Journal of Geometry and Physics 74 (2013) 328–341

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

On warped product manifolds satisfying some curvature conditions

J. Chojnacka-Dulas [a], R. Deszcz [a,*], M. Głogowska [a], M. Prvanović [b]

[a] Department of Mathematics, Wrocław University of Environmental and Life Sciences, Grunwaldzka 53, 50-357 Wrocław, Poland

[b] Mathematical Institute, Serbian Academy of Sciences and Arts, Kneza Mihaila 36, 11001 Beograd, p.p. 367, Serbia

ARTICLE INFO

Article history:
Received 29 October 2011
Accepted 10 August 2013
Available online 26 August 2013

Dedicated to the memory of Professor Barbara Rabijewska

MSC:
primary 53B20
53B30
53B50
secondary 53C25

Keywords:
quasi-Einstein manifold
Warped product manifold
Generalized Robertson–Walker spacetime
Pseudosymmetric manifold
Ricci-pseudosymmetric manifold
Tachibana tensor

ABSTRACT

We determine curvature properties of pseudosymmetric type of certain warped product manifolds, and in particular of generalized Robertson–Walker spacetimes, with Einsteinian or quasi-Einsteinian fibre.

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1. Introduction

Let (M, g) be an n-dimensional, $n \geq 3$, semi-Riemannian manifold and let ∇ be its Levi-Civita connection. The manifold (M, g), $n \geq 3$, is said to be an *Einstein manifold* if its Ricci tensor S is proportional to g, i.e. on M we have $S = \frac{\kappa}{n} g$, where κ is the scalar curvature. More generally, (M, g), $n \geq 3$, is called a *quasi-Einstein manifold* if at every point $x \in M$ its Ricci tensor satisfies rank $(S - \alpha g) \leq 1$, for some $\alpha \in \mathbb{R}$, i.e. the condition $S = \alpha g + \epsilon w \otimes w$, $\epsilon = \pm 1$, for some $\alpha \in \mathbb{R}$ and $w \in T_x^* M$, holds at x (see, e.g., [1–5]). For precise definitions of the symbols used, we refer to Sections 2 and 3 of this paper, as well as to [6,7]. Quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations and investigation of quasi-umbilical hypersurfaces of conformally flat spaces. Quasi-Einstein hypersurfaces in semi-Riemannian spaces of constant curvature were studied among others in: [8,9,2,4,5], see also [10].

Another extension of the class of Einstein manifolds form *Ricci-symmetric manifolds*, i.e. manifolds for which we have $\nabla S = 0$. We also mention that an important subclass of the class of Ricci-symmetric manifolds form *locally symmetric manifolds*, i.e. manifolds for which we have $\nabla R = 0$. The conditions $\nabla R = 0$ and $\nabla S = 0$, imply $R \cdot R = 0$ and $R \cdot S = 0$,

* Corresponding author. Tel.: +48 71 357282; fax: +48 71 3280214.
E-mail addresses: ryszard.deszcz@up.wroc.pl (R. Deszcz), malgorzata.glogowska@up.wroc.pl (M. Głogowska).

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http://dx.doi.org/10.1016/j.geomphys.2013.08.007

respectively. Manifolds satisfying these conditions are called *semisymmetric* [11] and *Ricci-semisymmetric* (see, e.g., [12]), respectively. Further extension of the last two classes of manifolds form pseudosymmetric and Ricci-pseudosymmetric manifolds.

A manifold (M, g), $n \geq 3$, is said to be *pseudosymmetric* [13,12] if

$$R \cdot R = L_R\, Q(g, R) \tag{1.1}$$

holds on $U_R = \{x \in M \mid R \neq \frac{\kappa}{(n-1)n}\, G \text{ at } x\}$, where L_R is some function on this set. Every semisymmetric manifold is pseudosymmetric. The converse statement is not true. Pseudosymmetric manifolds are conformally flat or not. Pseudosymmetric manifolds with the constant function L_R are called *pseudosymmetric spaces of constant type* [14]. For instance, the type number two hypersurfaces in semi-Riemannian spaces of constant curvature are pseudosymmetric spaces of constant type [15]. We refer to [6,15,1,13,16,12,17,18,2,19–24] for results on pseudosymmetric manifolds. A geometric interpretation of (1.1) is given in [23] (see also [24]). We recall that the warped product $\overline{M} \times_F \widetilde{N}$ of one dimensional manifold $(\overline{M}, \overline{g})$, $\overline{g}_{11} = \pm 1$, and a semi-Riemannian space of constant curvature $(\widetilde{N}, \widetilde{g})$, $\dim \widetilde{N} = n - 1 \geq 3$, is a conformally flat manifold. Moreover, (1.1) holds on $U_R \subset \overline{M} \times_F \widetilde{N}$ [13].

A manifold (M, g), $n \geq 3$, is said to be *Ricci-pseudosymmetric* [25,12,26], if

$$R \cdot S = L_S\, Q(g, S) \tag{1.2}$$

holds on $U_S = \{x \in M \mid S \neq \frac{\kappa}{n}\, g \text{ at } x\}$, where L_S is some function on this set. Ricci-pseudosymmetric manifolds with the constant function L_S are called *Ricci-pseudosymmetric spaces of constant type* [5]. For instance, the Cartan hypersurfaces are Ricci-pseudosymmetric spaces of constant curvature [5]. We refer to [6,25,16,8,18,26,4,27,28,5,29] for results on Ricci-pseudosymmetric manifolds. A geometric interpretation of (1.2) is given in [29].

It is well-known that the following identity with cyclic sum

$$\sum_{(X_1,X_2),(X_3,X_4),(X,Y)} (R \cdot R)(X_1, X_2, X_3, X_4, X, Y) = 0, \tag{1.3}$$

holds on every semi-Riemannian manifold (M, g), where $X_1, \ldots, X_4, X, Y$ are vector fields on M. (1.3) is called the *Walker identity*. We also can investigate semi-Riemannian manifolds satisfying the condition

$$\sum_{(X_1,X_2),(X_3,X_4),(X,Y)} (R \cdot C)(X_1, X_2, X_3, X_4, X, Y) = 0. \tag{1.4}$$

It is easy to check that every pseudosymmetric manifold satisfies (1.4). More generally, we prove (see Theorem 2.3 of this paper) that (1.4) holds on any Ricci-pseudosymmetric manifold. We also have the following conditions:

$$\sum_{(X_1,X_2),(X_3,X_4),(X,Y)} (C \cdot R)(X_1, X_2, X_3, X_4, X, Y) = 0, \tag{1.5}$$

$$\sum_{(X_1,X_2),(X_3,X_4),(X,Y)} (R \cdot C - C \cdot R)(X_1, X_2, X_3, X_4, X, Y) = 0. \tag{1.6}$$

The conditions (1.4)–(1.6) are equivalent [7, Proposition 4.1]. We mention that hypersurfaces in spaces of constant curvature satisfying (1.4)–(1.6) were investigated in [7,4,28]. In this paper we investigate warped products, of dimension ≥ 4, satisfying (1.4).

Section 3 contains basical formulas on warped product manifolds. That section also contains preliminary results on warped products satisfying (1.4).

Warped products play an important role in Riemannian geometry (see, e.g., [30,31]), as well as in the general relativity theory (see, e.g., [32,31]). Many well-known spacetimes considered in the general relativity, i.e. solutions of the Einstein field equations, are warped products, e.g., the Schwarzschild spacetime, the Kottler spacetime, the Reissner–Nordström spacetime and the Reissner–Nordström–de Sitter spacetime. The mentioned spacetimes are pseudosymmetric. We refer to [33–35,22,36] for results on pseudosymmetric spacetimes. We recall that the warped product $\overline{M} \times_F N$ of $(\overline{M}, \overline{g})$, $\dim \overline{M} = 1, \overline{g}_{11} = -1$, and a 3-dimensional Riemannian space of constant curvature $(N, \widetilde{g})$, is said to be a *Robertson–Walker spacetime* (see, e.g., [32,31,37]). Robertson–Walker spacetimes are quasi-Einstein conformally flat manifolds. They also are pseudosymmetric [13]. A natural extension of the class of the Robertson–Walker spacetimes form warped products $\overline{M} \times_F \widetilde{N}$ of $(\overline{M}, \overline{g})$, $\dim \overline{M} = 1, \overline{g}_{11} = -1$, and a Riemannian manifold $(\widetilde{N}, \widetilde{g})$, $\dim \widetilde{N} = n - 1 \geq 3$. Such warped products are called *generalized Robertson–Walker spacetimes* [38–40].

