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Infinitesimal deformations of naturally graded filiform Leibniz algebras

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ABSTRACT

In the present paper we describe infinitesimal deformations of complex naturally graded filiform Leibniz algebras. It is known that any n-dimensional filiform Lie algebra can be obtained by a linear integrable deformation of the naturally graded algebra $F_n^3(0)$. We establish that in the same way any n-dimensional filiform Leibniz algebra can be obtained by an infinitesimal deformation of the filiform Leibniz algebras F_n^1, F_n^2 and $F_n^3(\alpha)$. Moreover, we describe the linear integrable deformations of the above-mentioned algebras with a fixed basis of HL^2 in the set of all n-dimensional Leibniz algebras. Among these deformations one new rigid algebra has been found.

1. Introduction

Deformations of different algebraic and analytic objects are important aspect if one studies their properties. They characterize the local behavior in a small neighborhood in the variety of a given type objects. A geometric picture of the deformations is obtained by considering the variety M of all those bilinear maps (products) of the underlying vector space into itself which satisfy the conditions defining the variety. An algebra structure of the variety represents a point m of M, and deformations of the structure are represented by points of M near to m. Thus the study of the deformations of these algebras is a special case of the study of the local geometric properties of varieties.

Classical deformation theory of associative and Lie algebras began with the works of Gerstenhaber [1] and Nijenhuis–Richardson [2] in the 1960s. They studied the one-parameter deformations and established the connection between the cohomology and infinitesimal deformations of Lie algebra. After these works formal deformation theory was generalized in different categories. In fact, in the last fifty years, the deformation theory has played an important role in algebraic geometry. The main goal is the classification of families of geometric objects when the classifying space (the so called moduli space) is a reasonable geometric space. In particular, each point of our moduli space corresponds to one geometric object (class of isomorphism). The theory of deformations is one of the most effective approaches in the investigation of solvable and nilpotent Lie algebras (see for example, [3–6]).

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In this paper we study the infinitesimal deformations of some nilpotent Leibniz algebras. Recall, that Leibniz algebras are a generalization of Lie algebras [7,8] and it is natural to apply the theory of deformations to the study of these algebras. Particularly, the problems which were studied in [3,5,6] and others can be considered from the point of view of Leibniz algebras.

According to algebraic geometry an algebraic variety is a union of irreducible components. The closures of orbits of rigid algebras give the irreducible components of the variety. That is why the finding of rigid algebras is a crucial problem from the geometrical point of view.

Due to [9] we can apply the general principles for deformations and rigidity of Leibniz algebras. Namely, it is proved that nullity of the second cohomology group ($HL^2(L, L) = 0$) gives a sufficient condition for rigidity. In addition, it is established that Leibniz algebras for which every formal deformation is equivalent to a trivial deformation are rigid.

One of the inherent properties of finite-dimensional Leibniz algebras consists of the existence of nilpotent single-generated Leibniz algebras (so-called null-filiform algebras), which are Leibniz algebras of maximal nilindex. It is known that in each dimension all of those algebras are isomorphic to the algebra NF_n [10] and this algebra is rigid in the variety of n-dimensional nilpotent Leibniz algebras (denoted by $Leib_n$). In [11] infinitesimal deformations of the algebra NF_n are studied. It was proved that any single-generated Leibniz algebra (which is solvable) is a linear integrable deformation of NF_n. Moreover, it is shown that the closure of the set of all single generated Leibniz algebras forms an irreducible component of $Leib_n$.

Firstly the notion of filiform algebra was introduced by M. Vergne in [12] as an algebra of maximal nilindex in the variety of Lie algebras. Namely, naturally graded filiform Lie algebras are classified and it is proved that any filiform Lie algebra is represented by a linear integrable deformation of special filiform Lie algebra.

In [10] for Leibniz algebras by the approach of M. Vergne similar description was obtained. In particular, there are only three naturally graded filiform Leibniz algebras (F_n^1, F_n^2 and $F_n^3(\alpha)$, $\alpha \in \{0; 1\}$) up to isomorphism.

This paper is structured as follows: In Section 2 we give the necessary definitions and facts. Section 3 is divided into three subsections: Section 3.1 deals with the second group Leibniz cohomology of the algebra F_n^1 and the description of some linear integrable deformations of F_n^1 in the variety of $Leib_n$. Among these deformations we indicate unknown till now rigid Leibniz algebra. In Section 3.2 we describe infinitesimal deformations of the algebra F_n^2 and its linear integrable deformations with respect to chosen basis of $HL^2(F_n^2, F_n^2)$. In Section 3.3 for the algebra $F_n^3(0)$ we establish that Lie infinitesimal deformations together with three indicated Leibniz infinitesimal deformations form the space of all Leibniz infinitesimal deformations.

Throughout the paper we consider finite-dimensional vector spaces and algebras over the field of complex numbers. Moreover, in the multiplication table of a Leibniz algebra the omitted products and in the expansion of 2-cocycles the omitted values are assumed to be zero.

2. Preliminaries

In this section we give necessary definitions and known results.

Definition 2.1. A Leibniz algebra over F is a vector space L equipped with a bilinear map, called bracket,

$$[-, -] : L \times L \to L$$

satisfying the Leibniz identity:

$$[x, [y, z]] = [[x, y], z] - [[x, z], y],$$

for all $x, y, z \in L$.

The set $Ann_r(L) = \{x \in L : [y, x] = 0, \ \forall y \in L\}$ is called *the right annihilator of a Leibniz algebra L*. Note that $Ann_r(L)$ is an ideal of L and for any $x, y \in L$ the elements $[x, x], [x, y] + [y, x] \in Ann_r(L)$.

We call a vector space M a module over L if there are two bilinear maps:

$$[-, -] : L \times M \to M \quad \text{and} \quad [-, -] : M \times L \to M$$

satisfying the following three axioms

$$[m, [x, y]] = [[m, x], y] - [[m, y], x],$$
$$[x, [m, y]] = [[x, m], y] - [[x, y], m],$$
$$[x, [y, m]] = [[x, y], m] - [[x, m], y],$$

for any $m \in M, x, y \in L$.

Given a Leibniz algebra L, let $C^n(L, M)$ be the space of all F-linear homogeneous mappings $L^{\otimes n} \to M, n \geq 0$ and $C^0(L, M) = M$.

Let $d^n : C^n(L, M) \to C^{n+1}(L, M)$ be an F-homomorphism defined by

$$(d^n f)(x_1, \dots, x_{n+1}) := [x_1, f(x_2, \dots, x_{n+1})] + \sum_{i=2}^{n+1}(-1)^i[f(x_1, \dots, \widehat{x_i}, \dots, x_{n+1}), x_i]$$
$$+ \sum_{1 \leq i < j \leq n+1} (-1)^{j+1} f(x_1, \dots, x_{i-1}, [x_i, x_j], x_{i+1}, \dots, \widehat{x_j}, \dots, x_{n+1}),$$

where $f \in C^n(L, M)$ and $x_i \in L$. Since the derivative operator $d = \sum_{i \geq 0} d^i$ satisfies the property $d \circ d = 0$, the n-th cohomology group is well defined and

$$HL^n(L, M) = ZL^n(L, M)/BL^n(L, M),$$

where the elements $ZL^n(L, M)$ and $BL^n(L, M)$ are called *n-cocycles* and *n-coboundaries*, respectively.

The elements $f \in BL^2(L, L)$ and $\varphi \in ZL^2(L, L)$ are defined as follows

$$f(x, y) = [d(x), y] + [x, d(y)] - d([x, y]) \quad \text{for some linear map } d \tag{2.1}$$

and

$$(d^2\varphi)(x, y, z) = [x, \varphi(y, z)] - [\varphi(x, y), z] + [\varphi(x, z), y] + \varphi(x, [y, z]) - \varphi([x, y], z) + \varphi([x, z], y) = 0. \tag{2.2}$$

A 2-cocycle is called infinitesimal deformation.

