^R(e_1\otimes D)(e_1\wedge e_3) &=& e_1\wedge e_2, & \rho_2^R(e_3\otimes D)(e_1\wedge e_3) &=& -e_2\wedge e_3.
\end{array}$$

Thus, the Leibniz algebra on the space of fundamental objects is also a Lie algebra.

(iii) The corresponding Leibniz algebra on the space of fundamental objects is a matched pair of $\wedge^2\mathfrak{g}$ and $\mathfrak{g}\otimes\mathbb{K}D$, where $\wedge^2\mathfrak{g}$ is an abelian 3-dimensional Leibniz algebra with the basis $\{e_1\wedge e_2, e_1\wedge e_3, e_2\wedge e_3\}$ and $\mathfrak{g}\otimes\mathbb{K}D$ is a 3-dimensional Lie algebra which is isomorphic to the one given by (13). Here $\rho_1^L=\rho_1^R=0$, and nontrivial ρ_2^L, ρ_2^R are given by

$$\begin{array}{rclrcl}
\rho_2^L(e_1\otimes D)(e_2\wedge e_3) &=& e_1\wedge e_2, & \rho_2^L(e_2\otimes D)(e_1\wedge e_3) &=& -e_1\wedge e_2,\\
\rho_2^L(e_3\otimes D)(e_1\wedge e_2) &=& 2e_1\wedge e_2, & \rho_2^L(e_3\otimes D)(e_1\wedge e_3) &=& e_1\wedge e_3,\\
\rho_2^L(e_3\otimes D)(e_2\wedge e_3) &=& e_2\wedge e_3, & \rho_2^R(e_1\otimes D)(e_2\wedge e_3) &=& e_1\wedge e_2,\\
\rho_2^R(e_2\otimes D)(e_1\wedge e_3) &=& -e_1\wedge e_2, & \rho_2^R(e_3\otimes D)(e_1\wedge e_3) &=& -e_1\wedge e_3,\\
\rho_2^R(e_3\otimes D)(e_2\wedge e_3) &=& -e_2\wedge e_3. & & &
\end{array}$$

(iv) The corresponding Leibniz algebra on the space of fundamental objects is a matched pair of $\wedge^2\mathfrak{g}$ and $\mathfrak{g}\otimes\mathbb{K}D$, where $\wedge^2\mathfrak{g}$ is an abelian 3-dimensional Leibniz algebra with the basis $\{e_1\wedge e_2, e_1\wedge e_3, e_2\wedge e_3\}$ and $\mathfrak{g}\otimes\mathbb{K}D$ is a 3-dimensional Lie algebra which is isomorphic to the one given by (14). Here $\rho_1^L=\rho_1^R=0$, and nontrivial ρ_2^L, ρ_2^R are given by

$$\begin{array}{rclrcl}
\rho_2^L(e_1\otimes D)(e_1\wedge e_3) &=& -e_1\wedge e_2, & \rho_2^L(e_2\otimes D)(e_1\wedge e_3) &=& -e_1\wedge e_2,\\
\rho_2^L(e_2\otimes D)(e_2\wedge e_3) &=& \alpha e_1\wedge e_2, & \rho_2^L(e_3\otimes D)(e_1\wedge e_2) &=& e_1\wedge e_2,\\
\rho_2^L(e_3\otimes D)(e_1\wedge e_3) &=& e_2\wedge e_3, & \rho_2^L(e_3\otimes D)(e_2\wedge e_3) &=& (\alpha e_1+e_2)\wedge e_3,\\
\rho_2^R(e_1\otimes D)(e_1\wedge e_3) &=& e_1\wedge e_2, & \rho_2^R(e_1\otimes D)(e_2\wedge e_3) &=& e_1\wedge e_2,\\
\rho_2^R(e_2\otimes D)(e_2\wedge e_3) &=& -\alpha e_1\wedge e_2, & \rho_2^R(e_3\otimes D)(e_1\wedge e_3) &=& -e_2\wedge e_3,\\
\rho_2^R(e_3\otimes D)(e_2\wedge e_3) &=& -(\alpha e_1+e_2)\wedge e_3. & & &
\end{array}$$