The warped product $\overline{M} \times_F \widetilde{N}$ of $(\overline{M}, \overline{g})$, $\dim \overline{M} = 1, \overline{g}_{11} = \pm 1$, and a semi-Riemannian Einstein manifold $(\widetilde{N}, \widetilde{g})$, $\dim \widetilde{N} = n - 1 \geq 2$, satisfies on $U_S \subset \overline{M} \times_F \widetilde{N}$ the condition (1.2) [26, Corollary 3.2]. Moreover, we have (Theorem 4.1(i)):

$$\operatorname{rank}\left(S - \left(\frac{\kappa}{n-1} - L_S\right) g\right) = 1. \tag{1.7}$$

This means that such warped product is a quasi-Einstein manifold. In addition, if $(\widetilde{N}, \widetilde{g})$ is not of constant curvature and $n \geq 5$, then the tensor $R \cdot C - C \cdot R$ is expressed by a linear combination of the Tachibana tensors $Q(S, R)$ and $Q(g, R)$. Precisely, on $U_C \cap U_S$ we have (Theorem 4.1(ii))

$$(n-2)\,(R \cdot C - C \cdot R) = Q(S, R) - L_S\, Q(g, R), \tag{1.8}$$

where $U_C = \{x \in \overline{M} \times_F \widetilde{N} \mid C \neq 0 \text{ at } x\}$. Generalized Robertson–Walker spacetimes of dimension ≥ 5, with not of constant curvature Einsteinian fibre, satisfy (1.2), (1.4), (1.7) and (1.8) (Theorem 4.2). In that section we also present necessary and sufficient conditions for the warped product with one dimensional base manifold and non-Einstein fibre to be a quasi-Einstein manifold (Theorem 4.4).

In Section 5 we present an example of a warped product manifold with a p-dimensional base, $2 \leq p \leq n-2$, which is a non-Einstein and non-conformally flat Ricci-pseudosymmetric manifold satisfying

$$(n-2)\,(R \cdot C - C \cdot R) = \frac{1}{n-p}\left(\frac{(p-1)\kappa}{n-1} - p(n-1)L_S\right) Q(g, R) + Q(S, R). \tag{1.9}$$

In the last section we present necessary and sufficient conditions for a warped product manifold with a p-dimensional base, $2 \leq p \leq n-3$, and an $(n-p)$-dimensional, not of constant curvature Einsteinian fibre, to be a manifold satisfying (1.4) (Proposition 6.1). We also investigate warped product manifolds with p-dimensional base, $1 \leq p \leq n-3$, and $(n-p)$-dimensional non-Einsteinian fibre, satisfying (1.4). We prove that such manifolds are Ricci-pseudosymmetric (Theorem 6.3).

2. Preliminaries

Throughout this paper all manifolds are assumed to be connected paracompact manifolds of class C^∞. Let (M, g) be an n-dimensional, $n \geq 3$, semi-Riemannian manifold and let $\Xi(M)$ be the Lie algebra of vector fields on M. We define on M the endomorphisms $X \wedge_A Y$ and $\mathcal{R}(X, Y)$ of $\Xi(M)$, respectively, by

$$(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y,$$
$$\mathcal{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

where A is a symmetric $(0, 2)$-tensor on M and $X, Y, Z \in \Xi(M)$. The Ricci tensor S, the Ricci operator $\mathcal{S}$ and the scalar curvature κ of (M, g) are defined by $S(X, Y) = \operatorname{tr}\{Z \to \mathcal{R}(Z, X)Y\}$, $g(\mathcal{S}X, Y) = S(X, Y)$ and $\kappa = \operatorname{tr} \mathcal{S}$, respectively. The endomorphism $\mathcal{C}(X, Y)$ is defined by

$$\mathcal{C}(X, Y)Z = \mathcal{R}(X, Y)Z - \frac{1}{n-2}\left(X \wedge_g \mathcal{S}Y + \mathcal{S}X \wedge_g Y - \frac{\kappa}{n-1} X \wedge_g Y\right) Z.$$

Now the $(0, 4)$-tensor G, the Riemann–Christoffel curvature tensor R and the Weyl conformal curvature tensor C of (M, g) are defined by

$$G(X_1, X_2, X_3, X_4) = g((X_1 \wedge_g X_2)X_3, X_4),$$
$$R(X_1, X_2, X_3, X_4) = g(\mathcal{R}(X_1, X_2)X_3, X_4),$$
$$C(X_1, X_2, X_3, X_4) = g(\mathcal{C}(X_1, X_2)X_3, X_4),$$

respectively, where $X_1, X_2, \ldots \in \Xi(M)$. Let $\mathcal{B}(X, Y)$ be a skew-symmetric endomorphism of $\Xi(M)$ and B a $(0, 4)$-tensor associated with $\mathcal{B}(X, Y)$ by

$$B(X_1, X_2, X_3, X_4) = g(\mathcal{B}(X_1, X_2)X_3, X_4). \tag{2.1}$$

The tensor B is said to be a *generalized curvature tensor* if

$$B(X_1, X_2, X_3, X_4) + B(X_2, X_3, X_1, X_4) + B(X_3, X_1, X_2, X_4) = 0,$$
$$B(X_1, X_2, X_3, X_4) = B(X_3, X_4, X_1, X_2).$$

Let $\mathcal{B}(X, Y)$ be a skew-symmetric endomorphism of $\Xi(M)$ and B the tensor defined by (2.1). We extend the endomorphism $\mathcal{B}(X, Y)$ to derivation $\mathcal{B}(X, Y)\cdot$ of the algebra of tensor fields on M, assuming that it commutes with contractions and $\mathcal{B}(X, Y) \cdot f = 0$, for any smooth function f on M. For a $(0, k)$-tensor field T, $k \geq 1$, we define the $(0, k+2)$-tensor $B \cdot T$ by

$$\begin{aligned}(B \cdot T)(X_1, \ldots, X_k, X, Y) &= (\mathcal{B}(X, Y) \cdot T)(X_1, \ldots, X_k)\\ &= -T(\mathcal{B}(X, Y)X_1, X_2, \ldots, X_k) - \cdots - T(X_1, \ldots, X_{k-1}, \mathcal{B}(X, Y)X_k).\end{aligned}$$

In addition, if A is a symmetric $(0, 2)$-tensor then we define the $(0, k+2)$-tensor $Q(A, T)$ by

$$\begin{aligned}Q(A, T)(X_1, \ldots, X_k, X, Y) &= (X \wedge_A Y \cdot T)(X_1, \ldots, X_k)\\ &= -T((X \wedge_A Y)X_1, X_2, \ldots, X_k) - \cdots - T(X_1, \ldots, X_{k-1}, (X \wedge_A Y)X_k).\end{aligned}$$

In this manner we obtain the $(0, 6)$-tensors $B \cdot B$ and $Q(A, B)$. Setting in the above formulas $\mathcal{B} = \mathcal{R}$ or $\mathcal{B} = \mathcal{C}$, $T = R$ or $T = C$ or $T = S$, $A = g$ or $A = S$, we get the tensors $R \cdot R$, $R \cdot C$, $C \cdot R$, $R \cdot S$, $Q(g, R)$, $Q(S, R)$, $Q(g, C)$ and $Q(g, S)$.

Let B_{hijk}, T_{hijk}, and A_{ij} be the local components of generalized curvature tensors B and T and a symmetric $(0,2)$-tensor A on M, respectively, where $h, i, j, k, l, m, p, q \in \{1, 2, \ldots, n\}$. The local components $(B\cdot T)_{hijklm}$ and $Q(A,T)_{hijklm}$ of the tensors $B\cdot T$ and $Q(A,T)$ are the following

$$\begin{aligned}(B\cdot T)_{hijklm} &= g^{pq}(T_{pijk}B_{qhlm} + T_{hpjk}B_{qilm} + T_{hipk}B_{qjlm} + T_{hijp}B_{qklm}),\\ Q(A,T)_{hijklm} &= A_{hl}T_{mijk} + A_{il}T_{hmjk} + A_{jl}T_{himk} + A_{kl}T_{hijm} - A_{hm}T_{lijk} - A_{im}T_{hljk} - A_{jm}T_{hilk} - A_{km}T_{hijl}.\end{aligned} \tag{2.2}$$

For a symmetric $(0,2)$-tensor E and a $(0,k)$-tensor T, $k \geq 2$, we define their Kulkarni–Nomizu product $E \wedge T$ by [41]

$$\begin{aligned}(E\wedge T)(X_1,X_2,X_3,X_4;Y_3,\ldots,Y_k) &= E(X_1,X_4)T(X_2,X_3,Y_3,\ldots,Y_k) + E(X_2,X_3)T(X_1,X_4,Y_3,\ldots,Y_k)\\ &\quad - E(X_1,X_3)T(X_2,X_4,Y_3,\ldots,Y_k) - E(X_2,X_4)T(X_1,X_3,Y_3,\ldots,Y_k).\end{aligned}$$

Clearly, the tensors R, C, G and $E\wedge F$, where E and F are symmetric $(0,2)$-tensors are generalized curvature tensors. For a symmetric $(0,2)$-tensor E we define the $(0,4)$-tensor $\overline{E}$ by $\overline{E} = \frac{1}{2}E\wedge E$. We have $\overline{g} = G = \frac{1}{2}g\wedge g$. We note that the Weyl tensor C can be presented in the form

$$C = R - \frac{1}{n-2}g\wedge S + \frac{\kappa}{(n-2)(n-1)}G. \tag{2.3}$$

We also have (see, e.g., [9, Section 3])

$$Q(E, E\wedge F) = -Q(F,\overline{E}). \tag{2.4}$$

Now (2.3) and (2.4) yield

$$Q(g,C) = Q(g,R) + \frac{1}{n-2}Q(S,G). \tag{2.5}$$

Using the above definitions we can prove the following.

