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New periodic solutions for some planar $N+3$-body problems with Newtonian potentials

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ABSTRACT

For some planar Newtonian $N+3$-body problems, we use variational minimization methods to prove the existence of new periodic solutions satisfying that N bodies chase each other on a curve, and the other 3 bodies chase each other on another curve. From the definition of orbit spaces in our paper, we can find that they are new solutions which are also different from all the examples of Ferrario and Terracini (2004).

1. Introduction and main results

In recent years, many authors used methods of minimizing the Lagrangian action on a symmetric space to study the periodic solutions for Newtonian N-body problem [1–36]. Especially, A. Chenciner–R. Montgomery [13] proved the existence of the remarkable figure eight type periodic solution for Newtonian three-body problem with equal masses, C. Marchal [24] studied the fixed-ends (Bolza) problem for Newtonian N-body problem and proved that the minimizer for the Lagrangian action has no interior collision; A. Chenciner [9], D. Ferario and S. Terracini [19] simplified and developed C. Marchal's important works; S.Q. Zhang [32], S.Q. Zhang, Q. Zhou [33–36] decomposed the Lagrangian action for N-body problem into some sum for two-body problem and compared the lower bound for the lagrangian action on test orbits with the upper bound on collision set to avoid collisions under some cases. Motivated by the works of A. Chenciner and R. Montgomery, C. Simó, C. Marchal, S.Q. Zhang and Q. Zhou, K.C. Chen [5–8] studied some planar N-body problems and got some new planar non-collision periodic and quasi-periodic solutions.

The equations for the motion of the Newtonian N-body problem are:

$$m_i\ddot{q}_i=\frac{\partial U(q)}{\partial q_i},\quad i=1,\ldots,N,\tag{1.1}$$

where $q_i\in\mathbb{R}^k$ denotes the position of m_i, and the potential function is:

$$U=\sum_{1\le i<j\le N}\frac{m_im_j}{|q_i-q_j|}.$$

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It is well known that critical points of the action functional f:

$$f(q)=\int_0^T(\frac{1}{2}\sum_{i=1}^{N}m_i|\dot{q}_i|^2+U(q))dt,\quad q\in E, \tag{1.2}$$

are T periodic solutions of the N-body problem (1.1), where

$$E=\{q=(q_1,q_2,\ldots,q_N)\,|\,q_i(t)\in W^{1,2}(\mathbb{R}/T\mathbb{Z},\mathbb{R}^k),\ \sum_{i=1}^{N}m_iq_i(t)=0,\ q_i(t)\neq q_j(t),\forall i\neq j,\forall t\in\mathbb{R}\}, \tag{1.3}$$

$$W^{1,2}(\mathbb{R}/T\mathbb{Z},\mathbb{R}^k)=\{x(t)\,|\,x(t)\in L^2(\mathbb{R},\mathbb{R}^k),\dot{x}(t)\in L^2(\mathbb{R},\mathbb{R}^k),x(t+T)=x(t)\}. \tag{1.4}$$

Definition 1.1. Let $\Gamma:x(t),\ t\in[a,\,b]$ be a given oriented continuous closed curve, and p a point of the plane, not on the curve. Then the mapping $\varphi:\Gamma\to S^1$ given by

$$\varphi(x(t))=\frac{x(t)-p}{|x(t)-p|},\quad t\in[a,b],$$

is defined to be the position mapping of the curve Γ relative to p. When the point on Γ goes around the given oriented curve once, its image point $\varphi(x)$ will go around S^1 in the same direction with Γ a number of times. When moving counter-clockwise or clockwise, we set the sign $+$or $-$, and we denote it by $deg(\Gamma,\ p)$. If p is the origin, we denote it by $deg(\Gamma)$.

C.H. Deng and S.Q. Zhang [17], X. Su and S.Q. Zhang [29] study periodic solutions for some planar $N+2$-body problems, they defined the following orbit spaces:

$$\begin{aligned}\Lambda_0=\{q\in E_0\,|\,&q_i(t+\frac{T}{r})=O(\frac{2\pi}{r})q_i(t),\quad i=1,\ldots,N+2;\\&q_{i+1}(t)=q_i(t+\frac{T}{N}),\ i=1,\ldots,N-1,\ q_1(t)=q_N(t+\frac{T}{N});\\&q_i(t+\frac{T}{N})=q_i(t),\ i=N+1,\ N+2,\forall\,t>0\}\end{aligned} \tag{1.5}$$

and

$$\begin{aligned}\Lambda=\{q\in\Lambda_0\,|\,&q_i(t)\neq q_j(t),\ \forall i\neq j,\forall t\in\mathbb{R};\\&deg(q_i(t)-q_j(t))=1,\ 1\le i\neq j\le N,deg(q_{N+1}(t)-q_{N+2}(t))=k_1\},\end{aligned} \tag{1.6}$$

where

$$E_0=\{q=(q_1,q_2,\ldots,q_{N+2})\,|\,q_i(t)\in W^{1,2}(\mathbb{R}/T\mathbb{Z},\mathbb{R}^2),\ \sum_{i=1}^{N+2}m_iq_i(t)=0\}, \tag{1.7}$$

$$O(\theta)=\begin{pmatrix}\cos\theta&-\sin\theta\\ \sin\theta&\cos\theta\end{pmatrix}.$$

Motivated by their work, we consider $N+3$-body problems ($N>3$, N and 3 are coprime), the equations of the motion are:

$$m_i\ddot{q}_i(t)=\frac{\partial U(q)}{\partial q_i},\ \ i=1,\ \ldots,\ N+3. \tag{1.8}$$

We define the following orbit spaces:

$$\begin{aligned}\Lambda_1=\{q\in E_1\,|\,&q_i(t+\frac{T}{r})=O(\frac{2\pi d}{r})q_i(t),\quad i=1,\ \ldots,\ N+3;\\&q_{i+1}(t)=q_i(t+\frac{T}{N}),\ i=1,\ \ldots,\ N,\ q_1(t)=q_N(t+\frac{T}{N});\\&q_{N+j}(t)=q_{N+j-1}(t+\frac{T}{3}),\ j=2,\ 3,\ q_{N+1}(t)=q_{N+3}(t+\frac{T}{3});\\&q_i(t+\frac{T}{3})=q_i(t),\ i=1,\ldots,N;\\&q_j(t+\frac{T}{N})=q_j(t),\ j=N+1,N+2,N+3\},\end{aligned} \tag{1.9}$$

and

$$\begin{aligned}\Lambda_2=\{q\in\Lambda_1|&q_i(t)\neq q_j(t),\forall i\neq j,\forall t\in R;\\&deg(q_i(t)-q_j(t))=k_1,\ 1\le i<j\le N;\\&deg(q_{i'}(t)-q_{j'}(t))=k_2,N+1\le i'<j'\le N+3\},\end{aligned}$$

where

$$E_1 = \{q = (q_1, q_2, \dots, q_{N+3}) | q_i(t) \in W^{1,2}(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^2),\ \sum_{i=1}^{N+3} m_i q_i(t) = 0\}. \tag{1.10}$$

Notice that r, k_1, k_2, d satisfy the following compatible conditions:

$$k_1 = d(mod\, r), k_2 = d(mod\, r), k_1 = 3s_1, k_2 = Ns_2, s_1, s_2 \in \mathbb{Z}. \tag{1.11}$$