(v) The corresponding Leibniz algebra on the space of fundamental objects is a matched pair of $\wedge^2\mathfrak{g}$ and $\mathfrak{g}\otimes\mathbb{K}D$, where $\wedge^2\mathfrak{g}$ is an abelian 3-dimensional Leibniz algebra with the basis

$\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$ and $\mathfrak{g} \otimes \mathbb{K}D$ is a 3-dimensional Lie algebra which is isomorphic to the one given by (15). Here $\rho_1^L = \rho_1^R = 0$, and nontrivial ρ_2^L, ρ_2^R are given by

$$\begin{array}{rclrcl}
\rho_2^L(e_2 \otimes D)(e_2 \wedge e_3) & = & e_1 \wedge e_2, & \rho_2^L(e_3 \otimes D)(e_2 \wedge e_3) & = & e_1 \wedge e_3, \\
\rho_2^L(e_1 \otimes D)(e_1 \wedge e_3) & = & -e_1 \wedge e_2, & \rho_2^L(e_3 \otimes D)(e_1 \wedge e_3) & = & e_2 \wedge e_3, \\
\rho_2^L(e_1 \otimes D)(e_1 \wedge e_2) & = & -e_1 \wedge e_3, & \rho_2^L(e_2 \otimes D)(e_1 \wedge e_2) & = & -e_2 \wedge e_3, \\
\rho_2^R(e_2 \otimes D)(e_2 \wedge e_3) & = & -e_1 \wedge e_2, & \rho_2^R(e_3 \otimes D)(e_2 \wedge e_3) & = & -e_1 \wedge e_3, \\
\rho_2^R(e_1 \otimes D)(e_1 \wedge e_3) & = & e_1 \wedge e_2, & \rho_2^R(e_3 \otimes D)(e_1 \wedge e_3) & = & -e_2 \wedge e_3, \\
\rho_2^R(e_1 \otimes D)(e_1 \wedge e_2) & = & e_1 \wedge e_3, & \rho_2^R(e_2 \otimes D)(e_1 \wedge e_2) & = & e_2 \wedge e_3.
\end{array}$$

Thus, the Leibniz algebra on the space of fundamental objects is also a Lie algebra.

5 Nambu-Poisson structures

In this section, we analyze the Nambu-Poisson structure associated to a generalized derivation extension. Let $(\mathfrak{g}, [\cdot,\cdot,\cdot]_{\mathfrak{g}})$ be a 3-Lie algebra such that it induces a linear Nambu-Poisson structure $\pi_{\mathfrak{g}}$ on $\mathfrak{g}^*$. Let $D : \wedge^2\mathfrak{g} \longrightarrow \mathfrak{g}$ be a generalized derivation, and π_D the corresponding linear Poisson structure on $\mathfrak{g}^*$. Let $\{e_1, \cdots, e_n\}$ be a basis of $\mathfrak{g}$, which can be viewed as coordinate functions on $\mathfrak{g}^*$. Then π_D is given by

$$\pi_D = \sum_{i<j} D(e_i, e_j)\frac{\partial}{\partial e_i} \wedge \frac{\partial}{\partial e_j}.$$

It is obvious that $\{e_1, \cdots, e_n, D\}$ constitute a basis of $\mathfrak{g} \oplus \mathbb{K}D$. $\frac{\partial}{\partial D}$ is a constant vector field on $(\mathfrak{g} \oplus \mathbb{K}D)^*$ satisfying $\frac{\partial D}{\partial D} = 1$ and $\frac{\partial e_i}{\partial D} = 0$.