Lemma 2.1. *Let (M,g), $n \geq 3$, be a semi-Riemannian manifold.*

(i) [8,41] *Let E_1, E_2 and F be symmetric $(0,2)$-tensors at a point $x \in M$. Then at x we have*

$$E_1\wedge Q(E_2,F) + E_2\wedge Q(E_1,F) = -Q(F, E_1\wedge E_2).$$

If $E = E_1 = E_2$ then

$$E\wedge Q(E,F) = -Q(F,\overline{E}). \tag{2.6}$$

(ii) *(cf., [5, Lemma 3.1]) If E is a symmetric, not proportional to g, $(0,2)$-tensor at a point $x \in M$ then at this point the following conditions are equivalent:* (a) $\operatorname{rank}(E-\alpha g) = 1$, $\alpha \in \mathbb{R}$, *and* (b)

$$\frac{1}{2}E\wedge E - \alpha g\wedge E + \alpha^2 G = \frac{1}{2}(E-\alpha g)\wedge(E-\alpha g) = 0, \quad \alpha\in\mathbb{R}.$$

(iii) *(cf., [3, p. 20]) Let E be a symmetric, not proportional to g, $(0,2)$-tensor at a point $x \in M$. If at x we have* $\operatorname{rank}(E-\alpha_1 g) = 1$, $\alpha_1 \in \mathbb{R}$, *and* $\operatorname{rank}(E-\alpha_2 g) = 1$, $\alpha_2 \in \mathbb{R}$, *then $\alpha_1 = \alpha_2$.*

For a tensor T of type $(0,6)$ on M we denote by

$$\sum_{(X_1,X_2),(X_3,X_4),(X_5,X_6)} T(X_1,X_2,X_3,X_4,X_5,X_6)$$

the sum $T(X_1,\ldots,X_6) + T(X_3,\ldots,X_6,X_1,X_2) + T(X_5,X_6,X_1,\ldots,X_4)$.

Proposition 2.2. (i) *(cf., [13, Lemma 1]) For a symmetric $(0,2)$-tensor A and a generalized curvature tensor B on a semi-Riemannian manifold (M,g), $n \geq 3$, we have*

$$\sum_{(X_1,X_2),(X_3,X_4),(X,Y)} Q(A,B)(X_1,X_2,X_3,X_4,X,Y) = 0. \tag{2.7}$$

(ii) [7, *Proposition* 4.1] *On any semi-Riemannian manifold (M,g), $n \geq 4$, the following three identities are equivalent to each other:* (1.4)–(1.6).

Let $Q(A,T)$ be the $(0,k+2)$-tensor defined by the symmetric $(0,2)$-tensor A and a $(0,k)$-tensor T, $k \geq 2$. The tensor $Q(A,T)$ is called the *Tachibana tensor of the tensors A and T*, or shortly the *Tachibana tensor* (see, e.g., [42], also see [23,29]). By making use of (2.4) we can check that the following identities are satisfied on any semi-Riemannian manifold (M,g)

$$Q(g, g\wedge S) = -Q(S,G) \quad\text{and}\quad Q(S, g\wedge S) = -\frac{1}{2}Q(g, S\wedge S).$$

Thus on any semi-Riemannian manifolds (M, g), $n \geq 4$, we can consider the following $(0, 6)$-Tachibana tensors formed by the tensors g, R and S (see, e.g., [42]): $Q(g, R)$, $Q(S, R)$, $Q(g, g \wedge S)$ and $Q(S, g \wedge S)$.

Investigations of curvature properties of Einstein manifolds, as well as hypersurfaces in semi-Riemannian spaces of constant curvature lead to the following condition (see, e.g., [41, Theorem 4.1]; [7, Proposition 4.2]; [36])

$$R \cdot C - C \cdot R \text{ is a linear combination of } Q(g, R),\ Q(S, R), \quad Q(g, g \wedge S) \quad \text{and} \quad Q(S, g \wedge S). \tag{$*$}$$

We also refer to [10] for a survey of results on semi-Riemannian manifolds satisfying $(*)$. We note that $(*)$, in view of Proposition 2.2(i), implies (1.6). But this, by Proposition 2.2(ii), yields (1.4).

On any semi-Riemannian manifold (M, g) we have (see, e.g., [7])

$$(n-2)(R \cdot C - C \cdot R) = Q(S, R) - \frac{\kappa}{n-1} Q(g, R) - g \wedge (R \cdot S) + P,$$

where the $(0, 6)$-tensor P is defined by

$$\begin{aligned} P(X_1, X_2, X_3, X_4, X, Y) = {} & g(X, X_1)R(\mathcal{S}(Y), X_2, X_3, X_4) - g(Y, X_1)R(\mathcal{S}(X), X_2, X_3, X_4) + g(X, X_2)R(X_1, \mathcal{S}(Y), X_3, X_4) \\ & - g(Y, X_2)R(X_1, \mathcal{S}(X), X_3, X_4) + g(X, X_3)R(X_1, X_2, \mathcal{S}(Y), X_4) - g(Y, X_3)R(X_1, X_2, \\ & \mathcal{S}(X), X_4) + g(X, X_4)R(X_1, X_2, X_3, \mathcal{S}(Y)) - g(Y, X_4)R(X_1, X_2, X_3, \mathcal{S}(X)). \end{aligned}$$

The local expression of the above identity is the following

$$\begin{aligned} (n-2)(R \cdot C - C \cdot R)_{hijklm} = {} & Q(S, R)_{hijklm} - \frac{\kappa}{n-1} Q(g, R)_{hijklm} - g_{ij}(R \cdot S)_{hklm} - g_{hk}(R \cdot S)_{ijlm} + g_{ik}(R \cdot S)_{hjlm} \\ & + g_{hj}(R \cdot S)_{iklm} + g_{hl}A_{mijk} - g_{hm}A_{lijk} - g_{il}A_{mhjk} + g_{im}A_{lhjk} \\ & + g_{jl}A_{mkhi} - g_{jm}A_{lkhi} - g_{kl}A_{mjhi} + g_{km}A_{ljhi}, \end{aligned} \tag{2.8}$$

where $A_{mijk} = S_{mr}g^{rs}R_{sijk}$. We finish this section with the following.

Theorem 2.3. *On any Ricci-pseudosymmetric semi-Riemannian manifold (M, g), $n \geq 4$, the conditions (1.4)–(1.6) hold on $U_S \subset M$.*

Proof. Using (1.3) and (2.3) we obtain the following identities

$$\begin{aligned} (n-2) \sum_{(X_1,X_2),(X_3,X_4),(X,Y)} (R \cdot C)(X_1, X_2, X_3, X_4, X, Y) = {} & (n-2) \sum_{(X_1,X_2),(X_3,X_4),(X,Y)} (R \cdot R)(X_1, X_2, X_3, X_4, X, Y) \\ & - \sum_{(X_1,X_2),(X_3,X_4),(X,Y)} (g \wedge (R \cdot S))(X_1, X_2, X_3, X_4, X, Y), \\ (n-2) \sum_{(X_1,X_2),(X_3,X_4),(X,Y)} (R \cdot C)(X_1, X_2, X_3, X_4, X, Y) = {} & - \sum_{(X_1,X_2),(X_3,X_4),(X,Y)} (g \wedge (R \cdot S))(X_1, X_2, X_3, X_4, X, Y). \end{aligned} \tag{2.9}$$

From this, by making use of (1.2), (2.6) and (2.7), we obtain on U_S

$$\begin{aligned} (n-2) \sum_{(X_1,X_2),(X_3,X_4),(X,Y)} (R \cdot C)(X_1, X_2, X_3, X_4, X, Y) = {} & -L_S \sum_{(X_1,X_2),(X_3,X_4),(X,Y)} (g \wedge Q(g, S))(X_1, X_2, X_3, X_4, X, Y) \\ = {} & L_S \sum_{(X_1,X_2),(X_3,X_4),(X,Y)} Q(S, G)(X_1, X_2, X_3, X_4, X, Y) = 0, \end{aligned}$$

which together with Proposition 2.2 completes the proof.

3. Warped products

Let $(\overline{M}, \overline{g})$ and $(\widetilde{N}, \widetilde{g})$, $\dim \overline{M} = p$, $\dim \widetilde{N} = n - p$, $1 \leq p < n$, be semi-Riemannian manifolds covered by systems of charts $\{\overline{U}; x^a\}$ and $\{\widetilde{V}; y^\alpha\}$, respectively. Let $F : \overline{M} \to \mathbb{R}^+$ be a positive smooth function on $\overline{M}$. The warped product $\overline{M} \times_F \widetilde{N}$ of $(\overline{M}, \overline{g})$ and $(\widetilde{N}, \widetilde{g})$ is the product manifold $\overline{M} \times \widetilde{N}$ with the metric $g = \overline{g} \times_F \widetilde{g}$ defined by $\overline{g} \times_F \widetilde{g} = \pi_1^* \overline{g} + (F \circ \pi_1)\, \pi_2^* \widetilde{g}$, where $\pi_1 : \overline{M} \times \widetilde{N} \longrightarrow \overline{M}$ and $\pi_2 : \overline{M} \times \widetilde{N} \longrightarrow \widetilde{N}$ are the natural projections on $\overline{M}$ and $\widetilde{N}$, respectively. Let $\{\overline{U} \times \widetilde{V}; x^1, \ldots, x^p, x^{p+1} = y^1, \ldots, x^n = y^{n-p}\}$ be a product chart for $\overline{M} \times \widetilde{N}$. The local components of the metric $g = \overline{g} \times_F \widetilde{g}$ with respect to this chart are the following: $g_{hk} = \overline{g}_{ab}$ if $h = a$ and $k = b$, $g_{hk} = F\, \widetilde{g}_{\alpha\beta}$ if $h = \alpha$ and $k = \beta$, and $g_{hk} = 0$ otherwise, where $a, b, c, \ldots \in \{1, \ldots, p\}$, $\alpha, \beta, \gamma, \ldots \in \{p+1, \ldots, n\}$ and $h, i, j, k \ldots \in \{1, 2, \ldots, n\}$. We will denote by bars (resp., by tildes) tensors formed from $\overline{g}$ (resp., $\widetilde{g}$).

