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Multiplicity of solutions to the Yamabe equation on warped products

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ABSTRACT

Let (M^m, g_M) be a closed (compact, without boundary) connected manifold with positive scalar curvature and (F^k, h) a closed connected manifold with constant scalar curvature ($m \geq 3$ and $k > 3$). By a Theorem of Dobarro and Lami Dozo (1987), there are weights $f : M \to \mathbb{R}^+$ such that the warped product $(M^m \times F^k, g_M + f^2h)$ has constant scalar curvature. We construct paths of warped product metrics $(M^m \times F^k, g_M + f_\epsilon^2 h)$, $\epsilon \in (0, \epsilon_0)$, ϵ_0 small, with constant scalar curvature, that exhibit multiplicity of solutions to the Yamabe equation. Moreover, in the case that (F^k, h) has a flat metric we add the constraints of unit volume and fixed constant scalar curvature to the construction of paths of warped metrics $(M^m \times F^k, g_M + f^2 h_\epsilon)$, $\epsilon \in (0, \epsilon_0)$, that exhibit multiplicity. We use techniques from bifurcation theory along with spectral theory for warped products.

1. Introduction

Given a closed (compact, without boundary) Riemannian manifold (N, h), the solution of the Yamabe problem (cf. in [12]) gives a metric $\bar{h}$ for N, of constant scalar curvature and unit volume in the conformal class of h. These metrics are critical points of the Hilbert–Einstein functional restricted to conformal classes. The minima of the restricted functional are always realized. This was proved by the combined efforts of H. Yamabe [24], N. S. Trudinger [22], T. Aubin [1] and R. Schoen [19], giving the solution of the Yamabe problem. It is of interest to ask for other metrics of constant scalar curvature in the conformal class, as they are critical points of the Hilbert–Einstein functional on conformal classes, that are not necessarily minimizers. Uniqueness of a metric of constant scalar curvature of unit volume in a conformal class is known to be true in some special cases: when the scalar curvature is non-positive, by the maximum principle; if h is an Einstein metric that is not isometric to the round sphere, by a result of M. Obata [13]; and if the metric h is close in the $C^{2,\alpha}$ topology to an Einstein metric and $dim(N) \leq 7$ (or $dim(N) \leq 24$ and N is spin), by a result of L. L. De Lima, P. Piccione and M. Zedda [7]. On the other hand, a rich variety of constant scalar curvature metrics that are not necessarily minimizers have been studied recently, see for instance [5,10,16,17,20].

Classical results in Bifurcation theory, asserting the existence of bifurcation points in continuous paths of functionals and critical points, have been applied successfully to the Yamabe problem setting; giving proofs of multiplicity of constant scalar curvature metrics with unit volume in the same conformal class on many different types of manifolds, for example, on products of compact manifolds in [7], in the product manifold with a k-sphere [15], (and, more generally, on sphere bundles in [14]), also on collapsing Riemannian submersions in [3], on non-compact manifolds of the type $S^n \times \mathbb{R}^d$ and $S^n \times \mathbb{H}^d$ in [4] and on the product of a manifold of constant positive scalar curvature and a k-Torus in [18].

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Let (M^m, g_M), $m \geq 3$, be a closed, connected manifold with positive scalar curvature s_{g_M}, not necessarily constant and (F^k, h), $k > 3$, some closed manifold with constant scalar curvature s_h. In this article we will be interested in finding multiplicity of constant scalar curvature metrics on warped products $M^m \times_f F^k$, with the use of local bifurcation theory. Namely, we find bifurcation points in paths of metrics with constant scalar curvature on $M^m \times_f F^k$, for fixed g_M, and either varying f and fixing h, for the case $s_h \neq 0$, or varying h with f fixed, in the case $s_h = 0$.

Consider a path of metrics $\{G_t\}_{t\in[a,b]}$, of constant scalar curvature and unit volume for a fixed closed manifold N^m ($m \geq 3$). We call $t^* \in [a, b]$ a bifurcation instant for the path if there exist a sequence $t_n \in [a, b]$ and a sequence of Riemannian metrics $\bar{G}_n$ in the conformal class of G_{t_n} ($\bar{G}_n \neq G_{t_n}$), of constant scalar curvature and unit volume, such that $\lim_{n\to\infty} t_n = t^*$ and $\lim_{n\to\infty} \bar{G}_n = G_{t^*}$. We will call the corresponding metric, G_{t^*}, a bifurcation point. Hence around G_{t^*} we have multiplicity of solutions to the Yamabe equation.

Solutions to the Yamabe equation on the warped product of a compact manifold of positive scalar curvature with a compact manifold of constant scalar curvature were studied in [8] by F. Dobarro and E. Lami Dozo. It is shown there that, under these conditions, $s_{g_M} > 0$ and s_h constant, there are weights $f : M \to \mathbb{R}^+$ that make the warped product of constant scalar curvature.

Specifically, for warped products $(M^m \times F^k, g_M + f^2h)$ to have constant scalar curvature $\tilde{s}$, it must be satisfied

$$c_k \Delta_{g_M} u + s_{g_M} u + s_h u^q = \tilde{s} u \tag{1}$$

with $u = f^{(k+1)/2}$, $c_k = \frac{4k}{k+1}$, $q = \frac{k-3}{k+1}$ (Theorem 2.1 in [8]). Let $\mathbf{S}$ be the first non-zero eigenvalue of the operator $c_k \Delta_{g_M} + s_{g_M}$. That is

$$\mathbf{S} = \inf\{\int_M (c_k|\nabla v|^2 + s_{g_M} v^2) dV_{g_M};\ v \in H_1^2(M),\ \int_M v^2 dV_{g_M} = 1\}. \tag{2}$$

Note that since we are assuming that s_{g_M} is a positive scalar curvature for (M, g_M), the constant $\mathbf{S}$ will also be positive.

The resulting scalar curvature of the warped product $(M^m \times F^k, g_M + f^2h)$ may be any $\tilde{s}$, $\tilde{s} < \mathbf{S}$, if $s_h < 0$, and any $\tilde{s}$, $\tilde{s} \in (\mathbf{S}, \mathbf{S} + \delta)$, for some $\delta > 0$, if $s_h > 0$. Once $\tilde{s}$ is fixed, the weight f is unique. In the case $s_h = 0$, the resulting scalar curvature $\tilde{s}$ can only be equal to $\mathbf{S}$, and the weight f is unique up to constant factors. In any case, the weight f does not depend on the metric h of the second factor, save for the number s_h.

The positive eigenfunction u, associated with the first eigenvalue $\mathbf{S}$, satisfies the following equation,

$$c_k \Delta_{g_M} u + s_{g_M} u = \mathbf{S}\, u. \tag{3}$$

Therefore, taking $f = u^{2/(k+1)}$ the manifold $(M^m \times F^k, g_M + f^2h)$ has constant scalar curvature $\mathbf{S}$ for any Riemannian metric h with identically zero scalar curvature.

The spectrum of the Laplacian of warped product metrics on compact manifolds was studied thoroughly by N. Ejiri in [9]. It is shown there that given a base manifold (M^m, g_M) and a fiber (F^k, h) with eigenvalues $0 = \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_i \to \infty$, then the eigenvalues of the Laplacian of the warped product metric $(M \times F, g_M + f^2h)$ with weight $f \in C^\infty(M)$, $f > 0$, are given by the union of the sets of eigenvalues of the operators

$$L_{\lambda_i} = \Delta_M - \frac{k}{f}\nabla_{grad\, f} + \frac{\lambda_i}{f^2}, \tag{4}$$

for each λ_i, $i = 0, 1, 2, \ldots$. Note that for $\lambda_0 = 0$, the operator

$$L_0 = \Delta_M - \frac{k}{f}\nabla_{grad\, f}, \tag{5}$$

does not depend on the metric h at all.