A deformation of a Leibniz algebra L is a one-parameter family L_t of Leibniz algebras with the bracket

$$\mu_t = \mu_0 + t\varphi_1 + t^2\varphi_2 + \cdots,$$

where φ_i are 2-cochains, i.e., elements of $\mathrm{Hom}(L \otimes L, L) = C^2(L, L)$.

Two deformations $L_t,\ L'_t$ with the corresponding laws $\mu_t,\ \mu'_t$ are *equivalent* if there exists a linear automorphism $f_t = id + f_1 t + f_2 t^2 + \cdots$ of L, where f_i are elements of $C^1(L, L)$ such that the following equation holds

$$\mu'_t(x, y) = f_t^{-1}(\mu_t(f_t(x), f_t(y))) \quad \text{for } x, y \in L.$$

The Leibniz identity for the algebras L_t implies that the 2-cochain φ_1 is an infinitesimal deformation, i.e. $d^2\varphi_1 = 0$. If φ_1 vanishes identically, then the first non vanishing φ_i is an infinitesimal deformation.

If μ'_t is an equivalent deformation with cochains φ'_i, then $\varphi'_1 - \varphi_1 = d^1 f_1$, hence every equivalence class of deformations defines uniquely an element of $HL^2(L, L)$.

Note that the linear integrable deformation φ satisfies the condition

$$\varphi(x, \varphi(y, z)) - \varphi(\varphi(x, y), z) + \varphi(\varphi(x, z), y) = 0. \tag{2.3}$$

The linear reductive group $GL_n(F)$ acts on $Leib_n$ via change of basis, i.e.,

$$(g * \lambda)(x, y) = g\Big(\lambda\big(g^{-1}(x), g^{-1}(y)\big)\Big), \quad g \in GL_n(F),\ \lambda \in Leib_n.$$

The orbits $\mathrm{Orb}(-)$ under this action are the isomorphism classes of algebras. Recall, Leibniz algebras with open orbits are called *rigid*. Note that solvable (respectively, nilpotent) Leibniz algebras of the same dimension also form an invariant subvariety of the variety of Leibniz algebras under the mentioned action. We give a definition of degeneration.

Definition 2.2. It is said that an algebra λ degenerates to an algebra μ, if $\mathrm{Orb}(\mu)$ lies in the Zariski closure of $\mathrm{Orb}(\lambda)$, $\overline{\mathrm{Orb}(\lambda)}$. We denote this by $\lambda \to \mu$.

In the case of the field $\mathbb{F}$ be the complex numbers $\mathbb{C}$, we give an equivalent definition of degeneration.

Definition 2.3. Let $g : (0, 1] \to GL_n(V)$ be a continuous mapping. We construct a parameterized family of the Leibniz algebras $g_t = (V, [-, -]_t)$, $t \in (0, 1]$ isomorphic to L. For each t the new Leibniz bracket $[-, -]_t$ on V is defined via the old one as follows: $[x, y]_t = g_t[g_t^{-1}(x), g_t^{-1}(y)]$, $\forall x, y \in V$. If for any $x, y \in V$ there exists the limit

$$\lim_{t \to +0} [x, y]_t = \lim_{t \to +0} g_t[g_t^{-1}(x), g_t^{-1}(y)] =: [x, y]_0,$$

then $[-, -]_0$ is a well-defined Leibniz bracket. The Leibniz algebra $L_0 = (V, [-, -]_0)$ is called a degeneration of the algebra L.

For a Leibniz algebra L consider the following central lower series:

$$L^1 = L, \qquad L^{k+1} = [L^k, L^1], \quad k \geq 1.$$

Definition 2.4. An n-dimensional Leibniz algebra is said to be filiform if $\dim L^i = n - i,\ 2 \leq i \leq n$.

Now let us define a natural graduation for a filiform Leibniz algebra.

Definition 2.5. Given a filiform Leibniz algebra L, put $L_i = L^i/L^{i+1},\ 1 \leq i \leq n-1$, and $gr(L) = L_1 \oplus L_2 \oplus \cdots \oplus L_{n-1}$. Then $[L_i, L_j] \subseteq L_{i+j}$ and we obtain the graded algebra $gr(L)$. If $gr(L)$ and L are isomorphic, then we say that the algebra L is naturally graded.

In the following theorem we resume the classification of the naturally graded filiform Leibniz algebras given in [10,12].

Theorem 2.6. *Any complex n-dimensional naturally graded filiform Leibniz algebra is isomorphic to one of the following pairwise non isomorphic algebras:*

$$F_n^1: \ [x_i, x_1] = x_{i+1}, \quad 2 \le i \le n-1,$$
$$F_n^2: \ [x_i, x_1] = x_{i+1}, \quad 1 \le i \le n-2,$$
$$F_n^3(\alpha): \ \begin{cases} [x_i, x_1] = -[x_1, x_i] = x_{i+1}, & 2 \le i \le n-1, \\ [x_i, x_{n+1-i}] = -[x_{n+1-i}, x_i] = \alpha(-1)^{i+1}x_n, & 2 \le i \le n-1 \end{cases}$$

where $\alpha \in \{0, 1\}$ for even n and $\alpha = 0$ for odd n.

The following theorem decomposes all n-dimensional filiform Leibniz algebras into three families of algebras.

Theorem 2.7 (*[13]*)**.** *Any complex n-dimensional filiform Leibniz algebra admits a basis $\{x_1, x_2, \dots, x_n\}$ such that the table of multiplication of the algebra has one of the following forms:*

$$F_1 = \begin{cases} [x_i, x_1] = x_{i+1}, & 2 \le i \le n-1, \\ [x_1, x_2] = \theta x_n, & \\ [x_j, x_2] = \alpha_4 x_{j+2} + \alpha_5 x_{j+3} + \dots + \alpha_{n+2-j}x_n, & 2 \le j \le n-2, \end{cases}$$

$$F_2 = \begin{cases} [x_i, x_1] = x_{i+1}, & 1 \le i \le n-2, \\ [x_j, x_n] = \beta_3 x_{j+2} + \beta_4 x_{j+3} + \dots + \beta_{n-j}x_{n-1}, & 1 \le j \le n-3, \\ [x_n, x_n] = \gamma x_{n-1}, & \end{cases}$$

$$F_3 = \begin{cases} [x_i, x_1] = x_{i+1}, & 2 \le i \le n-1, \\ [x_1, x_i] = -x_{i+1}, & 3 \le i \le n-1, \\ [x_1, x_1] = \theta_1 x_n, & \\ [x_1, x_2] = -x_3 + \theta_2 x_n, & \\ [x_2, x_2] = \theta_3 x_n, & \\ [x_i, x_j] = -[x_j, x_i] \in lin\langle x_{i+j+1}, x_{i+j+2}, \dots, x_n\rangle, & 2 \le i < j \le n-1, \\ [x_i, x_{n+1-i}] = -[x_{n+1-i}, x_i] = \alpha(-1)^{i+1}x_n, & 2 \le i \le n-1 \end{cases}$$

where $\alpha \in \{0, 1\}$ for even n and $\alpha = 0$ for odd n.

In [11] we obtain that any single-generated Leibniz algebra has the following multiplication:

$$\widetilde{\mu}(\alpha_2, \alpha_3, \dots, \alpha_n) = \begin{cases} [x_i, x_1] = x_{i+1}, & 1 \le i \le n-1, \\ [x_n, x_1] = \sum\limits_{k=2}^{n} \alpha_k x_k. & \end{cases}$$

Note that any algebra of the family $\widetilde{\mu}(\alpha_2, \alpha_3, \dots, \alpha_n)$ is a linear integrable deformation of the algebra NF_n.
Let us introduce denotation

$$X = \overline{\bigcup_{\alpha_2,\dots,\alpha_n} \mathrm{Orb}(\widetilde{\mu}(\alpha_2, \alpha_3, \dots, \alpha_n))}.$$

Theorem 2.8 (*[11]*)**.** *X is an irreducible component of the variety $Leib_n$.*

3. Deformations of naturally graded filiform Leibniz algebras

In this section we calculate infinitesimal deformations of naturally graded filiform Leibniz algebras.

