



On nonlinear gauge theories

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ABSTRACT

In this note, we study non-linear gauge theories for principal bundles, where the structure group is replaced by a Lie groupoid. We follow the approach of Moerdijk–Mrčun and establish its relation with the existing physics literature. In particular, we derive a new formula for the gauge transformation which closely resembles and generalizes the classical formulas found in Yang Mills gauge theories.

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1. Introduction

Differential geometry of principal bundles is the natural language of gauge theories of Yang–Mills type. Since the late 50's this recognition spawned many fruitful ideas in both mathematical and physical sciences, ranging from the discovery of new topological invariants to the formulation of the Standard Model. The objects of the present study are principal groupoid bundles, where the structure Lie group is replaced in a suitable way by a Lie groupoid.

A nonlinear gauge theory is a model where the symmetry content is described by field dependent quantities which replace the Lie algebra structure constants. The language of Lie algebroids and groupoids provides a natural setting for this generalization [1]. An example is the Poisson Sigma Model ($P\sigma M$), a two dimensional topological field theory whose path integral quantization [2] yields Kontsevich's formula for deformation quantization [3]. Let Σ be a two dimensional surface (bounded, oriented), (M, π) a Poisson manifold: the $P\sigma M$ action is a functional on the space of vector bundle morphisms $T\Sigma \rightarrow T^*M$. Such maps decompose into pairs (X, η) , where X is a map between the basis and $\eta \in \Omega^1(\Sigma, X^*T^*M)$. The equations of motion for this classical theory are the conditions defining Lie algebroid morphisms from the standard Lie algebroid $T\Sigma$ to the cotangent Lie algebroid T^*M , whose Lie algebroid structure is defined by π .

Such a Poisson σ -model leads naturally to a construction of symplectic groupoid of M as shown by Cattaneo–Felder [4]. For an arbitrary Lie algebroid A , a similar construction leads to the Lie groupoid integrating A , which was extensively studied by Crainic–Fernandes leading to the solution of the problem of Lie's third theorem for Lie algebroids [5].

On the other hand, for a principal groupoid bundle, there is a natural notion of connections, namely horizontal distributions. Indeed, Moerdijk–Mrčun [6] introduced a definition of connections which reduces to the standard connections for usual principal bundles (with structure group being a Lie group).

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The purpose of this paper is to establish relations between the approach of Moerdijk–Mrčun and the one commonly adopted in non-linear gauge theory. In particular, we give an interpretation of gauge transformations in non-linear gauge theory in terms of principal groupoid bundles, and derive a new formula resembling the classical formula for the effect of gauge transformations on gauge fields [7–9,4]. For a different approach, see [10].

The paper is organized as follows. In Section 2, we recall the notion of principal bundle $P \rightarrow M$ with structure groupoid $G \rightrightarrows G_0$. Principal connections on a PGB are introduced in Section 3, as horizontal distributions H on the fiber bundle P satisfying suitable equivariance properties. Connections are then characterized as equivariant bundle maps $TP \rightarrow A$ satisfying the Moerdijk–Mrčun condition. Their curvature measures how far the corresponding bundle maps are from being Lie algebroid morphisms. Next, suitable gauge transformations are introduced. Finally, we discuss the case of trivial principal bundles.

2. Principal groupoid bundles

In this section we introduce principal groupoid bundles and fix the notation, we mainly follow [6,11]. For basic material on Lie groupoids and algebroids we refer to [12,13]. A very detailed account on principal groupoid bundles (PGB) together with a comprehensive bibliography can be found in [11].

2.1. Principal groupoid bundles

For a groupoid G , G_0 denotes its set of objects, $s, t : G \rightarrow G_0$ are the *source* and *target* maps, $1 : G_0 \rightarrow G$ is the unit map, the composition of arrows is defined for *composable pairs*, i.e. $(g, h) \in G \times G$ such that $t(g) = s(h)$. The group of local bisections is denoted by $\mathcal{B}(G)$. Given $\gamma \in \mathcal{B}(G)$ a *left (right) translation* is indicated with L_γ (R_γ). For source (s) and target (t) fibers we use the notations $G_x := s^{-1}(x)$ and $G^y := t^{-1}(y)$.

A right action of a Lie groupoid G on a manifold E along the moment map $\epsilon : E \rightarrow G_0$ is a smooth map:

$$\mu : E_\epsilon \times_s G \rightarrow E : (p, g) \mapsto pg,$$

such that, whenever defined, the following identities hold: $\epsilon(pg) = t(g)$, $p \cdot 1_{\epsilon(p)} = p$ and $p(gh) = (pg)h$. Here, $E_\epsilon \times_s G$ denotes the fibered product $\{(p, g) \in E \times G | \epsilon(p) = s(g)\}$.

Denote with $(A, [\ , \], \rho)$ the Lie algebroid of G , where $\rho = t_*$ is the anchor map.

Definition 2.1. A *principal groupoid bundle* P over a manifold M is a surjective submersion $\pi : P \rightarrow M$ equipped with a smooth right action μ along $\epsilon : P \rightarrow G_0$ which is fiberwise, such that the map:

$$(pr_1, \mu) : P_\epsilon \times_s G \rightarrow P_\pi \times_\pi P : (p, g) \mapsto (p, pg),$$

is a diffeomorphism.

A principal groupoid bundle

$$\begin{array}{ccc} P & & G \\ \pi \downarrow & \searrow \epsilon & \downarrow \\ M & & G_0 \end{array}$$

is sometimes referred to as $(M \xleftarrow{\pi} P \circlearrowright_\epsilon G)$.

2.2. Division map

The projection onto G of the inverse map of (pr_1, μ) is called *division map*, δ_P .

$$\delta_P : P_\pi \times_\pi P \rightarrow G.$$

In words, the right multiplication by the arrow $\delta_P(p, q)$ sends p to q :

$$p \cdot \delta_P(p, q) = q.$$

Some properties:

Proposition 2.2. For all $(p, q) \in P_\pi \times_\pi P$, the division map δ_P satisfies the following properties:

- (a) $\delta_P(p, p) = 1_{\epsilon(p)}$;
- (b) $(s \circ \delta_P)(p, q) = \epsilon(p)$, $(t \circ \delta_P)(p, q) = \epsilon(q)$;
- (c) $\delta_P(p, q)^{-1} = \delta_P(q, p)$;
- (d) $\delta_P(p, qg) = \delta_P(p, q)g$.