Since N and 3 are coprime, we have $(N, 3) = 1$. In this paper, we also require r and 3 coprime, so $(r, 3) = 1$.
We get the following theorem:

Theorem 1.1. (1) *Consider the seven-body problems (1.8) of equal masses, for $r = 7,\ k_1 = 3,\ k_2 = -4,\ d = 3$, then the global minimizer of f on $\bar{\Lambda}_2$ is a non-collision periodic solution of (1.8).*

(2) *Consider the eight-body problems (1.8) of equal masses, for $r = 8,\ k_1 = 3,\ k_2 = -5,\ d = 3$, then the global minimizer of f on $\bar{\Lambda}_2$ is a non-collision periodic solution of (1.8).*

(3) *Consider the ten-body problems (1.8) of equal masses, for $r = 10,\ k_1 = 3,\ k_2 = -7,\ d = 3$, then the global minimizer of f on $\bar{\Lambda}_2$ is a non-collision periodic solution of (1.8).*

2. Some lemmas

Lemma 2.1 (Eberlein–Shmulyan [37]). *A Banach space X is reflexive if and only if any bounded sequence in X has a weakly convergent subsequence.*

Lemma 2.2 ([37]). *Let X be a real reflexive Banach space, $M \subset X$ is a weakly closed subset, $f : M \to R$ is weakly semi-continuous. If f is coercive, that is, $f(x) \to +\infty$ as $\|x\| \to +\infty$, then $f(x)$ attains its infimum on M.*

Lemma 2.3 ([38]). *Let G be a group acting orthogonally on a Hilbert space H. Define the fixed point space $F_G = \{x \in H | g \cdot x = x, \forall g \in G\}$, if $f \in C^1(H, R)$ and satisfies $f(g \cdot x) = f(x)$ for any $g \in G$ and $x \in H$, then the critical point of f restricted on F_G is also a critical point of f on H.*

Lemma 2.4 ([39]). *Let $q \in W^{1,2}(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^n)$ and $\int_0^T q(t)\,\mathrm{d}t = 0$, then we have*
(i). Poincare–Wirtinger's inequality:

$$\int_0^T |\dot{q}(t)|^2\,\mathrm{d}t \geq \left(\frac{2\pi}{T}\right)^2 \int_0^T |q(t)|^2\,\mathrm{d}t. \tag{2.1}$$

(ii). Sobolev's inequality:

$$\max_{0\leq t\leq T} |q(t)| = \|q\|_\infty \leq \sqrt{\frac{T}{12}} \Big(\int_0^T |\dot{q}(t)|^2 \mathrm{d}t\Big)^{1/2}. \tag{2.2}$$

Lemma 2.5 (Gordon [21]). *(1) Let $x(t) \in W^{1,2}([t_1, t_2], R^k)$ and $x(t_1) = x(t_2) = 0$, Then for any $a > 0$, we have*

$$\int_{t_1}^{t_2} (\frac{1}{2}|\dot{x}|^2 + \frac{a}{|x|})dt \geq \frac{3}{2}(2\pi)^{\frac{2}{3}} a^{\frac{2}{3}} (t_2 - t_1)^{\frac{1}{3}}. \tag{2.3}$$

[23] (2) *Let $x(t) \in W^{1,2}(R/TZ, R^k)$, $\int_0^T x dt = 0$, then for any $a > 0$, we have*

$$\int_0^T (\frac{1}{2}|\dot{x}(t)|^2 + \frac{a}{|x|})dt \geq \frac{3}{2}(2\pi)^{\frac{2}{3}} a^{\frac{2}{3}} T^{\frac{1}{3}}. \tag{2.4}$$

3. The proof of Theorem 1.1

we consider the system (1.8) of equal masses. Without loss of generality, we suppose that the masses $m_1 = m_2 = \cdots = m_{N+3} = 1$, and the period $T = 1$.

Define $G = \mathbb{Z}_r \times \mathbb{Z}_3 \times \mathbb{Z}_N$ and the group action $g = \langle g_1 \rangle \times \langle g_2 \rangle \times \langle g_3 \rangle$ on the space E_1:

$$g_1(q_1(t), \dots, q_{N+3}(t)) = (O(-\frac{2\pi d}{r})q_1(t + \frac{1}{r}), \dots, O(-\frac{2\pi d}{r})q_{N+3}(t + \frac{1}{r})) \tag{3.1}$$

$$\begin{aligned} &g_2(q_1(t), \dots, q_{N+3}(t)) \\ &= (q_1(t + \frac{1}{3}), \dots, q_N(t + \frac{1}{3}), q_{N+3}(t + \frac{1}{3}), q_{N+1}(t + \frac{1}{3}), q_{N+2}(t + \frac{1}{3})) \end{aligned} \tag{3.2}$$

$$g_3(q_1(t),\ldots,q_{N+3}(t))$$
$$=(q_N(t+\frac{1}{N}),q_1(t+\frac{1}{N})\ldots,q_{N-1}(t+\frac{1}{N}),q_{N+1}(t+\frac{1}{N}),q_{N+2}(t+\frac{1}{N}),q_{N+3}(t+\frac{1}{N})). \tag{3.3}$$

This implies that Λ_1 is the fixed point space of g on E_1. Furthermore, for any g_i and $q\in E_1$, we have $f(g_i\cdot q)=f(q)$ for $i=1,2,3$. Then the Palais symmetry principle [38] implies that the critical point of f restricted on Λ_1 is also a critical point of f on E_1.

Lemma 3.1. *The limit curve $q(t)=(q_1(t),q_2(t),\ldots,q_{N+3}(t))\in\partial\Lambda_2$ of a sequence $q^l(t)=(q^l_1(t),q^l_2(t),\ldots,q^l_{N+3}(t))\in\Lambda_2$ may either have collisions between some two point masses or has the same winding number (i.e. $deg(q_i(t)-q_j(t))=k_1$, $1\le i\ne j\le N$; $deg(q_{i'}(t)-q_{j'}(t))=k_2$, $N+1\le i'\ne j'\le N+3$).*

The proof is similar to that of Lemma 3.2 in [18], we omit it.
Then by Lemma 3.1, we have

Lemma 3.2. *The critical point of minimizing the Lagrangian functional f restricted on Λ_2 is also a critical point of f on E_1, then it is also the solution of (1.8).*

By $q_i(t)=O(-\frac{2\pi d}{r})q_i(t+\frac{1}{r})(i=1,\ldots,N+3)$, we have

$$\int_0^1 q_i(t)dt=0.$$

Then by Poincare–Wirtinger's inequality [39], we have

$$\int_0^1|\dot q_i(t)|^2dt\ge(2\pi)^2\int_0^1|q_i(t)|^2dt.$$

Hence $f(q)$ is coercive on $\bar\Lambda_2$. It is easy to see that $\bar\Lambda_2$ is a weakly closed subset. Fatou's lemma implies that $f(q)$ is a weakly lower semi-continuous, then $f(q)$ attains $\inf\{f(q)|q\in\bar\Lambda_2\}$.

In the following, we prove that the minimizer of f is a non-collision solution of the system (1.8).

Since $\sum_{i=1}^{N+3}q_i=0$, by the Lagrangian identity, we have

$$f(q)=\frac{1}{N+3}\sum_{1\le i<j\le N+3}\int_0^1(\frac{1}{2}|\dot q_i-\dot q_j|^2+\frac{N+3}{|q_i-q_j|})dt. \tag{3.4}$$

Notice that each term on the right hand side of (3.4) is a Lagrangian action for a suitable two body problem, which is a key step for the lower bound estimate on the collision set.