As our first result, we construct families of warped metrics with infinitely many bifurcation points if the fiber manifold has positive, zero or negative constant scalar curvature.

Theorem 1.1. *Let (M^m, g_M) and (F^k, h) be closed connected Riemannian manifolds ($m \geq 3$ and $k > 3$) with s_{g_M} positive and s_h constant. Assume that $\frac{\mathbf{S}}{m+k-1} \notin Spec\{L_0\}$, then there exists a path of warped product metrics $G_\epsilon = g_M + f_\epsilon^2 h$, with $\epsilon \in (0, \epsilon_0)$, for some $\epsilon_0 > 0$, of constant scalar curvature that has infinitely many (discrete) bifurcation points.*

Recall that $\mathbf{S}$ is given by Eq. (2). Even though the metrics described in Theorem 1.1 will be shown to have constant scalar curvature close to $\mathbf{S}$, we will see that the volumes of the metrics will grow to infinity as $\epsilon \to 0$, so that the metrics where bifurcation points occur are of high energy, for the restricted Hilbert–Einstein functional.

On the other hand, if the metric of the fiber manifold is flat, then we may construct paths of metrics with constant scalar curvature $\mathbf{S}$ and unit volume, with infinitely many bifurcation points. Hence, these metrics may be low energy for the restricted Hilbert–Einstein functional.

Theorem 1.2. *Let (M^m, g_M), (F^k, h) be closed connected Riemannian manifolds of unit volume, $m \geq 3$, $k > 3$, with positive scalar curvature s_{g_M} and flat metric h, respectively. Let $f \in C^\infty(M)$ be the unique weight such that $(M^m \times F^k, g_M + f^2h)$ is of constant scalar curvature $\mathbf{S}$ and unit volume, for any closed, unit volume flat manifold (F^k, h). If $\frac{\mathbf{S}}{m+k-1} \notin Spec\{L_0\}$, then there is a path of unit volume warped product metrics $G_\epsilon = g_M + f^2h_\epsilon$, on $M^n \times F^k$, with $\epsilon \in (0, 1)$, and with each h_ϵ a flat metric of unit volume, such that there are infinitely many, discrete, bifurcation points G_{ϵ^*}, $\epsilon^* \in (0, 1)$.*

Recall that existence and uniqueness of the weight f in Theorem 1.2 is guaranteed by Theorem 3.1 in [8]. Moreover, f does not depend on the metrics h_ϵ as they all have constant scalar curvature zero. Note that if (M^m, g_M) is of constant scalar curvature s_{g_M} and of unit volume, then we have that $f = 1$ and we recover some cases of product manifolds studied previously [4,18]. In the case of product manifolds, the hypothesis $\frac{\mathbf{S}}{m+k-1} \notin Spec\{L_0\}$ simplifies to $\frac{s_{g_M}}{m+k-1} \notin Spec\{\Delta_{g_M}\}$.

We remark that the hypothesis $\frac{\mathbf{S}}{m+k-1} \notin Spec\{L_0\}$ is necessary, given that in the path of warped metrics only the metric of the second factor is varying. Consider for example the manifold $(M^m, g_M) = (\mathbb{S}^2 \times T^2, g_1 + g_2)$, where g_1 is the round metric and g_2 is a flat metric on T^2 such that $\lambda = \frac{2}{7}$ is an eigenvalue of Δ_{g_2} and the product metric $g_1 + g_2$ has unit volume (such a metric g_2 on T^2 is easy to construct, by choosing the appropriate size of the lattice Γ, such that $T^2 = \mathbb{R}^2/\Gamma$; see the end of Section 3 for some details on flat metrics for T^n and the eigenvalues of their Laplacians). Note that $s_{g_M} = 2$ and $\lambda = \frac{2}{7} \in Spec\{\Delta_{g_M}\}$. Consider now the manifold $(\mathbb{S}^2 \times T^2 \times T^4, g_M + h)$, with h any flat metric of unit volume on T^4. It follows that $g_M + h$ is of constant scalar curvature 2 and unit volume. We also have $\frac{s_{g_M}}{4+4-1} = \frac{2}{7} \in Spec\{\Delta_{g_M}\}$. This implies that regardless of the path of unit volume flat metrics h_t, $t \in (0, 1)$, we choose for T^4, the metrics $G_t = g_M + h_t$ are going to be degenerated in the sense of Morse Theory (see Section 2 for details). This means that we will always be missing a necessary condition (see Proposition 2.1) to ensure the existence of bifurcation points in these paths. Small perturbations of this example illustrate that this is also the case for warped products.

For our first result, our strategy will be to vary the weight f_ϵ so that the constant scalar curvature $\tilde{s}_\epsilon$ of the path of warped product metrics satisfies $\tilde{s}_\epsilon \to \mathbf{S}$ as $\epsilon \to 0$, and then study the spectra of the Laplacian of the resulting metrics G_ϵ, as $\epsilon \to 0$, in order to prove the existence of bifurcation points in the path of metrics.

For the second result, our strategy will be to vary the flat metric h on F^k in order to get a family of warped product metrics of constant scalar curvature and unit volume, and then prove the existence of a bifurcation point in this family. This goes along the general direction of some results in [18] and [4], on multiplicity of constant scalar curvature metrics on the product of a manifold with positive constant scalar curvature and a flat manifold.

2. Variational formulation and bifurcation

We recall the variational framework for the Yamabe problem, we refer the reader to [12,20] for details. Given a closed smooth manifold M^n, $n \geq 3$, for any metric g, its $H_1^2(M)$ conformal class is given by $[g] = \{\varphi^{\frac{4}{n-2}} g : \varphi \in H_1^2(M), \varphi > 0\}$. Note that the $H_1^2(M)$ conformal class can be canonically identified with the space of positive $H_1^2(M)$ functions. We thus denote by $[g]_1$ the set of positive $H_1^2(M)$ functions, φ, normalized so that $\int_M \varphi^{\frac{2n}{n-2}} dV_g = 1$. That is, those functions φ such that $\varphi^{\frac{4}{n-2}} g \in [g]$ and $\varphi^{\frac{4}{n-2}} g$ has unit volume.

It happens that $[g]_1$ is a submanifold of $H_1^2(M)$, and at $\varphi_0 = 1$,

$$T_{\varphi_0}[g]_1 \approx \{\psi \in H_1^2(M) : \int_M \psi dV_g = 0\}.$$

Let S_g denote the scalar curvature of g. Consider the Hilbert–Einstein functional, $\mathcal{A}(g) = Vol(M, g)^{\frac{2-n}{n}} \int_M S_g dV_g$, restricted to the space of metrics of volume 1 in the same $H_1^2(M)$ conformal class, which can be canonically identified with the space of functions $[g]_1$. Thus we may write,

$$\mathcal{A}|_{[g]_1} : [g]_1 \to \mathbb{R},$$

$$\mathcal{A}|_{[g]_1}(\varphi) = \int_M \left(\frac{4(n-1)}{(n-2)} \langle \nabla\varphi, \nabla\varphi \rangle_g + S_g \varphi^2 \right) dV_g, \quad \varphi \in [g]_1.$$

φ is a critical point of $\mathcal{A}|_{[g]_1}$ if and only if φ is smooth and $\tilde{g} = \varphi^{\frac{4}{n-2}} g$ has constant scalar curvature. If g has constant scalar curvature (i.e. $\varphi_0 = 1$ is a critical point) then the second derivative of $\mathcal{A}$ at $\varphi_0 = 1$ is a symmetric bilinear form,

$$d^2(\mathcal{A})(\varphi_0)(\psi_1, \psi_2) = \frac{(n-1)(n-2)}{2} \int_M \left(\langle \nabla\psi_1, \nabla\psi_2 \rangle_g - \frac{S_g}{n-1} \psi_1 \psi_2 \right) dV_g,$$

which can be represented by the self-adjoint elliptic operator

$$\mathcal{J}(\psi) = \frac{(n-1)(n-2)}{2} \left(\Delta_g \psi - \frac{S_g}{n-1} \psi \right),$$

called the Jacobi operator. Let g be a metric such that 1 is a critical point of $\mathcal{A}$ restricted to $[g]_1$. We say that g is nondegenerate, in the usual sense of Morse theory, if and only if either $S_g = 0$ or if $\frac{S_g}{n-1} \notin Spec\{\Delta_g\}$.