The local components R_{hijk} of the curvature tensor R and the local components S_{hk} of the Ricci tensor S of $\overline{M} \times_F \widetilde{N}$ which may not vanish identically are the following (see, e.g., [17,43]):

$$
\begin{aligned}
&R_{abcd} = \overline{R}_{abcd},\\
&R_{\alpha bc\beta} = -\frac{1}{2}\, T_{bc}\, \widetilde{g}_{\alpha\beta},\\
&R_{\alpha\beta\gamma\delta} = F\left(\widetilde{R}_{\alpha\beta\gamma\delta} - \frac{\Delta_1 F}{4F}\, \widetilde{G}_{\alpha\beta\gamma\delta}\right), \qquad (3.1)\\
&S_{ab} = \overline{S}_{ab} - \frac{n-p}{2F}\, T_{ab},\\
&S_{\alpha\beta} = \widetilde{S}_{\alpha\beta} - \left(\frac{\operatorname{tr} T}{2} + (n-p-1)\,\frac{\Delta_1 F}{4F}\right)\widetilde{g}_{\alpha\beta}, \qquad (3.2)\\
&T_{ab} = \overline{\nabla}_b F_a - \frac{1}{2F}\, F_a F_b, \qquad \Delta_1 F = \Delta_{1\overline{g}} F = \overline{g}^{ab} F_a F_b, \qquad F_a = \frac{\partial F}{\partial x^a}. \qquad (3.3)
\end{aligned}
$$

where T denotes the $(0,2)$-tensor with local components T_{ab} and $\operatorname{tr} T = \operatorname{tr}_{\overline{g}} T = \overline{g}^{ab} T_{ab}$. The scalar curvature κ of $\overline{M} \times_F \widetilde{N}$ is expressed by

$$
\kappa = \overline{\kappa} + \frac{\widetilde{\kappa}}{F} - \frac{n-p}{F}\left(\operatorname{tr} T + (n-p-1)\,\frac{\Delta_1 F}{4F}\right). \qquad (3.4)
$$

Using (3.1) and (3.2) we can check that the local components of the tensors $R \cdot S$ and $Q(g,S)$ of $\overline{M} \times_F \widetilde{N}$

$$
\begin{aligned}
&(R\cdot S)_{hijk} = g^{rs}(S_{ri}R_{shjk} + S_{hr}R_{sijk}),\\
&Q(g,S)_{hijk} = g_{hj}S_{ik} + g_{ij}S_{hk} - g_{ik}S_{hj} - g_{hk}S_{ij}, \qquad (3.5)
\end{aligned}
$$

which may not vanish identically are the following (see, e.g., [25])

$$
\begin{aligned}
&(R\cdot S)_{abcd} = (\overline{R}\cdot S)_{abcd} = (\overline{R}\cdot \overline{S})_{abcd} - \frac{n-p}{2F}\,(\overline{R}\cdot T)_{abcd},\\
&(R\cdot S)_{a\alpha\beta b} = \frac{1}{2F}\,(T_{ab}S_{\alpha\beta} - D_{ab}g_{\alpha\beta}), \qquad D_{ab} = \overline{g}^{ef}S_{ae}T_{bf},\\
&(R\cdot S)_{\alpha\beta\gamma\delta} = (\widetilde{R}\cdot \widetilde{S})_{\alpha\beta\gamma\delta} - \frac{\Delta_1 F}{4F}\, Q(\widetilde{g},\widetilde{S})_{\alpha\beta\gamma\delta}, \qquad (3.6)\\
&Q(g,S)_{abcd} = Q(\overline{g},\overline{S})_{abcd} - \frac{n-p}{2F}\, Q(\overline{g},T)_{abcd},\\
&Q(g,S)_{a\alpha\beta b} = g_{\alpha\beta}S_{ab} - g_{ab}S_{\alpha\beta},\\
&Q(g,S)_{\alpha\beta\gamma\delta} = F\, Q(\widetilde{g},\widetilde{S})_{\alpha\beta\gamma\delta}. \qquad (3.7)
\end{aligned}
$$

Let (1.4) holds on $\overline{M} \times_F \widetilde{N}$ and let $\mathcal{U}$ be a coordinate domain of a point $x \in \subset \overline{M} \times_F \widetilde{N}$. Now (1.4), by (1.3) and (2.9), yields (cf., [44, p. 91])

$$
\begin{aligned}
0 \;=\;& g_{ij}(R\cdot S)_{hklm} + g_{hk}(R\cdot S)_{ijlm} - g_{hj}(R\cdot S)_{iklm} - g_{ik}(R\cdot S)_{hjlm} + g_{jm}(R\cdot S)_{klhi} + g_{kl}(R\cdot S)_{jmhi}\\
&- g_{jl}(R\cdot S)_{kmhi} - g_{km}(R\cdot S)_{jlhi} + g_{li}(R\cdot S)_{mhjk} + g_{mh}(R\cdot S)_{lijk} - g_{lh}(R\cdot S)_{mijk} - g_{mi}(R\cdot S)_{lhjk}, \qquad (3.8)\\
&(n-2)(R\cdot S)_{hklm} + g_{kl}V_{hm} - g_{km}V_{lh} + g_{lh}V_{km} - g_{hm}V_{kl}\\
&+ (R\cdot S)_{klhm} + (R\cdot S)_{hmlk} + (R\cdot S)_{mklh} + (R\cdot S)_{hlkm} = 0, \qquad (3.9)\\
&(n-4)\left((R\cdot S)_{hijk} + \frac{1}{n}\, Q(g,V)_{hijk}\right) = 0, \qquad (3.10)
\end{aligned}
$$

where V is a symmetric $(0,2)$-tensor of local components

$$
\begin{aligned}
V_{kl} \;&=\; g^{hj}(R\cdot S)_{hklj} = g^{hj}g^{pq}(S_{pk}R_{qhlj} + S_{hp}R_{qklj})\\
&=\; -S_{kl}^2 + S^{rs}R_{rkls}. \qquad (3.11)
\end{aligned}
$$

We note that

$$
\operatorname{tr} V = g^{rs}V_{rs} = -g^{rs}S_{rs}^2 + g^{kl}S^{rs}R_{rkls} = 0, \qquad (3.12)
$$

and in a consequence

$$
g^{\varepsilon\mu}V_{\varepsilon\mu} = -g^{ef}V_{ef}. \qquad (3.13)
$$

From (3.9), for $h = a$, $k = \alpha$, $l = \beta$ and $m = b$ we get

$$(n-1)\,(R\cdot S)_{a\alpha\beta b} + (R\cdot S)_{b\alpha\beta a} + g_{\alpha\beta}V_{ab} - g_{ab}V_{\alpha\beta} = 0, \tag{3.14}$$
$$(n-1)(R\cdot S)_{b\alpha\beta a} + (R\cdot S)_{a\alpha\beta b} + g_{\alpha\beta}V_{ab} - g_{ab}V_{\alpha\beta} = 0.$$

The last two equations yield

$$(R\cdot S)_{a\alpha\beta b} + (R\cdot S)_{b\alpha\beta a} = -\frac{2}{n}\,(g_{\alpha\beta}V_{ab} - g_{ab}V_{\alpha\beta}).$$

Applying this into (3.14) we find

$$(R\cdot S)_{a\alpha\beta b} = -\frac{1}{n}\,(g_{\alpha\beta}V_{ab} - g_{ab}V_{\alpha\beta}). \tag{3.15}$$

This, in virtue of (3.6), implies

$$n\,T_{ab}S_{\alpha\beta} - 2F\,g_{ab}V_{\alpha\beta} + (2F\,V_{ab} - n\,D_{ab})g_{\alpha\beta} = 0. \tag{3.16}$$

In particular, from (3.16) it follows that

$$D_{ab} = D_{ba}. \tag{3.17}$$

From (3.8), for $h = a$, $i = \alpha$, $j = \beta$, $k = \gamma$, $l = \delta$ and $m = b$, we get

$$g_{ab}(R\cdot S)_{\delta\alpha\beta\gamma} = -\frac{1}{2F}\,T_{ab}Q(g,S)_{\delta\alpha\beta\gamma}. \tag{3.18}$$

Contracting (3.18) with g^{ab} and using (3.6) we find

$$(\widetilde{R}\cdot\widetilde{S})_{\delta\alpha\beta\gamma} = \left(\frac{\Delta_1 F}{4F} - \frac{\operatorname{tr}T}{2p}\right)Q(\widetilde{g},\widetilde{S})_{\delta\alpha\beta\gamma}. \tag{3.19}$$