We estimate the infimum of the action functional on the collision set. Since the symmetry for a two-body problem implies that the Lagrangian action on a collision solution is greater than that on the non-collision solution, and the more collisions there are, the greater the Lagrangian is. We only assume that the two bodies collide at some moment t_0, without loss of generality, let $t_0=0$, we will sufficiently use the symmetries of collision orbits.

since $q\in\bar\Lambda_2$, we have

$$q_i(t+\frac{1}{r})=O(\frac{2\pi d}{r})q_i(t),\ i=1,\ldots,N+3; \tag{3.5}$$
$$q_{i+1}(t)=q_i(t+\frac{1}{N}),i=1,\ldots,N-1,\ \ q_1(t)=q_N(t+\frac{1}{N}); \tag{3.6}$$
$$q_{N+2}(t)=q_{N+1}(t+\frac{1}{3}),\ q_{N+3}(t)=q_{N+2}(t+\frac{1}{3}),\ q_{N+1}(t)=q_{N+3}(t+\frac{1}{3}); \tag{3.7}$$
$$q_i(t+\frac{1}{3})=q_i(t),\ i=1,\ldots,N; \tag{3.8}$$
$$q_j(t+\frac{1}{N})=q_j(t),\ j=N+1,N+2,N+3. \tag{3.9}$$

Case 1: q_1,q_2 collide at $t=0$.
By (3.5), we can deduce q_1,q_2 collide at $t=\frac{i}{r}$, $i=0,\ldots,r-1$.
Furthermore, by (3.8), we can deduce q_1,q_2 collide at

$$t=\frac{i}{r},\ \frac{i}{r}+\frac{1}{3},\ \frac{i}{r}+\frac{2}{3}(mod\ 1). \tag{3.10}$$

From (3.6) and (3.10), we have

$$q_2,q_3\text{ collide at }\frac{i}{r}+\frac{N-1}{N},\ \ \frac{i}{r}+\frac{1}{3}+\frac{N-1}{N},\ \ \frac{i}{r}+\frac{2}{3}+\frac{N-1}{N}(mod\ 1),\ \ i=0,\ldots,r-1,$$
$$q_3,q_4\text{ collide at }\frac{i}{r}+\frac{N-2}{N},\ \ \frac{i}{r}+\frac{1}{3}+\frac{N-2}{N},\ \ \frac{i}{r}+\frac{2}{3}+\frac{N-2}{N}(mod\ 1),\ \ i=0,\ldots,r-1,$$

$$\vdots$$

$$q_{N-1}, q_N \text{ collide at } \frac{i}{r}+\frac{2}{N},\ \frac{i}{r}+\frac{1}{3}+\frac{2}{N},\ \frac{i}{r}+\frac{2}{3}+\frac{2}{N}(mod\,1),\ i=0,\dots,r-1,$$

$$q_N, q_1 \text{ collide at } \frac{i}{r}+\frac{1}{N},\ \frac{i}{r}+\frac{1}{3}+\frac{1}{N},\ \frac{i}{r}+\frac{2}{3}+\frac{1}{N}(mod\,1),\ i=0,\dots,r-1.$$

Lemma 3.3. *$\forall\ 0\le i, j\le r-1,\ 0\le k\le 2,\ (i-j)^2+k^2\ne 0$, we have*

$$\frac{i}{r}\ne\frac{j}{r}+\frac{k}{3}(mod\,1) \tag{3.11}$$

Proof. If there exist $0\le i_0, j_0\le r-1$, $0\le k_0\le 2$, $(i_0-j_0)^2+k_0^2\ne 0$ such that

$$\frac{i_0}{r}=\frac{j_0}{r}+\frac{k_0}{3}(mod\,1).$$

Then we have

$$1|(\frac{j_0}{r}+\frac{k_0}{3}-\frac{i_0}{r}).$$

Since

$$\frac{j_0}{r}+\frac{k_0}{3}-\frac{i_0}{r}\ge-\frac{r-1}{r}=-1+\frac{1}{r}>-1,$$

and

$$\frac{j_0}{r}+\frac{k_0}{3}-\frac{i_0}{r}\le\frac{r-1}{r}+\frac{2}{3}<2,$$

we can deduce

$$\frac{j_0}{r}+\frac{k_0}{3}-\frac{i_0}{r}=0 \quad\text{or}\quad \frac{j_0}{r}+\frac{k_0}{3}-\frac{i_0}{r}=1.$$

If $\frac{j_0}{r}+\frac{k_0}{3}-\frac{i_0}{r}=0$, then $3(i_0-j_0)=k_0r$. When $k_0=0$, we get $i_0=j_0$, which is a contradiction with our assumptions on the i_0, j_0, k_0; when $k_0\ne 0$, notice $0<k_0\le 2$, we can deduce $3|r$, which is a contradiction since $(r,3)=1$.

If $\frac{j_0}{r}+\frac{k_0}{3}-\frac{i_0}{r}=1$, then $3(j_0-i_0)=(3-k_0)r$. When $k_0=0$, we get $r=j_0-i_0$, which is a contradiction since $-r+1\le j_0-i_0\le r-1$; when $k_0\ne 0$, notice $1\le 3-k_0<3$, we can deduce $3|r$, which is also a contradiction since $(r,3)=1$. □

By (3.10) and Lemma 3.3, we know that q_1, q_2 collide at

$$t_i=\frac{i}{3r},\ i=0,\dots,3r-1. \tag{3.12}$$

Then by (2.3) and (3.12), we have

$$\begin{aligned}&\int_0^1(\frac{1}{2}|\dot q_1(t)-\dot q_2(t)|^2+\frac{N+3}{|q_1(t)-q_2(t)|})dt\\ &=\sum_{i=0}^{3r-1}\int_{t_i}^{t_{i+1}}(\frac{1}{2}|\dot q_1(t)-\dot q_2(t)|^2+\frac{N+3}{|q_1(t)-q_2(t)|})dt\\ &\ge\frac{3}{2}\times(2\pi)^{\frac{2}{3}}(N+3)^{\frac{2}{3}}3r(\frac{1}{3r})^{\frac{1}{3}}.\end{aligned} \tag{3.13}$$

From (3.6) and (3.12), we have

$$q_2, q_3 \text{ collide at } \frac{i}{3r}+\frac{N-1}{N}(mod\,1),\ \ i=0,\dots,3r-1,$$

$$q_3, q_4 \text{ collide at } \frac{i}{3r}+\frac{N-2}{N}(mod\,1),\ \ i=0,\dots,3r-1,$$

$$\vdots$$

$$q_{N-1}, q_N \text{ collide at } \frac{i}{3r}+\frac{2}{N}(mod\,1),\ \ i=0,\dots,3r-1, \tag{3.14}$$

$$q_N, q_1 \text{ collide at } \frac{i}{3r}+\frac{1}{N}(mod\,1),\ \ i=0,\dots,3r-1. \tag{3.15}$$

Lemma 3.4. *$\forall\, 0 \le i, i' \le 3r-1$, $1 \le j, j' \le N-1$, $(i-i')^2 + (j-j')^2 \ne 0$, we have*

$$\frac{i}{3r} + \frac{j}{N} \ne \frac{i'}{3r} + \frac{j'}{N} (mod\, 1). \tag{3.16}$$

The proof is similar to Lemma 3.3.