The Morse index η_g of a critical point g is the number of negative eigenvalues of the Jacobi operator; that is, the number of eigenvalues of the Laplace–Beltrami operator, Δ_g, counted with multiplicity, that are less than $\frac{S_g}{n-1}$.

We now take a look at local bifurcation theory, in particular, applied to solutions for the Yamabe problem. By a classical result in variational bifurcation theory, given a continuous path of smooth functionals and of critical points, there is a bifurcating branch issuing from the given path at each point where the Morse index changes. See for example the non equivariant bifurcation Theorem in the work of J. Smoller and A. G. Wasserman [21], see also the work of H. Kielhöfer [11]. We refer, for instance, to Theorem 3.3, in [7], by De Lima, Piccione and Zedda, for an application to the Yamabe problem setting:

Proposition 2.1 ([7, Theorem 3.3]). *Let M^n be a compact manifold with $n \geq 3$ and $\{g_t\}_{t\in[a,b]}$ a C^1-path of Riemannian metrics on M with constant scalar curvature. Let S_t denote the scalar curvature and η_t the number of eigenvalues of Δ_{g_t} counted with multiplicity that are less than $\frac{S_t}{n-1}$. If $\eta_a \neq \eta_b$ and $\frac{S_a}{n-1} = 0$ or $\frac{S_a}{n-1} \notin Spec\{\Delta_{g_a}\}$ and $\frac{S_b}{n-1} = 0$ or $\frac{S_b}{n-1} \notin Spec\{\Delta_{g_b}\}$, then there exists a bifurcation instant $t_* \in (a, b)$ for the path of metrics $\{g_t\}_{t\in[a,b]}$.*

3. Spectral theory and Proof of Theorem 1.2

Given a compact manifold (M^m, g_M) of positive scalar curvature, and a compact manifold (F^k, h) of constant scalar curvature, we fix a weight $f \in C^\infty(M)$ that makes the warped product $(M^m \times F^k, g_M + f^2h)$ of constant scalar curvature (see [8] for details on f) and study the spectra of the Laplacian of the metric $g_M + f^2h$. If $s_h = 0$, then f is unique as soon as one requires also volume one for the resulting metric $g_M + f^2h$.

The eigenvalues α_i of the Laplacian of a compact Riemannian manifold are discrete, of finite multiplicity and accumulate only at infinity: $\alpha_i \to \infty$ as $i \to \infty$.

Let $0 = \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_i \to \infty$ denote the eigenvalues of the Laplacian of the metric h. For fixed λ_i denote the eigenvalues of the elliptic operator L_{λ_i} from Eq. (4) by $\mu_i^0 < \mu_i^1 \leq \cdots \leq \mu_i^j \to \infty$. The eigenvalues of $\Delta_{g_M+f^2h}$ are exactly the list $\{\mu_i^j\}$ for $i, j = 0, 1, 2, \ldots$. We refer the reader to [9,23] for details.

An application of the min–max principle to the operator L_{λ_i} and Hölder's inequality, yields the following comparison between μ_i^0 and λ_i, for each $i \in \mathbb{N}$.

Proposition 3.1 ([23, Theorem 2]). *For a warped product of compact connected manifolds $(M^m \times F^k, g_M + f^2h)$, we have $\mu_i^0 \frac{\|f\|_k^2}{\|1\|_k^2} \leq \lambda_i$, for $i \in \mathbb{N}$. Equality holds if and only if f is constant.*

Proof. For $\psi, \varphi \in C^\infty(M)$ we denote by $\langle\psi, \varphi\rangle$, the scalar product given by $\langle\psi, \varphi\rangle = \int_M \left(\int_F \psi\varphi\, f^k dV_h\right) dV_{g_M}$. Fixing λ_i and f, the operator L_{λ_i} acts on $\varphi \in C^\infty(M)$ as

$$L_{\lambda_i}\varphi = \Delta_M\, \varphi - \frac{k}{f}\nabla_{grad\, f}\, \varphi + \frac{\lambda_i\varphi}{f^2}.$$

We apply the min–max principle to L_{λ_i}. In particular, for $\varphi = 1$, we have

$$\mu_i^0 \leq \frac{\langle L_{\lambda_i}(1), 1\rangle}{\langle 1, 1\rangle} = \left(\int_M \left(\int_F L_{\lambda_i}(1)\, f^k dV_h\right) dV_{g_M}\right)\left(\frac{1}{\int_M \left(\int_F f^k dV_h\right) dV_{g_M}}\right)$$

$$= \left(\int_M \left(\int_F \frac{\lambda_i}{f^2}\, f^k dV_h\right) dV_{g_M}\right)\left(\frac{1}{\int_M \left(\int_F f^k dV_h\right) dV_{g_M}}\right) = \lambda_i\, \frac{\int_M f^{k-2} dV_{g_M}}{\int_M f^k dV_{g_M}}.$$

Then, using Hölder's inequality,

$$\mu_i^0 \leq \lambda_i\, \frac{\int_M f^{k-2} dV_{g_M}}{\int_M f^k dV_{g_M}} \leq \lambda_i\, \frac{\left(\int_M f^k dV_{g_M}\right)^{(k-2)/k} \left(\int_M dV_{g_M}\right)^{2/k}}{\left(\int_M f^k dV_{g_M}\right)} = \lambda_i \frac{\|1\|_k^2}{\|f\|_k^2}. \qquad \square$$

For a given λ, we may denote the eigenvalues of the operator $L_\lambda = L(\lambda)$ defined in Eq. (4), by $\mu_\lambda^0 \leq \mu_\lambda^1 \leq \cdots \leq \mu_\lambda^j \leq \cdots$, for $j \in \mathbb{N}$. In the following, we generalize a result of K. Tsukada and see that in this sense, μ_λ^j is increasing as a function of λ, for any $j \geq 0$.

Proposition 3.2 ([23, Lemma 1]). *Let $0 \leq \lambda < \lambda'$, then $\mu_\lambda^j < \mu_{\lambda'}^j$, for $j \geq 0$.*

Proof. Let $Gr_j(C^\infty(M))$ be the j-dimensional Grassmannian in $C^\infty(M)$. Recall the min–max characterization of μ_λ^j:

$$\mu_\lambda^j = \inf_{V\in Gr_{j+1}(C^\infty(M))}\ \sup_{v\in V\setminus\{0\}} \frac{\langle L_\lambda v, v\rangle}{\langle v, v\rangle}.$$

Note that

$$\int_M (f^k v)(\Delta_M v)\, dV_{g_M}$$
$$= \int_M (\nabla v, \nabla(f^k v))_{g_M} dV_{g_M} = \int_M f^k(\nabla v, \nabla v)_{g_M} dV_{g_M} + \int_M k f^{k-1} v(\nabla v, \nabla f)_{g_M} dV_{g_M}. \qquad (6)$$

We compute

$$\langle L_\lambda v, v\rangle = \int_M \left(\int_F v(\Delta_M v - \frac{k}{f}(\nabla v, \nabla f)_{g_M} + \frac{\lambda}{f^2} v) f^k dV_h \right) dV_{g_M}$$

$$= \left(\int_M \left(f^k v \Delta_M v - k f^{k-1} v(\nabla v, \nabla f)_{g_M} + \lambda f^{k-2} v^2\right) dV_{g_M} \right) (Vol(F, h)),$$

and using Eq. (6), we obtain

$$\langle L_\lambda v, v\rangle = \left(\int_M f^k |\nabla v|_g^2 dV_{g_M} + \lambda \int_M f^{k-2} v^2 dV_{g_M} \right) (Vol(F, h)). \tag{7}$$

Since f is positive, this implies that

$$\langle L_\lambda v, v\rangle \leq \langle L_{\lambda'} v, v\rangle,$$

if $0 \leq \lambda \leq \lambda'$. Also, $\langle L_\lambda v, v\rangle = \langle L_{\lambda'} v, v\rangle$, if and only if $\lambda = \lambda'$.