Now (3.6), by (3.19), turns into

$$(R\cdot S)_{\delta\alpha\beta\gamma} = -\frac{\operatorname{tr}T}{2p}\,Q(\widetilde{g},\widetilde{S})_{\delta\alpha\beta\gamma}. \tag{3.20}$$

Next, applying (3.7) and (3.20) into (3.18), we obtain

$$\left(T_{ab} - \frac{\operatorname{tr}T}{p}\,\overline{g}_{ab}\right)Q(\widetilde{g},\widetilde{S})_{\delta\alpha\beta\gamma} = 0, \tag{3.21}$$

$$\left(T_{ab} - \frac{\operatorname{tr}T}{p}\,\overline{g}_{ab}\right)\left(\widetilde{S}_{\alpha\beta} - \frac{\widetilde{\kappa}}{n-p}\,\widetilde{g}_{\alpha\beta}\right) = 0. \tag{3.22}$$

Further, (3.22) by transvection with $\overline{g}^{af}S_{fd}$, yields

$$\left(D_{ab} - \frac{\operatorname{tr}T}{p}\,S_{ab}\right)\left(\widetilde{S}_{\alpha\beta} - \frac{\widetilde{\kappa}}{n-p}\,\widetilde{g}_{\alpha\beta}\right) = 0. \tag{3.23}$$

From (3.8), for $h = c$, $i = a$, $j = b$, $k = \alpha$, $l = \beta$ and $m = d$, we get

$$g_{\alpha\beta}(R\cdot S)_{bdac} = \frac{1}{2F}\,(S_{\alpha\beta}Q(\overline{g},T)_{bdac} - g_{\alpha\beta}Q(\overline{g},D)_{bdac}). \tag{3.24}$$

Contracting (3.24) with $g^{\alpha\beta}$ we get

$$(R\cdot S)_{bdac} = \frac{1}{2F}\left(\frac{g^{\varepsilon\mu}S_{\varepsilon\mu}}{n-p}\,Q(\overline{g},T)_{bdac} - Q(\overline{g},D)_{bdac}\right). \tag{3.25}$$

We finish this section with the following.

Proposition 3.1. *Let $\overline{M}\times_F\widetilde{N}$ be the warped product with $1\le p\le n-1$. Let $\mathcal{U}\subset U_C\cap U_S\subset\overline{M}\times_F\widetilde{N}$ be a coordinate domain of a point $x\in U_C\cap U_S$. If*

$$(R\cdot S)_{abcd} = 0, \tag{3.26}$$
$$(R\cdot S)_{\alpha\beta\gamma\delta} = 0, \tag{3.27}$$
$$(R\cdot S)_{a\alpha\beta b} = \rho\,g_{ab}g_{\alpha\beta}, \tag{3.28}$$

are satisfied on $\mathcal{U}$, with some function ρ, then (1.4) holds on this set.

Proof. Using (3.26)–(3.28) we can check that (3.8) holds on $\mathcal{U}$. Now (2.9), by (3.8), reduces to (1.4).

4. Generalized Robertson–Walker spacetimes

We consider the warped product manifold $\overline{M} \times_F \widetilde{N}$ with an 1-dimensional base $(\overline{M}, \overline{g})$, $\overline{g}_{11} = \varepsilon = \pm 1$, and an $(n-1)$-dimensional Einsteinian fibre $(\widetilde{N}, \widetilde{g})$, $n \geq 5$, not of constant curvature. It is well-known if the fibre $(\widetilde{N}, \widetilde{g})$, $\dim \widetilde{N} = 3$, is an Einsteinian manifold then it is also of constant curvature and as a consequence $\overline{M} \times_F \widetilde{N}$ is conformally flat. We also mention that the manifold $\overline{M} \times_F \widetilde{N}$, $\dim \overline{M} = 1$, $\dim \widetilde{N} = 3$, satisfies [45, Theorem 4.1]

$$R \cdot R - Q(S, R) = \frac{\operatorname{tr} T}{F} Q(g, C). \tag{4.1}$$

Let $x \in U_C \cap U_S \subset \overline{M} \times_F \widetilde{N}$. Now on a coordinate domain $\mathcal{U} \subset U_C \cap U_S$ of x, Eqs. (3.1), (3.2) and (3.4) turn into

$$\begin{aligned}
&R_{\alpha 11 \beta} = -\frac{1}{2} T_{11} g^{11} g_{11} \widetilde{g}_{\alpha\beta} = L_S g_{11} g_{\alpha\beta},\\
&R_{\alpha\beta\gamma\delta} = F \left(\widetilde{R}_{\alpha\beta\gamma\delta} - \frac{\Delta_1 F}{4F^2} G_{\alpha\beta\gamma\delta} \right),\\
&S_{11} = -\frac{n-1}{2F} T_{11} = -\frac{n-1}{2F} T_{11} g^{11} g_{11}\\
&\quad = (n-1) L_S g_{11},\\
&S_{\alpha\beta} = \left(\frac{\widetilde{\kappa}}{(n-1)F} + L_S - (n-2) \frac{\Delta_1 F}{4F^2} \right) g_{\alpha\beta},\\
&\frac{\kappa}{n-1} - L_S = \frac{\widetilde{\kappa}}{(n-1)F} + L_S - (n-2) \frac{\Delta_1 F}{4F^2},\\
&L_S = -\frac{\operatorname{tr} T}{2F},
\end{aligned} \tag{4.3}$$

respectively. Further (4.3) yields

$$S_{hk} - \left(\frac{\kappa}{n-1} - L_S \right) g_{hk} = \varepsilon n \left(L_S - \frac{\kappa}{(n-1)n} \right) w_h w_k, \tag{4.4}$$

where $w_1 = 1$ and $w_2 = w_3 = \cdots = w_n = 0$ are the local components of an 1-form w on $\mathcal{U}$. Thus (1.7) holds on $\mathcal{U}$. Further, using (4.2) and (4.3) we can check that on $\mathcal{U}$ we have

$$\begin{aligned}
S_{hr} g^{rs} R_{sijk} = &\left(\frac{\kappa}{n-1} - L_S \right) (R_{hijk} - L_S G_{hijk})\\
&+ L_S (g_{ij} S_{hk} - g_{ik} S_{hj}),
\end{aligned} \tag{4.5}$$

which by (3.5) yields (1.2). Moreover, (2.8), by making use of (1.2), (2.6) and (4.5), turns into (1.8). We remark that in view of Proposition 3.1 of [41], (1.8) is an immediate consequence of (4.5).

The above presented results, Corollary 3.2 of [26] and Theorem 2.3 yield the following.

Theorem 4.1. *Let $\overline{M} \times_F \widetilde{N}$ be the warped product manifold with an 1-dimensional base $(\overline{M}, \overline{g})$, $\overline{g}_{11} = \varepsilon = \pm 1$, and an $(n-1)$-dimensional Einsteinian fibre $(\widetilde{N}, \widetilde{g})$, $n \geq 3$. Then we have:* (i) *the conditions (1.2), with L_S defined by (4.3), (1.4) and (1.7) hold on $U_S \subset \overline{M} \times_F \widetilde{N}$,* (ii) *the condition (1.8) holds on $U_C \cap U_S \subset \overline{M} \times_F \widetilde{N}$, provided that $(\widetilde{N}, \widetilde{g})$ is not of constant curvature and $n \geq 5$.*

From the last result it follows immediately the following theorem.

Theorem 4.2. *Let $\overline{M} \times_F \widetilde{N}$ be the generalized Robertson–Walker spacetime with an $(n-1)$-dimensional Riemannian, not of constant curvature, Einsteinian fibre $(\widetilde{N}, \widetilde{g})$, $n \geq 5$. Then (1.2), with L_S defined by (4.3), (1.4), (1.7) and (1.8) hold on $U_C \cap U_S \subset \overline{M} \times_F \widetilde{N}$,*

Remark 4.3. In [4, Proposition 2.1] it was proved that on every quasi-Einstein hypersurface in a semi-Riemannian space of constant curvature the conditions: (1.2), (1.7) and (1.8) are satisfied. Moreover, an example of a quasi-Einstein non-pseudosymmetric Ricci-pseudosymmetric warped product, of dimension ≥ 5, which locally can be realized as a hypersurface in a semi-Riemannian space of constant curvature is given in [4, Example 4.1].