Remark 3.1. From Lemma 3.4, $\forall\, 0 \le i, i' \le r-1,\ \ 1 \le j, j' \le N-1,\ \ 0 \le k, k' \le 2, (i-i')^2 + (j-j')^2 + (k-k')^2 \ne 0$, we have

$$\frac{i}{r} + \frac{j}{N} + \frac{k}{3} \ne \frac{i'}{r} + \frac{j'}{N} + \frac{k'}{3} (mod\, 1).$$

By (2.3), Lemma 3.4 and (3.15), we have

$$\begin{aligned} &\int_0^1 (\frac{1}{2}|\dot{q}_{j+1}(t) - \dot{q}_{j+2}(t)|^2 + \frac{N+3}{|q_{j+1}(t) - q_{j+2}(t)|})dt \\ &\ge \frac{3}{2} \times (2\pi)^{\frac{2}{3}} (N+3)^{\frac{2}{3}} 3r (\frac{1}{3r})^{\frac{1}{3}},\ \ (j = 1, \ldots, N-2), \end{aligned} \tag{3.17}$$

$$\begin{aligned} &\int_0^1 (\frac{1}{2}|\dot{q}_N(t) - \dot{q}_1(t)|^2 + \frac{N+3}{|q_N(t) - q_1(t)|})dt \\ &\ge \frac{3}{2} \times (2\pi)^{\frac{2}{3}} (N+3)^{\frac{2}{3}} 3r (\frac{1}{3r})^{\frac{1}{3}}. \end{aligned} \tag{3.18}$$

Let

$$\begin{aligned} M_1 = &\sum_{j=0}^{N-2} \int_0^1 (\frac{1}{2}|\dot{q}_{j+1}(t) - \dot{q}_{j+2}(t)|^2 + \frac{N+3}{|q_{j+1}(t) - q_{j+2}(t)|})dt + \\ &\int_0^1 (\frac{1}{2}|\dot{q}_N(t) - \dot{q}_1(t)|^2 + \frac{N+3}{|q_N(t) - q_1(t)|})dt. \end{aligned}$$

Notice that $\forall\, 1 \le i \le N,\ N+1 \le j \le N+3,\ \ \int_0^{\frac{1}{3}} q_i(t)dt = 0,\ \ \int_0^{\frac{1}{N}} q_j(t)dt = 0$, then by (3.13), (3.17), (3.18), (2.3), (2.4) we have

$$\begin{aligned} f(q) &= \frac{1}{N+3} \sum_{1 \le i < j \le N+3} \int_0^1 (\frac{1}{2}\, |\dot{q}_i(t) - \dot{q}_j(t)|^2 + \frac{N+3}{|q_i(t) - q_j(t)|})dt \\ &= \frac{1}{N+3} \{\ M_1 + [\sum_{1 \le i < j \le N} \int_0^1 (\frac{1}{2}\, |\dot{q}_i(t) - \dot{q}_j(t)|^2 + \frac{N+3}{|q_i(t) - q_j(t)|})dt - M_1\ ] + \\ &\qquad \sum_{1 \le i \le N, 1 \le j \le 3} \int_0^1 (\frac{1}{2}\, |\dot{q}_i(t) - \dot{q}_{N+j}(t)|^2 + \frac{N+3}{|q_i(t) - q_{N+j}(t)|})dt + \\ &\qquad \sum_{N+1 \le i < j \le N+3} \int_0^1 (\frac{1}{2}\, |\dot{q}_i(t) - \dot{q}_j(t)|^2 + \frac{N+3}{|q_i(t) - q_j(t)|})dt\ \} \\ &\ge \frac{3}{2} \times (\frac{4\pi^2}{N+3})^{\frac{1}{3}} [\, N \times 3r (\frac{1}{3r})^{\frac{1}{3}} + 3 \times (\frac{1}{3})^{\frac{1}{3}} (C_N^2 - N) + 3N + 3N (\frac{1}{N})^{\frac{1}{3}}\ ] \\ &\triangleq A. \end{aligned} \tag{3.19}$$

In the following cases, we firstly study the cases under N is even.

Case 2: $q_1, q_{k+2} (k = 1, \ldots, \frac{N}{2} - 2)$ collide at $t = 0$.

By (3.5), we can deduce $q_1, q_{k+2} (k = 1, \ldots, \frac{N}{2} - 2)$ collide at $t = \frac{i}{r},\ i = 0, \ldots, r-1$.

Then by (3.8), q_1, q_{k+2} collide at

$$t = \frac{i}{r},\ \ \frac{i}{r} + \frac{1}{3},\ \ \frac{i}{r} + \frac{2}{3} (mod\, 1),\ \ i = 0, \ldots, r-1. \tag{3.20}$$

From Lemma 3.3, we get q_1, q_{k+2} collide at

$$t = \frac{i}{3r}, i = 0, \ldots, 3r-1. \tag{3.21}$$

Then by (3.8), we have

$$
\begin{aligned}
&q_2, q_{k+3} \text{ collide at } t=\frac{i}{3r}+\frac{N-1}{N}(mod\, 1),\ i=0,\ldots,3r-1,\\
&q_3, q_{k+4} \text{ collide at } t=\frac{i}{3r}+\frac{N-2}{N}(mod\, 1),\ i=0,\ldots,3r-1,\\
&\vdots\\
&q_{N-k-1}, q_N \text{ collide at } t=\frac{i}{3r}+\frac{k+2}{N}(mod\, 1),\ i=0,\ldots,3r-1,\\
&q_{N-k}, q_1, \text{ collide at } t=\frac{i}{3r}+\frac{k+1}{N}(mod\, 1),\ i=0,\ldots,3r-1,\\
&q_{N-k+1}, q_2 \text{collide at } t=\frac{i}{3r}+\frac{k}{N}(mod\, 1),\ i=0,\ldots,3r-1,\\
&\vdots\\
&q_N, q_{k+1} \text{collide at } t=\frac{i}{3r}+\frac{1}{N}(mod\, 1),\ i=0,\ldots,3r-1.
\end{aligned}
\tag{3.22}
$$

Then by (2.3), (2.4), Lemma 3.3, Lemma 3.4, (3.21)–(3.22), we have

$$
\begin{aligned}
f(q) &\geq \frac{3}{2}\times\left(\frac{4\pi^2}{N+3}\right)^{\frac{1}{3}}\left[N\times 3r\left(\frac{1}{3r}\right)^{\frac{1}{3}}+3\times\left(\frac{1}{3}\right)^{\frac{1}{3}}(C_N^2-N)+3N+3N\left(\frac{1}{N}\right)^{\frac{1}{3}}\right]\\
&=A.
\end{aligned}
\tag{3.23}
$$

Case 3: $q_1, q_{\frac{N}{2}+1}$ collide at $t=0$.
By (3.5), (3.6), (3.8), $q_1, q_{\frac{N}{2}+1}$ collide at

$$
\begin{aligned}
t=&\frac{i}{r},\ \frac{i}{r}+\frac{1}{3},\ \frac{i}{r}+\frac{2}{3},\\
&\frac{i}{r}+\frac{\frac{N}{2}}{N},\ \frac{i}{r}+\frac{1}{3}+\frac{\frac{N}{2}}{N},\ \frac{i}{r}+\frac{2}{3}+\frac{\frac{N}{2}}{N}(mod\, 1), i=0,\ldots,r-1.
\end{aligned}
\tag{3.24}
$$