Finally, since for each $j \in \mathbb{N}$,

$$\mu_\lambda^j = \inf_{V \in Gr_{j+1}(C^\infty(M))} \sup_{v \in V\setminus\{0\}} \frac{\langle L_\lambda v, v\rangle}{\langle v, v\rangle},$$

then, $\mu_\lambda^j \leq \mu_{\lambda'}^j$, if $0 \leq \lambda \leq \lambda'$. And $\mu_\lambda^j = \mu_{\lambda'}^j$, if and only if $\lambda = \lambda'$. $\square$

The following proposition proves Theorem 1.2, assuming the existence on F^k of a path of flat metrics h_ϵ, $\epsilon \in (0, 1]$, with specific properties.

Proposition 3.3. *Let (M, g_M) and (F^k, h) be closed connected Riemannian manifolds with positive scalar curvature and a flat metric of unit volume, respectively. Suppose that on F^k there is a path of flat unit volume metrics h_ϵ, $\epsilon \in (0, 1]$, with $h_1 = h$. Denote the eigenvalues of the Laplacian of h_ϵ by $\lambda_i(\epsilon)$. Suppose that for each i, $i > 0$, $\lambda_i(\epsilon)$ is a nonconstant polynomial function of ϵ and $\frac{1}{\epsilon}$, and that there is some sequence $\{\lambda_{q_j}(\epsilon)\}$, $j \in \mathbb{N}$, with $\lambda_{q_j}(\epsilon) \to 0$ as $\epsilon \to 0$. If $\frac{\mathbf{S}}{m+k-1} \notin Spec\{L_0\}$, then there is a path of warped product metrics $G_\epsilon = g_M + f^2 h_\epsilon$, $\epsilon \in (0, 1)$, of constant scalar curvature* **S** *and unit volume, with infinitely many bifurcation points.*

Proof. Given the path of unit volume flat metrics h_ϵ, $\epsilon \in (0, 1]$, from the hypothesis, consider the path of warped product metrics $G_\epsilon = g_M + f^2 h_\epsilon$, $\epsilon \in (0, 1)$. Here f is the unique weight that makes G_ϵ of constant scalar curvature **S** and unit volume. As discussed before, such f does not depend on the metric h_ϵ and hence is the same through all the path G_ϵ, $\epsilon \in (0, 1)$.

Let $\epsilon_0 \in (0, 1)$. We will show that there exist $a, b \in (0, \epsilon_0)$ such that the path $\{G_\epsilon\}_{\epsilon \in [b,a]}$, contains a bifurcation point. Let $\bar{S} = \frac{\mathbf{S}}{m+k-1}$. Recall that in order to use Proposition 2.1 the path must begin with a metric G_a such that $\bar{S} \notin Spec\{\Delta_{G_a}\}$.

Let $\alpha \in (0, \epsilon_0)$, if $\bar{S} \notin Spec\{\Delta_{G_\alpha}\}$, we let $G_a = G_\alpha$, otherwise we do the following.

Suppose that $\bar{S} \in Spec\{\Delta_{G_\alpha}\}$. Let $\mu_{i_1}^{j_1}, \mu_{i_2}^{j_2}, \ldots, \mu_{i_n}^{j_n}$ be the finite set of eigenvalues of the Laplacian of G_α, equal to $\bar{S}$. Recall the hypothesis, $\bar{S} \notin Spec\{L_0\}$; this means that neither of $i_1, i_2, \ldots, i_n$ is equal to zero, for the given set of eigenvalues. Thus, by hypothesis, advancing the path (i.e. decreasing ϵ) modifies (increases or decreases) the values of each of the eigenvalues of the Laplacian of the metric on F^k, in particular, of $\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_n}$ (recall that for $i > 0$, the functions $\epsilon \to \lambda_i(\epsilon)$ are nonconstant polynomial functions of ϵ and $\frac{1}{\epsilon}$ by hypothesis). By Proposition 3.2, this in turn means that the values of $\mu_{i_1}^{j_1}, \mu_{i_2}^{j_2}, \ldots, \mu_{i_n}^{j_n}$, are modified, as they are increasing functions of $\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_n}$. Since the eigenvalues are discrete and the path is continuous, there exists in the path a metric G_a $(0 < a < \alpha)$ such that $\bar{S} \notin Spec\{\Delta_{G_a}\}$.

Now, let η_a be the Morse index of G_a, that is, the number of eigenvalues of the Laplacian of G_a, counted with multiplicity, such that they are less than $\bar{S}$. Let r be a positive integer such that $r > \eta_a$. Then, we advance our path $\{G_\epsilon\}_{\epsilon \leq a}$ (i.e. we decrease ϵ), starting from G_a, until G_{ϵ_1}, where $\epsilon_1 \in (0, a)$ is small enough so that the eigenvalue $\lambda_{q_r}(\epsilon)$ from the sequence $\{\lambda_{q_j}\}$ in the hypothesis satisfies

$$\frac{\|1\|_k^2}{\|f\|_k^2} \lambda_{q_r}(\epsilon_1) < \bar{S}.$$

Hence, using Proposition 3.1, we have

$$\mu_{q_r}^0(\epsilon_1) \leq \frac{\|1\|_k^2}{\|f\|_k^2} \lambda_{q_r}(\epsilon_1) < \bar{S}, \tag{8}$$

That is, G_{ϵ_1} has at least r eigenvalues $(\mu_{q_1}^0(\epsilon_1) \leq \mu_{q_2}^0(\epsilon_1) \leq \cdots \leq \mu_{q_r}^0(\epsilon_1))$ that are less than $\bar{S}$. This makes the Morse index of G_{ϵ_1}, strictly greater than that of G_a. To finish the path, we must find a final metric G_b, $b \in (0, \epsilon_1)$, such that $\bar{S} \notin Spec\{\Delta_{G_b}\}$, while keeping its Morse index greater than η_a. We achieve this by advancing our path a little more, as we did in the case of G_a.

Note that, by construction, advancing the path (i.e. decreasing ϵ) modifies (increases or decreases) all the eigenvalues of the Laplacian of h_{ϵ_1}. This in turn modifies those eigenvalues of the Laplacian of G_{ϵ_1} that were equal to $\bar{S}$, since, by Proposition 3.2, the eigenvalues μ^j_λ are strictly increasing, as functions of λ. Also, Eq. (8) is still valid for $\epsilon < \epsilon_1$, since for $\epsilon < \epsilon_1$,

$$\mu^0_{q_r}(\epsilon) \leq \frac{\|1\|^2_k}{\|f\|^2_k}\,\lambda_{q_r}(\epsilon) < \frac{\|1\|^2_k}{\|f\|^2_k}\,\lambda_{q_r}(\epsilon_1) < \bar{S}. \tag{9}$$

This means that if we advance the path, the eigenvalues $\mu^0_{q_1}(\epsilon), \mu^0_{q_2}(\epsilon), \ldots, \mu^0_{q_r}(\epsilon)$ would still be strictly less than $\bar{S}$, and hence the Morse index of G_ϵ would still be greater than r (and hence strictly greater than η_a) for $\epsilon \leq \epsilon_1$. As before, since the path is continuous and the eigenvalues are discrete, there exists a metric $G_b = g_M + f^2 h_b$, for $b \in (0, \epsilon_1)$, $\epsilon > 0$, such that $\bar{S} \notin Spec\{\Delta_{G_b}\}$ (and $\eta_b > \eta_a$). We have constructed a path of metrics, $\{G_\epsilon\}_{\epsilon\in[b,a]}$, that satisfies the conditions of Proposition 2.1, proving thus the existence of a bifurcation point somewhere in the path. Note that since ϵ_0 was arbitrary, we may repeat the process for $\epsilon \in (0, b)$ and find another bifurcation instant in an interval $(b', a') \subset (0, b)$.