We consider now the warped product manifold $\overline{M} \times_F \widetilde{N}$ with an 1-dimensional base $(\overline{M}, \overline{g})$, $\overline{g}_{11} = \varepsilon = \pm 1$, and an $(n-1)$-dimensional non-Einsteinian fibre $(\widetilde{N}, \widetilde{g})$, $n \geq 4$. We determine the necessary and sufficient conditions for such warped product to be a quasi-Einstein manifold. Let W_{hijk} be the local components of the $(0, 4)$-tensor W defined on $\overline{M} \times_F \widetilde{N}$ by

$$W = \frac{1}{2} (S - \tau_1 g) \wedge (S - \tau_1 g), \quad \tau_1 = -\frac{n-1}{2F} \operatorname{tr} T. \tag{4.6}$$

Applying in (4.6) the relations (3.2) and (3.3) we obtain on a coordinate domain $\mathcal{U} \subset U_S \subset \overline{M} \times {}_F\widetilde{N}$ of a point $x \in U_S$

$$\begin{aligned} W_{1\alpha\beta 1} &= S_{11}S_{\alpha\beta} - \tau_1 (g_{11}S_{\alpha\beta} + g_{\alpha\beta}S_{11}) + \tau_1^2 g_{11}g_{\alpha\beta} \\ &= (S_{11} - \tau_1 g_{11})(S_{\alpha\beta} - \tau_1 g_{\alpha\beta}) \\ &= -\varepsilon \left(\tau_1 + \frac{n-1}{2F}\operatorname{tr} T\right)(\widetilde{S}_{\alpha\beta} - \tau_2 \widetilde{g}_{\alpha\beta}) = 0, \end{aligned} \tag{4.7}$$

$$\begin{aligned} W_{\alpha\beta\gamma\delta} &= S_{\alpha\delta}S_{\gamma\beta} - S_{\alpha\gamma}S_{\beta\delta} - \tau_1 (g \wedge S)_{\alpha\beta\gamma\delta} + \tau_1^2 G_{\alpha\beta\gamma\delta} \\ &= \widetilde{S}_{\alpha\delta}\widetilde{S}_{\gamma\beta} - \widetilde{S}_{\alpha\gamma}\widetilde{S}_{\beta\delta} - \left(\tau_1 F + \frac{1}{2}\operatorname{tr} T + \frac{(n-2)\varepsilon}{4F}(F')^2\right)(\widetilde{g} \wedge \widetilde{S})_{\alpha\beta\gamma\delta} \\ &\quad + \left((\tau_1 F)^2 + 2\tau_1 F\left(\frac{1}{2}\operatorname{tr} T + \frac{(n-2)\varepsilon}{4F}(F')^2\right) + \left(\frac{1}{2}\operatorname{tr} T + \frac{(n-2)\varepsilon}{4F}(F')^2\right)^2\right)\widetilde{G}_{\alpha\beta\gamma\delta} \\ &= (\widetilde{S}_{\alpha\delta} - \tau_2 \widetilde{g}_{\alpha\delta})(\widetilde{S}_{\beta\gamma} - \tau_2 \widetilde{g}_{\beta\gamma}) - (\widetilde{S}_{\alpha\gamma} - \tau_2 \widetilde{g}_{\alpha\gamma})(\widetilde{S}_{\beta\delta} - \tau_2 \widetilde{g}_{\beta\delta}), \end{aligned} \tag{4.8}$$

$$\tau_2 = \tau_1 F + \frac{1}{2}\operatorname{tr} T + \frac{(n-2)\varepsilon}{4F}(F')^2 = -\frac{(n-2)\varepsilon}{2F}(FF'' - (F')^2), \tag{4.9}$$

where $F'' = \frac{dF'}{dx^1}$ and $F' = \frac{dF}{dx^1}$.

Theorem 4.4. *Let $\overline{M} \times {}_F\widetilde{N}$, $\dim \overline{M} = 1$, $\overline{g}_{11} = \varepsilon = \pm 1$ and $\dim \widetilde{N} = n - 1 \geq 3$, be the warped product manifold such that the tensor $\widetilde{S} - \frac{\widetilde{\kappa}}{n-p}\widetilde{g}$ is non-zero at every point of $U_S \subset \overline{M} \times {}_F\widetilde{N}$.*

(i) *If*

$$\operatorname{rank}(S - \tau_1 g) = 1 \tag{4.10}$$

holds on U_S, where τ_1 is defined by (4.6), then on this set we have

$$\operatorname{rank}(\widetilde{S} - (n-2)C_1\widetilde{g}) = 1, \quad C_1 = \text{const.}, \tag{4.11}$$

$$FF'' - (F')^2 + 2\varepsilon C_1 F = 0. \tag{4.12}$$

(ii) *If (4.11) and (4.12) are satisfied on U_S then (4.10) holds on this set.*

Proof. (i) (4.10), in view of Lemma 2.1(ii), is equivalent to $W = 0$. Thus, by (4.8), we obtain on a coordinate domain $\mathcal{U} \subset U_S$ of a point $x \in U_S$

$$(\widetilde{S}_{\alpha\delta} - \tau_2 \widetilde{g}_{\alpha\delta})(\widetilde{S}_{\beta\gamma} - \tau_2 \widetilde{g}_{\beta\gamma}) - (\widetilde{S}_{\alpha\gamma} - \tau_2 \widetilde{g}_{\alpha\gamma})(\widetilde{S}_{\beta\delta} - \tau_2 \widetilde{g}_{\beta\delta}) = 0, \tag{4.13}$$

$$\widetilde{S}_{\alpha\delta}\widetilde{S}_{\beta\gamma} - \widetilde{S}_{\alpha\gamma}\widetilde{S}_{\beta\delta} - \tau_2 (\widetilde{g} \wedge \widetilde{S})_{\alpha\beta\gamma\delta} + \tau_2^2 \widetilde{G}_{\alpha\beta\gamma\delta} = 0, \tag{4.14}$$

where τ_2 is defined by (4.9). Contracting (4.14) with $\widetilde{g}^{\alpha\delta}$ and $\widetilde{g}^{\beta\gamma}$ we get

$$\tau_2^2 = \frac{1}{(n-2)(n-1)}(\operatorname{tr}\widetilde{S}^2 - \widetilde{\kappa}^2 + 2(n-2)\widetilde{\kappa}\tau_2).$$

Therefore we can express (4.14) in the form

$$\widetilde{S}_{\alpha\delta}\widetilde{S}_{\beta\gamma} - \widetilde{S}_{\alpha\gamma}\widetilde{S}_{\beta\delta} + \frac{\operatorname{tr}\widetilde{S}^2 - \widetilde{\kappa}^2}{(n-2)(n-1)}\widetilde{G}_{\alpha\beta\gamma\delta} = \tau_2 \left(\widetilde{g} \wedge \left(\widetilde{S} - \frac{\widetilde{\kappa}}{n-1}\widetilde{g}\right)\right)_{\alpha\beta\gamma\delta},$$

which means that on U_S we have

$$\frac{1}{2}\widetilde{S} \wedge \widetilde{S} + \frac{\operatorname{tr}\widetilde{S}^2 - \widetilde{\kappa}^2}{(n-2)(n-1)}\widetilde{G} = \tau_2 \widetilde{g} \wedge \left(\widetilde{S} - \frac{\widetilde{\kappa}}{n-1}\widetilde{g}\right). \tag{4.15}$$

From (4.15), via (4.9), it follows that τ_2 must be a constant, i.e. for instance, on U_S we have $\tau_2 = (n-2)C_1$, where C_1 is a constant. Therefore (4.9) turns into (4.12) (cf., [21, Eq. (36)]). Now we can present (4.13) in the form $\frac{1}{2}(\widetilde{S} - (n-2)C_1\widetilde{g}) \wedge (\widetilde{S} - (n-2)C_1\widetilde{g}) = 0$. This, in view of Lemma 2.1(ii), completes the proof of (i).

(ii) Let W be the tensor defined by (4.6). By making use of (4.11) and (4.12) we get $W_{\alpha\beta\gamma\delta} = 0$. Thus on U_S we have $W = 0$, which in view of Lemma 2.1(ii) and (iii), leads to (4.10), completing the proof of (ii).

Remark 4.5. The following functions satisfy (4.12) (cf., [21, Eq. (37)])

$$F(x^1) = \varepsilon C_1 \left(x^1 + \frac{\varepsilon c}{C_1}\right)^2, \quad \varepsilon C_1 > 0,$$
$$F(x^1) = \frac{c}{2}\left(\exp\left(\pm\frac{b}{2}x^1\right) - \frac{2\varepsilon C_1}{b^2 c}\exp\left(\mp\frac{b}{2}x^1\right)\right)^2, \quad c > 0,\ b \neq 0,$$
$$F(x^1) = \frac{2\varepsilon C_1}{c^2}(1 + \sin(cx^1 + b)), \quad \varepsilon C_1 > 0,\ c \neq 0, \tag{4.16}$$

where b and c are constants and x^1 belongs to a suitable non-empty open interval of $\mathbb{R}$.

5. An example of a warped product satisfying (1.9)

In this section we present an example of a Ricci-pseudosymmetric warped product, with the base manifold of dimension $2 \leq p \leq n-2$, and Einsteinian fibre, for which the tensor $R \cdot C - C \cdot R$ is expressed by a linear combination of the tensors $Q(S, R)$ and $Q(g, R)$, precisely, (1.9) is satisfied.