3.1. Infinitesimal deformations of the algebra F_n^1

In order to achieve the purpose of the subsection we need the matrix form of a derivation of the filiform Leibniz algebra F_n^1 [14]:

$$\begin{pmatrix} \alpha_1 & 0 & 0 & 0 & \cdots & 0 & \alpha_n \\ 0 & \beta_2 & \beta_3 & \beta_4 & \cdots & \beta_{n-1} & \beta_n \\ 0 & 0 & \alpha_1+\beta_2 & \beta_3 & \cdots & \beta_{n-2} & \beta_{n-1} \\ 0 & 0 & 0 & 2\alpha_1+\beta_2 & \cdots & \beta_{n-3} & \beta_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & (n-2)\alpha_1+\beta_2 \end{pmatrix}. \quad (3.1)$$

Due to (2.1) it is easy to see that $\dim BL^2(F_n^1, F_n^1) = n^2 - n - 1$.

The following proposition presents the general form of the Leibniz infinitesimal deformation of the algebra F_n^1.

Proposition 3.1. *An arbitrary infinitesimal deformation φ of F_n^1 has the following form:*

$$\begin{cases} \varphi(x_1,x_1)=\sum\limits_{k=2}^{n}\alpha_{1,k}x_k, & \varphi(x_j,x_1)=\sum\limits_{k=1}^{n}\alpha_{j,k}x_k, \qquad 2\le j\le n-1, \\ \varphi(x_n,x_1)=\sum\limits_{k=2}^{n}\alpha_{n,k}x_k, & \varphi(x_1,x_2)=\gamma_1x_1+\gamma_nx_n, \\ \varphi(x_i,x_2)=((i-2)\gamma_1+\beta_2)x_i+\sum\limits_{k=3}^{n+2-i}\beta_kx_{k+i-2}, & 2\le i\le n, \\ \varphi(x_i,x_3)=-(\alpha_{2,1}+\gamma_1)x_{i+1}, & 2\le i\le n-1, \\ \varphi(x_i,x_{j+1})=-\alpha_{j,1}x_{i+1}, & 2\le i\le n-1,\ 3\le j\le n-1. \end{cases}$$

Proof. Using the property of infinitesimal deformations for $(d^2\varphi)(x_i,x_j,x_k)=0$ with $1\le i\le n$ and $2\le j,\ k\le n$, we obtain $[x_i,\varphi(x_j,x_k)]=0$, which implies $\varphi(x_j,x_k)\in\langle x_2,x_3,\dots,x_n\rangle$.

Similarly, the equation $(d^2\varphi)(x_i,x_1,x_1)=0$ leads to $[x_i,\varphi(x_1,x_1)]=0$, consequently we have $\varphi(x_1,x_1)\in\langle x_2,x_3,\dots,x_n\rangle$.

From the condition $(d^2\varphi)(x_i,x_j,x_1)=0$ with $1\le i\le n$ and $2\le j\le n$, we derive

$$[x_i,\varphi(x_j,x_1)]-[\varphi(x_i,x_j),x_1]+\varphi(x_i,[x_j,x_1])+\varphi([x_i,x_1],x_j)=0. \tag{3.2}$$

Similarly, from the condition $(d^2\varphi)(x_i,x_1,x_k)=0$ with $2\le k\le n$, we have

$$[x_i,\varphi(x_1,x_k)]+[\varphi(x_i,x_k),x_1]-\varphi([x_i,x_1],x_k)=0. \tag{3.3}$$

The equality (3.3) with $i=1,k=2$ deduces $[\varphi(x_1,x_2),x_1]=\varphi([x_1,x_1],x_2)-[x_1,\varphi(x_1,x_2)]=0$, hence we can assume $\varphi(x_1,x_2)=\gamma_1x_1+\gamma_nx_n$ for some parameters γ_1,γ_n.

From the equality (3.2) with $i=1,2\le j\le n-1$, we have $\varphi(x_1,x_{j+1})=[\varphi(x_1,x_j),x_1]=0$.

Summarizing the equalities (3.2) and (3.3) we obtain

$$\begin{cases} \varphi(x_i,x_3)=-[x_i,\varphi(x_1,x_2)+\varphi(x_2,x_1)], \\ \varphi(x_i,x_{j+1})=-[x_i,\varphi(x_j,x_1)], \qquad 3\le j\le n-1, \\ [x_i,\varphi(x_n,x_1)]=0. \end{cases} \tag{3.4}$$

We set

$$\varphi(x_j,x_1)=\sum_{k=1}^{n}\alpha_{j,k}x_k,\quad 1\le j\le n,\qquad \varphi(x_2,x_2)=\sum_{k=2}^{n}\beta_kx_k.$$

Applying Eqs. (3.2)–(3.4) we derive $a_{1,1}=a_{n,1}=0$ and

$$\varphi(x_i,x_2)=((i-2)\gamma_1+\beta_2)x_i+\sum_{k=3}^{n+2-i}\beta_kx_{k+i-2},\quad 2\le i\le n,$$

$$\varphi(x_i,x_3)=-(\alpha_{2,1}+\gamma_1)x_{i+1},\quad 2\le i\le n-1,\qquad \varphi(x_i,x_{j+1})=-\alpha_{j,1}x_{i+1},\quad 2\le i\le n-1,\ 3\le j\le n-1. \quad \square$$

Using Proposition 3.1 we indicate a basis of the space $ZL^2(F_n^1,F_n^1)$.

Theorem 3.2. *The following cochains:*

$$\varphi_{j,1}(2\le j\le n-1):\begin{cases}\varphi_{j,1}(x_j,x_1)=x_1,\\ \varphi_{j,1}(x_i,x_{j+1})=-x_{i+1}, \quad 2\le i\le n-1,\end{cases}$$

$$\varphi_{j,k}(1\le j\le n,\ 2\le k\le n):\big\{\varphi_{j,k}(x_j,x_1)=x_k,$$

$$\psi_j(2\le j\le n):\big\{\psi_j(x_i,x_2)=x_{j+i-2},\quad 2\le i\le n-j+2,$$

$$\xi_1:\begin{cases}\xi_1(x_1,x_2)=x_1,\\ \xi_1(x_i,x_2)=(i-2)x_i, \quad 3\le i\le n,\\ \xi_1(x_i,x_3)=-x_{i+1}, \quad 2\le i\le n-1,\end{cases}$$

$$\xi_2:\{\xi_2(x_1,x_2)=x_n$$

form a basis of the space $ZL^2(F_n^1,F_n^1)$.

Corollary 3.3. $\dim(ZL^2(F_n^1, F_n^1)) = n^2 + n - 1$.

Below, we describe a basis of the subspace $BL^2(F_n^1, F_n^1)$ in terms of $\varphi_{j,k}$, ψ_j, ξ_1 and ξ_2.

Proposition 3.4. *The cocycles*

$$\eta_{j,k}: \begin{cases} \eta_{1,k-1} = \varphi_{1,k}, & 3 \le k \le n, \\ \eta_{2,1} = \psi_3, & \\ \eta_{j,1} = \varphi_{j-1,1}, & 3 \le j \le n, \\ \eta_{j,k} = \varphi_{j-1,k}, & 3 \le j \le k \le n, \\ \eta_{j,k} = \varphi_{j-1,k} - \varphi_{j,k+1}, & 3 \le k < j \le n, \end{cases}$$

form a basis of $BL^2(F_n^1, F_n^1)$.

Proof. Consider the endomorphisms $f_{j,k}$ defined as follows:

$$\begin{cases} f_{2,1}(x_2) = x_1, & \\ f_{1,k}(x_1) = x_k, & 2 \le k \le n-1, \\ f_{j,k}(x_j) = x_k, & 3 \le j \le n,\ 1 \le k \le n. \end{cases}$$

According to (3.1) it implies that $f_{j,k}$ are the complemented linear maps to derivations in $C^1(F_n^1, F_n^1)$. Therefore, $d^1 f_{j,k}$ form a basis of the space $BL^2(F_n^1, F_n^1)$, where $d^1 f_{j,k} = f_{j,k}([x, y]) - [f_{j,k}(x), y] - [x, f_{j,k}(y)]$.