In order to obtain an infinite (discrete) sequence of bifurcation instants we can repeat the argument above for a sequence of intervals $[b_l, a_l]$, $l \in \mathbb{N}$, with $a_1 = a$, $b_1 = b$ and $a_l > b_l > a_{l+1} > b_{l+1} > 0$ for each $l \in \mathbb{N}$. □

There exist paths of flat metrics h_ϵ, $\epsilon \in (0, 1)$, on the k-Torus that satisfy the hypothesis of Proposition 3.3, see for instance Example 3.5 and [18]. Given a closed flat manifold (F^k, h), it is a classical result that (F^k, h) is isometric to the orbit space $\mathbb{R}^k/\pi$ of a free action on $\mathbb{R}^k$ of the fundamental group π of F^k. Such groups are called Bieberbach groups. By studying the moduli space of flat metrics, R. Bettiol and P. Piccione constructed in [4] paths of unit volume flat metrics, on any flat manifold (F^k, h), that satisfy the hypothesis of Proposition 3.3.

Proposition 3.4 ([4, Proposition 4.3]). *Any closed flat manifold (F^k, h) of unit volume admits a real analytic family h_ϵ, $\epsilon \in (0, 1]$, $h_1 = h$, of flat metrics with unit volume, such that there is some sequence $\{\lambda_{q_j}\}$, $j \in \mathbb{N}$, of eigenvalues of the Laplacian, with $\lambda_{q_j}(\epsilon) \to 0$ as $\epsilon \to 0$. Moreover, the eigenvalues of the Laplacian of h_ϵ, are nonconstant polynomials in ϵ and $\frac{1}{\epsilon}$.*

Proof. Proposition 4.3 in [4] establishes the existence of a family h_ϵ of unit volume flat metrics, with $\epsilon \in (0, 1]$ and $h_1 = h$, such that $diam(F^k, h_\epsilon) \to \infty$, as $\epsilon \to 0$. Since $Ric_{h_\epsilon} \geq 0$ on (F^k, h_ϵ), we may follow the proof of Proposition 4.4 in [4] and use the classical eigenvalue estimate of S. Y. Cheng in terms of diameter (Corollary 2.2 in [6]):

$$\lambda_i(\epsilon) \leq 2i^2\frac{k(k+4)}{(diam(F^k, h_\epsilon))^2}. \tag{10}$$

It follows from (10) that for each $i \in \mathbb{N}$, we may always find $\epsilon > 0$ small enough so that the first i eigenvalues of the Laplacian of h_ϵ are arbitrarily small. This yields the sequence $\{\lambda_{q_j}(\epsilon)\}$, $j \in \mathbb{N}$, of eigenvalues such that $\lambda_{q_j}(\epsilon) \to 0$ as $\epsilon \to 0$. The claim that the eigenvalues of the Laplacian of h_ϵ are nonconstant polynomials in ϵ and $\frac{1}{\epsilon}$ is also explicit in Section 4.4 of [4]. □

Proof of Theorem 1.2. *To prove the theorem we use the metrics h_ϵ provided by Proposition 3.4. Note that $G_\epsilon = g_M + f^2 h_\epsilon$ verifies the hypothesis of Proposition 3.3, where f is the unique weight that makes $g_M + f^2 h$ of constant scalar curvature $\mathbf{S}$ and unit volume. f is fixed through the whole path and depends only on (M, g_M) and k. Since h_ϵ is flat and of unit volume for each $0 < \epsilon < 1$, we have that $(M \times F^k, G_\epsilon)$ is of unit volume and constant scalar curvature $\mathbf{S}$ for each $\epsilon \in (0, 1)$.* □

We now construct an example of a path of warped product metrics with infinitely many bifurcation points, which is a direct consequence of Proposition 3.4 and Theorem 1.2.

Example 3.5. Let (M, g_M) be compact with $s_{g_M} > 0$. Any k-Torus (T^k, h) endowed with a unit volume flat metric admits a family h_ϵ of flat metrics with unit volume, $\epsilon \in (0, 1)$, such that there is a sequence of eigenvalues of the Laplacian of h_ϵ, $\{\lambda_{q_i}(\epsilon)\}$, such that $\lambda_{q_i}(\epsilon) \to 0$ as $\epsilon \to 0$. As a consequence if $\frac{\mathbf{S}}{m+k-1} \notin Spec\{L_0\}$, then there is a path of warped product metrics $G_\epsilon = g_M + f^2 h_\epsilon$, $\epsilon \in (0, 1)$, of unit volume and constant scalar curvature $\mathbf{S}$, with infinitely many bifurcation points, where f is the unique weight that makes $g_M + f^2 h$ of constant scalar curvature $\mathbf{S}$ and unit volume.

We will make an explicit construction of the path of metrics G_ϵ of Example 3.5 in the last part of this section. We first recall some elementary facts about the eigenvalues of the Laplacian of flat metrics with unit volume on T^k; we refer the reader, for example, to [2]. Recall that a lattice $\Gamma \subset \mathbb{R}^k$ is a set consisting of linear combinations with integer coefficients of a basis of $\mathbb{R}^k$. We may thus associate to a lattice Γ, a matrix $B \in GL(k)$, with its columns given by a basis $(v_1, v_2, \ldots, v_k)$ of $\mathbb{R}^k$. Moreover, since we will only deal with metrics of unit volume, we will only consider basis B, such that $|det(B)| = 1$. Recall that a flat metric on T^k is given by $\mathbb{R}^k/\Gamma$, for some lattice $\Gamma \subset \mathbb{R}^k$, where Γ acts by isometries on $\mathbb{R}^k$. Recall also that (T^k, g_Γ) and $(T^k, g_{\Gamma'})$ are isometric if and only if there exists an isometry F of the Euclidean space such that $F(\Gamma) = \Gamma'$.

Given a lattice Γ with associated basis B, its dual lattice Γ^* is the lattice associated to a basis B^*, which is the dual of the basis B. In practice, this means that if B is a matrix of a basis associated to Γ, then the inverse of its transpose, $(B^T)^{-1}$, is a matrix of a basis associated to Γ^*. Now, given a lattice Γ, with dual lattice Γ^*, the eigenvalues of the Laplacian of the

Riemannian metric for $T^k = \mathbb{R}^k/\Gamma$, are given by $\lambda_i = 4\pi^2\|\beta_i\|^2$, where $\beta_i \in \Gamma^*$. In the following, we will denote by $\mathcal{M}$ the set of unit volume flat metrics on T^k. Given a metric h_ϵ in $\mathcal{M}$, we will denote by $\lambda_i(\epsilon)$ the eigenvalues of its Laplacian and by G_ϵ the warped product metric $G_\epsilon = g_M + f^2 h_\epsilon$. Recall that f is the unique weight that makes $(M \times T^k, G_\epsilon)$ of unit volume and positive constant scalar curvature $\mathbf{S}$; neither f nor $\mathbf{S}$ depend on ϵ. Given a metric $h_1 \in \mathcal{M}$ with associated matrix $B = (v_1, v_2, \ldots, v_k)$, we denote its dual matrix by $(B^T)^{-1}$, with columns $(w_1, w_2, \ldots, w_k)$, i.e. $(B^T)^{-1} = (w_1, w_2, \ldots, w_k)$. With this in mind, for $0 < \epsilon < 1$, consider the path of flat, unit volume metrics $h_\epsilon \in \mathcal{M}$, given by the family of matrices

$$B_\epsilon = (\epsilon^{k-1} v_1, \frac{1}{\epsilon} v_2, \ldots, \frac{1}{\epsilon} v_k), \tag{11}$$

(hence $(B_\epsilon^T)^{-1} = (\frac{1}{\epsilon^{k-1}} w_1, \epsilon w_2, \ldots, \epsilon w_k)$). Consider the corresponding path $\{G_\epsilon\}_{0<\epsilon<1}$, with $G_\epsilon = g_M + f^2 h_\epsilon$. Recall that we are assuming $\frac{\mathbf{S}}{m+k-1} \notin Spec\{L_0\}$.