Simplify (3.24), we get $q_1, q_{\frac{N}{2}+1}$ collide at

$$
t=\frac{i}{r}+\frac{j}{6},\ i=0,\ldots,r-1,\ j=0,\ldots,5
\tag{3.25}
$$

Lemma 3.5. $\forall\, 0\leq i,i'\leq r-1, 0\leq j,j'\leq 5, (i-i')^2+(j-j')^2\neq 0$, *we have*

$$
\frac{i}{r}+\frac{j}{6}\neq\frac{i'}{r}+\frac{j'}{6}(mod\, 1)
\tag{3.26}
$$

Proof. If there exist $0\leq i_0,i_1\leq r-1,\ 0\leq j_0,j_1\leq 5, (i_0-i_1)^2+(j_0-j_1)^2\neq 0$ such that

$$
\frac{i_0}{r}+\frac{j_0}{6}=\frac{i_1}{r}+\frac{j_1}{6}(mod 1)
\tag{3.27}
$$

Since

$$
\begin{aligned}
&\frac{i_1}{r}+\frac{j_1}{6}-\frac{i_0}{r}-\frac{j_0}{6}\geq -\frac{r-1}{r}-\frac{5}{6}>-2,\\
&\frac{i_1}{r}+\frac{j_1}{6}-\frac{i_0}{r}-\frac{j_0}{6}\leq \frac{r-1}{r}+\frac{5}{6}<2,
\end{aligned}
$$

then we deduce

$$
\frac{i_1}{r}+\frac{j_1}{6}-\frac{i_0}{r}-\frac{j_0}{6}=-1,\ \text{ or }\ \frac{i_1}{r}+\frac{j_1}{6}-\frac{i_0}{r}-\frac{j_0}{6}=0,\ \text{ or }\ \frac{i_1}{r}+\frac{j_1}{6}-\frac{i_0}{r}-\frac{j_0}{6}=1.
$$

If $\frac{i_1}{r}+\frac{j_1}{6}-\frac{i_0}{r}-\frac{j_0}{6}=-1$, we have $r(6+j_1-j_0)=6(i_0-i_1)$. When $i_0=i_1$, which is a contradiction since $r(6+j_1-j_0)\neq 0$; when $i_0\neq i_1$ and $j_0=j_1$, we can deduce $r=i_0-i_1$, which is a contradiction since $-r+1\leq i_0-i_1\leq r-1$; when $i_0\neq i_1$ and $j_0\neq j_1$, we can deduce $6|r$, which is a contradiction since $(r,3)=1$.

We can use similar arguments to prove $\frac{i_1}{r}+\frac{j_1}{6}-\frac{i_0}{r}-\frac{j_0}{6}\neq 0$ and $\frac{i_1}{r}+\frac{j_1}{6}-\frac{i_0}{r}-\frac{j_0}{6}\neq 1$. $\square$

From (3.5) and (3.26), we can deduce $q_1, q_{\frac{N}{2}+1}$ collide at

$$t_i = \frac{i}{6r}, \;\; r = 0, \ldots, 6r-1. \tag{3.28}$$

Then by (2.3) and (3.28), we have

$$\begin{aligned}&\int_0^1 (\frac{1}{2}|\dot{q}_1(t) - \dot{q}_{\frac{N}{2}+1}(t)|^2 + \frac{N+3}{|q_1(t) - q_{\frac{N}{2}+1}(t)|})dt \\ &= \sum_{i=0}^{6r-1} \int_{t_i}^{t_{i+1}} (\frac{1}{2}|\dot{q}_1(t) - \dot{q}_{\frac{N}{2}+1}(t)|^2 + \frac{N+3}{|q_1(t) - q_{\frac{N}{2}+1}(t)|})dt \\ &\geq \frac{3}{2} \times (2\pi)^{\frac{2}{3}}(N+3)^{\frac{2}{3}} 6r(\frac{1}{6r})^{\frac{1}{3}}.\end{aligned} \tag{3.29}$$

By (3.6), (3.28), we have

$$\begin{aligned}&q_2, q_{\frac{N}{2}+2}, \text{ collide at } t = \frac{i}{6r} + \frac{\frac{N}{2}-1}{N}, \; i = 0, \ldots, 6r-1, \\ &q_3, q_{\frac{N}{2}+3}, \text{ collide at } t = \frac{i}{6r} + \frac{\frac{N}{2}-2}{N}, \; i = 0, \ldots, 6r-1, \\ &\vdots \\ &q_{\frac{N}{2}}, q_N \text{ collide at } t = \frac{i}{6r} + \frac{1}{N}, \; i = 0, \ldots, 6r-1.\end{aligned} \tag{3.30}$$

Lemma 3.6. *$\forall\, 0 \leq i, i' \leq 6r-1,\ 1 \leq j, j' \leq \frac{N}{2} - 1,\ (i-i')^2 + (j-j')^2 \neq 0$, we have*

$$\frac{i}{6r} + \frac{j}{N} \neq \frac{i'}{6r} + \frac{j'}{N}. \tag{3.31}$$

The proof is similar to Lemma 3.5.

By (2.3), Lemma 3.6 and (3.30)–(3.31), we have

$$\begin{aligned}&\int_0^1 (\frac{1}{2}|\dot{q}_{j+1}(t) - \dot{q}_{\frac{N}{2}+j+1}(t)|^2 + \frac{N+3}{|q_{j+1}(t) - q_{\frac{N}{2}+j+1}(t)|})dt \\ &\geq \frac{3}{2} \times (2\pi)^{\frac{2}{3}}(N+3)^{\frac{2}{3}} 6r(\frac{1}{6r})^{\frac{1}{3}} \quad (j = 1, \ldots, \frac{N}{2} - 1).\end{aligned} \tag{3.32}$$

Let

$$M_2 = \sum_{j=0}^{\frac{N}{2}-1} \int_0^1 (\frac{1}{2}\,|\dot{q}_{j+1}(t) - \dot{q}_{\frac{N}{2}+j+1}(t)|^2 + \frac{N+3}{|q_{j+1}(t) - q_{\frac{N}{2}+j+1}(t)|})dt.$$

Then from (2.3), (2.4), Lemma 3.6, (3.29) and (3.32), we obtain

$$\begin{aligned}f(q) &= \frac{1}{N+3} \sum_{1\leq i<j\leq N+3} \int_0^1 (\frac{1}{2}\,|\dot{q}_i(t) - \dot{q}_j(t)|^2 + \frac{N+3}{|q_i(t) - q_j(t)|})dt \\ &= \frac{1}{N+3}\{\, M_2 + [\sum_{1\leq i<j\leq N} \int_0^1 (\frac{1}{2}\,|\dot{q}_i(t) - \dot{q}_j(t)|^2 + \frac{N+3}{|q_i(t) - q_j(t)|})dt - M_2] + \\ &\qquad \sum_{1\leq i\leq N, 1\leq j\leq 3} \int_0^1 (\frac{1}{2}\,|\dot{q}_i(t) - \dot{q}_{N+j}(t)|^2 + \frac{N+3}{|q_i(t) - q_{N+j}(t)|})dt + \\ &\qquad \sum_{N+1\leq i<j\leq N+3} \int_0^1 (\frac{1}{2}\,|\dot{q}_i(t) - \dot{q}_j(t)|^2 + \frac{N+3}{|q_i(t) - q_j(t)|})dt \,\} \\ &\geq \frac{3}{2} \times (\frac{4\pi^2}{N+3})^{\frac{1}{3}} [\, \frac{N}{2} \times 6r(\frac{1}{6r})^{\frac{1}{3}} + 3 \times (\frac{1}{3})^{\frac{1}{3}}(C_N^2 - \frac{N}{2}) + 3N + 3N(\frac{1}{N})^{\frac{1}{3}}\,] \\ &\triangleq B.\end{aligned} \tag{3.33}$$

Finally, we study the cases under N is odd.