When there is no risk of confusion we write δ for δ_P .

2.3. Trivializations

A Lie groupoid G is itself a principal groupoid bundle $(G_0 \xleftarrow{s} G \circlearrowright_t G)$, the right action is given by the groupoid multiplication. It is usually denoted with \mathcal{U}_G and called *unit bundle of the Lie groupoid* G . The division map $\delta_{\mathcal{U}_G}$ is given by:

$$G_s \times_s G \ni (g, h) \mapsto g^{-1}h \in G,$$

hence the terminology.

We define the *trivial G -bundle over M w.r.t. a given map $M \xrightarrow{f} G_0$* as the pullback bundle

$$f^*\mathcal{U}_G = \{(m, g) \in M \times G \mid f(m) = s(g)\}$$

endowed with the momentum map

$$\epsilon : f^*\mathcal{U}_G \rightarrow G : (m, g) \mapsto t(g)$$

and the action

$$\mu : f^*\mathcal{U}_G \times_s G : ((m, g_1), g_2) \mapsto (m, g_1 g_2).$$

Proposition 2.3. Every principal G -bundle $(M \xleftarrow{\pi} P \circlearrowright_\epsilon G)$ is locally diffeomorphic to a trivial bundle.

Remark 2.4. Contrary to the classical case, for a principal groupoid bundle a trivialization includes as a priori data a (local) section σ_i , which is used to specify the map ϵ_i along which the unit bundle \mathcal{U}_G is pulled back. Indeed, a section completely defines a trivialization.

2.4. Transition maps

Let $U_i, U_j \subset M$ be open subsets with nonempty intersection $U_{ij} = U_i \cap U_j$. Let Φ_i and Φ_j be the local trivializations corresponding to the sections $U_i \xrightarrow{\sigma_i} P$ and $U_j \xrightarrow{\sigma_j} P$. Then $\Phi_{ji} := \Phi_j^{-1} \circ \Phi_i$:

$$\Phi_{ji} : (\epsilon_i^*\mathcal{U}_G)|_{U_{ij}} \rightarrow (\epsilon_j^*\mathcal{U}_G)|_{U_{ij}} : (m, g) \mapsto (m, \delta_P(\sigma_j(m), \sigma_i(m))g),$$

is a diffeomorphism.

The *transition map* from U_i to U_j is the map:

$$\varphi_{ji} : U_{ij} \rightarrow G : m \mapsto \delta_P(\sigma_j(m), \sigma_i(m)).$$

For an open cover $\{U_i\}$ of M , $\{\varphi_i\}$ is a non abelian 1-cocycle, i.e. it satisfies the identity $\varphi_{ij}\varphi_{jk} = \varphi_{ik}$, on the triple intersection U_{ijk} . It is a standard result that PGs over M are classified up to isomorphism by the non abelian cohomology group $H^1(M, G)$.

Given a section $U_j \xrightarrow{\sigma_j} P$ and the transition map φ_{ji} , the section $U_{ij} \xrightarrow{\sigma_i} P$ is given for all m in U_{ij} by:

$$\sigma_i(m) = \sigma_j(m)\varphi_{ji}(m).$$

3. Horizontal distribution on principal groupoid bundles

3.1. Connections

Recall that a *horizontal distribution* on a fiber bundle $P \xrightarrow{\pi} M$ is a subbundle $H \hookrightarrow TP$ such that $TP = \ker \pi_* \oplus H$.

Let $(M \xleftarrow{\pi} P \circlearrowright_\epsilon G)$ be a principal groupoid bundle. A horizontal distribution defines a *connection 1-form* $\alpha \in \Omega^1(P, \epsilon^*A)$, which is obtained by assigning to each vector its vertical component.

To construct explicitly the connection 1-form, one can use the division map δ . When restricted to the second coordinate for a fixed $p \in P$, it gives rise to a diffeomorphism $\delta(p, \cdot) := \delta_p : P_{\pi(p)} \rightarrow G_{\epsilon(p)}$. Let $X_p \in T_p P$, then the connection form $\alpha \in \Omega^1(P, \epsilon^*A)$ associated to the distribution H is:

$$\alpha(X_p) = (\delta_p)_{*,p} \text{Ver}(X_p),$$

where $\text{Ver}(X_p)$ is the projection of X_p on $\ker \pi_*$ uniquely defined by the splitting $TP = H \oplus \ker \pi_*$. Equivalently α can be thought of as a bundle map $TP \rightarrow A$ over the momentum map $\epsilon : P \rightarrow G_0$:

Lemma 3.1. Let $(M \xleftarrow{\pi} P \circlearrowright_\epsilon G)$ be a principal groupoid bundle. There is one-one correspondence between the following:

- (a) horizontal distributions $H \subset TP$;
- (b) 1-forms $\alpha \in \Omega^1(P, \epsilon^*A)$;

(c) bundle maps $\alpha^\sharp : TP \rightarrow A$ over the momentum map $\epsilon : P \rightarrow G_0$.

We say that a horizontal distribution satisfies the *Moerdijk–Mrčun condition* if $\epsilon_* H = 0$ [6]. In this case the tangent action of G is well defined.