It should be noted that

$$\begin{cases} d^1 f_{1,k} = -\varphi_{1,k+1}, & 2 \le k \le n-1, \\ d^1 f_{2,1} = -\psi_3, & \\ d^1 f_{j,1} = \varphi_{j-1,1}, & 3 \le j \le n, \\ d^1 f_{j,k} = \varphi_{j-1,k} - \varphi_{j,k+1}, & 3 \le j \le n,\ 2 \le k \le n-1, \\ d^1 f_{j,n} = \varphi_{j-1,n}, & 3 \le j \le n. \end{cases}$$

From the condition $d^1 f_{j,k} + d^1 f_{j+1,k+1} + \cdots + d^1 f_{n+j-k,n} = \varphi_{j-1,k}$ for $3 \le j \le k \le n$, we conclude that the maps $\eta_{j,k}$ form a basis of $BL^2(F_n^1, F_n^1)$. $\square$

Corollary 3.5. *The adjoint classes* $\overline{\psi_2}, \overline{\xi_1}, \overline{\xi_2}, \overline{\varphi_{1,2}}, \overline{\varphi_{n,k}}$ $(2 \le k \le n)$ *and* $\overline{\psi_j}$ $(4 \le j \le n)$ *form a basis of* $HL^2(F_n^1, F_n^1)$. *Consequently,* $\dim HL^2(F_n^1, F_n^1) = 2n$.

Since every non-trivial equivalence class of deformations defines uniquely an element of $HL^2(L, L)$, due to Corollary 3.5 it is sufficient to consider a linear deformation

$$\mu_t = F_n^1 + t\varphi,$$

where $\varphi = c_1\xi_1 + c_2\xi_2 + a_1\varphi_{1,2} + \sum_{k=2}^n a_k\varphi_{n,k} + b_2\psi_2 + \sum_{k=4}^n b_k\psi_k$.

If $t \neq 0$, then we can assume $t = 1$ and the linear deformation μ_1 we shall denote by μ:

$$\mu: \begin{cases} [x_1, x_1] = a_1 x_2, & \\ [x_i, x_1] = x_{i+1}, & 2 \le i \le n-1, \\ [x_n, x_1] = \sum\limits_{k=2}^{n} a_k x_k, & \\ [x_1, x_2] = c_1 x_1 + c_n x_n, & \\ [x_i, x_2] = ((i-2)c_1 + b_2)x_i + \sum\limits_{k=4}^{n+2-i} b_k x_{k+i-2}, & 2 \le i \le n, \\ [x_i, x_3] = -c_1 x_{i+1}, & 2 \le i \le n-1. \end{cases} \tag{3.5}$$

In the next proposition we clarify under which conditions on the parameters a_i, b_i, c_1 and c_n the algebras of the family μ are the Leibniz algebras.

Proposition 3.6. *A linear integrable deformation of the algebra F_n^1 consists of the first class of filiform Leibniz algebra F_1 and the following Leibniz algebras*

$$\lambda(a_1, \dots, a_n): \begin{cases} [x_1, x_1] = a_1 x_2, & \\ [x_i, x_1] = x_{i+1}, & 2 \le i \le n-1, \\ [x_n, x_1] = \sum\limits_{k=2}^{n} a_k x_k, & \end{cases} \qquad R: \begin{cases} [x_i, x_1] = x_{i+1}, & 2 \le i \le n-1, \\ [x_1, x_2] = x_1, & \\ [x_i, x_2] = (i-2)x_i, & 3 \le i \le n, \\ [x_i, x_3] = -x_{i+1}, & 2 \le i \le n-1. \end{cases}$$

Proof. Verifying Leibniz identity for the algebra μ we obtain the following restrictions:

$$b_2 = 0, \qquad c_1 a_k = 0, \qquad c_n a_k = 0, \qquad b_i a_k = 0, \quad 4 \le i \le n, \ 1 \le k \le n.$$

If $a_i \neq 0$ for some i, then we deduce $c_1 = c_n = 0, \ b_i = 0, \ 4 \le i \le n$. So, the family of algebras $\lambda(a_1, a_2, \dots, a_n)$ is obtained.

If $a_i = 0$ for all i, then the table of multiplication of the family μ has the form:

$$\begin{cases} [x_i, x_1] = x_{i+1}, & 2 \le i \le n-1, \\ [x_1, x_2] = c_1 x_1 + c_n x_n, & \\ [x_i, x_2] = (i-2)c_1 x_i + \sum\limits_{k=4}^{n+2-i} b_k x_{k+i-2}, & 2 \le i \le n, \\ [x_i, x_3] = -c_1 x_{i+1}, & 2 \le i \le n-1. \end{cases}$$

In the case $c_1 = 0$ we get the family of filiform Leibniz algebras F_1.

If $c_1 \neq 0$, then taking the following transformation:

$$x_1' = x_1 + \frac{c_n}{c_1(3-n)} x_n, \qquad x_i' = \frac{1}{c_1} x_i + \sum_{j=i+2}^{n} A_{j-i+2} x_j, \quad 2 \le i \le n-2, \qquad x_{n-1}' = \frac{1}{c_1} x_{n-1}, \qquad x_n' = \frac{1}{c_1} x_n,$$

with

$$A_4 = -\frac{b_4}{2c_1^2}, \qquad A_5 = -\frac{b_5}{3c_1^2}, \qquad A_i = -\frac{1}{(i-2)c_1}\Big(\frac{b_i}{c_1} + \sum_{j=4}^{i-2} A_j b_{i+2-j}\Big), \quad 6 \le i \le n,$$

we obtain the algebra R. $\square$

Below we establish that the family λ is in X.

Proposition 3.7. *$\lambda(a_1, a_2, \dots, a_n) \in X$ for any values of the parameters a_i.*

Proof. If in the family λ the parameter $a_1 \neq 0$, then by taking the change of basic elements as follows: $x_1' = x_1, \ x_i' = a_1 x_i, \ 2 \le i \le n$, we can assume $a_1 = 1$ and the family $\lambda(1, a_2, \dots, a_n)$ is the same family as the one of $\widetilde{\mu}(\alpha_2, \alpha_3, \dots, \alpha_n)$. Thus, $\lambda(1, a_2, \dots, a_n) \in X$.

If $a_1 = 0$, then in the case $a_2 \neq 0$ by changing of basis in the following way:

$$x_1' = x_1 + \frac{1}{a_2}\left(x_n - \sum_{k=3}^{n} a_k x_{k-1}\right), \qquad x_i' = x_i, \quad 2 \le i \le n,$$

we have $\lambda(0, a_2, \dots, a_n) \simeq \widetilde{\mu}(\alpha_2, \alpha_3, \dots, \alpha_n)$. Thus, $\lambda(0, a_2, \dots, a_n) \in X$.

Let us suppose $a_1 = a_2 = 0$, then by choosing the transformations g_t as follows $g_t(x_1) = x_1, \ g_t(x_i) = t x_i, \ 2 \le i \le n$, we derive

$$\lim_{t \to 0} g_t * \widetilde{\mu}(0, \alpha_3, \dots, \alpha_n) = \lambda(0, 0, a_3, \dots, a_n),$$

which implies $\lambda(0, 0, a_3, \dots, a_n) \in X$. $\square$

The interesting properties of the algebra R are given in the following assertions, proofs of which can be given by direct computations.

Proposition 3.8. *Any derivation of the algebra R has the matrix form:*

$$\begin{pmatrix} \alpha & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \beta & 0 & \cdots & 0 & 0 \\ 0 & 0 & \alpha & \beta & \cdots & 0 & 0 \\ 0 & 0 & 0 & 2\alpha & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (n-3)\alpha & \beta \\ 0 & 0 & 0 & 0 & \cdots & 0 & (n-2)\alpha \end{pmatrix}.$$

Taking into account that the operators of right multiplications R_{x_1} and R_{x_2} of the algebra R are linear independent inner derivations, we conclude that any derivation of the algebra R is inner.

