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On the geometry of twisted symmetries: Gauging and coverings

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ABSTRACT

We consider the theory of *twisted symmetries* of differential equations, in particular λ and μ-symmetries, and discuss their geometrical content. We focus on their interpretation in terms of gauge transformations on the one hand, and of coverings on the other one.

1. Introduction

The Geometry of Differential Equations has been a constant topic in the research by Josiph Krasil'shich, and a great deal of this has been devoted to (the symmetry approach to) the study of *symmetries of differential equations*.

These were first considered systematically by Sophus Lie, who laid down the theory of point and contact symmetries. This theory was later on generalized in several ways by many authors (including JK). The basic idea by Lie is that once we know how the basic (independent and dependent, possibly allowing first derivatives to transform in a special way) variables transform, we also know how higher derivative transform:this corresponds to the concept of *prolongation* of a vector field, which is thus lifted from the phase manifold M to the associated *jet bundle* J^kM or $J^\infty M$, of finite or infinite order [1,7,22,39,40,43].

In most of the generalizations of Lie-point and contact symmetries, this feature is preserved: one considers more general types of vector fields in M (e.g. generalized vector fields), but the action these induce in J^kM or $J^\infty M$ is still obtained from the action in M by means of the standard *prolongation* operation — and hence the standard prolongation formula.

There is, however, a class of generalizations for which this does not hold true; these were first considered by Muriel and Romero [27,28] in the specific case of scalar ODEs,[1] and in this case one speaks of λ*-symmetries* or of C^∞-symmetries;

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[1] From the point of view of the general theory built afterwards, this is a degenerate case in many ways; which made not so immediate to understand the underlying Geometry.

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in the general case they are known as *twisted symmetries* [16,18]. For these, the very prolongation operation is modified, so that the (twisted) prolongation of a vector field in M to $J^k M$ or $J^\infty M$ does not describe its action on (standard) derivatives. This notwithstanding, twisted symmetries turn out to be "as useful as standard ones" in reducing or solving nonlinear differential equations (both ODEs and PDEs) and are thus of great interest both from the abstract and geometrical point of view and from the concrete and applicative one.

Over the years, we have (separately) worked on this topic, and shown relations of it with two subjects which are also central in the scientific interests of Josiph Krasil'shich; that is, the theory of *coverings* [22,24] on the one hand, and that of *gauge transformations* [4,13,38] one the other one (for the relations with twisted symmetries, see [2,3] and [14,15,17] respectively).

The purpose of this paper is to review, and partially reconcile, these two points of view on twisted symmetries, and their relations with relevant geometric structures.

2. Symmetries of differential equations

We assume the reader is familiar with symmetry of differential equations; the purpose of this section is thus mainly to fix notation.

We will consider differential equations[2] with independent variables x^i ($i = 1, \ldots, p$) and dependent variables u^a ($a = 1, \ldots, q$); partial derivatives will be denoted by u^a_J, where J is a multi-index $J = \{j_1, \ldots, j_p\}$ of order $|J| = j_1 + \cdots + j_p$ and

$$u^a_J \ = \ \frac{\partial^{|J|} u^a}{\partial x_1^{j_1} \ldots \partial x_p^{j_p}} \tag{1}$$

(here and somewhere in the following we moved downstairs the vector index of the x for typographical convenience). We denote by $u_{(k)}$ the set of all partial derivatives of order k, and by $u_{[n]}$ the set of all partial derivatives of order $k \le n$. We also denote by $\widetilde{J} = (J, i)$ the multi-index with entries $\widetilde{j}_k = j_k + \delta_{ik}$.

The x are local coordinates in a manifold B, while u are local coordinates in a manifold U; we consider the phase manifold $M = B \times U$, which has a natural structure of bundle (M, π, B) over B with fiber U.

We also associate to M its Jet bundles $J^n M$, which associate to any point (x, u) the set of equivalence classes of sections being mutually tangent of order n; these are described in local coordinates by $(x, u, u_{(1)}, \ldots, u_{(n)})$. Note that $J^n M$ should be thought as equipped with a *contact structure*, generated by the contact forms

$$\vartheta^a_J \ := \ \mathrm{d} u^a_J \ - \ u^a_{J,i} \, dx^i \ . \tag{2}$$

A (uni-valued) function $u = f(x)$ corresponds to a section γ_f of (M, π, B); this is just the graph of f,

$$\gamma_f \ = \ \{ (x,u) \in B \times U \ : \ u = f(x) \} \ .$$

We will denote the set of sections of M as $\Sigma(M)$, and $\gamma_f \in \Sigma(M)$.

If we assign $u = f(x)$, we are implicitly assigning also all of its derivatives; thus $\gamma_f \in \Sigma(M)$ also identifies prolongations (of any order) $\gamma_f^{(n)} \in \Sigma(J^n M)$; in multi-index notation,

$$\gamma_f^{(n)} \ = \ \{ (x, u_{[n]}) \in J^n M \ : \ u_J = (\partial_J f)(x) \ , \ |J| \le n \} \ .$$

These can be thought of as sections of $(J^n M, \pi_n, B)$.

If we consider a differential equation[3] of order n, say

$$\Delta \ := \ F^\ell (x, u, u_{(1)}, \ldots, u_{(n)}) = 0 \quad (\ell = 1, \ldots, L) \tag{3}$$

(we always assume F to be smooth in all of its arguments) this identifies a manifold in $J^n M$, called the *solution manifold* S_Δ; if Δ is non-degenerate, this is a manifold of codimension s.

A function $u = f(x)$ is a solution to Δ if and only if

$$\gamma_f^{(n)} \ \subset \ S_\Delta \ \subset \ J^n M \ .$$

This also means that vector fields Y in $J^n M$ which are both tangent to S_Δ *and* preserve the contact structure map solutions into solutions.

The condition to preserve the contact structure can be stated more precisely as follows: if Θ is the Cartan ideal generated by the ϑ^a_J, then Y preserves the contact structure if

$$\mathcal{L}_Y (\Theta) \ \subseteq \ \Theta \ ,$$

[2] For the moment, ODEs or PDEs will not make a difference, and differential equations, are always possibly vector ones, i.e. systems; similarly, functions are always possibly vector ones — albeit in some cases we will use vector indices explicitly to avoid possible confusion.

[3] Note that by this we always mean possibly a system of equations, ODEs or PDEs.

i.e. if for any $\omega \in \Theta$ we have $\mathcal{L}_Y(\omega) \in \Theta$. In view of the properties of Cartan ideals, this is the case if and only if $\mathcal{L}_Y(\vartheta^a_J) \in \Theta$, i.e. if and only if there are functions $T^{aK}_{bJ} \in \mathbf{C}^\infty(J^nM, R)$ such that

$$\mathcal{L}_Y(\vartheta^a_J) \ = \ T^{aK}_{bJ} \ \vartheta^b_K \ .$$

By a standard computation, this is the case if and only if the coefficients of the vector field

$$Y \ = \ \xi^i \, \frac{\partial}{\partial x^i} \ + \ \psi^a_J \, \frac{\partial}{\partial u^a_J}$$

satisfy the *prolongation formula*

$$\psi^a_{J,i} \ = \ D_i \psi^a_J \ - \ u^a_{J,k} \, (D_i \xi^k) \ . \tag{4}$$

Note that – setting $\psi^a_0 = \varphi^a$ – this means that Y is the prolongation of the vector field on M

$$X \ = \ \xi^i \, \partial_i \ + \ \varphi^a \, \partial_a \ ;$$

this is a well defined vector field in M provided

$$\xi^i \ = \ \xi^i(x,u) \ , \quad \varphi^a \ = \ \varphi^a(x,u) \ ;$$

we will assume this to be the case,[4] and in this case we also write

$$Y \ = \ X^{(n)}$$

to emphasize that the vector field we are considering in J^nM is the *prolongation* of the vector field X in M.