Definition 3.2. A principal connection on a principal groupoid bundle $(M \xleftarrow{\pi} P \circlearrowleft_\epsilon G)$ is a horizontal distribution satisfying the Moerdijk–Mrčun condition and such that:

$$H_{p \cdot g} = (R_g)_* H_p, \quad \forall (p, g) \in P_\epsilon \times_s G.$$

Proposition 3.3. Let $(M \xleftarrow{\pi} P \circlearrowleft_\epsilon G)$ be a principal groupoid bundle. Then

(a) A horizontal distribution H satisfies the Moerdijk–Mrčun condition if and only if $\alpha^\sharp : TP \rightarrow A$ as defined in Lemma 3.1 is a bundle map over ϵ such that

$$\rho \circ \alpha^\sharp = \epsilon_*.$$

(b) H is a principal connection if and only if, in addition, the connection 1-form α is equivariant, i.e. for any local bisection γ of G :

$$R_\gamma^* \alpha = \text{Ad}_{\gamma^{-1}} \alpha.$$

Proof. (a) Let $p \in P$ and X be an arbitrary vector field on P . $\text{Ver}(X_p)$ can be written as:

$$\text{Ver}(X_p) = \left. \frac{d}{d\tau} \right|_0 (p \cdot g_\tau),$$

where $\tau \mapsto g_\tau$ is a path in G such that $s(g_\tau) = \epsilon(p)$ and $g_0 = 1_{\epsilon(p)}$. Notice that

$$\begin{aligned} \alpha(X_p) &= (\delta_p)_{*,p} \text{Ver}(X_p) \\ &= (\delta_p)_{*,p} \left. \frac{d}{d\tau} \right|_0 (p \cdot g_\tau) \\ &= \left. \frac{d}{d\tau} \right|_0 \delta(p, p \cdot g_\tau) \\ &= \left. \frac{d}{d\tau} \right|_0 (\delta(p, p) \cdot g_\tau) = \left. \frac{d}{d\tau} \right|_0 g_\tau. \end{aligned}$$

Thus the horizontal part of X_p is

$$\text{Hor}(X_p) = X_p - \left. \frac{d}{d\tau} \right|_0 (p \cdot g_\tau),$$

and one can apply $\epsilon_* : TP \rightarrow TG_0$ to both sides to obtain:

$$\begin{aligned} \epsilon_* \text{Hor}(X_p) &= \epsilon_* X_p - \epsilon_* \left. \frac{d}{d\tau} \right|_0 (p \cdot g_\tau) \\ &= \epsilon_* X_p - \left. \frac{d}{d\tau} \right|_0 \epsilon(p \cdot g_\tau) \\ &= \epsilon_* X_p - \left. \frac{d}{d\tau} \right|_0 t(g_\tau) \\ &= \epsilon_* X_p - \rho \circ \alpha(X_p). \end{aligned}$$

Therefore:

$$\epsilon_* H = 0 \iff \epsilon_* = \rho \circ \alpha.$$

(b) This is obvious. \square

3.2. Curvature

We denote the horizontal and vertical parts of a vector $v \in TP$ by $\text{Hor}(v)$ and $\text{Ver}(v)$ respectively. Recall that the Ehresmann curvature of a horizontal distribution H , on the bundle $P \xrightarrow{\phi} M$, is the 2-form on P , valued in the vertical space $\ker \phi_*$, which is defined by

$$\omega(u, v) = -\text{Ver}([\text{Hor}(\tilde{u}), \text{Hor}(\tilde{v})]), \quad (1)$$

where $u, v \in T_p P$ and \tilde{u}, \tilde{v} are vector fields on P such that $\tilde{u}_p = u$ and $\tilde{v}_p = v$. It is easy to check that the right hand side of Eq. (1) is well defined, i.e. independent of the choice of the vector fields \tilde{u}, \tilde{v} .

Using the action of the Lie groupoid G , one can identify the vertical space $\ker \phi_* \subset T_p P$ with $A|_{\epsilon p}$. Therefore, the Ehresman curvature can be seen as a 2-form $\Omega \in \Omega^2(P, \epsilon^* A)$, or equivalently a bundle map

$$\Omega^\# : \wedge^2 TP \rightarrow A$$

over $\epsilon : P \rightarrow G_0$.

Given $\alpha \in \Omega^1(P, \epsilon^* A)$, let

$$F_\alpha = d_P \circ \alpha^\# - \alpha^\# \circ d_A : \Omega^1(A) \rightarrow \Omega^2(P),$$

where $d_P : \Omega^1(P) \rightarrow \Omega^2(P)$ and $d_A : \Gamma(A^*) \rightarrow \Gamma(\wedge^2 A^*)$ are the differentials of the Lie algebroids TP and A respectively. The following result relates the curvature to F_α .

Proposition 3.4. Assume that H is a horizontal distribution satisfying the Moerdijk–Mrčun condition with curvature Ω . Then

$$(\Omega^\#)^* = F_\alpha,$$

where $\alpha \in \Omega^1(P, \epsilon^* A)$ is as in Lemma 3.1 and both sides are considered as maps $\Omega^1(A) \rightarrow \Omega^2(P)$.

Proof. Consider $X_p \in T_p P$. Take its image $\alpha(X_p)$ and extend it to a smooth section \mathcal{X} of A . Let $\vec{\mathcal{X}} \in \Gamma(VP)$ be the fundamental vector field on P generated by \mathcal{X} . Thus $\alpha^\#(\vec{\mathcal{X}}) = \mathcal{X}_{\epsilon(p)}$. Take $U \subset P$ such that there is a smooth horizontal extension $\widetilde{\text{Hor } X_p} \in \Gamma(H|_U)$ of $\text{Hor } X_p$. Choose an open neighborhood V of p in P with $V \subset \bar{V} \subset U$, and a function $f \in C^\infty(P, \mathbb{R})$ such that $\text{supp}(f) \subset U$ and $f|_V = 1$. Then the vector field

$$X = \vec{\mathcal{X}} + f \cdot \widetilde{\text{Hor } X_p} \in \Gamma(TP)$$

is α -related to \mathcal{X} . Similarly, given $Y_p \in T_p P$ we construct a pair of α -related vector fields Y and \mathcal{Y} on an open neighborhood of p , such that the vector field Y extends the tangent vector $Y(p)$. For any $\lambda \in A^*$, one has:

$$\begin{aligned} (F_\alpha \cdot \lambda)(X \wedge Y)_p &= \langle (\alpha^\#)^* d_A \lambda, X_p \wedge Y_p \rangle - \langle d_P((\alpha^\#)^* \cdot \lambda), X_p \wedge Y_p \rangle \\ &= \langle d_A \lambda, \alpha^\# \cdot X_p \wedge \alpha^\# \cdot Y_p \rangle - \langle d_P((\alpha^\#)^* \cdot \lambda), X_p \wedge Y_p \rangle \\ &= \langle d_A \lambda, \mathcal{X} \wedge \mathcal{Y} \rangle_{\epsilon(p)} - \langle d_P((\alpha^\#)^* \cdot \lambda), X_p \wedge Y_p \rangle \\ &= \rho(\mathcal{X})_{\epsilon(p)} \langle \lambda, \mathcal{Y} \rangle - \rho(\mathcal{Y})_{\epsilon(p)} \langle \lambda, \mathcal{X} \rangle - \langle \lambda, [\mathcal{X}, \mathcal{Y}]_A \rangle_{\epsilon(p)} \\ &\quad - X_p \langle (\alpha^\#)^* \cdot \lambda, Y \rangle + Y_p \langle (\alpha^\#)^* \cdot \lambda, X \rangle + \langle (\alpha^\#)^* \cdot \lambda, [X, Y]_p \rangle \\ &= \langle \lambda, \alpha \cdot [X, Y]_p - [\mathcal{X}, \mathcal{Y}]_A|_{\epsilon(p)} \rangle. \end{aligned}$$

The last equality follows from the Moerdijk–Mrčun condition and from

$$\epsilon^* \langle \lambda, \mathcal{X} \rangle = \langle \alpha^* \lambda, X \rangle.$$

We conclude that

$$(F_\alpha \cdot \lambda)(X \wedge Y)_p = \langle \lambda, \alpha^\# \cdot [X, Y]_m - [\mathcal{X}, \mathcal{Y}]_A|_{\epsilon(p)} \rangle.$$

Finally, we have

$$\begin{aligned} [X, Y] &= \left[\vec{\mathcal{X}} + f \cdot \widetilde{\text{Hor } X_p}, \vec{\mathcal{Y}} + g \cdot \widetilde{\text{Hor } Y_p} \right] \\ &= \left[\vec{\mathcal{X}}, \vec{\mathcal{Y}} \right] + \vec{\mathcal{X}}(g) \widetilde{\text{Hor } Y_p} - \vec{\mathcal{Y}}(f) \widetilde{\text{Hor } X_p} + \left[f \cdot \widetilde{\text{Hor } X_p}, g \cdot \widetilde{\text{Hor } Y_p} \right] \end{aligned}$$

and

$$\alpha[X, Y] = \alpha \left[\vec{\mathcal{X}}, \vec{\mathcal{Y}} \right] + \alpha \left[f \cdot \widetilde{\text{Hor } X_p}, g \cdot \widetilde{\text{Hor } Y_p} \right] = [\mathcal{X}, \mathcal{Y}]_A + \Omega(X, Y). \quad \square$$

Recall that a bundle map between Lie algebroids $f : B \rightarrow C$ is a Lie algebroid morphism if and only if its transpose is a chain map $f^* : (\Gamma(\wedge^* C), d_C) \rightarrow (\Gamma(\wedge^* B), d_B)$.

Corollary 3.5. As above, let H be a horizontal distribution. Then H is a flat connection satisfying the Moerdijk–Mrčun condition if and only if $\alpha^\# : TP \rightarrow A$ is a Lie algebroid morphism.

H is a flat principal connection if and only if, in addition, α is equivariant.

3.3. Gauge transformations

The next step is the introduction of a suitable concept of *gauge transformation*. Suppose now P is trivialized by a section $\sigma : M \rightarrow P$. Define the pullback connection 1-form $\theta_\sigma := \sigma^* \alpha \in \Omega^1(M, \epsilon_\sigma^* A)$, which is a $\epsilon_\sigma^* A$ -valued one-form on M . Also, its induced bundle map $\theta_\sigma^\# : TM \rightarrow A$ over $\epsilon_\sigma := \epsilon \circ \sigma : M \rightarrow G_0$ is simply the composition of $\sigma_* : TM \rightarrow TP$ with $\alpha^\# : TP \rightarrow A$.

$$\begin{array}{ccccc}
 & TP & & & \\
 \sigma_* \nearrow & \downarrow & \searrow \alpha^\# & & \\
 TM & \xrightarrow{\theta_\sigma^\#} & A & & \\
 \downarrow & \downarrow \sigma & \downarrow \epsilon & & \\
 M & \xrightarrow{\epsilon_\sigma} & G_0 & &
 \end{array}$$

One can also pull-back the Ehresmann curvature Ω via $\sigma : M \rightarrow P$ to get a 2-form $\Omega_\sigma \in \Omega^2(M, \epsilon_\sigma^* A)$.

Let $\mathcal{B}(G)$ be the group of (local) bisections of G . For any $\gamma \in C^\infty(M, \mathcal{B}(G))$, $\sigma_\gamma := \sigma \cdot \gamma$ is another section of the principal bundle $P \rightarrow M$. A natural question is: how are θ_σ and Ω_σ related to θ_{σ_γ} and Ω_{σ_γ} ?

Theorem 3.6. *We have the following relations:*

$$\theta_{\sigma_\gamma} = \text{Ad}_{\gamma^{-1}} \theta_\sigma + \gamma^{-1} \gamma', \quad (2)$$

$$\Omega_{\sigma_\gamma} = \text{Ad}_{\gamma^{-1}} \Omega_\sigma. \quad (3)$$

First we need to postulate the meaning of the transformation law (2) for the connection form. By thinking of $C^\infty(M, \mathcal{B}(G))$ as the *gauge group* \mathcal{G} , Eq. (2) is exactly the physicists' shorthand notation for the gauge transformed vector potential.