Proposition 3.9. *Any infinitesimal deformation of the algebra R has the following form:*

$$\begin{cases}
\varphi(x_1,x_1)=a_{1,1}x_1+a_{1,1}x_3+\sum\limits_{k=4}^{n}a_{1,k}x_k, & \\
\varphi(x_i,x_1)=\sum\limits_{k=1}^{n}a_{i,k}x_k, & 2\le i\le n-1,\\
\varphi(x_n,x_1)=-\sum\limits_{k=3}^{n-1}\sum\limits_{j=1}^{k-2}a_{n-j,k-j}x_k+\Big(\dfrac{(n-1)(n-2)}{2}a_{1,1}-\sum\limits_{k=2}^{n-1}a_{k,k}\Big)x_n, & \\
\varphi(x_1,x_2)=b_{1,1}x_1-a_{1,1}x_2+\sum\limits_{k=4}^{n-1}(n-3)a_{1,k+1}x_k, & \\
\varphi(x_2,x_2)=b_{2,1}x_1+b_{2,1}x_3+\sum\limits_{k=4}^{n}b_{2,k}x_k, & \\
\varphi(x_3,x_2)=a_{2,2}x_2+b_{1,1}x_3+\sum\limits_{k=4}^{n}(b_{2,k-1}-(k-3)a_{2,k})x_k, & \\
\varphi(x_i,x_2)=(i-3)a_{i-1,1}x_1+(i-2)b_{1,1}x_i-\Big(\dfrac{(i-2)(i-3)}{2}a_{1,1}-\sum\limits_{k=2}^{i-1}a_{k,k}\Big)x_{i-1} & \\
\quad+\sum\limits_{k=2}^{i-2}(i-k)\sum\limits_{j=1}^{k-1}a_{i-j,k+1-j}x_k+\sum\limits_{k=i+1}^{n}\Big(b_{2,k+2-i}+(i-k)\sum\limits_{j=2}^{i-1}a_{j,k+1+j-i}\Big)x_k, & 4\le i\le n,\\
\varphi(x_1,x_3)=-a_{2,2}x_1-a_{1,1}x_3-\sum\limits_{k=4}^{n}a_{1,k}x_k, & \\
\varphi(x_2,x_3)=-a_{2,1}x_1-a_{2,2}x_2-(b_{1,1}+a_{2,1})x_3-\sum\limits_{k=4}^{n}a_{2,k}x_k, & \\
\varphi(x_i,x_3)=(i-2)(a_{1,1}-a_{2,2})x_i+(a_{2,3}-a_{2,1}-b_{1,1})x_{i+1}-\sum\limits_{k=1}^{n}a_{i,k}x_k, & 3\le i\le n-1,\\
\varphi(x_n,x_3)=\sum\limits_{k=3}^{n-1}\sum\limits_{j=1}^{k-2}a_{n-j,k-j}x_k-\Big(\dfrac{(n-2)(n-3)}{2}a_{1,1}+(n-3)a_{2,2}-\sum\limits_{j=4}^{n-1}a_{k,k}\Big)x_n, & \\
\varphi(x_1,x_j)=-a_{j-1,2}x_1, & 4\le j\le n,\\
\varphi(x_2,x_j)=(a_{j-2,2}-a_{j-1,1}+a_{j-1,3})x_3, & 4\le j\le n,\\
\varphi(x_i,x_j)=-(i-2)a_{j-1,2}x_i+(a_{j-2,2}-a_{j-1,1}+a_{j-1,3})x_{i+1}, & 3\le i\le n-1,\ 4\le j\le n,\\
\varphi(x_n,x_j)=-(n-2)a_{j-1,2}x_n, & 4\le j\le n.
\end{cases}$$

Corollary 3.10. *The algebra R is rigid.*

Proof. Due to Proposition 3.8 we have $\dim Der\,R=2$. Therefore, $\dim BL^2(R,R)=n^2-2$. From Proposition 3.9 we conclude that $\dim ZL^2(R,R)=n^2-2$, hence $HL^2(R,R)=0$. Applying the result of the paper [9] on rigidity of Leibniz algebras which satisfy the condition $HL^2(R,R)=0$ we complete the proof. □

3.2. Infinitesimal deformations of the algebra F_n^2

Further we shall use the result in [14] which describes the derivations of the filiform Leibniz algebra F_n^2. Namely, any derivation of F_n^2 has the following matrix form:

$$\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{n-1} & \alpha_n\\
0 & 2\alpha_1 & \alpha_2 & \cdots & \alpha_{n-2} & 0\\
0 & 0 & 3\alpha_1 & \cdots & \alpha_{n-2} & 0\\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots\\
0 & 0 & 0 & \cdots & (n-1)\alpha_1 & 0\\
0 & 0 & 0 & \cdots & \beta_1 & \beta_2
\end{pmatrix}. \tag{3.6}$$

This matrix implies that $\dim Der(F_n^2)=n+2$ and $\dim BL^2(F_n^2,F_n^2)=n^2-n-2$.

Proposition 3.11. *An arbitrary infinitesimal deformation φ of F_n^2 has the following form:*

$$\begin{cases} \varphi(x_j, x_1) = \sum\limits_{k=1}^{n} \alpha_{j,k} x_k, & 1 \le j \le n-2, \\ \varphi(x_{n-1}, x_1) = \sum\limits_{k=2}^{n} \alpha_{n-1,k} x_k, \qquad \varphi(x_n, x_1) = \sum\limits_{k=1}^{n} \alpha_{n,k} x_k, & \\ \varphi(x_i, x_{j+1}) = -\alpha_{j,1} x_{i+1}, & 1 \le i \le n-2,\ 1 \le j \le n-2, \\ \varphi(x_1, x_n) = -\alpha_{n,1} x_1 + \sum\limits_{k=2}^{n} \beta_k x_k, & \\ \varphi(x_i, x_n) = -i\alpha_{n,1} x_i + \sum\limits_{k=2}^{n-i} \beta_k x_{k+i-1}, & 2 \le i \le n-1, \\ \varphi(x_n, x_n) = \gamma_1 x_{n-1} + \gamma_n x_n. & \end{cases}$$

Proof. The proof of this proposition is carrying out by applying similar arguments as in the proof of Proposition 3.1. $\square$

Using Proposition 3.11 we indicate a basis of the space $ZL^2(F_n^2, F_n^2)$.

Theorem 3.12. *The following cochains:*

$$\varphi_{j,1}(1 \le j \le n-2) : \begin{cases} \varphi_{j,1}(x_j, x_1) = x_1, \\ \varphi_{j,1}(x_i, x_{j+1}) = -x_{i+1}, \quad 1 \le i \le n-2, \end{cases}$$

$$\varphi_{j,k}(1 \le j \le n,\ 2 \le k \le n) : \{\varphi_{j,k}(x_j, x_1) = x_k,$$

$$\psi_1 : \begin{cases} \psi_1(x_n, x_1) = x_1, \\ \psi_1(x_i, x_n) = -ix_i, \quad 1 \le i \le n-1, \end{cases}$$

$$\psi_j(2 \le j \le n-1) : \{\psi_j(x_i, x_n) = x_{j+i-1},\ 1 \le i \le n-j,$$

$$\psi_n : \{\psi_n(x_1, x_n) = x_n,$$

$$\psi_{n+1} : \{\psi_{n+1}(x_n, x_n) = x_{n-1},$$

$$\psi_{n+2} : \{\psi_{n+2}(x_n, x_n) = x_n$$

form a basis of $ZL^2(F_n^2, F_n^2)$.

Corollary 3.13. $\dim(ZL^2(F_n^2, F_n^2)) = n^2 + n$.

Next, we describe a basis of the subspace $BL^2(F_n^2, F_n^2)$ by means of $\varphi_{j,k},\ \psi_j$.