Example 5.1. (i) Let $(\overline{M}, \overline{g})$ be a 2-dimensional semi-Riemannian manifold or a semi-Riemannian of constant curvature of dimension ≥ 3, i.e. $\dim \overline{M} = p \geq 2$, and let F be a smooth positive function on $\overline{M}$ satisfying

$$T = \frac{\operatorname{tr} T}{p}\,\overline{g}, \tag{5.1}$$

where the $(0, 2)$-tensor T is defined by (3.3). Then Eq. (5.1) yields

$$0 = \frac{1}{2}\left(T - \frac{\operatorname{tr} T}{p}\,\overline{g}\right) = \frac{1}{2}\,T - \frac{\operatorname{tr} T}{2pF}\,F\,\overline{g} = \frac{1}{2}\,T + FL_S\,\overline{g},$$
$$L_S = -\frac{\operatorname{tr} T}{2pF}. \tag{5.2}$$

Further, let $(\widetilde{N}, \widetilde{g})$ be a 2-dimensional semi-Riemannian manifold or a semi-Riemannian Einstein manifold of dimension ≥ 3, i.e. $\dim \widetilde{N} = n - p \geq 2$. We consider the warped product $\overline{M} \times_F \widetilde{N}$. Let $x \in U_C \cap U_S \subset \overline{M} \times_F \widetilde{N}$ and let $\mathcal{U} \subset U_C \cap U_S$ be a coordinate domain of x. We have on $\mathcal{U}$

$$\overline{R} = \frac{\overline{\kappa}}{(p-1)p}\,\overline{G}, \qquad \widetilde{S} = \frac{\widetilde{\kappa}}{n-p}\,\widetilde{g}. \tag{5.3}$$

Now (3.2), by making use of (5.2) and (5.3), yields

$$S_{ab} = c_1\,\overline{g}_{ab}, \qquad c_1 = \frac{\overline{\kappa}}{p} - \frac{(n-p)\operatorname{tr} T}{2pF} = \frac{\overline{\kappa}}{p} + (n-p)L_S,$$
$$S_{\alpha\beta} = c_2\,g_{\alpha\beta},$$
$$c_2 = \frac{\widetilde{\kappa}}{(n-p)F} - \frac{\operatorname{tr} T}{2F} - (n-p-1)\,\frac{\Delta_1 F}{4F^2}$$
$$= \frac{\widetilde{\kappa}}{(n-p)F} - (n-p-1)\,\frac{\Delta_1 F}{4F^2} + p\,L_S. \tag{5.4}$$

Using (5.4) we obtain

$$S_{ae}g^{ef}R_{fbcd} = c_1\,R_{abcd}, \qquad S_{ae}g^{ef}R_{f\alpha\beta b} = c_1\,R_{a\alpha\beta b},$$
$$S_{\alpha\varepsilon}g^{\varepsilon\mu}R_{\mu ab\beta} = c_2\,R_{\alpha ab\beta}, \qquad S_{\alpha\varepsilon}g^{\varepsilon\mu}R_{\mu\beta\gamma\delta} = c_2\,R_{\alpha\beta\gamma\delta}. \tag{5.5}$$

Applying (5.2) into (3.1) we obtain

$$R_{\alpha bc\beta} = -\frac{1}{2F}\,T_{bc}\,g_{\alpha\beta} = L_S\,g_{bc}g_{\alpha\beta} = L_S\,G_{\alpha bc\beta}. \tag{5.6}$$

We note that (5.5) and (5.6) imply (3.26)–(3.28).

Let W be the $(0, 4)$-tensor on $U_C \cap U_S$ with the local components W_{hijk} defined by

$$W_{hijk} = -S_{hr}g^{rs}R_{sijk} + c_2\,(R_{hijk} - L_S\,G_{hijk}) + L_S\,(g_{ij}S_{hk} - g_{ik}S_{hj}).$$

Using (5.3)–(5.6), we can easily check that

$$W_{abcd} = (c_2 - c_1)\left(\frac{\overline{\kappa}}{(p-1)p} - L_S\right) G_{abcd}, \tag{5.7}$$

and that other local components of W vanish identically on $\mathcal{U}$.

(ii) Under some additional assumptions imposed on the base manifold $(\overline{M}, \overline{g})$ the tensor W is a zero tensor. Let $S^p(r)$, $p \geq 2$, be the p-dimensional standard sphere with the standard metric $\overline{g}$ and $\overline{\nabla}$ its Levi-Civita connection. Let f be a non-zero function defined on $S^p(r)$ satisfying the equation [46]

$$\overline{\nabla}(df) + \frac{1}{r^2} f\, \overline{g} = 0. \tag{5.8}$$

Let $\overline{M}$ be a connected open subset of $S^p(r)$ such that $f > 0$ at every point of this set. We define on $\overline{M}$ the function F by $F = f^2$ (cf., Examples 3.1 and 3.2 of [27]). We can check that the tensor T, defined by (3.3), satisfies $\frac{1}{2} T + \frac{1}{r^2} F\, \overline{g} = 0$. Since $\frac{1}{r^2} = \frac{\overline{\kappa}}{(p-1)p}$, the last equation and (5.2) give $\frac{1}{2} T + F L_S\, \overline{g} = 0$ and

$$L_S = \frac{\overline{\kappa}}{(p-1)p}, \qquad \frac{1}{2} \operatorname{tr} T = -pFL_S. \tag{5.9}$$

Thus (5.7) reduces to $W_{abcd} = 0$, and in a consequence, on $\mathcal{U}$ we have

$$S_{hr} g^{rs} R_{sijk} = c_2\,(R_{hijk} - L_S\, G_{hijk}) + L_S\,(g_{ij} S_{hk} - g_{ik} S_{hj}). \tag{5.10}$$

Thus (1.2) holds on $\mathcal{U}$. Now (5.4), by making use of (3.4) and (5.9), yield

$$c_1 = (n-1)L_S, \qquad c_2 = \frac{1}{n-p}(\kappa - (n-1)pL_S). \tag{5.11}$$

Further, using (2.3), (5.3), (5.4), (5.6), (5.9) and (5.11), we can express the local components C_{hijk} of the Weyl conformal tensor C of $\overline{M} \times_F \widetilde{N}$ which may not vanish identically by

$$\begin{aligned}
C_{abcd} &= \frac{n}{n-2}\left(\frac{\kappa}{(n-1)n} - L_S\right) G_{abcd},\\
C_{a\alpha\beta b} &= -\frac{(p-1)n}{(n-p)(n-2)}\left(\frac{\kappa}{(n-1)n} - L_S\right) G_{a\alpha\beta b},\\
C_{\alpha\beta\gamma\delta} &= R_{\alpha\beta\gamma\delta} - \frac{1}{(n-p)(n-2)}\left(\frac{(n+p-2)\kappa}{n-1} - 2(n-1)pL_S\right) G_{\alpha\beta\gamma\delta}.
\end{aligned}$$

If $c_1 \neq c_2$ at a point $x \in \overline{M} \times_F \widetilde{N}$ then $S - \frac{\kappa}{n} g$ and C are non-zero at this point (cf. [27, Examples 3.1 and 3.2]). This means that $U_C \cap U_S \subset \overline{M} \times_F \widetilde{N}$ is a non-empty set. On this set we have: (1.2) with L_S given by (5.9), $S_{ab} = c_1 g_{ab}$ and $S_{\alpha\beta} = c_2 g_{\alpha\beta}$ with c_1 and c_2 given by (5.11). In addition, (5.10) and (5.11), in view of Proposition 3.1 of [41], lead to (1.9). This, by Proposition 2.2, yields (1.4).

6. Warped products satisfying (1.4)

Let $\overline{M} \times_F \widetilde{N}$ be the warped product with a p-dimensional base $(\overline{M}, \overline{g})$, $2 \leq p \leq n-3$, and an $(n-p)$-dimensional Einsteinian fibre $(\widetilde{N}, \widetilde{g})$, which is not of constant curvature. Moreover, let (1.4) be satisfied on $U_C \cap U_S \subset \overline{M} \times_F \widetilde{N}$ and let $\mathcal{U} \subset U_C \cap U_S$ be a coordinate domain of $x \in U_C \cap U_S$. Since $n \geq 5$, (3.10) reduces on $\mathcal{U}$ to

$$(R \cdot S)_{hijk} = -\frac{1}{n} Q(g, V)_{hijk}. \tag{6.1}$$

From (3.9), for $h = a$, $k = b$, $l = c$ and $m = d$ we get

$$(n-2)\,(R \cdot S)_{abcd} + g_{bc} V_{ad} - g_{bd} V_{ca} + g_{ac} V_{bd} - g_{ad} V_{bc} + (R \cdot S)_{cbad} + (R \cdot S)_{adcb} + (R \cdot S)_{dbca} + (R \cdot S)_{acbd} = 0.$$

Thus, by making use of (3.10), we obtain

$$2(n-1)\, Q(g, V)_{abcd} + Q(g, V)_{acbd} + Q(g, V)_{cbad} + Q(g, V)_{adcb} + Q(g, V)_{dbca} = 0.$$

Contracting this with g^{ad} we find

$$V_{ab} = \frac{g^{ef} V_{ef}}{p} g_{ab}. \tag{6.2}$$

From (6.1) we also have $(R\cdot S)_{abcd} = -\frac{1}{n}\,Q(g,V)_{abcd}$, which by an application of (6.2), reduces to (3.26), which together with (3.6) yields

$$\left(\overline{R}\cdot(\overline{S}-\frac{n-p}{2F}\,T)\right)_{abcd}=0. \tag{6.3}$$

We note that (3.6) and (5.3) imply (3.27). Contracting (3.16) with g^{ab} and using (6.2) we find

$$V_{\alpha\beta}=\frac{g^{\varepsilon\mu}V_{\varepsilon\mu}}{n-p}\,g_{\alpha\beta}. \tag{6.4}$$

By our assumptions we have

$$S_{\alpha\beta}=c_3\,g_{\alpha\beta},\quad c_3=\frac{\widetilde{\kappa}}{(n-p)F}-\frac{\operatorname{tr}T}{2F}-(n-p-1)\,\frac{\Delta_1 F}{4F^2}. \tag{6.5}$$

Applying now (6.2), (6.4) and (6.5) into (3.16) we obtain

$$T_{ab}S_{\alpha\beta}-D_{ab}g_{\alpha\beta}=\frac{2F}{n}\left(\frac{g^{\varepsilon\mu}V_{\varepsilon\mu}}{n-p}-\frac{g^{ef}V_{ef}}{p}\right)g_{ab}g_{\alpha\beta},$$

$$c_3\,T_{ab}-D_{ab}=\frac{2F}{n}\left(\frac{g^{\varepsilon\mu}V_{\varepsilon\mu}}{n-p}-\frac{g^{ef}V_{ef}}{p}\right)g_{ab}.$$

Now (3.6) yields (3.28) with $\rho=\frac{1}{n}\left(\frac{g^{\varepsilon\mu}V_{\varepsilon\mu}}{n-p}-\frac{g^{ef}V_{ef}}{p}\right)$.