Case 2′: $q_1, q_{k+2}(k=1,\dots,\frac{N+1}{2}-2)$ collide at $t=0$.
By (3.5), (3.8), $q_1, q_{k+2}(k=1,\dots,\frac{N+1}{2}-2)$ collide at

$$t=\frac{i}{r},\ \frac{i}{r}+\frac{1}{3},\ \frac{i}{r}+\frac{2}{3}(mod\,1),\ i=0,\dots,r-1, \tag{3.34}$$

from Lemma 3.3, we get $q_1, q_{k+2}(k=1,\dots,\frac{N+1}{2}-2)$ collide at

$$t=\frac{i}{3r},\ i=0,\dots,3r-1, \tag{3.35}$$

then by (3.6), we have

$$\begin{aligned}
&q_2, q_{k+3} \text{ collide at } t=\frac{i}{3r}+\frac{N-1}{N}(mod\,1),\ i=0,\dots,3r-1,\\
&q_3, q_{k+4} \text{ collide at } t=\frac{i}{3r}+\frac{N-2}{N}(mod\,1),\ i=0,\dots,3r-1,\\
&\vdots\\
&q_{N-k-1}, q_N \text{ collide at } t=\frac{i}{3r}+\frac{k+2}{N}(mod\,1),\ i=0,\dots,3r-1,\\
&q_{N-k}, q_1, \text{ collide at } t=\frac{i}{3r}+\frac{k+1}{N}(mod\,1),\ i=0,\dots,3r-1,\\
&q_{N-k+1}, q_2 \text{collide at } t=\frac{i}{3r}+\frac{k}{N}(mod\,1),\ i=0,\dots,3r-1,\\
&\vdots\\
&q_N, q_{k+1} \text{collide at } t=\frac{i}{3r}+\frac{1}{N}(mod\,1),\ i=0,\dots,3r-1.
\end{aligned} \tag{3.36}$$

Then by (2.3), (2.4), Lemma 3.4, (3.35) and (3.36), we have

$$\begin{aligned}
f(q)\geq&\frac{3}{2}\times(\frac{4\pi^2}{N+3})^{\frac{1}{3}}[N\times 3r(\frac{1}{3r})^{\frac{1}{3}}+3\times(\frac{1}{3})^{\frac{1}{3}}(C_N^2-N)+3N+3N(\frac{1}{N})^{\frac{1}{3}}]\\
&=A.
\end{aligned} \tag{3.37}$$

Case 4: q_{N+1}, q_1 collide at $t=0$.
By (3.5), we have
q_{N+1}, q_1 collide at

$$t=\frac{i}{r},\quad i=0,\dots,r-1. \tag{3.38}$$

Then by (2.3) and (3.37), we have

$$\begin{aligned}
&\int_0^1(\frac{1}{2}|\dot{q}_1(t)-\dot{q}_{N+1}(t)|^2+\frac{N+3}{|q_1(t)-q_{N+1}(t)|})dt\\
&\quad=\sum_{i=0}^{r-1}\int_{t_i}^{t_{i+1}}(\frac{1}{2}|\dot{q}_1(t)-\dot{q}_{N+1}(t)|^2+\frac{N+3}{|q_1(t)-q_{N+1}(t)|})dt\\
&\quad\geq\frac{3}{2}\times(4\pi^2)(N+3)^{\frac{2}{3}}r(\frac{1}{r})^{\frac{1}{3}}.
\end{aligned} \tag{3.39}$$

From (3.38), (3.5)–(3.9), we can obtain
q_{N+2}, q_1, collide at $t=\frac{i}{r}+\frac{2}{3}(mod\,1)$, $\quad q_{N+3}, q_1$ collide at $t=\frac{i}{r}+\frac{1}{3}(mod\,1)$, $i=0,\dots,r-1$,
q_{N+1}, q_2 collide at $\frac{i}{r}+\frac{N-1}{N}(mod\,1)$, $\quad q_{N+2}, q_2$ collide at $\frac{i}{r}+\frac{N-1}{N}+\frac{2}{3}(mod\,1)$, $\quad q_{N+3}, q_2$ collide at $\frac{i}{r}+\frac{N-1}{N}+\frac{1}{3}(mod\,1)$, $i=0,\dots,r-1$,
$\vdots$
q_{N+1}, q_{N-1} collide at $\frac{i}{r}+\frac{2}{N}(mod\,1)$, $\quad q_{N+2}, q_{N-1}$ collide at $\frac{i}{r}+\frac{2}{N}+\frac{2}{3}(mod\,1)$, $\quad q_{N+3}, q_{N-1}$ collide at $\frac{i}{r}+\frac{2}{N}+\frac{1}{3}(mod\,1)$, $i=0,\dots,r-1$,
q_{N+1}, q_N collide at $\frac{i}{r}+\frac{1}{N}(mod\,1)$, $\quad q_{N+2}, q_N$ collide at $\frac{i}{r}+\frac{1}{N}+\frac{2}{3}(mod\,1)$, $\quad q_{N+3}, q_N$ collide at $\frac{i}{r}+\frac{1}{N}+\frac{1}{3}(mod\,1)$, $i=0,\dots,r-1$.

Then by (2.3), Lemma 3.3 and Remark 3.1, we have $\forall\, 0 \le i \le r-1,\ 1 \le j \le 3$,

$$\int_0^1 (\frac{1}{2}|\dot{q}_i(t) - \dot{q}_{N+j}(t)|^2 + \frac{N+3}{|q_i(t) - q_{N+j}(t)|})dt \geq \frac{3}{2} \times (4\pi^2)(N+3)^{\frac{2}{3}} r(\frac{1}{r})^{\frac{1}{3}}. \tag{3.40}$$

By (2.3), (2.4) and (3.40), we have

$$\begin{aligned} f(q) &= \frac{1}{N+3} \sum_{1\le i<j\le N+3} \int_0^1 (\frac{1}{2}\, |\dot{q}_i(t) - \dot{q}_j(t)|^2 + \frac{N+3}{|q_i(t) - q_j(t)|})dt \\ &= \frac{1}{N+3}(\sum_{\substack{1\le i\le N\\ 1\le j\le 3}} \int_0^1 (\frac{1}{2}|\dot{q}_i(t) - \dot{q}_{N+j}(t)|^2 + \frac{N+3}{|q_i(t) - q_{N+j}(t)|})dt + \\ &\qquad \sum_{1\le i<j\le N} \int_0^1 (\frac{1}{2}\, |\dot{q}_i(t) - \dot{q}_j(t)|^2 + \frac{N+3}{|q_i(t) - q_j(t)|})dt + \\ &\qquad \sum_{N+1\le i<j\le N+3} \int_0^1 (\frac{1}{2}\, |\dot{q}_i(t) - \dot{q}_j(t)|^2 + \frac{N+3}{|q_i(t) - q_j(t)|})dt\ ) \\ &\geq \frac{3}{2} \times (\frac{4\pi^2}{N+3})^{\frac{1}{3}} [\, 3N \times r(\frac{1}{r})^{\frac{1}{3}} + 3 \times (\frac{1}{3})^{\frac{1}{3}} C_N^2 + 3N(\frac{1}{N})^{\frac{1}{3}}\,] \\ &\triangleq C. \end{aligned} \tag{3.41}$$

Case 5: q_{N+1}, q_{N+2} collide at $t = 0$.