The first term is easily interpreted as an adjoint action: for any $m \in M$, $\text{Ad}_{\gamma^{-1}(m)}$ is the automorphism of the Lie algebroid A induced by the adjoint action of the bisection $\gamma^{-1}(m)$. The bundle map $(\text{Ad}_{\gamma^{-1}} \theta_\sigma)^\# : A \rightarrow A$ over $\epsilon_{\sigma, \gamma} := t \circ R_\gamma \circ \epsilon_\sigma$ maps $X_m \in T_m M$ to

$$\text{Ad}_{\gamma^{-1}(m)}(\theta(X_m)) \in A_{t(\epsilon(\sigma(m)) \cdot \gamma(m))}.$$

The second term is interpreted in terms of the Maurer Cartan form on $\mathcal{B}(G)$. Let $T^s G = \ker s_*$. We define the tangent space of $\mathcal{B}(G)$ at σ as the space of maps $\Gamma(G_0, \sigma^* T^s G)$ and the tangent bundle as

$$T\mathcal{B}(G) = \{(\sigma, X_\sigma) | \sigma \in \mathcal{B}(G), X_\sigma \in \Gamma(G_0, \sigma^* T^s G)\}.$$

The Maurer Cartan form θ_{MC} is defined as follows:

$$\theta_{MC} : T\mathcal{B}(G) \rightarrow \Gamma(A) : (\sigma, X_\sigma) \mapsto (L_\sigma^{-1})_* X_\sigma.$$

Using $\gamma \in C^\infty(M, \mathcal{B}(G))$ and $\epsilon_\sigma : M \rightarrow G_0$, one can define a bundle map $\Theta_{(\gamma, \epsilon_\sigma)} : TM \rightarrow A$ over $\epsilon_{\sigma, \gamma}$ as follows:

$$\Theta_{(\gamma, \epsilon_\sigma)} : TM \rightarrow A : X_m \mapsto \underbrace{((L_{\gamma(m)}^{-1})_* (\underbrace{\gamma_* X_m}_{T_{\gamma(m)} \mathcal{B}(G)}))}_{\Gamma(A)} |_{\epsilon_{\sigma, \gamma}(m)}.$$

That is to say, $\Theta_{(\gamma, \epsilon_\sigma)} = \gamma^* \theta_{MC} |_{\epsilon_{\sigma, \gamma}}$ is the pullback of θ_{MC} via γ evaluated along $\epsilon_{\sigma, \gamma}$.

Proof of Theorem 3.6. Take $\gamma \in \mathcal{B}(G)$ and let σ be a section of $\phi : P \rightarrow M$. Set $\sigma_i = \sigma$ and $\sigma_j = \sigma_i \cdot \gamma$. Also, let Φ_i and Φ_j denote the trivializations associated to σ_i and σ_j . We note that $\sigma_i(m) = \sigma_j(m) \cdot \delta(\sigma_j(m), \sigma_i(m))$. We now set to derive the gauge transformation formula (2), i.e. the expression for $\theta_j = \sigma_j^* \alpha$ in terms of $\theta_i = \sigma_i^* \alpha$. Let $X_m \in T_m U$.

$$\begin{aligned}
 \theta_j(X_m) &= (\sigma_j^* \alpha)(X_m) \\
 &= [(\Phi_i^{-1} \circ \Phi_j)^* \alpha](\sigma_{j,*} X_m) \\
 &= (\Phi_i^* \alpha)((\Phi_i^{-1} \circ \sigma_j)_* X_m).
 \end{aligned} \quad (4)$$

However

$$\Phi_i^{-1} \circ \sigma_j(m) = \Phi_i^{-1} \circ \Phi_j(m, e) = (m, \delta(\sigma_i(m), \sigma_j(m))),$$

and $\sigma_j(m) = \sigma_i(m) \cdot \gamma(m) |_{\epsilon(\sigma_i(m))}$ implies that

$$\delta(\sigma_i(m), \sigma_j(m)) = \delta(\sigma_i(m), \sigma_i(m)) \cdot \gamma(m) [\epsilon(\sigma_i(m))] = \gamma(m) [\epsilon_i(m)].$$

To compute the derivative of $\Phi_i^{-1} \circ \sigma_j$ at a vector $X_m \in TM$, we pick a path $\tau \mapsto x(\tau)$ such that $\dot{x}(0) = X_m$. Then

$$\begin{aligned} (\Phi_i^{-1} \circ \sigma_j)_* X_m &= \left(X_m, \frac{d}{d\tau} \Big|_0 \gamma(x(\tau))[\epsilon_i(x(\tau))] \right) \\ &= (X_m, \gamma(m)_* \epsilon_{i,*} X_m + (\gamma_* X_m)(\epsilon_i(m))). \end{aligned}$$

Resuming the computation (4):

$$\begin{aligned} \theta_j(X_m) &= (\Phi_i^* \alpha)((\Phi_i^{-1} \circ \sigma_j)_* X_m) \\ &= (\Phi_i^* \alpha)(X_m, \gamma(m)_* \epsilon_{i,*} X_m) + (\gamma_* X_m)(\epsilon_i(m)) \\ &= (\Phi_i^* \alpha)(X_m, \gamma(m)_* \epsilon_{i,*} X_m) + (\Phi_i^* \alpha)(0_m, (\gamma_* X_m)(\epsilon_i(m))). \end{aligned} \quad (5)$$

Notice that $(X_m, \gamma(m)_* \epsilon_{i,*} X_m) \in T\epsilon_i^* \mathcal{U}_G$ as

$$s_* \cdot \gamma(m)_* \epsilon_{i,*} X_m = \text{id}_* \cdot \epsilon_{i,*} X_m = \epsilon_{i,*} X_m$$

and that the vector $(\gamma_* X_m)(\epsilon_i(m))$ is tangent to an s -fiber, therefore (5) is well defined.

The second contribution can be computed as follows:

$$\begin{aligned} (\Phi_i^* \alpha)(0_m, (\gamma_* X_m)(\epsilon_i(m))) &= \alpha \left(\frac{d}{d\tau} \Phi_i(m), \gamma(x(\tau))(\epsilon_i(m)) \right) \\ &= \alpha \left(\frac{d}{d\tau} \sigma_i(m) \cdot \gamma(x(\tau))(\epsilon_i(m)) \right). \end{aligned}$$