The path of metrics G_ϵ in this example satisfies the hypothesis of Proposition 3.3. For example, the eigenvalues $\{\lambda_{q_j}\}$ associated with the last columns of $(B_\epsilon^T)^{-1}$: $\epsilon w_2, \epsilon w_3, \ldots, \epsilon w_k$, satisfy $\lambda_{q_j}(\epsilon) \to 0$, as $\epsilon \to 0$. Also, all nonzero eigenvalues are nonconstant polynomial functions of ϵ and $\frac{1}{\epsilon}$. In fact, the metrics are of unit volume. It follows from Proposition 3.3 that the path of warped product metrics G_ϵ exhibits infinitely many bifurcation points G_{ϵ^*}, $\epsilon^* \in (0, 1)$.

4. Proof of Theorem 1.1

Let (M^m, g_M) and (F^k, h), $m \geq 3$, $k > 3$, be closed connected manifolds with positive scalar curvature s_{g_M} and constant scalar curvature s_h as before.

Recall that $\mathbf{S}$ is the principal eigenvalue of the operator $c_k \Delta_{g_M} + s_{g_M}$. Let u_0 be the positive eigenfunction associated with $\mathbf{S}$ such that $\max u_0 = 1$. For warped products $(M^m \times F^k, g_M + f^2 h)$ to have constant scalar curvature $\tilde{s}$ it must be satisfied

$$c_k \Delta_{g_M} u + s_{g_M} u + s_h u^q = \tilde{s} u \tag{12}$$

with $u = f^{(k+1)/2}$, $c_k = \frac{4k}{k+1}$, $q = \frac{k-3}{k+1}$ (Theorem 2.1 in [8]).

By Theorem 3.3 in [8], if $s_h > 0$ then there is some $\epsilon_0 > 0$ such that for every $\epsilon \in (0, \epsilon_0)$, there exists a unique positive solution u_ϵ to Eq. (12) with $\tilde{s} = \mathbf{S} + \epsilon$. On the other hand, if $s_h < 0$, by Theorem 3.2 in [8], for every $\epsilon > 0$ there exists a unique positive solution u_ϵ to Eq. (12) with $\tilde{s} = \mathbf{S} - \epsilon$. If $s_h = 0$, by Theorem 3.1 in [8], a unique positive solution (up to constant factors) to Eq. (12) exists and only for $\tilde{s} = \mathbf{S}$.

Let $M_\epsilon = \max u_\epsilon$ and $\varphi_\epsilon = \frac{u_\epsilon}{M_\epsilon}$.

The idea of the proof is to show that if $s_h \neq 0$, then, as $\epsilon \to 0$, $M_\epsilon \to \infty$ and $\varphi_\epsilon \to u_0$ in $L^r(M)$, for some specific r. We will see that the first implication would let us accumulate eigenvalues of the Laplacian below $\frac{\mathbf{S}}{m+k-1}$, giving place to jumps in the Morse index. The second implication, would indicate that the eigenvalues of the Laplacian of $G_\epsilon = g_M + u_\epsilon^{\frac{4}{k+1}} h$ are getting closer to those of $g_M + u_0^{\frac{4}{k+1}} \frac{1}{\epsilon^2} h$, as $\epsilon \to 0$. Together with the hypothesis $\frac{\mathbf{S}}{m+k-1} \notin L_0$, this would allow us to start and finish paths of metrics on non-degenerated metrics, in the Morse sense, ensuring the existence of bifurcation points.

We start with the following.

Claim 1. If $s_h \neq 0$, $M_\epsilon \to \infty$, as $\epsilon \to 0$.

Proof. We divide the proof of this claim in cases.

Case 1. $s_h > 0$.

By Theorem 3.3 in [8], given $s_h > 0$, there is some $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, there is a unique positive solution u_ϵ to Eq. (12) with $\tilde{s} = \mathbf{S} + \epsilon$. In the proof of (Theorem 3.3, [8]) was shown that these solutions u_ϵ are obtained as a bifurcation from infinity and the claim that $M_\epsilon \to \infty$ as $\epsilon \to 0$ is explicit.

Case 2. $s_h < 0$

Consider the operator $(c_k \Delta_{g_M} + s_{g_M} - \mathbf{S} + \epsilon)$. Note that for any $R > 0$,

$$(c_k \Delta_{g_M} + s_{g_M} - \mathbf{S} + \epsilon) R\, u_0 \geq \epsilon\, R\, u_0.$$

It follows that $R\, u_0$ is a supersolution of Eq. (1) if

$$(c_k \Delta_{g_M} + s_{g_M} - \mathbf{S} + \epsilon) R\, u_0 \geq \epsilon\, R u_0 \geq |s_h| R^q u_0^q$$

that is, if $\epsilon \geq |s_h| R^{q-1} u_0^{q-1}$.

Then, given $\epsilon > 0$ let $R_\epsilon = \left(\frac{\epsilon}{|s_h|}\right)^{\frac{1}{q-1}}$. Since $\max u_0 = 1$, it follows from the previous arguments that $R_\epsilon u_0$ is a supersolution with $R_\epsilon \to \infty$ as $\epsilon \to 0$, since $q - 1 = \frac{-4}{k+1} < 0$. This implies that a solution u_ϵ satisfies

$$R_\epsilon u_0 \leq u_\epsilon. \tag{13}$$

We conclude that $M_\epsilon \to \infty$, as $\epsilon \to 0$. $\square$

Claim 2. $\varphi_\epsilon = \frac{u_\epsilon}{M_\epsilon} \to u_0$, as $\epsilon \to 0$, weakly in $H_1^2(M)$ (and thus strongly in $L^p(M)$ for $p \in (2, \frac{2m}{m-2})$).

Proof. We use the same notation as before. Note that

$$\Delta_{g_M}\varphi_\epsilon + s_\epsilon\varphi_\epsilon = -\frac{s_h}{c_k}\varphi_\epsilon^q M_\epsilon^{q-1} \tag{14}$$

with $s_\epsilon = \frac{s_{g_M}-(\mathbf{S}+\epsilon)}{c_k}$ or $s_\epsilon = \frac{s_{g_M}-(\mathbf{S}-\epsilon)}{c_k}$ depending on whether $s_h > 0$ or $s_h < 0$.

Multiplying (14) by φ_ϵ and integrating on M we get

$$\int_M (|\nabla\varphi_\epsilon|^2 + s_\epsilon\varphi_\epsilon^2)dV_{g_M} = -\frac{s_h}{c_k}\int_M \varphi_\epsilon^{q+1}M_\epsilon^{q-1}dV_{g_M}. \tag{15}$$

Note that the right hand side of Eq. (15) tends to zero as ϵ goes to zero, since

$$0 < \int_M \varphi_\epsilon^{q+1}M_\epsilon^{q-1}dV_{g_M} \le M_\epsilon^{-\frac{4}{k+1}}\, Vol(M, g_M)$$

and $M_\epsilon^{-\frac{4}{k+1}} \to 0$ as $\epsilon \to 0$.