The above presented considerations and Proposition 3.1 lead to the following.

Proposition 6.1. *Let $\overline{M}\times_F\widetilde{N}$, $2\le p\le n-3$, be the warped product manifold with Einsteinian fibre $(\widetilde{N},\widetilde{g})$ which is not of constant curvature. Let $\mathcal{U}\subset U_C\cap U_S\subset\overline{M}\times_F\widetilde{N}$ be a coordinate domain of a point $x\in U_C\cap U_S$. If (1.4) is satisfied on $\mathcal{U}$ then (3.26)–(3.28) hold on this set. Conversely, if (3.26)–(3.28) are satisfied on $\mathcal{U}$ then (1.4) holds on this set.*

We note that the warped product manifold constructed in the previous section satisfies (1.4) and (3.26)–(3.28).

We consider now warped products satisfying (1.4) with non-Einstein fibre.

Proposition 6.2. *Let (1.4) be satisfied on $\overline{M}\times_F\widetilde{N}$, $1\le p\le n-3$. If at $x\in U_S\subset\overline{M}\times_F\widetilde{N}$ the tensor $\widetilde{S}-\frac{\widetilde{\kappa}}{n-p}\,\widetilde{g}$ is non-zero then on some open set $\mathcal{U}\subset\overline{M}\times_F\widetilde{N}$ we have*

$$\text{(a)}\ \frac{\Delta_1 F}{4F}-\frac{\operatorname{tr}T}{2p}=\text{const.},\qquad \text{(b)}\ R\cdot S=-\frac{\operatorname{tr}T}{2pF}\,Q(g,S). \tag{6.6}$$

In addition, if $n=4$ then (4.1) and

$$R\cdot R=-\frac{\operatorname{tr}T}{2F}\,Q(g,R), \tag{6.7}$$

hold on $\mathcal{U}$. Moreover, at every point of $\mathcal{U}$ we have

$$\operatorname{rank}(\widetilde{S}-\tau\,\widetilde{g})=1,\quad \tau\in\mathbb{R}. \tag{6.8}$$

Proof. Without loss of generality we can assume that $\mathcal{U}$ is a coordinate domain of x. Since at every point of $\mathcal{U}$ we have $\widetilde{S}\neq\frac{\widetilde{\kappa}}{n-p}\,\widetilde{g}$, (3.19) implies immediately (6.6)(a). Next, using (3.22) and (3.23) we obtain on $\mathcal{U}$

$$T=\frac{\operatorname{tr}T}{p}\,\overline{g},\qquad D=\frac{\operatorname{tr}T}{p}\,S. \tag{6.9}$$

Now (3.16) turns into

$$-\frac{1}{n}\left(g_{\alpha\beta}V_{ab}-g_{ab}V_{\alpha\beta}\right)=\frac{\operatorname{tr}T}{2pF}\left(g_{ab}S_{\alpha\beta}-g_{\alpha\beta}S_{ab}\right).$$

Applying this into (3.15) we find $(R\cdot S)_{a\alpha\beta b}=-\frac{\operatorname{tr}T}{2pF}\,Q(g,S)_{a\alpha\beta b}$. Clearly, (3.18) takes the form $(R\cdot S)_{\delta\alpha\beta\gamma}=-\frac{\operatorname{tr}T}{2pF}\,Q(g,S)_{\delta\alpha\beta\gamma}$. In addition, (3.25) by making use of (6.9), gives $(R\cdot S)_{bdac}=-\frac{\operatorname{tr}T}{2pF}\,Q(g,S)_{bdac}$. Thus (6.6)(b) holds on $\mathcal{U}$.

We consider now the case $n=4$. Therefore, we also have $p=1$. (6.6)(b), in view of Proposition 4 of [16], implies (6.7). As it was mentioned in Section 4, (4.1) is satisfied on every 4-dimensional warped product manifold with 1-dimensional base. Now (6.7) and (4.1), together with (2.5), lead to

$$Q\left(S+\frac{3\operatorname{tr}T}{2F}\,g,R+\frac{\operatorname{tr}T}{2F}\,G\right)=0. \tag{6.10}$$

If rank $(S+\frac{3\operatorname{tr}T}{2F}g)=1$ at a point $x\in\mathcal{U}$ then (3.2) turns into (6.8). Let now rank $(S+\frac{3\operatorname{tr}T}{2F}g)>1$ at a point $x\in\mathcal{U}$. Then at this point we have (see, e.g. [2, Lemma 3.1])

$$R+\frac{\operatorname{tr}T}{2F}G=\lambda\left(S+\frac{3\operatorname{tr}T}{2F}g\right)\wedge\left(S+\frac{3\operatorname{tr}T}{2F}g\right),\quad \lambda\in\mathbb{R}\setminus\{0\}. \tag{6.11}$$

Now (3.1), (3.2) and (6.11) lead to

$$\widetilde{R}_{\alpha\beta\gamma\delta}=\tau_3\,(\widetilde{g}_{\alpha\delta}\widetilde{S}_{\beta\gamma}+\widetilde{g}_{\beta\gamma}\widetilde{S}_{\alpha\delta}-\widetilde{g}_{\alpha\gamma}\widetilde{S}_{\beta\delta}-\widetilde{g}_{\beta\delta}\widetilde{S}_{\alpha\gamma})+\frac{\lambda}{F}\,(\widetilde{S}_{\alpha\delta}\widetilde{S}_{\beta\gamma}-\widetilde{S}_{\alpha\gamma}\widetilde{S}_{\beta\delta})+\tau_4\,\widetilde{G}_{\alpha\beta\gamma\delta},\qquad \tau_3,\,\tau_4\in\mathbb{R}, \tag{6.12}$$

which by the identity

$$\widetilde{C}_{\alpha\beta\gamma\delta}=\widetilde{R}_{\alpha\beta\gamma\delta}-(\widetilde{g}_{\alpha\delta}\widetilde{S}_{\beta\gamma}+\widetilde{g}_{\beta\gamma}\widetilde{S}_{\alpha\delta}-\widetilde{g}_{\alpha\gamma}\widetilde{S}_{\beta\delta}-\widetilde{g}_{\beta\delta}\widetilde{S}_{\alpha\gamma})+\frac{\widetilde{\kappa}}{2}\,\widetilde{G}_{\alpha\beta\gamma\delta}=0,$$

turns into

$$\widetilde{S}_{\alpha\delta}\widetilde{S}_{\beta\gamma}-\widetilde{S}_{\alpha\gamma}\widetilde{S}_{\beta\delta}=\tau_1\,(\widetilde{g}_{\alpha\delta}\widetilde{S}_{\beta\gamma}+\widetilde{g}_{\beta\gamma}\widetilde{S}_{\alpha\delta}-\widetilde{g}_{\alpha\gamma}\widetilde{S}_{\beta\delta}-\widetilde{g}_{\beta\delta}\widetilde{S}_{\alpha\gamma})+\tau_2\,\widetilde{G}_{\alpha\beta\gamma\delta},\quad \tau_1,\,\tau_2\in\mathbb{R}. \tag{6.13}$$

From this, in view of Lemma 3.1 of [47], it follows that (6.8) holds at x, which completes the proof.

As an immediate consequence of Proposition 6.2 we have the following.

Theorem 6.3. *Let (1.4) be satisfied on $\overline{M}\times_F\widetilde{N}$, $1\le p\le n-3$. Then we have:* (i) *if at every $x\in U_S\subset\overline{M}\times_F\widetilde{N}$ the tensor $\widetilde{S}-\frac{\widetilde{\kappa}}{n-p}\widetilde{g}$ is non-zero then $\overline{M}\times_F\widetilde{N}$ is a Ricci-pseudosymmetric manifold,* (ii) *if $n=4$ and at every $x\in U_S\subset\overline{M}\times_F\widetilde{N}$ the tensor $\widetilde{S}-\frac{\widetilde{\kappa}}{3}\widetilde{g}$ is non-zero then $\overline{M}\times_F\widetilde{N}$ is a pseudosymmetric manifold and $(\widetilde{N},\widetilde{g})$ is a quasi-Einstein manifold.*

Acknowledgements

R. Deszcz and M. Głogowska are supported by a grant of the Wroclaw University of Environmental and Life Science (Poland). R. Deszcz is also supported by a grant of the Technische Universität Berlin (Germany).

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