Proposition 3.14. *The cocycles*

$$\eta_{j,k} : \begin{cases} \eta_{j,1} = \varphi_{j-1,1} - \varphi_{j,2}, & 2 \le j \le n-1, \\ \eta_{j,k} = \varphi_{j-1,k}, & 2 \le j \le k \le n-1, \\ \eta_{j,k} = \varphi_{j-1,k} - \varphi_{j,k+1}, & 2 \le k < j \le n-1, \\ \eta_{j,n} = \varphi_{j-1,n}, & 2 \le j \le n-1, \\ \eta_{n,1} = \varphi_{n,2} + \psi_2, & \\ \eta_{n,k} = \varphi_{n,k+1}, & 2 \le k \le n-2, \end{cases}$$

form a basis of $BL^2(F_n^2, F_n^2)$.

Corollary 3.15. *The adjoint classes $\overline{\varphi_{n,n}},\ \overline{\varphi_{n-1,k}}$ $(2 \le k \le n)$ and $\overline{\psi_j}$ $(1 \le j \le n+2)$ form a basis of $HL^2(F_n^2, F_n^2)$. Consequently,* $\dim HL^2(F_n^2, F_n^2) = 2n + 2$.

In the next proposition we clarify that basic elements of $ZL^2(F_n^2, F_n^2)$ satisfy the condition (2.3).

Proposition 3.16. *The infinitesimal deformations $\varphi_{j,k}$ $(1 \le j \le n, 2 \le k \le n)$, ψ_j $(1 \le j \le n-1)$ and ψ_{n+1} satisfy the condition (2.3), however $\psi_n,\ \psi_{n+2}$, and $\varphi_{j,1}$ $(1 \le j \le n-2)$ do not satisfy the condition (2.3).*

Proof. The proof of this proposition is straightforward. $\square$

Since every non-trivial equivalence class of deformations defines uniquely an element of $HL^2(L, L)$, due to Corollary 3.15 it is sufficient to consider the linear deformation

$$\nu_t = F_n^2 + t\varphi,$$

where $\varphi = c_1\varphi_{n,n} + \sum_{k=2}^{n} a_k\varphi_{n-1,k} + \sum_{k=1}^{n+2} b_k\psi_k$.

Without loss of generality, we can assume $t = 1$.

Then we have the following table of multiplications

$$\nu : \begin{cases} [x_i, x_1] = x_{i+1}, & 1 \le i \le n-2, \\ [x_{n-1}, x_1] = \sum\limits_{k=2}^{n} a_k x_k, & \\ [x_n, x_1] = b_1 x_1 + c_1 x_n, & \\ [x_1, x_n] = -b_1 x_1 + \sum\limits_{k=2}^{n} b_k x_k, & \\ [x_i, x_n] = -ib_1 x_i + \sum\limits_{k=2}^{n-i} b_k x_{k+i-1}, & 2 \le i \le n-1, \\ [x_n, x_n] = b_{n+1}x_{n-1} + b_{n+2}x_n. & \end{cases}$$

From the equalities

$$0 = [x_n, [x_n, x_n]] = [x_n, b_{n+1}x_{n-1} + b_{n+2}x_n] = b_{n+2}(b_{n+1}x_{n-1} + b_{n+2}x_n)$$

we get $b_{n+2} = 0$.

Proposition 3.17. *Any linear integrable deformation of the algebra F_n^2 admits a basis $\{x_1, x_2, \dots, x_n\}$ such that its table of multiplication has the form of the families $F_1,\ F_2,\ \widetilde{\mu}(a_2, \dots, a_n),\ \widetilde{\mu}(a_2, \dots, a_{n-1}) \oplus \mathbb{C}$ and*

$$\nu_1(a_2, a_3, \dots, a_{n-1}) : \begin{cases} [x_i, x_1] = x_{i+1}, & 1 \le i \le n-2, \\ [x_{n-1}, x_1] = \sum\limits_{k=2}^{n-1} a_k x_k, & \sum\limits_{k=2}^{n-1} a_k = 1, \\ [x_n, x_1] = x_n, & \end{cases}$$

$$\nu_2 : \begin{cases} [x_i, x_1] = x_{i+1}, & 1 \le i \le n-2, \\ [x_n, x_1] = x_1, & \\ [x_i, x_n] = -ix_i, & 1 \le i \le n-1, \end{cases}$$

$$\nu_3(b_2, b_3, \dots, b_{n-1}) : \begin{cases} [x_i, x_1] = x_{i+1}, & 1 \le i \le n-2, \\ [x_{n-1}, x_1] = -x_{n-1}, & \\ [x_n, x_1] = -x_n, & \\ [x_1, x_n] = x_n + \sum\limits_{k=2}^{n-1} b_k x_k, & \\ [x_i, x_n] = \sum\limits_{k=2}^{n-i} b_k x_{k+i-1}, & 2 \le i \le n-2, \end{cases}$$

where we can assume that the first non-zero element of the vector $(b_2, b_3, \dots, b_{n-1})$ is equal to 1,

$$\nu_4 : \begin{cases} [x_i, x_1] = x_{i+1}, & 1 \le i \le n-2, \\ [x_{n-1}, x_1] = -2x_{n-1}, & \\ [x_n, x_1] = -x_n, & \\ [x_1, x_n] = x_n, & \\ [x_n, x_n] = x_{n-1}, & \end{cases}$$

$$\nu_5(a_2, a_3, \dots, a_{n-1}) : \begin{cases} [x_i, x_1] = x_{i+1}, & 1 \le i \le n-2, \\ [x_{n-1}, x_1] = \sum\limits_{k=2}^{n-1} a_k x_k, & \\ [x_n, x_1] = -x_n, & \\ [x_1, x_n] = x_n. & \end{cases}$$

Proof. Note that $\{x_2, x_3, \dots, x_{n-1}\} \in Ann_r(\nu)$. If $x_n \in Ann_r(\nu)$, then we have $b_k = 0$, $1 \le k \le n+1$.

If $a_n \neq 0$ then we have the class of single generated algebras $\widetilde{\mu}(a_2, \dots, a_n)$.

If $a_n = 0$ and $c_1 = 0$, then we have the split algebra $\widetilde{\mu}(a_2, \dots, a_{n-1}) \oplus \mathbb{C}$.

If $a_n = 0$ and $c_1 \neq 0$, then by scaling the element x_n we can suppose $c_1 = 1$. In fact, in the case of $\sum_{k=2}^{n-1} a_k \neq 1$ this algebra is also single-generated (we can choose the element $x_1 + x_n$ as the generator). It is easy to see that when $\sum_{k=2}^{n-1} a_k = 1$ we obtain the two-generated family of algebras $\nu_1(a_2, a_3, \dots, a_{n-1})$.

Now we consider $x_n \notin Ann_r(\nu)$. It implies $a_n = 0,\ c_1 = -b_n$ and the table of multiplication of ν has the form:

$$\nu: \begin{cases} [x_i, x_1] = x_{i+1}, & 1 \le i \le n-2, \\ [x_{n-1}, x_1] = \sum\limits_{k=2}^{n-1} a_k x_k, & \\ [x_n, x_1] = b_1 x_1 - b_n x_n, & \\ [x_1, x_n] = -b_1 x_1 + \sum\limits_{k=2}^{n} b_k x_k, & \\ [x_i, x_n] = -i b_1 x_i + \sum\limits_{k=2}^{n-i} b_k x_{k+i-1}, & 2 \le i \le n-1, \\ [x_n, x_n] = b_{n+1} x_{n-1}. & \end{cases}$$

Consider the Leibniz identity

$$\begin{aligned} [x_1, [x_n, x_1]] &= [[x_1, x_n], x_1] - [[x_1, x_1], x_n] = \left[-b_1 x_1 + \sum_{k=2}^{n} b_k x_k, x_1\right] - [x_2, x_n] \\ &= -b_1 x_2 + \sum_{k=2}^{n-2} b_k x_{k+1} + b_{n-1} \sum_{k=2}^{n-1} a_k x_k + b_n (b_1 x_1 - b_n x_n) - \left(-2 b_1 x_2 + \sum_{k=2}^{n-2} b_k x_{k+1}\right) \\ &= b_1 x_2 + b_{n-1} \sum_{k=2}^{n-1} a_k x_k + b_n (b_1 x_1 - b_n x_n). \end{aligned}$$