If such a vector field is tangent to S_Δ, i.e.

$$X^{(n)} \ : \ S_\Delta \to \mathrm{T}S_\Delta \ , \tag{5}$$

we say that X is a Lie-point symmetry for Δ. (More precisely, X is then the generator of a one-parameter local group of symmetries; but this slight abuse of notation is commonplace in the literature, and we will adhere to it.)

If Δ is written as in Eq. (3), then the condition that X is a Lie-point symmetry can be expressed as

$$X^{(n)} \, [F^\mu]_{F=0} \ = \ 0 \ . \tag{6}$$

Remark 1. Note that in (6) we are only requiring the invariance of the level set $\mathbf{F} = 0$, *not* of all the level sets $\mathbf{F} = \mathbf{c}$; in the latter case, we would speak of *strong symmetries*. $\odot$

3. Coverings and nonlocal symmetries

We consider the notion of (first order) *covering* of a differential equation; here we discuss it in terms of coordinates, for the sake of brevity; see [22,24] for an intrinsic discussion.

Together with independent variables $x \in B$ and dependent ones $u \in U$, with local coordinates respectively $(x^1, \ldots, x^p)$ in B and $(u^1, \ldots, u^q)$ in U, we consider auxiliary variables $w \in W$, with W a smooth manifold with local coordinates $(w^1, \ldots, w^r)$.

Then the system of m equations

$$\Delta \ := \ F^a(x, u, u_{(1)}, \ldots, u_{(n)}) \ = \ 0 \quad (a = 1, \ldots, m) \tag{7}$$

is augmented to a system $\widetilde{\Delta}$ of $m + s$ equations with a new set of $s = r \cdot p$ auxiliary first order equations

$$w^\mu_i \ = \ H^\mu_i(x, u, w) \ . \tag{8}$$

This also means that the total derivative operators, which in J^nM are

$$D_i \ := \ \frac{\partial}{\partial x^i} \ + \ u^a_i \, \frac{\partial}{\partial u^a} \ + \ u^a_{ij} \, \frac{\partial}{\partial u^a_j} \ + \cdots ,$$

are now modified into total derivative operators acting in a larger space,

$$\widetilde{D}_i \ = \ D_i \ + \ w^\mu_i \, \frac{\partial}{\partial w^\mu} \ .$$

Note that Eqs. (8) have a compatibility condition; that is, we should require

$$\widetilde{D}_i \, H^\mu_j \ = \ \widetilde{D}_j \, H^\mu_i \quad \forall \mu = 1, \ldots, r \ , \quad \forall i, j = 1, \ldots, p \ . \tag{9}$$

The relevant – interesting and applicable – case occurs when these compatibility conditions (9) just amount to the original equations (7). In this case indeed the original system Δ is properly embedded in the system $\widetilde{\Delta}$, or – seen the other way round – $\widetilde{\Delta}$ is a *covering* of the original system Δ.

[4] In other words, here we are not considering contact or generalized vector fields and symmetries.

Example 1. Consider the Gibbons–Tsarev equation [20]

$$u_{xx} \ + \ u_t \, u_{xt} \ - \ u_x \, u_{tt} \ + \ 1 \ = \ 0 \ ; \tag{10}$$

A covering for this is provided by the equations [23,41]

$$\begin{aligned} w_t \ &= \ \frac{1}{u_x \ + \ u_t \, w \ - \ w^2} \ := \ H_{(t)} \ , \\ w_x \ &= \ \frac{w \ - \ u_t}{u_x \ + \ u_t \, w \ - \ w^2} \ := \ H_{(x)} \ . \end{aligned}$$

Indeed, if we compute $D_t H_{(x)} - D_x H_{(t)}$ and substitute for w_t and w_x according to the above equations, we obtain

$$\frac{1 - u_{tt} u_x + u_t u_{xt} + u_{xx}}{[u_x + (u_t - w) w]^2} \ ,$$

and immediately recognize that this vanishes if and only if (10) holds. ⊙

Example 2. Consider the Burgers equation[5]

$$u_t \ = \ u_{xx} \ + \ u \, u_x \ . \tag{11}$$

A covering of the Burgers equation is provided by adding the auxiliary equations written in matrix form as

$$\frac{\partial W}{\partial x} \ = \ A \, W \ , \quad \frac{\partial W}{\partial t} \ = \ B \, W \ , \tag{12}$$

where we have defined the 2×2 real matrices

$$\begin{aligned} W \ &= \ \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} \ ; \\ A \ &= \ \begin{pmatrix} 4\eta & 2u + 4\eta \\ 2u - 4\eta & -4\eta \end{pmatrix} \ , \quad B \ = \ \begin{pmatrix} 2u\eta & u^2 + 2u_x + 2u\eta \\ u^2 + 2u_x - 2u\eta & -2u\eta \end{pmatrix} \ . \end{aligned}$$

In fact, computing $\chi = D_t[AW] - D_x[BW]$ and then substituting according to (12), we immediately obtain that $\chi = 0$ if and only if (11) holds. ⊙

Coming back to the general discussion, we can now look for standard symmetries of the augmented equation $\widetilde{\Delta}$. These will be vector fields to be prolonged in the standard way in the augmented space: thus if $\widetilde{X}$ is a vector field in $\widetilde{M} = M \times W = B \times U \times W$, given in local coordinates by

$$\begin{aligned} \widetilde{X} \ &= \ \xi^i(x,u,w) \, \frac{\partial}{\partial x^i} \ + \ \varphi^a(x,u,w) \, \frac{\partial}{\partial u^a} \ + \ \eta^\mu(x,u,w) \, \frac{\partial}{\partial w^\mu} \\ &\equiv \ \xi^i \, \partial_i \ + \ \varphi^a \, \partial_a \ + \ \eta^\mu \, \partial_\mu \ , \end{aligned} \tag{13}$$

its prolongation $\widetilde{Y} \ = \ \widetilde{X}^{(n)}$ will be a vector field

$$\begin{aligned} \widetilde{Y} \ &= \ \xi^i \, \frac{\partial}{\partial x^i} \ + \ \psi^a_J \, \frac{\partial}{\partial u^a_J} \ + \ \chi^\mu_J \, \frac{\partial}{\partial w^\mu_J} \\ &\equiv \ \xi^i \, \partial_i \ + \ \psi^a_J \, \partial^J_a \ + \ \chi^\mu_J \, \partial^J_\mu \ , \end{aligned} \tag{14}$$

where J are multi-indices, $\psi^a_0 := \varphi^a$, $\chi^\mu_0 := \eta^\mu$, and the coefficients follow the standard prolongation rule, i.e. (recalling the total derivative operators are now the $\widetilde{D}_i$)

$$\begin{aligned} \psi^a_{J,i} &= \widetilde{D}_i \psi^a_J \ - \ u^a_{J,k} \, \widetilde{D}_i \xi^k \ , \\ \chi^\mu_{J,i} &= \widetilde{D}_i \chi^\mu_J \ - \ w^\mu_{J,k} \, \widetilde{D}_i \xi^k \ . \end{aligned}$$

If such a vector field on $J^n \widetilde{M}$ is tangent to the solution manifold for the system $\widetilde{\Delta}$, i.e. if $\widetilde{X}$ is a symmetry for $\widetilde{\Delta}$, then the restriction of $\widetilde{X}$ to M will in general be a *nonlocal symmetry* for the equation Δ [22,24].

It should be noted that if we just look at the restriction of $\widetilde{Y}$ to $J^n M$, this is

$$Y \ = \ \xi^i \, \frac{\partial}{\partial x^i} \ + \ \psi^a_J \, \frac{\partial}{\partial u^a_J} \ \equiv \ \xi^i \, \partial_i \ + \ \psi^a_J \, \partial^J_a \ ;$$

5 As well known, this is mapped into the heat equation $v_t = v_{xx}$ by the Hopf–Cole transformation. Note also that sometimes the equation is written in a slightly different (potential) form, i.e. as $w_t = w_{xx} + (1/2) w_x^2$; taking the x derivative of this we get $w_{xt} = w_{xxx} + w_x w_{xx}$; setting now $u = w_x$ we get (11).

the coefficients ψ^a_J do now appear to follow – from the point of view of $J^n M$ – the modified prolongation rule

$$\begin{aligned}\psi^a_{J,i} &= \left(D_i \psi^a_J \ - \ u^a_{J,k} D_i \xi^k\right) \ + \ w^\mu_i \ \left(\partial_\mu \psi^a_J \ - \ u^a_{J,k} (\partial_\mu \xi^k)\right) \\ &= \left(D_i \psi^a_J \ - \ u^a_{J,k} D_i \xi^k\right) \ + \ H^\mu_i \ \left(\partial_\mu \psi^a_J \ - \ u^a_{J,k} (\partial_\mu \xi^k)\right) \ .\end{aligned} \tag{15}$$

In the second line, we have used (8).