Using $x(0) = m$, $\gamma(x(0))(\epsilon_i(m)) = \delta(\sigma_i(m), \sigma_j(m))$, the definition of α and the fact that $\frac{d}{d\tau} \sigma_i(m) \cdot \gamma(x(\tau))[\epsilon_i(m)]|_0$ is vertical, one further obtains:

$$\begin{aligned} (\Phi_i^* \alpha)(0_m, (\gamma_* X_m)(\epsilon_i(m))) &= \alpha \left(\frac{d}{d\tau} \sigma_j(m) \cdot \delta(\sigma_j(m), \sigma_i(m)) \gamma(x(\tau))[\epsilon_i(m)] \Big|_0 \right) \\ &= \frac{d}{d\tau} \delta(\sigma_i(m), \sigma_j(m))^{-1} \cdot \gamma(x(\tau))[\epsilon_i(m)] \Big|_0 \\ &= \frac{d}{d\tau} L_{\gamma(m)[\epsilon_i(m)]}^{-1} (\gamma(x(\tau))[\epsilon_i(m)]) \Big|_0 = \Theta_{(\gamma, \epsilon_i)} X_m. \end{aligned}$$

For the first contribution, we claim:

$$(\Phi_i^* \alpha)(X_m, \gamma(m)_* \epsilon_{i,*} X_m) = \text{Ad}_{\gamma^{-1}(m)} \cdot \theta_i(X_m).$$

Indeed:

$$\begin{aligned} (\Phi_i^* \alpha)(X_m, \gamma(m)_* \epsilon_{i,*} X_m) &= \alpha \left(\frac{d}{d\tau} \Big|_0 \Phi_i(x(\tau)), \gamma(m)(\epsilon_i(x(\tau))) \right) \\ &= \alpha \left(\frac{d}{d\tau} \Big|_0 R_{\gamma(m)} \sigma_i(x(\tau)) \right). \end{aligned}$$

On the other hand:

$$\begin{aligned} \text{Ad}_{\gamma^{-1}(m)} \cdot \theta_i(X_m) &= \text{Ad}_{\gamma^{-1}(m)} \sigma_i^* \alpha(X_m) \\ &= \text{Ad}_{\gamma^{-1}(m)} \alpha(\sigma_{i,*} X_m) \\ &= \alpha((R_{\gamma(m)})_* \sigma_{i,*} X_m), \end{aligned}$$

which concludes the derivation of Eq. (2). \square

Consider now a local section $\sigma_i : U \rightarrow P|_U$, it defines uniquely a trivialization $(\epsilon_i^* \mathcal{U}_G, \Phi_i)$:

$$\Phi_i : \epsilon_i^* \mathcal{U}_G \rightarrow P|_U : (m, g) \mapsto \sigma_i(m)g.$$

Given $\gamma \in C^\infty(M, \mathcal{B}(G))$, one can obtain a new section $\sigma_j := \sigma_i \star \gamma$:

$$\sigma_j : m \mapsto \sigma_i(m) \star \gamma(m),$$

and the corresponding new trivialization $(\epsilon_j^* \mathcal{U}_G, \Phi_j)$. Each section defines a local connection form as pullback, $\theta_i := \sigma_i^* \alpha$ and $\theta_j := \sigma_j^* \alpha$. In this context, the gauge transformation formula is the expression for θ_j in terms of θ_i .

Corollary 3.7. *With the notations above, the gauge transformation formula holds:*

$$\begin{aligned}\theta_j &= \text{Ad}_{\gamma^{-1}} \theta_i + \gamma^{-1} \gamma' \\ \Omega_j &= \text{Ad}_{\gamma^{-1}} \Omega_i.\end{aligned}$$

3.4. Trivial principal bundles

We now consider trivial principal bundles. It is in this setting that our general construction reduces to the standard non linear gauge theories found in the literature [7–9,4].

Let $f : M \rightarrow G_0$ and let $f^* \mathcal{U}_G$ be the trivial G -bundle over M w.r.t. f . As seen in Sections 3.1 and 3.3, we have the following.

Proposition 3.8. (a) *A principal connection on $f^* \mathcal{U}_G$ is determined by a bundle map $\theta^\sharp : TM \rightarrow A$ over $f : M \rightarrow G_0$ satisfying the anchor condition $\rho \circ \theta^\sharp = f_*$.*

(b) *Principal flat connections are in one-one correspondence with Lie algebroid morphisms: $TM \rightarrow A$ over $f : M \rightarrow G_0$.*

This motivated the following:

Definition 3.9. By the curvature of a bundle map $\theta^\sharp : TM \rightarrow A$, we mean the graded map $F_\theta : \Omega^\bullet(A) \rightarrow \Omega^{\bullet+1}(M)$ defined by:

$$F_\theta = d_M \circ (\theta^\sharp)^* - (\theta^\sharp)^* \circ d_A.$$

The curvature therefore measures the failure of θ^\sharp being a Lie algebroid morphism. By Proposition 3.4 and Eq. (3), if θ^\sharp satisfies the anchor condition $f_* = \rho \circ \theta^\sharp$, then $\text{Ad}_{\gamma^{-1}} F_\theta = F_{\theta_\gamma}$.

In fact, we can prove that this relation always holds without any assumption on the anchor.

Theorem 3.10. *The curvature F_θ is always gauge invariant, i.e.*

$$\text{Ad}_{\gamma^{-1}} F_\theta = F_{\theta_\gamma}. \quad (6)$$

Proof. As an algebra, $\Omega^\bullet(A)$ is generated by $\Omega^{\leq 1}(A) = C^\infty(G_0) \oplus \Omega^1(A)$. Since F_θ is a derivation, it suffice to check Eq. (6) on the generators.

We consider degree 0 first. Let $f \in C^\infty(G_0)$ and $X_m \in T_m M$.