Then by (3.5), (3.9), we deduce

q_{N+1}, q_{N+2} collide at

$$t = \frac{i}{r} + \frac{j}{N}(mod\, 1),\ i = 0, \dots, r-1,\ j = 0, \dots, N-1. \tag{3.42}$$

From Remark 3.1, and (3.42), we can deduce q_{N+1}, q_{N+2} collide at

$$t_i = \frac{i}{Nr}, \quad i = 0, \dots, Nr-1. \tag{3.43}$$

Then we have

$$\begin{aligned} &\int_0^1 (\frac{1}{2}|\dot{q}_{N+1}(t) - \dot{q}_{N+2}(t)|^2 + \frac{N+3}{|q_{N+1}(t) - q_{N+2}(t)|})dt \\ &= \sum_{i=0}^{Nr-1} \int_{t_i}^{t_{i+1}} (\frac{1}{2}|\dot{q}_{N+1}(t) - \dot{q}_{N+2}(t)|^2 + \frac{N+3}{|q_{N+1}(t) - q_{N+2}(t)|})dt \\ &\geq \frac{3}{2} \times (4\pi^2)(N+3)^{\frac{2}{3}} Nr(\frac{1}{Nr})^{\frac{1}{3}}. \end{aligned} \tag{3.44}$$

By (3.7), we deduce q_{N+2}, q_{N+3}, collide at

$$t = \frac{i}{Nr} + \frac{2}{3}, \quad i = 0, \dots, Nr-1, \tag{3.45}$$

q_{N+3}, q_{N+1} collide at

$$t = \frac{i}{Nr} + \frac{1}{3}, \quad i = 0, \dots, Nr-1. \tag{3.46}$$

Then by (2.3), Remark 3.1, (3.45), and (3.46), we have

$$\int_0^1 (\frac{1}{2}|\dot{q}_{N+2}(t) - \dot{q}_{N+3}(t)|^2 + \frac{N+3}{|q_{N+2}(t) - q_{N+3}(t)|})dt \geq \frac{3}{2} \times (4\pi^2)(N+3)^{\frac{2}{3}} Nr(\frac{1}{Nr})^{\frac{1}{3}} \tag{3.47}$$

$$\int_0^1 (\frac{1}{2}|\dot{q}_{N+3}(t) - \dot{q}_{N+1}(t)|^2 + \frac{N+3}{|q_{N+3}(t) - q_{N+1}(t)|})dt \geq \frac{3}{2} \times (4\pi^2)(N+3)^{\frac{2}{3}} Nr(\frac{1}{Nr})^{\frac{1}{3}}. \tag{3.48}$$

So, by (2.3), (2.4), we get

$$
\begin{aligned}
f(q) &= \frac{1}{N+3}\sum_{1\le i<j\le N+3}\int_0^1 (\frac{1}{2}\,|\dot{q}_i(t)-\dot{q}_j(t)|^2+\frac{N+3}{|q_i(t)-q_j(t)|})dt \\
&= \frac{1}{N+3}(\sum_{N+1\le i<j\le N+3}\int_0^1 (\frac{1}{2}\,|\dot{q}_i(t)-\dot{q}_j(t)|^2+\frac{N+3}{|q_i(t)-q_j(t)|})dt+ \\
&\qquad \sum_{\substack{1\le i\le N\\ 1\le j\le 3}}\int_0^1 (\frac{1}{2}|\dot{q}_i(t)-\dot{q}_{N+j}(t)|^2\frac{N+3}{|q_i(t)-q_{N+j}(t)|})dt+ \\
&\qquad \sum_{1\le i<j\le N}\int_0^1 (\frac{1}{2}\,|\dot{q}_i(t)-\dot{q}_j(t)|^2+\frac{N+3}{|q_i(t)-q_j(t)|})dt) \\
&\ge \frac{3}{2}\times(\frac{4\pi^2}{N+3})^{\frac{1}{3}}[\,3\times Nr(\frac{1}{Nr})^{\frac{1}{3}}+3\times(\frac{1}{3})^{\frac{1}{3}}C_N^2+3N\,] \\
&\triangleq D.
\end{aligned}
\tag{3.49}
$$

When N is odd, let $\tilde{A}=\inf\{A,\ C,\ D\}$, then on the collision set, the action functional $f\ge\tilde{A}$.
When N is even, let $\tilde{B}=\inf\{A,\ B,\ C,\ D\}$, then on the collision set, the action functional $f\ge\tilde{B}$.
(1) Take $N=4$, $d=3$, $r=7$, $k_1=3$, $k_2=-4$.
We choose the following function as the test function:
Let $a>0,\ \ b>0$, and

$$
\begin{aligned}
q_i &= a(\cos(6\pi t+\frac{2\pi(i-1)}{4}),\sin(6\pi t+\frac{2\pi(i-1)}{4})),\ \ i=1,\ldots,4,\\
q_j &= b(\cos(-8\pi t+\frac{2\pi(j-5)}{3}),\sin(-8\pi t+\frac{2\pi(j-5)}{3})),\ \ j=5,6,7.
\end{aligned}
$$

We choose $a=0.2300,\ \ b=0.0880$,then

$$
A\approx 144.6215,\ B\approx 138.9586,\ C\approx 170.7479,\ \ D\approx 139.2196,\ \ \tilde{B}=138.9586,
$$
$$
f(q)\approx 135.5123<\tilde{B}.
$$

This proves that the minimizer of $f(q)$ on the closure $\bar{\Lambda}_2$ is a non-collision solution of the seven-body problem.
(2) Take $N=5$, $d=3$, $r=8$, $k_1=3$, $k_2=-5$.
We choose the following function as the test function:
Let $a>0,\ \ b>0$, and

$$
\begin{aligned}
q_i &= a(\cos(6\pi t+\frac{2\pi(i-1)}{5}),\ \sin(6\pi t+\frac{2\pi(i-1)}{5})),\ \ i=1,\ldots,5,\\
q_j &= b(\cos(-10\pi t+\frac{2\pi(j-6)}{3}),\ \sin(-10\pi t+\frac{2\pi(j-6)}{3})),\ \ j=6,7,8.
\end{aligned}
$$

We choose $a=0.2450,\ \ b=0.0760$, then

$$
A\approx 193.5057,\ C\approx 181.0305,\ \ D\approx 228.7437,\ \ \tilde{A}=181.0305,
$$
$$
f(q)\approx 175.2312<\tilde{A}.
$$

This proves that the minimizer of $f(q)$ on the closure $\bar{\Lambda}_2$ is a non-collision solution of the eight-body problem.
(3) Take $N=7$, $d=3$, $r=10$, $k_1=3$, $k_2=-7$.
We choose the following function as the test function:
Let $a>0,\ b>0$, and

$$
\begin{aligned}
q_i &= a(\cos(6\pi t+\frac{2\pi(i-1)}{7}),\sin(6\pi t+\frac{2\pi(i-1)}{7})),\ \ i=1,\ldots,7,\\
q_j &= b(\cos(-14\pi t+\frac{2\pi(j-8)}{3}),\sin(-14\pi t+\frac{2\pi(j-8)}{3})),\ \ j=8,9,10.
\end{aligned}
$$

We choose $a=0.2500,\ \ b=0.0640$, then

$$
A\approx 305.0645,\ C\approx 274.1354,\ \ D\approx 360.6557,\ \ \tilde{A}=274.1354,
$$
$$
f(q)\approx 266.6297<\tilde{A}.
$$

This proves that the minimizer of $f(q)$ on the closure $\bar{\Lambda}_2$ is a non-collision solution of the ten-body problem.