On the other hand

$$[x_1, [x_n, x_1]] = [x_1, b_1 x_1 - b_n x_n] = b_1 x_2 - b_n \sum_{k=2}^{n-1} b_k x_k + b_n (b_1 x_1 - b_n x_n).$$

Comparing the coefficients at the basic elements, we deduce

$$b_{n-1} a_k = -b_n b_k, \quad 2 \le k \le n-1.$$

Consider the Leibniz identity $[x_i, [x_n, x_1]]$ with $2 \le i \le n-2$,

$$\begin{aligned} [x_i, [x_n, x_1]] &= [[x_i, x_n], x_1] - [[x_i, x_1], x_n] = \left[-i b_1 x_i + \sum_{k=2}^{n-i} b_k x_{k+i-1}, x_1\right] - [x_{i+1}, x_n] \\ &= -i b_1 x_{i+1} + \sum_{k=2}^{n-i-1} b_k x_{k+i} + b_{n-i} \sum_{k=2}^{n-1} a_k x_k - \left(-(i+1) b_1 x_{i+1} + \sum_{k=2}^{n-i-1} b_k x_{k+i}\right) \\ &= b_1 x_{i+1} + b_{n-i} \sum_{k=2}^{n-1} a_k x_k. \end{aligned}$$

On the other hand, we have

$$[x_i, [x_n, x_1]] = [x_i, b_1 x_1 - b_n x_n] = b_1 x_{i+1} - b_n \left(-i b_1 x_i + \sum_{k=2}^{n-i} b_k x_{k+i-1}\right).$$

Comparing the coefficients at the basic elements, we derive

$$\begin{cases} b_{n-i} a_k = 0, & 2 \le i \le n-2,\ 2 \le k \le i-1, \\ b_{n-i} a_i = i b_n b_1, & 2 \le i \le n-2, \\ b_{n-i} a_k = -b_n b_{k-i+1}, & 2 \le i \le n-2,\ i+1 \le k \le n-1. \end{cases}$$

Similarly, from the following equalities

$$\begin{aligned}[x_{n-1},[x_n,x_1]] &= [[x_{n-1},x_n],x_1]-[[x_{n-1},x_1],x_n] = -(n-1)b_1[x_{n-1},x_1]-\Big[\sum_{k=2}^{n-1}a_kx_k,x_n\Big]\\
&= -(n-1)b_1\sum_{k=2}^{n-1}a_kx_k-\sum_{k=2}^{n-1}a_k\left(-kb_1x_k+\sum_{s=2}^{n-k}b_sx_{s+k-1}\right)\\
&= b_1\sum_{k=2}^{n-2}(-n+1+k)a_kb_1x_k-\Big(\sum_{k=2}^{n-2}a_kb_{n-k}\Big)x_{n-1}\\
&= b_1\sum_{k=2}^{n-2}(-n+1+k)a_kb_1x_k-\sum_{k=2}^{n-2}ib_1b_nx_{n-1}\\
&= b_1\sum_{k=2}^{n-2}(-n+1+k)a_kb_1x_k-\frac{n(n-3)b_1b_n}{2}x_{n-1}\end{aligned}$$

and

$$[x_{n-1},[x_n,x_1]] = [x_{n-1},b_1x_1-b_nx_n] = b_1\sum_{k=2}^{n-1}a_kx_k+(n-1)b_nb_1x_{n-1},$$

we obtain

$$\begin{cases}(n-k)a_kb_1=0, & 2\le k\le n-2,\\ \left(a_{n-1}+\dfrac{(n+1)(n-2)b_n}{2}\right)b_1=0.\end{cases}$$

Consider

$$\begin{aligned}[x_n,[x_n,x_1]] &= [[x_n,x_n],x_1]-[[x_n,x_1],x_n] = b_{n+1}[x_{n-1},x_1]-[b_1x_1-b_nx_n,x_n]\\
&= b_{n+1}\sum_{k=2}^{n-1}a_kx_k-b_1\left(-b_1x_1+\sum_{k=2}^{n}b_kx_k\right)+b_nb_{n+1}x_{n-1}\\
&= b_1^2x_1+\sum_{k=2}^{n-2}(b_{n+1}a_k-b_1b_k)x_k+(b_{n+1}a_{n-1}-b_1b_{n-1}+b_nb_{n+1})x_{n-1}-b_1b_nx_n.\end{aligned}$$

On the other hand, we have

$$[x_n,[x_n,x_1]] = [x_n,b_1x_1-b_nx_n] = b_1^2x_1-b_nb_{n+1}x_{n-1}-b_1b_nx_n.$$

Comparing the appropriate coefficients, we conclude

$$\begin{cases}b_1b_k=b_{n+1}a_k, & 2\le k\le n-2,\\ b_1b_{n-1}=b_{n+1}(a_{n-1}+2b_n).\end{cases}$$

Let us summarize the above restrictions:

$$\begin{cases}b_{n-1}a_k=-b_nb_k, & 2\le k\le n-1,\\
b_{n-i}a_k=0, & 2\le i\le n-2,\ 2\le k\le i-1,\\
b_{n-i}a_i=ib_nb_1, & 2\le i\le n-2,\\
b_{n-i}a_k=-b_nb_{k-i+1}, & 2\le i\le n-2,\ i+1\le k\le n-1,\\
(n-k)a_kb_1=0, & 2\le k\le n-2,\\
\left(a_{n-1}+\dfrac{(n+1)(n-2)b_n}{2}\right)b_1=0, & \\
b_1b_k=b_{n+1}a_k, & 2\le k\le n-2,\\
b_1b_{n-1}=b_{n+1}(a_{n-1}+2b_n). & \end{cases}$$

We need to consider the following cases.

Case 1. Let $b_1 \neq 0$. Then $a_k = 0,\ 2 \leq k \leq n-1$ and $b_k = 0,\ 2 \leq k \leq n$. In this case we have the following table of multiplication

$$\begin{cases} [x_i, x_1] = x_{i+1}, & 1 \leq i \leq n-2, \\ [x_n, x_1] = b_1 x_1, & \\ [x_i, x_n] = -ib_1 x_i, & 1 \leq i \leq n-1, \\ [x_n, x_n] = b_{n+1} x_{n-1}, & 1 \leq i \leq n-1. \end{cases}$$

Taking the change $x_n' = \frac{1}{b_1} x_n + \frac{b_{n-1}}{b_1(n-1)} x_{n-1}$ we obtain the algebra ν_2.

Case 2. Let $b_1 = 0$. Then the above restrictions are reduced to the following one:

$$\begin{cases} b_{n-1} a_k = -b_n b_k, & 2 \leq k \leq n-1, \\ b_{n-i} a_k = 0, & 2 \leq i \leq n-2,\ 2 \leq k \leq i, \\ b_{n-i} a_k = -b_n b_{k-i+1}, & 2 \leq i \leq n-2,\ i+1 \leq k \leq n-1, \\ b_{n+1} a_k = 0, & 2 \leq k \leq n-2, \\ b_{n+1}(a_{n-1} + 2b_n) = 0. & \end{cases} \tag{3.7}$$

- If there exists some $b_i \neq 0$, where $2 \leq i \leq n-1$, then from (3.7) we obtain

  $$a_i = 0, \quad 2 \leq i \leq n-1, \qquad a_{n-1} = -b_n, \qquad b_n b_{n+1} = 0.$$

  – Let us suppose $b_n = 0$. Then we have the multiplication

  $$\begin{cases} [x_i, x_1] = x_{i+1}, & 1 \leq i \leq n-2, \\ [x_i, x_n] = \sum\limits_{k=2}^{n-i} b_k x_{k+i-1}, & 1 \leq i \leq n-2, \\ [x_n, x_n] = b_{n+1} x_{n-1}. & \end{cases}$$

  This family of algebras represents the families F_1 and F_2. Namely, if $b_2 \neq 0$, then we obtain the family F_1 and if $b_2 = 0$, then we get the family F_2.