Remark 2. Note that if the H^μ_i in (8) are such that their solutions w^μ can be expressed as a local function of the u — which in particular is the case if we allow the H^μ_i to depend also on the x-derivatives of the u, e.g. $h^\mu_i = c^\mu_a u^a_i$, or if the H^μ_i depend only on the x but not on the u — then the above formulas still yield local (albeit not following the standard prolongation formula) prolonged vector fields. ⊙

Finally, we note that one could as well consider *generalized symmetries*; that is – with the shorthand notation introduced in (13) – vector fields

$$\widehat{X} \ = \ \xi^i \, \partial_i \ + \ \varphi^a \, \partial_a \ + \ \eta^\mu \, \partial_\mu$$

where the functions ξ, φ, η depend not only on (x, u, w) but also on derivatives of u and w up to some order. If the dependence on derivatives is only in the η^μ, and this is limited to derivatives[6] of u, i.e. if we have

$$\widehat{X} \ = \ \xi^i(x,u,w) \, \frac{\partial}{\partial x^i} \ + \ \varphi^a(x,u,w) \, \frac{\partial}{\partial u^a} \ + \ \eta^\mu(x,u,w;u_x,u_{xx},\ldots) \, \frac{\partial}{\partial w^\mu} \ , \tag{16}$$

then we speak of *semi-classical symmetries*. This will play a special role in the following, see Section 7.

4. Twisted symmetries

All different symmetries, Lie-point, non-local, generalized etc., considered in the literature share the same fundamental aspect: there is an action in M, and this is lifted – i.e. prolonged – to Jet bundles $J^n M$ requiring the prolonged vector field preserves the contact structure; this requirement is embodied by the prolongation formula.

It was then rather surprising that in 2001 Muriel and Romero [27,28] proposed a different type of generalization, where the prolongation formula itself was modified. Starting with these work (see also [26,29–37]), several kinds of *twisted symmetries* have been considered in the literature [16,18].

For these, one considers a Lie-point vector field X in M, but the prolongation operation is deformed in a way which depends on some kind of auxiliary object. In different realizations this can be a scalar function (λ-symmetries [27,28]), a matrix-valued one-form satisfying the horizontal Maurer–Cartan equations — i.e. a set of matrices satisfying a compatibility condition (μ-symmetries [8]) — or also a matrix acting in an auxiliary space (σ-symmetries [9]).[7]

It should also be stressed that twisted symmetries are more easily used for *higher order* differential equations (ordinary or partial), while the case of first order equations is in some sense degenerate from this point of view, and presents several additional problems.

Here we provide a sketchy discussion of different types of twisted symmetries; the reader can consult e.g. [16,18] for further detail and a review.

4.1. λ-symmetries

The first type of twisted symmetries to be introduced was λ-symmetries (the name C^∞ symmetries also appears in the literature). These were originally introduced to deal with scalar ODEs of any order, and the name "λ-symmetries" refers to the auxiliary C^∞ function $\lambda(t, x, \dot{x})$ defining the twisted prolongation, which in this case is called λ-prolongation. In fact, this is recursively defined as

$$\begin{aligned}\psi^a_{(k+1)} &= D_x \psi^a_{(k)} \ - \ u^a_{(k+1)} \, D_x \xi \ + \ \lambda \, \left(\psi^a_{(k)} \ - \ u^a_{(k)} \, \xi\right) \\ &= (D_x \ + \ \lambda) \psi^a_{(k)} \ - \ u^a_{(k+1)} \, (D_x \ + \ \lambda) \xi \ .\end{aligned} \tag{17}$$

We will denote the λ-prolongation of order k of the vector field X in M as $X^{(k)}_\lambda$.

The vector field X in M is said to be a λ-*symmetry* of the equation Δ (of order k) if

$$X^{(k)}_\lambda \ : \ S_\Delta \ \to \mathrm{T} S_\Delta \ . \tag{18}$$

Note that in general a vector field is a λ-symmetry of a given equation *only* for a specific choice of the function λ.

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[6] Note that if the auxiliary equations are first order, this is automatically true.

[7] An actual "twisting" only occurs in the latter cases, not for λ-symmetries — where one has instead a "stretching" — but it is convenient to use this collective name in all cases where the prolongation operation is modified [16,18].

Remark 3. In general, the commutator of the λ-prolongations of two vector fields X, Y in M is *not* the λ-prolongation of their commutator, i.e. if $Z = [X, Y]$ then (in general, for $\lambda \neq 0$)

$$\left[X_\lambda^{(n)}, Y_\lambda^{(n)}\right] \neq Z_\lambda^{(n)} . \tag{19}$$

In fact, consider e.g. $X = x\partial_u$, $Y = u\partial_u$; in this case $Z = [X, Y] = X$, and $\delta := [X_\lambda^{(1)}, Y_\lambda^{(1)}] - Z_\lambda^{(1)} = x\lambda\partial_{u_x} \neq 0$. ⊙

We recall that reduction of ODEs is based on properties of differential invariants for a prolonged vector field. In particular, we know that once differential invariants of order zero and of order one – call them η and $\zeta^{(1)}$ – are known, then those of higher orders can be built by just applying total derivative operators; that is (denoting by x the independent variable)

$$\zeta^{(n+1)} := \frac{D_x \zeta^{(n)}}{D_x \eta}$$

is a differential invariant of order $(n + 1)$ if $\zeta^{(n)}$ is a DI of order n and η a DI of order zero. This property, which stems from the algebra of the prolongation operation, is also known as "invariant by differentiation property", or IBDP.

Lemma (*IBDP Lemma*)**.** *The IBDP holds for λ-prolonged vector fields.*

Proof. This follows from direct computation; see e.g. [27,28], or [16]. ⋄

Remark 4. It is the IBDP Lemma that makes λ-symmetries "as useful as standard ones", as discussed e.g. in [16,18]. ⊙

Remark 5. It was pointed out by Pucci and Saccomandi [42] that λ-prolonged vector fields can be characterized as the *only* vector fields in J^kM with the property that their integral lines are the same as the integral lines of some vector field which is the standard prolongation of some vector field in M. This remark was fully understood only some time after their paper, and was the basis for many of the following developments, discussed below. ⊙

4.2. μ-symmetries

The λ-prolongation is specifically designed to deal with ODEs (or systems thereof); a generalization of it aiming at tackling PDEs (or systems thereof) is the μ-prolongation. This can of course also be applied to ODEs and Dynamical Systems, as we will see below.

4.2.1. PDEs

Now the relevant object is not a single matrix, but an array of matrices Λ_i, one for each independent variable. These are better encoded as a ($GL(n, \mathbf{R})$-valued) *horizontal one-form*

$$\mu = \Lambda_i(x, u, u_x)\, \mathrm{d}x^i . \tag{20}$$

The matrices Λ_i should satisfy a compatibility condition, i.e.

$$D_i \Lambda_j - D_j \Lambda_i + \left[\Lambda_i, \Lambda_j\right] = 0 ; \tag{21}$$

this is immediately recognized as the *horizontal Maurer–Cartan equation*, or equivalently as a *zero-curvature condition* for the connection on TU identified by

$$\nabla_i = D_i + \Lambda_i . \tag{22}$$

If μ satisfies (21), we can define μ-prolongations in terms of a modified prolongation formula, called of course *μ-prolongation formula* (and which represents now an actual twisting of the familiar prolongation operation):

$$\begin{aligned} \psi^a_{J,i} &= D_i \psi^a_J - u^a_{J,k}\, D_i \xi^k + (\Lambda_i)^a{}_b \left(\psi^b_J - u^b_{J,k}\, \xi^k\right) \\ &= (D_i I + \Lambda_i)^a{}_b\, \psi^b_J - u^b_{J,k}\, (D_i I + \Lambda_i)^a{}_b\, \xi^k . \end{aligned} \tag{23}$$

We will denote the μ prolongation (of order k) of the vector field X in M as $X_\mu^{(k)}$. The vector field X in M is said to be a *μ-symmetry* of the equation Δ (of order k) if

$$X_\mu^{(k)} : S_\Delta \to \mathrm{T} S_\Delta . \tag{24}$$

Note that in general a vector field is a μ-symmetry of a given equation *only* for a specific choice of the one-form μ.