Theorem 5.1. *Let $(\mathfrak{g}, [\cdot,\cdot,\cdot]_{\mathfrak{g}})$ be a 3-Lie algebra such that it induces a linear Nambu-Poisson structure $\pi_{\mathfrak{g}}$ on $\mathfrak{g}^*$, and $D : \wedge^2\mathfrak{g} \longrightarrow \mathfrak{g}$ a generalized derivation on $\mathfrak{g}$. Then*

$$\pi_{\mathfrak{g}} + \frac{\partial}{\partial D} \wedge \pi_D \in \mathfrak{X}^3((\mathfrak{g} \oplus \mathbb{K}D)^*)$$

is the Nambu-Poisson structure corresponding to the 3-Lie algebra $(\mathfrak{g} \oplus \mathbb{K}D, [\cdot,\cdot,\cdot]_D)$ if and only if

$$\pi_D^\sharp(df) \wedge \pi_D = 0, \quad \forall f \in C^\infty(M).$$

Proof. $\pi_{\mathfrak{g}} + \frac{\partial}{\partial D} \wedge \pi_D$ is a Nambu-Poisson structure if and only if for all $\phi, \varphi \in C^\infty((\mathfrak{g} \oplus \mathbb{K}D)^*)$, there holds:

$$L_{(\pi_{\mathfrak{g}} + \frac{\partial}{\partial D} \wedge \pi_D)^\sharp(d\phi \wedge d\varphi)}(\pi_{\mathfrak{g}} + \frac{\partial}{\partial D} \wedge \pi_D) = 0. \tag{19}$$

For all $f, g \in C^\infty(\mathfrak{g}^*)$, by the fact $\pi_{\mathfrak{g}}$ is a Nambu-Poisson structure, we have

$$\begin{aligned}
& L_{(\pi_{\mathfrak{g}} + \frac{\partial}{\partial D} \wedge \pi_D)^\sharp(df \wedge dg)}(\pi_{\mathfrak{g}} + \frac{\partial}{\partial D} \wedge \pi_D) \\
= \; & L_{\pi_{\mathfrak{g}}^\sharp(df \wedge dg)}\pi_{\mathfrak{g}} + \frac{\partial}{\partial D} \wedge L_{\pi_{\mathfrak{g}}^\sharp(df \wedge dg)}\pi_D + L_{\langle \pi_D, df \wedge dg\rangle \frac{\partial}{\partial D}}\pi_{\mathfrak{g}} \\
= \; & \frac{\partial}{\partial D} \wedge (L_{\pi_{\mathfrak{g}}^\sharp(df \wedge dg)}\pi_D - \iota_{d\langle \pi_D, df \wedge dg\rangle}\pi_{\mathfrak{g}}).
\end{aligned}$$

For all $f \in C^\infty(\mathfrak{g}^*)$ and $\mu \in C^\infty((\mathbb{K}D)^*)$, we have

$$\begin{aligned} & L_{(\pi_{\mathfrak{g}}+\frac{\partial}{\partial D}\wedge\pi_D)^\sharp(d\mu\wedge df)}(\pi_{\mathfrak{g}} + \frac{\partial}{\partial D}\wedge\pi_D) \\ = \; & L_{\frac{\partial\mu}{\partial D}\pi_D^\sharp(df)}(\pi_{\mathfrak{g}} + \frac{\partial}{\partial D}\wedge\pi_D) \\ = \; & \frac{\partial\mu}{\partial D}L_{\pi_D^\sharp(df)}\pi_{\mathfrak{g}} - \frac{\partial^2\mu}{\partial D^2}\pi_D^\sharp(df)\wedge\pi_D + \frac{\partial\mu}{\partial D}\frac{\partial}{\partial D}\wedge L_{\pi_D^\sharp(df)}\pi_D. \end{aligned}$$

Therefore, $\pi_{\mathfrak{g}} + \frac{\partial}{\partial D}\wedge\pi_D$ is a Nambu-Poisson structure if and only if the following equalities hold:

$$L_{\pi_{\mathfrak{g}}^\sharp(df\wedge dg)}\pi_D - \iota_{d\langle\pi_D, df\wedge dg\rangle}\pi_{\mathfrak{g}} = 0, \tag{20}$$