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References

[1] G. Arioli, F. Gazzola, S. Terracini, Minimization properties of Hill's orbits and application to some N-body problems, Ann. Inst. H. Poincaré Anal. Non Lineairé 17 (2000) 617–650.
[2] V. Barutello, S. Terracini, Action minimizing orbits in the N-body problem with simple choreography constraint, Nonlinearity 17 (2004) 2015–2039.
[3] V. Barutello, D. Ferrario, S. Terracini, Symmetry groups of the planar three-body problem and action-minimizing trajectories, Arch. Ration. Mech. Anal. 190 (2008) 189–226.
[4] U. Bessi, V. Coti Zelati, Symmetries and noncollision closed orbits for planar n-body-type problems, Nonlinear Anal. 16 (1991) 587–598.
[5] K.C. Chen, Action minimizing orbits in the parellelogram four-body problem with equal masses, Arch. Ration. Mech. Anal. 158 (2001) 293–318.
[6] K.C. Chen, Binary decompositions for planar N-body problems and symmetric periodic solutions, Arch. Ration. Mech. Anal. 170 (2003) 247–276.
[7] K.C. Chen, Variational methods on periodic and quasi-periodic solutions for the N-body problems, Ergodic Theory Dynam. Systems 23 (2003) 1691–1715.
[8] K.C. Chen, Existence and minimizing properties of retrograde orbits to the three-body problem with various choices of masses, Ann. of Math. 167 (2008) 325–348.
[9] A. Chenciner, Action minimizing solutions of the Newtonian n-body problem, From homology to symmetry, ICM 2002, Vol. 3, 279–294, Vol. 1, 641–643.
[10] A. Chenciner, Collisions totales Mouvements complètement paraboliques et réduction des homothéties dans le problème des n corps, Regul. Chaotic Dyn. V.3 3 (1998) 93–106.
[11] A. Chenciner, Simple non-planar periodic solutions of the n-body problem, in: Proceedings of the NDDS Conference, Kyoto, 2002.
[12] A. Chenciner, N. Desolneux, Minima de l'intégrale d'action et équilibres relatifs de n corps, C. R. Acad. Sci. Paris, sér. I 327 (1998) 193.
[13] A. Chenciner, R. Montgomery, A remarkable periodic solutions of the three-body problem in the case of equal masses, Ann. of Math 152 (2000) 881–901.
[14] A. Chenciner, A. Venturelli, Minima de l'intégrale d'action du problème newtonien de 4 corps de masses égales dans $\mathbb{R}^3$: orbites " hip-hop", Celestial Mech. 77 (2000) 139–152.
[15] V. Coti Zelati, The periodic solutions of N-body type problems, Ann. Inst. H. Poincaré Anal. Non Lineairé 7 (1990) 477–492.
[16] M. Degiovanni, F. Giannoni, Dynamical systems with Newtonian type potentials, Ann. Sc. Norm. Super. Pisa 15 (1989) 467–494.
[17] C.H. Deng, S.Q. Zhang, New periodic solutions for planar N + 2-body problems, J. Geom. Phys. 61 (2011) 2369–2377.
[18] C.H. Deng, S.Q. Zhang, Q. Zhou, Rose solutions with three petals for planar 4-body problems, Sci. China Math. 53 (2010) 3085–3094.
[19] D. Ferrario, S. Terracini, On the existence of collisionless equivariant minimizers for the classical n-body problem, Invent. Math. 155 (2004) 305–362.
[20] D. Ferrario, Transitive decomposition of symmetry groups for the n-body problem, Adv. Math. 213 (2007) 763–784.
[21] W.B. Gordon, A minimizing property of Keplerian orbits, Amer. J. Math. 99 (1977) 961–971.
[22] W.B. Gordon, Conservative dynamical systems involving strong forces, Trans. Amer. Math. Soc. 204 (1975) 113–135.
[23] Y.M. Long, S.Q. Zhang, Geometric characterizations for variational minimization solutions of the 3-body problems, Act Math. Sinica 16 (2000) 579–592.
[24] C. Marchal, How the method of minimization of action avoids singularities, Cel. Mech. Dyn. Astr. 83 (2002) 325–353.
[25] R. Montgomery, The N-body problem, the braid group, and action-minimizing periodic solutions, Nonlinearity 11 (1998) 363–376.
[26] C. Moore, Braids in classical gravity, Phys. Rev. Lett. 70 (1993) 3675–3679.
[27] C. Simó, Dynamical properties of the figure eight solution of the three-body problem, in: Contemp. Math., vol. 292, AMS, Providence, RI, 2002, pp. 209–228.
[28] C. Simó, New families of solutions in N-body problems, Progress Math. 21 (2001) 101–115.
[29] X. Su, S.Q. Zhang, New periodic solutions for planar five-body and seven-body problems, Rep. Math. Phys. 70 (2012) 27–38.
[30] S. Terracini, A. Venturelli, Symmetric trajectories for the 2N-body problem with equal masses, Arch. Ration. Mech. Anal. 184 (2007) 465–493.
[31] A. Venturelli, Une caractérisation variationnelle des solutions de Lagrange du problème plan des trois corps, C. R. Math. Acad. Sci. Paris 332 (2001) 641–644.
[32] S.Q. Zhang, Periodic soluitons of N-body problems, in: K.C. Chang, Y.M. Long (Eds.), Progress in Nonlinear Analysis, World Scientific, 2000, pp. 423–443.
[33] S.Q. Zhang, Q. Zhou, A minimizing property of Lagrangian solutions, Acta Math. Sinica 17 (2001) 497–500.
[34] S.Q. Zhang, Q. Zhou, Variational methods for the choregraphy solution to the three-body problem, Sci. China 45 (2002) 594–597.
[35] S.Q. Zhang, Q. Zhou, Nonplanar and noncollision periodic solutions for N-body problems, Discrete Contin. Dyn. Syst. 10 (2004) 679–685.
[36] S.Q. Zhang, Q. Zhou, Y. Liu, New periodic solutions for 3-body problems, Cel. Mech. Dyn. Astr. 88 (2004) 365–378.
[37] G. Buttazzo, M. Giaquinta, S. Hildebrandt, One-Dimensional Variational Problems, Oxford University Press, 1998.
[38] R. Palais, The principle of symmetric criticality, Comm. Math. Phys. 69 (1979) 19–30.
[39] W.P. Ziemer, Weakly Differentiable Functions, Springer, 1989.