Remark 6. In λ-prolongations the prolongation operation is modified, but it acts separately on the different vectorial components in TU (and in TU_J). In μ-prolongations, instead, the different vector components of TU (and of TU_J) are "mixed" by the prolongation operation which thus operates a "twisting" among different components of the vector field; this is the origin of the name "twisted symmetries". Obviously, λ-symmetries are – even in the vector framework – a special case of μ-symmetries, with matrices Λ_i being multiple of the identity matrix through functions λ_i. ⊙

Remark 7. It is known that μ-symmetries (and hence λ-symmetries) are related to *nonlocal* symmetries [2,31,36]; we will discuss this relation below. $\odot$

4.2.2. ODEs

In the case of ODEs one just replaces the scalar function $\lambda : J^1M \to \mathbf{R}$ with a *matrix* function $\Lambda : J^1M \to \mathtt{Mat}(n)$ and define a "Λ-prolongation" [5,6] (which is just a special case of μ-prolongation, for $\mu = \Lambda dx$)

$$\begin{aligned} \psi^a_{(k+1)} &= D_x \psi^a_{(k)} \ - \ u^a_{(k+1)} \, D_x \xi \ + \ \Lambda^a_{\ b} \left(\psi^b_{(k)} \ - \ u^b_{(k)} \, \xi \right) \\ &= (D_x I \ + \ \Lambda)^a_{\ b} \, \psi^b_{(k)} \ - \ u^b_{(k+1)} \, (D_x I \ + \ \Lambda)^a_{\ b} \, \xi \ . \end{aligned} \tag{25}$$

In this ODE case we just have $\mu = \Lambda \, dx$ (only one component), and (21) is identically satisfied.

Remark 8. The IBDP property is in general *not* holding for μ-prolonged vector fields, not even in the ODEs framework; the exception is the case where the Λ_i are diagonal matrices. This means that in general μ-symmetries cannot be used to obtain a symmetry reduction of ODEs (see however Remark 9). $\odot$

4.2.3. Recursion formula

The μ-prolongation $X^{(k)}_\mu$, which we will now write in components as $X^{(k)}_\mu = \xi^i \partial_i + (\psi^a_J)_{(\mu)} \partial^J_a$, of a vector field X in M is defined through (23); however in some cases and applications it is relevant to characterize these in terms of the difference

$$F^a_J \ := \ \left(\psi^a_J \right)_\mu \ - \ \left(\psi^a_J \right)_0 \ . \tag{26}$$

It can be shown [8,19] that the F^a_J satisfy the recursion relation

$$F^a_{J,i} \ = \ \delta^a_b \, \left[D_i \, \left(\Gamma^J \right)^b_c \right] \, (D_i Q^c) \ + \ (\Lambda_i)^a_b \, \left[\left(\Gamma^J \right)^b_c \, (D_J Q^c) \ + \ D_J Q^b \right] \ , \tag{27}$$

where we have written

$$Q^a \ = \ \varphi^a \ - \ u^a_i \, \xi^i \ , \tag{28}$$

and the Γ^J are certain matrices whose detailed expression can be computed [8,19] but is not essential.

Remark 9. With the notation (28), the set I_X of X-invariant functions is characterized by $Q^a|_{I_X} = 0$. It follows from (27) that $X^{(k)}_\mu$ coincides with $X^{(k)}_0$ on I_X. This means that μ-symmetries are as good as standard Lie-point symmetries to obtain invariant solutions to differential equations — which is what we do when we have determined symmetries of PDEs. $\odot$

4.3. σ-symmetries

When dealing with symmetries of differential equations we often use them one at a time, in particular for ODEs — e.g. when we reduce the order of the equation. But in general we have a k-dimensional Lie algebra $\mathcal{G}$ of symmetries; the prolongation acts separately on each vector field in $\mathcal{G}$.

It turns out that a different kind of modification of the prolongation operation is possible when we consider a Lie algebra $\mathcal{G}$ of vector field, or more generally a system of vector fields which are in involution (in the sense of Frobenius); in this case the "twisting" corresponds to mixing the different vector fields in the prolongation operation. This approach has received the name of "σ-prolongation" and correspondingly one speaks of "σ-symmetries" [9–12]. This approach is specially suited to the study of dynamical systems.

We will not discuss this type of twisted symmetries here; the reader is referred to the original papers cited above or to the reviews [16,18].

5. Twisted prolongations and gauge groups

Let us consider the case where the fields $u^a = u^a(x)$, i.e. the dependent variables, take values in a vector space $U = R^q$; in this case M is a vector bundle.[8]

We can then operate an x-dependent change of frame in U; as well known, this means acting on our fields (and equations) by a *gauge transformation*.

When we deal with J^nM, there are natural coordinates u^a_J in it. Note that for a given multi-index J the variables $u_J = (u^1_J, \dots, u^q_J)$ can be seen as belonging to a vector space U_J isomorphic to U; we can then prolong the gauge transformation defined on U (more precisely, on the bundle (M, π, B)) to a gauge transformation in J^NM (more precisely, on the bundle (J^nM, π_n, B)) by acting in the same way on all the vector spaces U_J, $|J| = 0, \dots, n$.

8 The general case can be treated along the same lines; but as our considerations will be local, this would just lead to a heavier notation and discussion.

This induces an action on vector fields on M as well as on vector fields on J^nM; it is rather obvious that such an action is specially simple if we look at *vertical* vector fields, including the evolutionary representative

$$X_v \ = \ (\varphi^a \ - \ u^a_i \xi^i) \, \frac{\partial}{\partial u^a} \ := \ \phi^a(x,u,u_x) \, \frac{\partial}{\partial u^a}$$

of any Lie-point vector field

$$X \ = \ \xi^i(x,u) \, \frac{\partial}{\partial x^i} \ + \ \varphi^a(x,u) \, \frac{\partial}{\partial u^a}$$

in M.[9]

Let us thus consider vector fields X on M and their prolongations $X^{(n)}$ on J^nM, or better the evolutionary representatives X_v and their prolongations $X_v^{(n)}$; and let us consider the gauge transformed of these. Due to the *local* nature of the gauge transformation, the gauge transformed of $X_v^{(n)}$ is *not* the prolongation of the gauge transformed of X_v.

Let us denote the μ-prolongation operator defined in Section 4.2 as $\mathtt{Pr}_\mu$, with $\mathtt{Pr} = \mathtt{Pr}_0$ the standard prolongation operator, and denote by γ a given gauge transformation.