Therefore, φ_ϵ is bounded in $H_1^2(M)$. Since bounded sets in a Hilbert space are weakly precompact, this implies that a subsequence converges weakly to a function φ_0. This weak convergence implies

$$\int_M |\nabla\varphi_0|^2 dV_{g_M} = \lim_{\epsilon\to 0}\int_M \langle\nabla\varphi_\epsilon, \nabla\varphi_0\rangle dV_{g_M}$$

$$\le \limsup_{\epsilon\to 0}\left(\int_M |\nabla\varphi_\epsilon|^2 dV_{g_M}\right)^{1/2}\left(\int_M |\nabla\varphi_0|^2 dV_{g_M}\right)^{1/2}$$

Therefore

$$\int_M (|\nabla\varphi_0|^2 + s_0\varphi_0^2)dV_{g_M} \le \lim_{\epsilon\to 0}\int_M (|\nabla\varphi_\epsilon|^2 + s_\epsilon\varphi_\epsilon^2)dV_{g_M} = 0, \tag{16}$$

with $s_0 = \frac{s_{g_M}-\mathbf{S}}{c_k}$. Note also that since $\mathbf{S}$ is an infimum (Eq. (2)), we have

$$\mathbf{S}\int_M \varphi_0^2\, dV_{g_M} \le \int_M (c_k|\nabla\varphi_0|^2 + s_{g_M}\varphi_0^2)dV_{g_M}.$$

It follows that

$$0 \le \int_M (|\nabla\varphi_0|^2 + \frac{s_{g_M}-\mathbf{S}}{c_k}\,\varphi_0^2)dV_{g_M},$$

that is

$$0 \le \int_M (|\nabla\varphi_0|^2 + s_0\varphi_0^2)dV_{g_M}. \tag{17}$$

Inequalities (16) and (17) then imply that φ_0 is a weak solution to

$$\Delta\varphi_0 + s_0\varphi_0 = 0. \tag{18}$$

We also know by construction that $\varphi_0 \ge 0$ and $\max\varphi_0 = 1$. By standard regularity arguments $\varphi_0 > 0$ and is $C^\infty(M)$. This implies that $\varphi_0 = u_0$, the unique positive solution of Eq. (3) with $\max u_0 = 1$.

The strong convergence $\varphi_\epsilon = \frac{u_\epsilon}{M_\epsilon} \to u_0$ in $L^p(M)$ for $p \in (2, \frac{2m}{m-2})$ as $\epsilon \to 0$, then follows from the Sobolev embedding Theorem. This finishes the proof of the claim. □

Now let $f_\epsilon = u_\epsilon^{\frac{2}{k+1}}$. For the metric $G_\epsilon = g_M + f_\epsilon^2 h$, we denote the eigenvalues of the Laplacian of G_ϵ by $\mu_i^j(\epsilon)$, and for an eigenvalue λ of the Laplacian of h the operator in Eq. (4) turns into

$$L_\lambda(\epsilon) = \Delta_{g_M} - \frac{k}{f_\epsilon}\nabla_{grad\, f_\epsilon} + \frac{\lambda}{f_\epsilon^2}. \tag{19}$$

Claim 3. For fixed λ_i, and fixed $j \in \mathbb{N}$, an eigenvalue $\mu_i^j(\epsilon) \to \mu_0^j$ as $\epsilon \to 0$. Being μ_0^j an eigenvalue of the L_0 operator of Eq. (5), with $f = f_0 = u_0^{\frac{2}{k+1}}$.

Proof. Let $Gr_j(C^\infty(M))$ be the j-dimensional Grassmannian in $C^\infty(M)$. Recall the min–max characterization of μ_i^j:

$$\mu_i^j(\epsilon) = \inf_{V\in Gr_{j+1}(C^\infty(M))}\ \sup_{v\in V\setminus\{0\}} \frac{\langle L_{\lambda_i}v, v\rangle}{\langle v, v\rangle}.$$

Using Eq. (7) we obtain from Eq. (19) that

$$\langle L_{\lambda_i}(\epsilon)v, v\rangle = \left(\int_M f_\epsilon^k |\nabla v|_g^2 dV_{g_M} + \lambda_i \int_M f_\epsilon^{k-2} v^2 dV_{g_M}\right) Vol(F^k, h) \tag{20}$$

Let $\psi_\epsilon = \varphi_\epsilon^{\frac{2}{k+1}} = M_\epsilon^{-\frac{2}{k+1}} f_\epsilon$, and $f_0 = u_0^{\frac{2}{k+1}}$. Since $f_\epsilon^k = u_\epsilon^{\frac{2k}{k+1}}$, and since by Hölder's inequality, the norms $L^{\frac{2k}{k+1}}(M)$ and $L^{\frac{2(k-2)}{k+1}}(M)$ are dominated by $L^p(M)$ for $p \in (2, \frac{2m}{m-2})$, it follows that $\psi_\epsilon \to f_0$ in $L^k(M)$ and $L^{k-2}(M)$. In turn, this implies that, as $\epsilon \to 0$,

$$\int_M \psi_\epsilon^k dV_{g_M} \to \int_M f_0^k dV_{g_M} \tag{21}$$

and

$$\int_M \psi_\epsilon^{k-2} dV_{g_M} \to \int_M f_0^{k-2} dV_{g_M}. \tag{22}$$

Recall that $\langle v, v\rangle = \left(\int_M v^2 f_\epsilon^k dV_{g_M}\right) Vol(F^k, h)$. From the above discussion, for any $v \in V \setminus \{0\}$, $V \in Gr_j(C^\infty(M))$ we have, as $\epsilon \to 0$:

$$\frac{\int_M f_\epsilon^k |\nabla v|_g^2 dV_{g_M}}{\int_M f_\epsilon^k v^2 dV_{g_M}} = \frac{\int_M \psi_\epsilon^k |\nabla v|_g^2 dV_{g_M}}{\int_M \psi_\epsilon^k v^2 dV_{g_M}} \to \frac{\int_M f_0^k |\nabla v|_g^2 dV_{g_M}}{\int_M f_0^k v^2 dV_{g_M}}, \tag{23}$$

also

$$\frac{M_\epsilon^{-\frac{2(k-2)}{k+1}} \int_M f_\epsilon^{k-2} v^2 dV_{g_M}}{M_\epsilon^{-\frac{2k}{k+1}} \int_M f_\epsilon^k v^2 dV_{g_M}} = \frac{\int_M \psi_\epsilon^{k-2} v^2 dV_{g_M}}{\int_M \psi_\epsilon^k v^2 dV_{g_M}} \to \frac{\int_M f_0^{k-2} v^2 dV_{g_M}}{\int_M f_0^k v^2 dV_{g_M}},$$

that is, for some c independent of ϵ,

$$0 \le \frac{\int_M f_\epsilon^{k-2} v^2 dV_{g_M}}{\int_M f_\epsilon^k v^2 dV_{g_M}} \le M_\epsilon^{-\frac{4}{k+1}} c.$$

Then, since λ_i is fixed, as $\epsilon \to 0$,

$$\lambda_i \frac{\int_M f_\epsilon^{k-2} v^2 dV_{g_M}}{\int_M f_\epsilon^k v^2 dV_{g_M}} \to 0. \tag{24}$$

Using Eq. (20) we have

$$\frac{\langle L_{\lambda_i}(\epsilon)\, v, v\rangle}{\langle v, v\rangle} = \frac{\int_M f_\epsilon^k |\nabla v|_g^2 dV_{g_M}}{\int_M f_\epsilon^k v^2 dV_{g_M}} + \lambda_i \frac{\int_M f_\epsilon^{k-2} v^2 dV_{g_M}}{\int_M f_\epsilon^k v^2 dV_{g_M}}.$$