First we compute $\langle F_\gamma \cdot f, X_m \rangle$.

$$\begin{aligned}\langle F_\gamma \cdot f, X_m \rangle &= \langle d_M \theta_\gamma^* \cdot f, X_m \rangle - \langle \theta_\gamma^* d_A f, X_m \rangle \\ &= \langle \epsilon_\gamma^* df, X_m \rangle - \rho(\theta_\gamma \cdot X_m) \cdot f \\ &= X_m(f \circ \epsilon_{\sigma, \gamma}) - \rho(\text{Ad}_{\gamma^{-1}(m)} \cdot X_m) \cdot f - \rho((L_{\gamma(m)}^{-1})_* \gamma_* \cdot X_m|_{\epsilon_{\sigma, \gamma}(m)}) \cdot f.\end{aligned}$$

On the other hand $\langle \text{Ad}_{\gamma^{-1}} F \cdot f, X_m \rangle$ gives:

$$\begin{aligned}\langle \text{Ad}_{\gamma^{-1}} F \cdot f, X_m \rangle &= \langle F(\text{Ad}_{\gamma^{-1}(m)}^* f), X_m \rangle \\ &= \langle d_M \theta^*(\text{Ad}_{\gamma^{-1}(m)}^* f), X_m \rangle - \langle \theta^* d_A \text{Ad}_{\gamma^{-1}(m)}^* f, X_m \rangle \\ &= X_m(f \circ \text{Ad}_{\gamma^{-1}(m)} \circ \epsilon_\sigma) - \rho(\theta X_m)(f \circ \text{Ad}_{\gamma^{-1}(m)}) \\ &= X_m((\text{Ad}_{\gamma^{-1}(m)} \epsilon_\sigma)^* f) - \rho(\text{Ad}_{\gamma^{-1}(m)} \theta \cdot X_m) \cdot f.\end{aligned}$$

Therefore:

$$\langle (F_\gamma - \text{Ad}_{\gamma^{-1}} F) \cdot f, X_m \rangle = X_m(f \circ \epsilon_{\sigma, \gamma}) - X_m((\text{Ad}_{\gamma^{-1}(m)} \epsilon_\sigma)^* f) - \rho((L_{\gamma(m)}^{-1})_* \gamma_* \cdot X_m|_{\epsilon_{\sigma, \gamma}(m)}) \cdot f.$$

Notice that the functions $\epsilon_{\sigma, \gamma}$ and $\text{Ad}_{\gamma^{-1}(m)} \epsilon_\sigma$ are different:

$$\begin{aligned}\epsilon_{\sigma, \gamma} : x &\mapsto \text{Ad}_{\gamma^{-1}(x)} \epsilon_\sigma(x), \\ \text{Ad}_{\gamma^{-1}(m)} \epsilon_\sigma : x &\mapsto \text{Ad}_{\gamma^{-1}(m)} \epsilon_\sigma(x).\end{aligned}$$

Take a path $\tau \mapsto x_\tau$ in M such that $\frac{dx_\tau}{d\tau}|_0 = X_m$, then:

$$\begin{aligned}X_m(f \circ \epsilon_{\sigma, \gamma}) &= \frac{d}{d\tau} f(\text{Ad}_{\gamma^{-1}(m)} \epsilon_\sigma(x_\tau))|_0 + \frac{d}{d\tau} f(\text{Ad}_{\gamma^{-1}(x_\tau)} \epsilon_\sigma(m))|_0, \\ X_m((\text{Ad}_{\gamma^{-1}(m)} \epsilon_\sigma)^* f) &= \frac{d}{d\tau} f(\text{Ad}_{\gamma^{-1}(m)} \epsilon_\sigma(x_\tau))|_0.\end{aligned}$$

Hence:

$$\langle (F_\gamma - \text{Ad}_{\gamma^{-1}} F) \cdot f, X_m \rangle = \frac{d}{d\tau} f(\text{Ad}_{\gamma^{-1}(x_\tau)} \epsilon_\sigma(m))|_0 - \rho((L_{\gamma(m)}^{-1})^* \gamma_* X_m|_{\epsilon_{\sigma, \gamma}(m)}) \cdot f.$$

By definition of the adjoint action on the base G_0 :

$$\begin{aligned} \text{Ad}_{\gamma^{-1}(x_\tau)} \epsilon_\sigma(m) &= t(\epsilon_\sigma(m) \cdot \gamma(x_\tau)) \\ &= t(R_{\gamma(x_\tau)}(\epsilon_\sigma(m))) \\ &= t(\gamma(x_\tau)(\epsilon_\sigma(m))). \end{aligned}$$

On the other hand:

$$\begin{aligned} \rho((L_{\gamma(m)}^{-1})^* \gamma_* X_m|_{\epsilon_{\sigma, \gamma}(m)}) &= \frac{d}{d\tau} \Big|_0 (t \circ L_{\gamma(m)}^{-1})(\gamma(x_\tau)(\epsilon_\sigma(m))) \\ &= \frac{d}{d\tau} \Big|_0 t(\gamma(x_\tau)(\epsilon_\sigma(m))), \end{aligned}$$

thus:

$$\rho((L_{\gamma(m)}^{-1})^* \gamma_* X_m|_{\epsilon_{\sigma, \gamma}(m)}) \cdot f = \frac{d}{d\tau} \Big|_0 f(t(\gamma(x_\tau)(\epsilon_\sigma(m)))) = \frac{d}{d\tau} \Big|_0 f(\text{Ad}_{\gamma^{-1}(x_\tau)} \epsilon_\sigma(m)).$$

Since X_m is arbitrary, we conclude that:

$$(F_\gamma - \text{Ad}_{\gamma^{-1}} F) \cdot f = 0.$$

To check gauge invariance in degree 1 we choose $\lambda \in \Omega^1 A$. The following is just an unraveling of definitions:

$$\begin{aligned} (F_\gamma - \text{Ad}_{\gamma^{-1}} F) \lambda &= -(\text{Ad}_{\gamma^{-1}} \theta^*)^*(d_A \lambda) - \Theta_{(\gamma, \epsilon_\sigma)}(d_A \lambda) + d_M((\text{Ad}_{\gamma^{-1}} \theta)^* \cdot \lambda) \\ &\quad + d_M(\Theta_{(\gamma, \epsilon_\sigma)}^* \cdot \lambda) + \theta^* \circ d_A(\text{Ad}_{\gamma^{-1}}^* \cdot \lambda) - d_M \circ \theta^*(\text{Ad}_{\gamma^{-1}}^* \cdot \lambda) \\ &= \Theta_{(\gamma, \epsilon_\sigma)}^*(d_A \lambda) - d_M(\Theta_{(\gamma, \epsilon_\sigma)}^* \cdot \lambda). \end{aligned}$$