Then it turns out that the diagram (where now all vector fields are vertical, albeit this is not explicitly indicated in order to keep notation simple)

$$\begin{tikzcd} X \arrow[r, "\gamma"] \arrow[d, "\mathtt{Pr}_0"] & W \arrow[d, "\mathtt{Pr}_\mu"] \\ Y \arrow[r, "\gamma"] & Z \end{tikzcd} \tag{29}$$

is commutative, provided $\gamma = R^a_{\ b}(x,u)$ and μ are related by

$$\mu \ = \ R^a_{\ c} \left[D_i \, (R^{-1})^c_{\ b} \right] \, \mathrm{d}x^i \ := \ \Lambda_i \ \mathrm{d}x^i \ . \tag{30}$$

This is readily seen for first prolongations[10] just working in coordinates. We write

$$X \ = \ \phi^a \, \frac{\partial}{\partial u^a} \ , \quad W \ = \ (R^a_{\ b} \, \phi^b) \, \frac{\partial}{\partial u^a} \ ;$$

the (standard) first prolongations of these are respectively

$$\begin{aligned} Y \ &= \ X^{(1)} = \phi^a \, \frac{\partial}{\partial u^a} \ + \ (D_i \phi^a) \, \frac{\partial}{\partial u^a_i} \ , \\ Z \ &= \ W^{(1)} = (R^a_{\ b} \phi^b) \, \frac{\partial}{\partial u^a} \ + \ [D_i (R^a_{\ b} \phi^b)] \, \frac{\partial}{\partial u^a_i} \\ &= R^a_{\ b} \phi^b \, \frac{\partial}{\partial u^a} \ + \ R^a_{\ b} (D_i \phi^b) \, \frac{\partial}{\partial u^a_i} \ + \ (D_i R^a_{\ b}) \, \phi^b \, \frac{\partial}{\partial u^a_i} \\ &= R^a_{\ b} \left[\phi^b \, \frac{\partial}{\partial u^a} \ + \ (D_i \phi^b) \, \frac{\partial}{\partial u^a_i} \right] \ + \ \left[(D_i R^a_{\ \ell}) \, (R^{-1})^\ell_{\ m} \, R^m_{\ b} \, \phi^b \right] \, \frac{\partial}{\partial u^a_i} \ . \end{aligned}$$

On the other hand, it is immediate to see that the gauge transformed of Y is

$$\gamma(Y) \ = \ R^a_{\ b} \left[\phi^b \, \frac{\partial}{\partial u^a} \ + \ (D_i \phi^b) \, \frac{\partial}{\partial u^a_i} \right] \ ;$$

thus in order to have a commutative diagram we need to choose

$$\mu \ = \ - \, (D_i R) \, R^{-1} \, \mathrm{d}x^i \ = \ R \, D_i R^{-1} \, \mathrm{d}x^i \ .$$

In other words, the matrices Λ_i in the definition of the horizontal one-form μ must be chosen according to (30).

As mentioned above, this computation extends at once to higher order prolongations.

Remark 10. Note that the compatibility condition discussed in Section 4.2 is automatically satisfied. In fact, now

$$\begin{aligned} D_i \Lambda_j \ - \ D_j \Lambda_i \ &= D_i \, (R \, D_j R^{-1}) \ - \ D_j \, (R \, D_i R^{-1}) \\ &= (D_i R) (D_j R^{-1}) \ + \ R \, (D_i D_j R^{-1}) \ - \ (D_j R) (D_i R^{-1}) \ - \ R \, (D_j D_i R^{-1}) \\ &= (D_i R) (D_j R^{-1}) \ - \ (D_j R) (D_i R^{-1}) \ ; \\ \left[\Lambda_i , \, \Lambda_j \right] \ &= R \, (D_i R^{-1}) \cdot R \, (D_j R^{-1}) \ - \ R \, (D_j R^{-1}) \cdot R \, (D_i R^{-1}) \end{aligned}$$

---

[9] Note that, as well known, X_v is in general (that is, unless $\xi^i = 0$ for all $i = \ldots, p$) a generalized vector field, and the formalism of evolutionary representatives has full geometrical sense only when considering infinite jets $J^\infty M$ [22].

[10] And hence for higher ones as well, recalling that the $(n+1)$th prolongation is the first prolongation of the nth prolongation.

$$= -R[R^{-1}(D_iR)R^{-1}]R(D_jR^{-1}) + R[R^{-1}(D_jR)R^{-1}]R(D_iR^{-1})$$
$$= -(D_iR)(D_jR^{-1}) + (D_jR)(D_iR^{-1}) .$$

Thus the horizontal Maurer–Cartan equation (21) is satisfied. ⊙

We summarize our discussion in the form of the following statements (their proof is in fact given by the previous discussion):

Proposition 1. *Z be the μ-prolongation of the vertical vector field W, defined on (M, π, B), to J^nM. Then there are vertical vector fields X on M and Y on J^nM which are gauge-equivalent to W and Z respectively, and such that Y is the standard prolongation of X. The gauge transformation realizing this equivalence and the horizontal one-form μ in J^1M are related by* (30).

Corollary. *Let W be a μ-symmetry for a given differential equation Δ. Then there is a vector field X on M such that a gauge transform of its standard prolongation is tangent to $S_\Delta \subset J^nM$.*

6. Twisted prolongations and gauging of derivatives

A different approach, also based on gauge transformations, has been followed by Morando [25]. She noted that one can describe λ and μ symmetries in terms of gauge-deformed Lie and exterior derivatives. We will follow her work, and work directly with μ-prolongations and μ-symmetries; as already mentioned, this includes λ-prolongations and λ-symmetries as a special case.

In the case of μ-prolongations, the fundamental object is the closed differential horizontal one-form $\mu = \Lambda_i dx^i$. One can define a deformed exterior derivative d_μ acting on forms of any degree by

$$d_\mu \alpha := d\alpha + \mu \wedge \alpha . \tag{31}$$

It is immediate to check that $d_\mu^2 = 0$; thus d_μ allows to define a cohomology.

When $\mu = df$, with f a $\mathbf{C}^\infty$ function on B, we have

$$d_\mu \alpha = e^{-f} d(e^f \alpha) ;$$

in this sense the deformed exterior derivative d_μ corresponds to (a generalization of) a gauging of the standard exterior derivative d.

Similarly, one can consider a deformed Lie derivative $\mathcal{L}^\mu$. For X a vector field, the deformed Lie derivative $\mathcal{L}^\mu_X$ is defined to act on forms α and on vector fields Y by

$$\mathcal{L}^\mu_X(\alpha) = \mathcal{L}_X \alpha + \mu \wedge (X \lrcorner \alpha) ,$$

$$\mathcal{L}^\mu_X(Y) = \mathcal{L}_X(Y) - (Y \lrcorner \mu) X .$$

Again, if $\mu = df$ these read

$$\mathcal{L}^\mu_X(\alpha) = e^{-f} \mathcal{L}_{(e^f X)}(\alpha) ,$$
$$\mathcal{L}^\mu_X(Y) = e^{-f} \mathcal{L}_{(e^f X)}(Y) ,$$

so this corresponds to (a generalization of) a gauging of the standard Lie derivative $\mathcal{L}$.

Then, μ-prolonged vector fields can be characterized exactly in the same way as standardly prolonged ones, at the exception that the deformed Lie derivative plays the role of the standard one.

That is, we consider the contact forms $\vartheta^a_J = du^a_J - u^a_{J,i}dx^i$ and the Cartan ideal Θ generated by them. Then we have:

Proposition 2. *A vector field Y on J^nM is the μ-prolongation of the vector field X in M if and only if*
(a) it admits a projection on M, and this coincides with X;
(b) it satisfies

$$\mathcal{L}^\mu_Y(\Theta) \subseteq \Theta ,$$

i.e. for any a, J there are smooth functions $A^{\mu K}_{J,\beta}$ such that

$$\mathcal{L}^\mu_Y(\vartheta^a_J) = A^{\mu K}_{J\beta} \vartheta^\beta_K .$$

Proof. This is Theorem 4 in [25], and the reader is referred to there for a proof, extensions, and a discussion. ⋄

7. Twisted prolongations and coverings

The theory of coverings allows to provide a nonlocal interpretation of λ and more generally μ symmetries; that is, a (local) μ symmetry for a given equation corresponds to a standard *non-local* one for the same equation. This generalizes a property holding also for standard symmetries [22,24].

The idea is the following. If the auxiliary equations (8) are solved for w as a function of the x and u, say with

$$w^\mu \ = \ \Theta^\mu (x,u) \ , \tag{32}$$

then we can restrict the vector field $\widetilde{X}$, see (13), to the (x,u) space; this will be

$$\widetilde{X}_0 \ = \ \xi^i \, [x,u,\Theta(x,u)] \ \frac{\partial}{\partial x^i} \ + \ \varphi^a \, [x,u,\Theta(x,u)] \ \frac{\partial}{\partial u^a} \ . \tag{33}$$

But in general – albeit not always – the functions $\Theta^\mu(x,u)$ will contain *integrals* of x and u, as some trivial or less trivial example can easily show.