  – Let us assume now that $b_n \neq 0$. Then $b_{n+1} = 0$ and by scaling the basic elements, one can assume $b_n = 1$. Thus, we obtain the algebra $\nu_3(b_2, b_3, \ldots, b_{n-1})$.
- If $b_i = 0$ for all $2 \leq i \leq n-1$.

  – Let $b_{n+1} \neq 0$. Then $a_i = 0,\ 2 \leq i \leq n-1$ and $a_{n-1} = -2b_n$. In the case of $b_n = 0$, we have a filiform Leibniz algebra of the family F_2 and if $b_n \neq 0$, by scaling of appropriate basic elements, we can suppose $b_n = b_{n+1} = 1$. Thus, we get the algebra ν_4.

  – Let $b_{n+1} = 0$. Since $x_n \notin Ann_r(\nu)$ one can conclude $b_n \neq 0$. By scaling the basic elements, we can assume $b_n = 1$ and the algebra $\nu_5(a_2, a_3, \ldots, a_{n-1})$ is obtained. □

Below we give some remarks concerning the algebras $\nu_1 - \nu_5$.

Remark 3.18. (1) Since the following single-generated Leibniz algebra

$$\begin{cases} [x_i, x_1] = x_{i+1}, & 1 \leq i \leq n-2, \\ [x_{n-1}, x_1] = x_n + \sum\limits_{k=2}^{n-1} a_k x_k, & \\ [x_n, x_1] = x_n & \end{cases}$$

degenerates to the algebra $\nu_1(a_2, a_3, \ldots, a_{n-1})$ via the family of transformations g_t, where

$$g_t(x_n) = t x_n, \qquad g_t(x_i) = x_i, \quad 1 \leq i \leq n-1,$$

we conclude $\nu_1(a_2, \ldots, a_{n-1}) \in X$.

(2) Note that the algebra ν_2 is the unique (up to isomorphism) solvable Leibniz algebra with null-filiform nilradical [15]. Due to the work [16] the algebra ν_2 is rigid.

(3) The algebra ν_3 is a solvable Leibniz algebra with nilradical $N = \langle x_2, x_3, \ldots, x_n \rangle$, which has the table of multiplication:

$$[x_i, x_n] = \sum_{k=2}^{n-i} b_k x_{k+i-1}, \quad 2 \leq i \leq n-2.$$

In particular, if $b_2 \neq 0$, then N is a filiform algebra.

(4) The algebra ν_4 is a solvable Leibniz algebra with nilradical $N = \langle x_2, x_3, \ldots, x_n \rangle$, which is isomorphic to the direct sum of two-dimensional Leibniz algebra $[x_n, x_n] = x_{n-1}$ and $\mathbb{C}^{n-3}$. It should be noted that the algebra N is the algebra of level one [17].

(5) The algebra ν_5 is a solvable Leibniz algebra with an abelian nilradical $\langle x_2, x_3, \ldots, x_n \rangle$.

3.3. Infinitesimal deformations of the algebra F_n^3

In this subsection we give some additional information on Leibniz infinitesimal deformations of $F_n^3(0)$.

Recall, a Lie infinitesimal deformation φ is defined as a bilinear map which satisfies the equality (2.2) and the skew-symmetric condition $\varphi(x,y) = -\varphi(y,x)$ [18].

Thanks to [12,5], where the infinitesimal deformations of the algebra $F_n^3(0)$ in the varieties of nilpotent and all Lie algebras are described, respectively, it is sufficient to study infinitesimal Leibniz deformations which do not satisfy skew-symmetric condition.

Denote by $ZL^2(F_n^3(0), F_n^3(0))$ and $Z^2(F_n^3(0), F_n^3(0))$ the spaces of Leibniz and Lie infinitesimal deformations, respectively.

Theorem 3.19 ([5]). *If $Z^2(F_n^3(0), F_n^3(0))$ is a vector space of the infinitesimal deformations of $F_n^3(0)$ on the variety of n-dimensional Lie algebra Lie_n, then*

$$\dim Z^2(F_n^3(0), F_n^3(0)) = \begin{cases} 8, & \text{if } n=3, \\ 15, & \text{if } n=4, \\ \dfrac{(n-1)(3n-5)}{8} + n^2 - n - 1, & \text{if } n \text{ is odd and } n \geq 5, \\ \dfrac{n(3n-10)}{8} + n^2 - n + \left[\dfrac{n}{4}\right], & \text{if } n \text{ is even and } n \geq 5. \end{cases}$$

In the next proposition we present a general form of non-Lie Leibniz infinitesimal deformations of the algebra $F_n^3(0)$.

Proposition 3.20. *If $\varphi \in ZL^2(F_n^3(0), F_n^3(0))$, then*

$$\varphi(x_1,x_1) = \alpha x_n, \qquad \varphi(x_1,x_2) + \varphi(x_2,x_1) = \beta x_n, \qquad \varphi(x_2,x_2) = \gamma x_n,$$
$$\varphi(x_i,x_j) = -\varphi(x_j,x_i) \quad \text{for other } i \text{ and } j.$$

Proof. The proof is carried out by checking the infinitesimal deformation property on algebra $F_n^3(0)$. $\square$

Theorem 3.21. *The cochains ψ_1, ψ_2, ψ_3 defined as*

$$\psi_1(x_1,x_1) = x_n, \qquad \psi_2(x_1,x_2) = x_n, \qquad \psi_3(x_2,x_2) = x_n$$

complement the subspace $Z^2(F_n^3(0), F_n^3(0))$ to the space $ZL^2(F_n^3(0), F_n^3(0))$.

Corollary 3.22.

$$\dim ZL^2(F_n^3(0), F_n^3(0)) = \begin{cases} 11, & \text{if } n=3, \\ 18, & \text{if } n=4, \\ \dfrac{(n-1)(3n-5)}{8} + n^2 - n + 2, & \text{if } n \text{ is odd and } n \geq 5, \\ \dfrac{n(3n-10)}{8} + n^2 - n + 3 + \left[\dfrac{n}{4}\right], & \text{if } n \text{ is even and } n \geq 5. \end{cases}$$

It is known that any derivation of algebra $F_n^3(0)$ with $n = 3$ and $n \geq 4$ has the following matrix form [18]:

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & a_1+b_2 \end{pmatrix}, \qquad \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \cdots & \alpha_{n-1} & \alpha_n \\ 0 & \beta_2 & \beta_3 & \beta_4 & \cdots & \beta_{n-1} & \beta_n \\ 0 & 0 & \alpha_1+\beta_2 & \beta_3 & \cdots & \beta_{n-2} & \beta_{n-1} \\ 0 & 0 & 0 & 2\alpha_1+\beta_2 & \cdots & \beta_{n-3} & \beta_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & (n-2)\alpha_1+\beta_2 \end{pmatrix}.$$

Therefore,

$$\dim Der(F_n^3(0)) = \begin{cases} 6, & n=3, \\ 2n-1, & n \geq 4. \end{cases} \qquad \dim BL^2(F_n^3(0), F_n^3(0)) = \begin{cases} 3, & n=3, \\ (n-1)^2, & n \geq 4. \end{cases}$$

Consequence of Theorem 3.19 is the following result.

Corollary 3.23.

$$\dim HL^2(F_n^3(0), F_n^3(0)) = \begin{cases} 9, & \textit{if } n = 3, \\ 10, & \textit{if } n = 4, \\ \dfrac{(n-1)(3n-5)}{8} + n + 2, & \textit{if } n \textit{ is odd and } n \geq 5, \\ \dfrac{n(3n-10)}{8} + n + 3 + \left[\dfrac{n}{4}\right], & \textit{if } n \textit{ is even and } n \geq 5. \end{cases}$$

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