$$L_{\pi_D^\sharp(df)}\pi_{\mathfrak{g}} = 0, \tag{21}$$

$$L_{\pi_D^\sharp(df)}\pi_D = 0, \tag{22}$$

$$\pi_D^\sharp(df)\wedge\pi_D = 0. \tag{23}$$

First it is obvious that (22) is equivalent to that π_D is a Lie-Poisson structure corresponding to the Lie algebra structure D. That is, (22) is equivalent to Condition (c) in Definition 3.1. Then (21) is equivalent to Condition (a) in Definition 3.1. In fact, for linear functions $x, y, z, u \in \mathfrak{g}$ on $\mathfrak{g}^*$, we have

$$\begin{aligned} & \langle L_{\pi_D^\sharp(dx)}\pi_{\mathfrak{g}}, dy\wedge dz\wedge du\rangle \\ = \; & \pi_D^\sharp(dx)\langle\pi_{\mathfrak{g}}, dy\wedge dz\wedge du\rangle - \langle\pi_{\mathfrak{g}}, (L_{\pi_D^\sharp(dx)}dy)\wedge dz\wedge du\rangle \\ & -\langle\pi_{\mathfrak{g}}, dy\wedge(L_{\pi_D^\sharp(dx)}dz)\wedge du\rangle - \langle\pi_{\mathfrak{g}}, dy\wedge dz\wedge(L_{\pi_D^\sharp(dx)}du)\rangle \\ = \; & \pi_D^\sharp(dx)[y,z,u]_{\mathfrak{g}} - \langle\pi_{\mathfrak{g}}, d\langle\pi_D^\sharp(dx), dy\rangle\wedge dz\wedge du\rangle \\ & -\langle\pi_{\mathfrak{g}}, dy\wedge d\langle\pi_D^\sharp(dx), dz\rangle\wedge du\rangle - \langle\pi_{\mathfrak{g}}, dy\wedge dz\wedge d\langle\pi_D^\sharp(dx), du\rangle\rangle \\ = \; & D(x,[y,z,u]_{\mathfrak{g}}) - [D(x,y),z,u]_{\mathfrak{g}} - [y,D(x,z),u]_{\mathfrak{g}} - [y,z,D(x,u)]_{\mathfrak{g}}, \end{aligned}$$

which implies that (21) is equivalent to Condition (a). Finally, we have

$$\begin{aligned} & \langle L_{\pi_{\mathfrak{g}}^\sharp(dx\wedge dy)}\pi_D - \iota_{d\langle\pi_D, dx\wedge dy\rangle}\pi_{\mathfrak{g}}, dz\wedge du\rangle \\ = \; & \pi_{\mathfrak{g}}^\sharp(dx\wedge dy)\langle\pi_D, dz\wedge du\rangle - \langle\pi_D, L_{\pi_{\mathfrak{g}}^\sharp(dx\wedge dy)}dz\wedge du\rangle - \langle\pi_D, dz\wedge L_{\pi_{\mathfrak{g}}^\sharp(dx\wedge dy)}du\rangle \\ & -\langle\pi_{\mathfrak{g}}, d\langle\pi_D, dx\wedge dy\rangle\wedge dz\wedge du\rangle \\ = \; & [x,y,D(z,u)]_{\mathfrak{g}} - D([x,y,z]_{\mathfrak{g}},u) - D(z,[x,y,u]) - [D(x,y),z,u]_{\mathfrak{g}}, \end{aligned}$$

which implies that (20) is equivalent to Condition (b) in Definition 3.1. Thus, $\pi_{\mathfrak{g}} + \frac{\partial}{\partial D}\wedge\pi_D$ is a Nambu-Poisson structure if and only if (23) holds. It is straightforward to see that the corresponding 3-Lie algebra is $(\mathfrak{g}\oplus\mathbb{K}D, [\cdot,\cdot]_D)$. The proof is finished. ■

Remark 5.2. *Not every 3-Lie algebra, or more generally n-Lie algebra can give rise to a Nambu-Poisson structure on the dual space. However, it gives rise to a Filippov tensor which was introduced in [12]. A Filippov tensor is a Nambu-Poisson structure if and only if it is a decomposable. The condition in the above theorem guarantee that the corresponding Filippov tensor is decomposable.*

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