Example 3. Consider the equation

$$du/dx \ = \ f(x,u) \ = \ u \ ; \tag{34}$$

we add to this the equation

$$dw/dx \ = \ h(x,u,w) \ = \ u\,w \ ; \tag{35}$$

note that the latter is rewritten as $dw/w = udx$ and hence solved by

$$w(x) \ = \ \exp \left[\int u\,dx \right] \ . \tag{36}$$

Consider now Lie-point symmetries for the system (34), (35); these will be in the form (13). One of the symmetries of the system turns out to be[11]

$$\widetilde{X} \ = \ u\,w\,\partial_u \ + \ w\,\partial_w \ ;$$

by using (36), the restriction of this to the (x,u) space is

$$\widetilde{X}_0 \ = \ \left(u \, \exp \left[\int u\,dx \right] \right) \ \partial_u \ , \tag{37}$$

i.e. a non-local vector field. ⊙

Example 4 (*See [22, Section 6.1]*)**.** Let us consider again the Burgers equation

$$u_t \ = \ u_{xx} \ + \ u\,u_x \ .$$

Then we have symmetries

$$X_\alpha \ := \ (\alpha\,u \ - \ 2\,\alpha_x) \ \exp\left[-\frac{1}{2} \int u dx \right] \ \frac{\partial}{\partial u} \ ,$$

with $\alpha = \alpha(x,t)$ any solution to the heat equation $\alpha_t = \alpha_{xx}$.

If we look for solutions to the Burgers equation which are invariant under X_α, we have to solve for the system made of the Burgers equation and of the condition $X_\alpha[u] = 0$, i.e.

$$\begin{aligned} & u_t \ = \ u_{xx} \ + \ u\,u_x \\ & (\alpha\,u \ - \ 2\,\alpha_x) \ \exp\left[-\frac{1}{2} \int u dx \right] \ = \ 0 \ . \end{aligned}$$

The second equation requires $u = 2\alpha_x/\alpha$; plugging this into the first one, we obtain

$$\frac{2}{\alpha^2} \ [\alpha\,D_x\,(\alpha_t \ - \ \alpha_{xx}) \ - \ \alpha_x \ (\alpha_t \ - \ \alpha_{xx})] \ = \ 2\,D_x \left(\frac{\alpha_t \ - \ \alpha_{xx}}{\alpha} \right) \ .$$

In other words, the nonlocal symmetries X_α lead us to the Hopf–Cole transformation. ⊙

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[11] The action of this vector field is readily integrated to give $w(s) = k_1 e^s$, $u(s) = k_2 \exp[w[s]]$; the quantity ue^{-w} is thus invariant under $\widetilde{X}$.

7.1. λ-symmetries

Pretty much the same mechanism is at work also when one considers twisted rather than standard symmetries. In particular the situation is fully understood in the case of λ-symmetries (while no much work in the context of μ and σ-symmetries appears in the literature, see however the next subsection); in this context we have the following general result, which is Proposition 1 in [2].

Proposition 3. *Consider a given smooth function $\lambda = \lambda(x, u, u_x)$; consider moreover the ODE*

$$\Delta_0 \ := \ \frac{d^k u}{dx^k} \ = \ f(x, u, \ldots, u^{(k-1)})$$

and its covering $\widetilde{\Delta}$ consisting of the system

$$\begin{aligned} \frac{d^k u}{dx^k} &= f(x, u, \ldots, u^{(k-1)}) \\ \frac{dw}{dx} &= \lambda(x, u, u_x) \ . \end{aligned}$$

Then Δ admits a λ-symmetry X if and only if $\widetilde{\Delta}$ admits a semi-classical *symmetry $Y = \xi\partial_x + \varphi\partial_u + \eta\partial_w$ such that $[\partial_w, Y] = Y$. Moreover, X is the projection to the (x, u) space of the restriction of Y to the solution manifold for the auxiliary equation $dw/dx = \lambda(x, u, u_x)$, i.e. to*

$$w(x) \ = \ \int \lambda(x, u, u_x)\, dx \ .$$

Proof. For a detailed proof, the reader is referred to [2]. Here we give a sketch of it. For a given equation Δ_0, we consider the system $\widetilde{\Delta}$ consisting of it and of Δ_1 given by $w_x = \lambda(x, u, u_x)$. Suppose then that some Lie-point symmetry $X = \xi(x, u, w)\partial_x + \varphi(x, u, w)\partial_u + \eta(x, u, w)\partial_w$ for $\widetilde{\Delta}$ has been determined, and denote by Y the prolongation (of suitable order) of X. This means that

$$[Y(\Delta_0)]_{\{\Delta_0=0,\Delta_1=0\}} \ = \ 0 \ , \quad [Y(\Delta_1)]_{\{\Delta_0=0,\Delta_1=0\}} \ = \ 0 \ .$$

On the other hand, it is clear that $Y(\Delta_0)$ only involves the prolongation of $X_0 = \xi(x, u, w)\partial_x + \varphi(x, u, w)\partial_u$, call it $Y^{(0)}$. This is of the form

$$Y^{(0)} \ = \ \xi\, \partial_x \ + \ \sum_k \psi^{(k)} \, \frac{\partial}{\partial u^{(k)}} \ ,$$

where $\psi^{(0)} = \varphi$ and the $\psi^{(k)}$ obey the prolongation formula

$$\psi^{(k+1)} \ = \ D_x \psi^{(k)} \ - \ u^{(k+1)}\, D_x \xi \ . \tag{38}$$

It is convenient to rewrite the total derivative operator

$$D_x \ = \ \partial_x \ + \ \sum_k u^{(k+1)} \, \frac{\partial}{\partial u^{(k)}} \ + \ \sum_k w^{(k+1)} \, \frac{\partial}{\partial w^{(k)}}$$

in the form

$$D_x \ = \ D_x^{(0)} \ + \ D_x^{(1)} \ , \tag{39}$$

having defined

$$D_x^{(0)} \ = \ \partial_x \ + \ \sum_k u^{(k+1)} \, \frac{\partial}{\partial u^{(k)}} \ ; \quad D_x^{(1)} \ = \ \sum_k w^{(k+1)} \, \frac{\partial}{\partial w^{(k)}} \ . \tag{40}$$

With this notation, we rewrite Eq. (38) as

$$\psi^{(k+1)} \ = \ D_x^{(0)} \psi^{(k)} \ - \ u^{(k+1)}\, D_x^{(0)} \xi \ + \ D_x^{(1)} \psi^{(k)} \ - \ u^{(k+1)}\, D_x^{(1)} \xi \ . \tag{41}$$

If we assume that the condition $[Y(\Delta_1)]_{\{\Delta_0=0,\Delta_1=0\}} = 0$ is satisfied, the other condition $[Y(\Delta_0)]_{\{\Delta_0=0,\Delta_1=0\}} = 0$ can be rewritten solving explicitly Δ_1 as

$$[Y(\Delta_0)]_{\{\Delta_0=0, w=\int \lambda dx\}} \ = \ 0 \ .$$

This in turn can be written as

$$\left[\widehat{Y}(\Delta_0)\right]_{\{\Delta_0=0\}} \ = \ 0 \ ,$$

where the vector field $\widehat{Y}$ is defined by restricting the vector field Y to

$$w = \int \lambda(x,u,u_x)\,dx \tag{42}$$

and its differential consequences; note that under this restriction we get

$$D_x^{(1)} = \lambda\,\partial_w + (D_x\lambda)\,\partial_{w_x} + \cdots = \sum_\ell (D_x^\ell \lambda)\,\frac{\partial}{\partial w^{(\ell)}} . \tag{43}$$