It follows from (23) and (24) that, as $\epsilon \to 0$,

$$\frac{\langle L_{\lambda_i}(\epsilon)\, v, v\rangle}{\langle v, v\rangle} \to \frac{\int_M f_0^{k-2} v^2 dV_{g_M}}{\int_M f_0^k v^2 dV_{g_M}},$$

that is, as $\epsilon \to 0$,

$$\frac{\langle L_{\lambda_i}(\epsilon)\, v, v\rangle}{\langle v, v\rangle} \to \frac{\langle L_0\, v, v\rangle}{\langle v, v\rangle}.$$

Finally, since for each $j \in \mathbb{N}$,

$$\mu_i^j(\epsilon) = \inf_{V \in Gr_{j+1}(C^\infty(M))} \sup_{v \in V\setminus\{0\}} \frac{\langle L_{\lambda_i(\epsilon)} v, v\rangle}{\langle v, v\rangle},$$

we conclude $\mu_i^j(\epsilon) \to \mu_0^j$ as $\epsilon \to 0$. $\square$

For $s_h \neq 0$ let $G_\epsilon = g_M + f_\epsilon^2 h$ with $f_\epsilon = u_\epsilon^{2/(k+1)}$, for $\epsilon \in (0, \epsilon_0)$. Note that if $s_h > 0$, by u_ϵ we mean the unique solution for Eq. (12), with $\tilde{s}_\epsilon = \mathbf{S} + \epsilon$, $\epsilon \in (0, \epsilon_0)$. And if $s_h < 0$, by u_ϵ we mean the unique solution for Eq. (12), with $\tilde{s}_\epsilon = \mathbf{S} - \epsilon$, $\epsilon \in (0, \epsilon_0)$. There should be no confusion since if $s_h > 0$, Eq. (12) has no solutions with $\tilde{s}_\epsilon = \mathbf{S} - \epsilon$, $\epsilon \in (0, \epsilon_0)$. And if $s_h < 0$, Eq. (12) has no solutions with $\tilde{s}_\epsilon = \mathbf{S} + \epsilon$, $\epsilon \in (0, \epsilon_0)$. In the same way, note that G_ϵ has constant scalar curvature $\tilde{s}_\epsilon = \mathbf{S} + \epsilon$ or $\tilde{s}_\epsilon = \mathbf{S} - \epsilon$, depending on whether $s_h > 0$ or $s_h < 0$.

Claim 4. Given any $\epsilon_1 > 0$, there is some $c \in (0, \epsilon_1)$, such that $\frac{\tilde{s}_c}{m+k-1} \notin Spec\{\Delta_{G_c}\}$.

Proof. Since $Spec\{\Delta_{G_\epsilon}\}$ is discrete for each ϵ, the claim would be false only if there are some $i, j \in \mathbb{N}$ such that $\mu_i^j(\epsilon) \in Spec\{\Delta_{G_\epsilon}\}$ for some interval $(0, \epsilon_2)$, $0 < \epsilon_2 < \epsilon_1$.

But since $\frac{\tilde{s}_\epsilon}{m+k-1} \to \frac{\mathbf{S}}{m+k-1}$ and $\mu_i^j(\epsilon) \to \mu_0^j$, as $\epsilon \to 0$, if this interval $(0, \epsilon_2)$ existed, then, by continuity, there would be eigenvalues of L_0 equal to $\frac{\mathbf{S}}{m+k-1}$, which contradicts the hypothesis $\frac{\mathbf{S}}{m+k-1} \notin Spec\{L_0\}$. □

We are now ready to prove that the path G_ϵ has infinitely many bifurcation instants.

We begin with the case $s_h \neq 0$. Recall that $f_\epsilon = \psi_\epsilon M_\epsilon^{\frac{2}{k+1}}$, hence

$$\|f_\epsilon\|_k = M_\epsilon^{\frac{2}{k+1}} \|\psi_\epsilon\|_k.$$

By Claim 1 we have $M_\epsilon \to \infty$ and by (21), $\|\psi_\epsilon\|_k \to \|f_0\|_k$. It follows that

$$\|f_\epsilon\|_k \to \infty, \tag{25}$$

as $\epsilon \to 0$. By Claim 4 there exists $a \in (0, \epsilon_0)$ such that $\frac{\tilde{s}_a}{m+k-1}$ does not belong to the spectrum of Δ_{G_a}.

Let η_a be the number of eigenvalues, counted with multiplicity, that are less than $\frac{\tilde{s}_a}{m+k-1}$. Let $l \in \mathbb{N}$, such that $l > \eta_a$.

It follows from (25) that there is some $\epsilon_b \in (0, a)$ small enough so that

$$\frac{\|1\|_k}{\|f_{\epsilon_b}\|_k}\lambda_l < \frac{\tilde{s}_{\epsilon_b}}{m+k-1}$$

with λ_l the lth eigenvalue of Δ_h. Proposition 3.1 then yields

$$\mu_{\lambda_l}^0 \leq \frac{\|1\|_k}{\|f_{\epsilon_b}\|_k}\lambda_l < \frac{\tilde{s}_{\epsilon_b}}{m+k-1},$$

and by Proposition 3.2 we also have: $\mu_{\lambda_1}^0 \leq \mu_{\lambda_2}^0 \leq \cdots \leq \mu_{\lambda_l}^0 < \frac{\tilde{s}_{\epsilon_b}}{m+k-1}$.

This implies that at G_{ϵ_b} we have at least $l > \eta_a$ eigenvalues that are less than $\frac{\tilde{s}_\epsilon}{m+k-1}$. Note that taking any $\epsilon \in (0, \epsilon_b)$ still guarantees that at G_ϵ there are at least $l > \eta_a$ eigenvalues less than $\frac{\tilde{s}_\epsilon}{m+k-1}$. In order to finish the path we use Claim 4 to find some $b \in (0, \epsilon_b)$ such that the eigenvalues of the Laplacian of G_b, are different from $\frac{\tilde{s}_b}{m+k-1}$. By Proposition 2.1 we conclude that a bifurcation instant took place somewhere in $(b, a) \subset (0, \epsilon_0)$ for the path of constant scalar curvature metrics $G_\epsilon, \epsilon \in [b, a]$.

We may iterate this process to find infinitely many more bifurcation instants in $(0, b)$. This concludes the proof for the case $s_h \neq 0$.

Finally, we treat the case $s_h = 0$. In this case, Eq. (1) turns into

$$c_k \Delta_{g_M} u + s_{g_M} u = \mathbf{S} u. \tag{26}$$

Note that u_0 is a solution, and tu_0 is also a solution, for $t > 0$. For $\epsilon \in (0, 1]$, consider the metrics $G_\epsilon = g + (\frac{1}{\epsilon} f_0)^2 h = g + f_0^2 \frac{1}{\epsilon^2} h = g + f_0^2 h_\epsilon$, with $f_0 = u_0^{2/(k+1)}$, and $h_\epsilon = \frac{1}{\epsilon^2} h$. Note that this path of metrics has constant scalar curvature $\mathbf{S}$ for any $\epsilon > 0$ and each eigenvalue of the Laplacian of h_ϵ, satisfies $\lambda_i(\epsilon) \to 0$ as $\epsilon \to 0$. Also, they are nonconstant polynomial functions of ϵ and $\frac{1}{\epsilon}$. The $h_\epsilon = \frac{1}{\epsilon^2} h$ are scalar flat metrics for each ϵ and $h_1 = h$. Following the same arguments as in the proof of Proposition 3.3, we obtain that there are infinitely many bifurcation instants in $(0, 1)$ for G_ϵ.

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