We used the fact the $\text{Ad}_{\gamma^{-1}(m)}$ is an algebroid morphism for all m in M . Recall that $\Theta_{(\gamma, \epsilon_\sigma)}$ is the pullback of the Maurer Cartan form on $\mathcal{B}(G)$ via $\gamma : M \rightarrow \mathcal{B}(G)$ evaluated along $\epsilon_{\sigma, \gamma} : M \rightarrow G_0$. We now show that

$$\langle \Theta_{(\gamma, \epsilon_\sigma)}^*(d_A \lambda) - d_M(\Theta_{(\gamma, \epsilon_\sigma)}^* \cdot \lambda), X_m \wedge Y_m \rangle = 0, \quad (7)$$

for all $X, Y \in \mathfrak{X}(M)$ and $m \in M$. Using the definition of $\Theta_{(\gamma, \epsilon_\sigma)}$:

$$\Theta_{(\gamma, \epsilon_\sigma)} : X_m \mapsto (L_{\gamma(m)}^{-1})^* \gamma_* X_m|_{\epsilon_{\sigma, \gamma}(m)},$$

we compute separately the two terms. The first one yields:

$$\begin{aligned} \langle \Theta_{(\gamma, \epsilon_\sigma)}^*(d_A \lambda), X_m \wedge Y_m \rangle &= \rho(\Theta_{(\gamma, \epsilon_\sigma)} \cdot X_m) \left\langle \lambda, (L_{\gamma(m)}^{-1})^* \gamma_* Y_m \right\rangle - \rho(\Theta_{(\gamma, \epsilon_\sigma)} \cdot Y_m) \left\langle \lambda, (L_{\gamma(m)}^{-1})^* \gamma_* X_m \right\rangle \\ &\quad - \left\langle \lambda, \left[(L_{\gamma(m)}^{-1})^* \gamma_* X_m, (L_{\gamma(m)}^{-1})^* \gamma_* Y_m \right] \right\rangle_{\epsilon_{\sigma, \gamma}(m)}. \end{aligned}$$

The second term reads:

$$\begin{aligned} \langle d_M(\Theta_{(\gamma, \epsilon_\sigma)}^* \cdot \lambda), X_m \wedge Y_m \rangle &= \langle d_M(\Theta_{(\gamma, \epsilon_\sigma)}^* \cdot \lambda), X \wedge Y \rangle_m = X_m \langle \Theta_{(\gamma, \epsilon_\sigma)}^* \cdot \lambda, Y \rangle \\ &\quad - Y_m \langle \Theta_{(\gamma, \epsilon_\sigma)}^* \cdot \lambda, X \rangle - \left\langle \lambda, (L_{\gamma(m)}^{-1})^* \gamma_* [X, Y] \right\rangle_{\epsilon_{\sigma, \gamma}(m)}. \end{aligned}$$

Since solutions to the Maurer Cartan equations are preserved by pullbacks, for every $m \in M$ we have:

$$(L_{\gamma(m)}^{-1})^* \gamma_* [X, Y]_m - \left[(L_{\gamma(m)}^{-1})^* \gamma_* X_m, (L_{\gamma(m)}^{-1})^* \gamma_* Y_m \right] = X_m((L_{\gamma(m)}^{-1})^* \gamma_* Y) - Y_m((L_{\gamma(m)}^{-1})^* \gamma_* X).$$

Therefore, the L.H.S. of Eq. (7) is the sum of the two triples

$$\rho(\Theta_{(\gamma, \epsilon_\sigma)} \cdot X_m) \left\langle \lambda, (L_{\gamma(m)}^{-1})^* \gamma_* Y_m \right\rangle - X_m \langle \Theta_{(\gamma, \epsilon_\sigma)}^* \cdot \lambda, Y \rangle + \langle \lambda, X((L_{\gamma(m)}^{-1})^* \gamma_* Y) \rangle_{\epsilon_{\sigma, \gamma}(m)}$$

and

$$\rho(\Theta_{(\gamma, \epsilon_\sigma)} \cdot Y_m) \left\langle \lambda, (L_{\gamma(m)}^{-1})^* \gamma_* X_m \right\rangle - Y_m \langle \Theta_{(\gamma, \epsilon_\sigma)}^* \cdot \lambda, X \rangle + \langle \lambda, Y((L_{\gamma(m)}^{-1})^* \gamma_* X) \rangle_{\epsilon_{\sigma, \gamma}(m)}.$$

In the first triple, the first term is the derivative with respect to

$$\rho(\Theta_{(\gamma, \epsilon_\sigma)} X_m) = \rho((L_{\gamma(m)}^{-1})^* \gamma_* X_m|_{\epsilon_{\sigma, \gamma(m)}})$$

of the function

$$G_0 \in \mathfrak{g} \mapsto \left\langle \lambda, (L_{\gamma(m)}^{-1})^* \gamma_* Y_m \right\rangle_{\mathfrak{g}},$$

the second term is the derivative with respect to X_m of the function

$$M \in \mathfrak{x} \mapsto \left\langle \lambda, (L_{\gamma(x)}^{-1})^* \gamma_* Y_x \right\rangle_{\epsilon_{\sigma, \gamma(x)}}$$

and the third term is the derivative with respect to X_m of the function:

$$M \in \mathfrak{x} \mapsto \left\langle \lambda, (L_{\gamma(x)}^{-1})^* \gamma_* Y_x \right\rangle_{\epsilon_{\sigma, \gamma(m)}}.$$

Therefore they cancel by Leibniz rule. The same argument applies to the second triple and the theorem is proven. \square

By \mathcal{A}_0 , we denote the space of Lie algebroid morphisms $TM \rightarrow A$ on which $\mathcal{G} = C^\infty(M, \mathcal{B}(G))$ acts naturally. Therefore the *moduli space of flat connections* $\mathcal{A}_0/\mathcal{G}$ is well defined. This construction was studied in detail in [14].

A special case of this construction gives a gauge theoretical interpretation of the topological groupoid $G(A)$, which integrates A . In that case, M is the interval $I = [0, 1]$. The original approach involves the Weinstein groupoid of A which is the space of algebroid morphisms from TI to A modulo algebroid homotopies (see [5] for details). It is a proposition in [15] that two Lie algebroid morphisms are homotopic if and only if they are connected by the flow of an infinitesimal gauge transformation. Therefore the Weinstein groupoid of A can be interpreted as the moduli space of flat connections on the interval I .

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