Thus *if* $[\partial_w, Y] = Y$, it follows that φ and ξ are of the form

$$\varphi(x,u,w) = e^w\,\varphi_0(x,u)\,, \quad \xi(x,u,w) = e^w\,\xi_0(x,u)\,, \tag{44}$$

and then (41) reads just as the λ-prolongation formula.[12] $\diamond$

The situation can be summarized in a diagram:

$$\begin{array}{ccccc} \widetilde{\Delta} & \xrightarrow{\texttt{sym}} & \widetilde{X} & \xrightarrow{\texttt{Pr}_0} & \widetilde{Y} \\ \Big\downarrow{\scriptstyle \texttt{cov}} & & \Big\downarrow{\scriptstyle \Delta_1=0} & & \Big\downarrow{\scriptstyle \Delta_1=0} \\ \Delta_0 & \xrightarrow{\lambda-\texttt{sym}} & X^{(0)} & \xrightarrow{\texttt{Pr}_\lambda} & Y^{(0)} \end{array}$$

Here $\texttt{sym}$ (respectively, $\lambda - \texttt{sym}$) refers to the fact we determine a symmetry (a λ-symmetry) of the equation, $\texttt{cov}$ refers to the fact $\widetilde{\Delta}$ is a covering of Δ_0, and $\Delta_1 = 0$ refers to the restriction to the solution manifold for Δ_1 (and its differential consequences). Note here $\widetilde{X}$ must be of the form (44).

We will illustrate this result by an example, also taken from [2], which we consider in some detail.

Example 5. Consider the equation, or actually the class of equations,

$$\Delta := \quad u_{xx} = \frac{u_x^2}{u} + \left[m\,g(x)\,u_x + g'(x)\,u\right] u^m\,, \tag{45}$$

where $g(x)$ is a smooth function and $m \neq 0$ a real constant. This class of equations was studied by Gonzalez-Lopez [21], and for general $g(x)$ it has no Lie-point symmetries. On the other hand, it was shown by Muriel and Romero [28], and it is easily checked, that it always admits as λ-symmetry the vector field

$$X = \partial_u$$

provided one chooses

$$\lambda(x,u,u_x) = \frac{u_x}{u} + m\,g(x)\,u^m\,.$$

In fact, the second λ-prolongation of X will be

$$Y = \partial_u + \widehat{\psi}^{(1)}\,\partial_{u_x} + \widehat{\psi}^{(2)}\,\partial_{u_{xx}}\,,$$

with the coefficients $\psi^{(k)}$ satisfying the λ-prolongation formula, which in this case ($\xi = 0$) reads simply

$$\psi^{(k+1)} = D_x\psi^{(k)} + \lambda\,\psi^{(k)}\,,$$

and of course with $\psi^{(0)} = 1$. Thus we get

$$\psi^{(1)} = \lambda\,, \quad \psi^{(2)} = D_x\lambda + \lambda^2\,.$$

Thus, by explicit computation,

$$Y[\Delta] = \frac{u\,u_{xx} - u_x^2 - u^{m+1}\left[m\,g(x)\,u_x + u\,g'(x)\right]}{u^2}\,;$$

substituting for u_{xx} according to Δ — i.e. according to Eq. (45) — we get indeed

$$[Y[\Delta]]_{\Delta=0} = 0\,.$$

When we consider the system $\widetilde{\Delta}$ made by (45) and by the auxiliary equation

$$w_x = \lambda(x,u,u_x) \tag{46}$$

and look for standard Lie-point symmetries, say of the simplified form

$$\widetilde{X} = \varphi(x,u,w)\,\partial_u + \eta(x,u,w)\,\partial_w$$

12 Note that the same condition $[\partial_w, Y] = Y$ also implies $\eta(x,u,w) = e^w\eta_0(x,u)$.

it turns out that choosing

$$\varphi \ = \ e^w \ , \quad \eta \ = \ (m+1)\,\frac{e^w}{u} \ ,$$

or in other words

$$\widetilde{X} \ = \ e^w \ \left[\partial_u \ + \ \frac{m+1}{u}\,\partial_w \right] \ ,$$

we have a symmetry. This can be checked by standard computations.

On the other hand, (46) is solved by

$$w \ = \ \int \lambda(x,u,u_x)\,dx \ = \ \log(u) \ + \ m \int u(x)\,g(x)\,dx \ ; \tag{47}$$

thus the vector field $\widetilde{X}$ restricted to the solution to (46) and projected to the (x,u) space reads

$$\widehat{X} \ = \ \exp\left[\lambda\,dx\right]\ \partial_u \ ,$$

i.e. we have a non-local vector field.

Now if we look at the second prolongation of $\widetilde{X}$, we have

$$\begin{aligned}\widetilde{Y} = e^w & \left[\frac{\partial}{\partial u} \ + \ w_x\,\frac{\partial}{\partial u_x} \ + \ (w_x^2 + w_{xx})\,\frac{\partial}{\partial u_{xx}} \right] \\ & + \ e^w\,\frac{(m+1)}{u}\,\left[\frac{\partial}{\partial w} \ + \ \frac{u w_x - u_x}{u}\ \frac{\partial}{\partial w_x} \right. \\ & \left. + \ \frac{2u_x^2 - 2u u_x w_x - u u_{xx} + u^2(w_x^2 + w_{xx})}{u^2}\ \frac{\partial}{\partial w_{xx}} \right] \ .\end{aligned}$$

When we restrict to solutions to (46), i.e. substitute for w and its derivatives according to (47), and project to the (x,u,u_x,u_{xx}) space — i.e. to J^2M — we get

$$\widetilde{Y} \ = \ \left(\exp\left[\int \lambda dx \right] \right)\ \left[\frac{\partial}{\partial u} \ + \ \lambda\,\frac{\partial}{\partial u_x} \ + \ (\lambda^2 + D_x\lambda)\,\frac{\partial}{\partial u_{xx}} \right] \ . \tag{48}$$

By construction, this is tangent to the solution manifold for Δ, $\widetilde{Y} : S_\Delta \to \mathrm{T}S_\Delta$. But if this is the case, the same also applies to any vector field which is collinear to $\widetilde{Y}$, in particular to

$$\begin{aligned}\widehat{Y} &= \exp\left[- \int \lambda\,dx \right]\ \widehat{Y} \\ &= \frac{\partial}{\partial u} \ + \ \lambda\,\frac{\partial}{\partial u_x} \ + \ (\lambda^2 + D_x\lambda)\,\frac{\partial}{\partial u_{xx}} \ .\end{aligned} \tag{49}$$

This is the λ-prolongation of the vector field $\widehat{X} = \partial_u$. $\odot$

7.2. μ-symmetries

The discussion given above for λ-symmetries can be extended to μ-symmetries, provided we only consider *vertical* vector fields, both in the (x,u) space and in the augmented (x,u,w) one.

Thus to a PDE or system of PDEs Δ_0 of order n

$$\Delta_0 \ : \quad F^\ell(x,u,\ldots,u^{(n)}) \ = \ 0 \ , \qquad \ell = 1,\ldots,L \tag{50}$$

for $u = (u^1,\ldots,u^p)$ depending on the independent variables $x = (x^1,\ldots,x^q)$ we associate the auxiliary equations for $w = (w^1,\ldots,w^m)$ given by

$$\Delta_i^\beta \ : \quad w_i^\beta \ = \ h_i^\beta(x,u,w,u_x) \ , \tag{51}$$

where the functions h_i^β satisfy the compatibility condition

$$D_i h_j^\beta \ = \ D_j h_i^\beta \tag{52}$$

for all pairs $i,j = 1,\ldots,q$ and for all $\mu = 1,\ldots,m$. Note that now and in the following D_i denotes the total derivative w.r.t. x^i in the augmented space, i.e. taking care of both the u and the w variables.

We will then consider the system $\widetilde{\Delta}$ made of the original equation Δ_0 and of the auxiliary equations Δ_i^β. When looking for Lie-point symmetries of $\widetilde{\Delta}$, we will *only* be considering vertical vector fields, i.e. vector fields of the form

$$X \ = \ \phi^a(x,u,w)\,\frac{\partial}{\partial u^a} \ + \ \eta^\beta(x,u,w)\,\frac{\partial}{\partial w^\beta} \ . \tag{53}$$

In order to apply this to $\widetilde{\Delta}$, it suffice to consider prolongation to order n in the u derivatives but only to order one in the w derivatives; we will write this as

$$Y \ = \ \Psi^a_J \, \frac{\partial}{\partial u^a_J} \ + \ \chi^\beta_i \, \frac{\partial}{\partial w^\beta_i} \ , \tag{54}$$

where J is a multi-index of order $|J| \le n$, the index i runs on $1, \dots, q$, and sum over repeated indices and multi-indices is understood. Moreover we set $\Psi^a_0 = \Phi^a$, $\chi^\beta_0 = \eta^\beta$. We will also write, for later reference, the restriction of Y to the $J^n M$ bundle (with $M = B \times U$, and $x \in B$, $u \in U$ the manifolds in which x and u take values) as

$$Y_0 \ = \ \Psi^a_J \, \frac{\partial}{\partial u^a_J} \ .$$

Suppose that we are able to determine such a vector field which is a symmetry of $\widetilde{\Delta}$ and *moreover* such that

$$\phi^a(x,u,w) \ = \ G^a_{\ b}(w) \, \varphi^b(x,u) \ . \tag{55}$$

Then the coefficients in the first prolongation read

$$\Psi^a_i \ = \ D_i \phi^a \ = \ (D_i G^a_b) \, \varphi^b \ + \ G^a_b \, (D_i \varphi^b) \ .$$

As the matrix G only depends on w, while the vector φ only depends on (x,u) we can use the decomposition (39), (40), and rewrite this – in vector notation for ease of writing – as

$$\Psi_i \ = \ G \, (D^{(0)}_i \varphi) \ + \ G[G^{-1} \, (D^{(1)}_i G)] \, \varphi \ = \ G \, \left[(D^{(0)}_i \varphi) + (G^{-1} \, D_i G) \, \varphi \right] \ . \tag{56}$$

Defining the matrices M_i as $M_i := G^{-1} \left(D^{(1)}_i G \right)$, i.e. as

$$(M_i)^a_{\ b} \ = \ [G^{-1}(w)]^a_{\ c} \, \left[w^\beta_i \, \frac{\partial G^a_{\ b}(w)}{\partial w^\beta} \right] \ ,$$

this is also rewritten as

$$\Psi_i \ = \ G \, \left[(D^{(0)}_i \varphi) + M_i \, \varphi \right] \ . \tag{57}$$

Let us now take the restriction of this to the set of solutions to the auxiliary equations Δ^β_i. Here $w^\beta_i = h^\beta_i(x,u,w)$, and the w^β themselves are written in terms of the (x,u) variables — in general through expressions containing integrals of the u^a. We will also denote the restrictions of G and M to $\Delta^\beta_i = 0$ as

$$\widehat{G} \ := \ [G]_{\Delta^\beta_i = 0} \ , \quad \Lambda_i \ := \ [M_i]_{\Delta^\beta_i = 0} \ . \tag{58}$$

Note that the Λ_i satisfy (21) by construction.

With this notation, let us consider the restriction of Y to the solutions of Δ^β_i and let us project it on the $J^n M$ bundle; call the resulting vector field $\widehat{Y}$. We then have

$$\widehat{Y} \ = \ \widehat{\psi}^a_J \, \frac{\partial}{\partial u^a_J} \ ,$$

where the coefficients $\widehat{\psi}^a_J$ satisfy $\widehat{\psi}^a_0 = \widehat{\phi}^a = \widehat{G}^a_b \varphi^b$ and obey the prolongation formula

$$\widehat{\psi}^a_{J,i} \ = \ \widehat{G}^a_b \, \left[D^{(0)}_i \, \widehat{\psi}^b_J \ + \ (\Lambda_i)^a_b \, \widehat{\psi}^b_J \right] \ . \tag{59}$$

Thus, if we consider the vector field

$$\widehat{Z} \ = \ \widehat{G}^{-1} \, \widehat{Y} \ = \ (\widehat{G}^{-1})^a_b \, \widehat{\psi}^b_J \, \frac{\partial}{\partial u^a_J} \ ,$$

then this is the μ-prolongation of

$$X_0 \ = \ \varphi^a(x,u) \, (\partial / \partial u^a) \tag{60}$$

for the horizontal one-form

$$\mu \ = \ \Lambda_i(x,u,u_x) \, \mathrm{d}x^i \ . \tag{61}$$

In this case we could summarize our discussion in the form of a diagram similar to the one given above for λ-symmetries, i.e.

$$\begin{array}{ccccc} \widetilde{\Delta} & \xrightarrow{\mathtt{sym}} & \widetilde{X} & \xrightarrow{\mathtt{Pr}_0} & \widetilde{Y} \\ \Big\downarrow \mathtt{cov} & & \Big\downarrow \Delta^\beta_i = 0 & & \Big\downarrow \Delta^\beta_i = 0 \\ \Delta_0 & \xrightarrow{\mu-\mathtt{sym}} & X^{(0)} & \xrightarrow{\mathtt{Pr}_\mu} & Y^{(0)} \end{array}$$

where $\Delta_i^\beta = 0$ refers to the restriction to the solution manifold for the whole set of auxiliary equations Δ_i^β, and we have to require that the coefficient of the (x, u) variables in the vector field X are as above; note that we have not discussed the functional form of the η^β coefficients.[13]

It is maybe convenient to summarize our discussion as a formal statement (the previous discussion gives a proof of it).

Proposition 4. *Let the system made of Eqs. (50) and (51) — with functions h_i^β satisfying Eq. (52) — admit a Lie-point symmetry of the form (53), (55). Then Eq. (50) admits the μ-symmetry X_0 Eq. (60) with μ provided by Eq. (61).*

8. Conclusions

We have discussed *twisted symmetries*; these were introduced as a practical tool to obtain (generalized) symmetry-reduction and symmetry-invariant solutions for differential equations, but here we focused on their geometrical interpretation and meaning.

In particular we considered three different approaches to them, looking at them in different ways:

(a) consider these as standard prolongation under a local gauge transformation, which yields the deformed prolongation operator;
(b) consider these as prolongations obtained applying the standard prolongation operator but with gauge-deformed (exterior and Lie) derivatives;
(c) consider these as the image of standard prolongations in a covering space when projected to the original one.

It is quite clear that these different approaches are related to each other, and we will now sketchily discuss such relations; we hope to provide a more detailed discussion in a forthcoming work.

The approaches (a) and (b) are clearly and directly related, and are both based on considering gauge transformations. In the first case this is acting on vector fields which are prolonged in a standard way, i.e. on prolongation operation based on the requirement the Lie derivative of prolonged vector fields preserves the (Cartan) contact structure in $J^n M$, while in the second case the gauging is applied to the Lie derivative – and to the exterior derivatives appearing in the contact forms – themselves. Thus we are in a way considering "active" and "passive" gauging.

The relation with the approach by covering is less immediate. As we have seen, covering is based on considering new degrees of freedom (and corresponding auxiliary variables w^β), and new equations for this; the vector fields are prolonged in the standard way in the augmented phase space, but projecting this prolongation, or actually its restriction to the solutions of the auxiliary equations — to the original space and its prolongations results in a vector field which is equivalent to a vector field prolonged by the λ- or μ-prolongation formula.

Note that behind all of these approaches lies the basic remark – due originally to Pucci and Saccomandi [42] – that twisted prolongations are vector field collinear to standard prolongations (of different vector fields), which allows them to preserve the contact structure. This is essentially due to the very basic fact that in this only the *integral curves* of vector fields are relevant, and not the way the flow generated by the vector field itself runs along them.

In concrete application, one or the other of the different approaches reviewed here can be more convenient: in several cases, in particular if analyzing equations stemming from Physics, the gauge approach can yield more transparent results; on the other hand, the approach through the theory of covering makes a direct connection with *non-local symmetries*, which would be quite artificial in the gauge formalism.

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