Journal of Geometry and Physics 134 (2018) 119–132

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/geomphys

Generalised Seiberg–Witten equations and almost-Hermitian geometry

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ARTICLE INFO

Article history:
Received 15 November 2017
Received in revised form 1 August 2018
Accepted 18 August 2018
Available online xxxx

MSC:
primary 53C26
53B35

Keywords:
Spinor
Four-manifold
HyperKähler manifolds
Generalised Seiberg–Witten
Almost-complex geometry

ABSTRACT

In this article, we study a generalisation of the Seiberg–Witten equations, replacing the spinor representation with a hyperKähler manifold equipped with certain symmetries. Central to this is the construction of a (non-linear) Dirac operator acting on the sections of the non-linear fibre-bundle. For hyperKähler manifolds admitting a hyperKähler potential, we derive a transformation formula for the Dirac operator under the conformal change of metric on the base manifold.

As an application, we show that when the hyperKähler manifold is of dimension four, then, away from a singular set, the equations can be expressed as a second order PDE in terms of almost-complex structure on the base manifold, and a conformal factor. This extends a result of Donaldson to generalised Seiberg–Witten equations.

1. Introduction

Let X be a 4-dimensional, oriented, smooth, Riemannian manifold and let $Q \to X$ be a Spin-structure. A spinor bundle over X is a vector bundle associated to Q, with typical fibre $\mathbb{H}$. The idea for generalisation is to replace the spinor representation with a hyperKähler manifold (M, g_M, I_1, I_2, I_3) equipped with an isometric action of $\mathrm{Sp}(1)$ (or $\mathrm{SO}(3)$) which *permutes* the complex structures on M. We will often refer to M as the *target hyperKähler manifold*. The sections of the non-linear fibre-bundle now play the role of spinors. The interplay between the $\mathrm{Sp}(1)$ (or $\mathrm{SO}(3)$) action and the quaternionic structure on M allows one to define the Clifford multiplication. Composing the Clifford multiplication with the covariant derivative gives the generalised Dirac operator, which we denote by $\mathcal{D}$.

In order to define a generalisation of the Seiberg–Witten equations, we need additionally a twisting principal G-bundle $P_G \to X$, with a tri-Hamiltonian action of G on M. The action gives rise to a hyperKähler moment map $\mu : M \longrightarrow \mathfrak{sp}(1)^* \otimes \mathfrak{g}^*$. For a connection A on P_G and a spinor u, the 4-dimensional generalised Seiberg–Witten equations on X are the following system of equations

$$\begin{cases} \mathcal{D}_A u = 0 \\ F_A^+ - \mu \circ u = 0 \end{cases} \tag{1}$$

where $\mathcal{D}_A$ is a twisted Dirac operator for a connection A on P_G.

This non-linear generalisation of the Dirac operator is well-known to physicists and has been used in the study of gauged, non-linear σ-models [1]. The 3-dimensional version of Eqs. (1) was studied by Taubes [2] (see also [3]). The

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https://doi.org/10.1016/j.geomphys.2018.08.009
0393-0440/

4-dimensional generalisation was considered by Pidstrygach [4], Schumacher [5] and Haydys [6]. The moduli spaces of solutions to (1) makes for an interesting study, especially because of its application to gauge theories on manifolds with special holonomies (cf. [7,8]). Many well-known gauge-theoretic equations like the PU(2)-monopole equations [9], the Vafa–Witten equations [10], Pin(2)-monopole equations [11], the non-Abelian monopole equations [12], etc. can be treated as special cases of this generalisation.

It is possible to obtain the target hyperKähler manifold with requisite symmetries from *Swann's construction* [13,14]. Starting with a quaternionic Kähler manifold N of positive scalar curvature, Swann constructs a fibration $\mathcal{U}(N) \to N$, whose total space admits a hyperKähler structure. Such manifolds are characterised by the existence of a hyperKähler potential. Alternatively, the permuting Sp(1)-action extends to a homothetic action of $\mathbb{H}^*$. The bundle construction commutes with the hyperKähler quotient construction of Hitchin, Karlhede, Lindström and Roček [15] and the quaternionic Kähler quotient construction of Galicki and Lawson [16]. As a result, many examples of (finite dimensional) hyperKähler manifolds with homothetic $\mathbb{H}^*$-action can be obtained via hyperKähler reduction of $\mathbb{H}^n$.

With $M = \mathcal{U}(N)$, we derive a transformation formula for the generalised Dirac operator, under the conformal change of metric on the base manifold. Since $\mathcal{U}(N)$ admits a natural homothetic action of $\mathbb{R}^+$, this setting allows one to make sense of "weighted spinors".

Let $\pi_1 : P_{\mathrm{CO}(4)} \to X$ be the bundle of conformal frames with respect to the conformal class $[g_X]$ and $P_G \to X$ be a principal G-bundle over X. Assume that the action of G on M is tri-Hamiltonian. Let $\widetilde{\pi} : \widetilde{Q} \to X$ denote the conformal $\mathrm{Spin}^G(4)$-bundle, which is a double cover of $P_{\mathrm{CO}(4)} \times_X P_G$.

Theorem 1.1. *Let f be a smooth, real-valued function on X and let u be a (generalised) spinor. Consider the metric $g'_X := e^{2f} g_X$ in the conformal class $[g_X]$ and let φ' and φ be the Levi-Civita connections associated to g_X and g'_X respectively. For a fixed connection A on P_G, denote by A_φ and $A_{\varphi'}$ the corresponding lifts to $\widetilde{Q}$. Then, the associated generalised Dirac operators $\mathcal{D}_{A_\varphi}$ and $\mathcal{D}_{A_{\varphi'}}$ are related as*

$$\mathcal{D}_{A_{\varphi'}}(\mathscr{B}u) = \mathscr{B}\Big(de^{-5/2\pi_1^* f}\mathcal{D}_{A_\varphi}(e^{3/2\pi_1^* f}u)\Big) \tag{2}$$

where, $\mathscr{B}$ is the lift of the automorphism $B : P_{\mathrm{CO}(4)} \longrightarrow P_{\mathrm{CO}(4)}$, given by $p \longmapsto e^{-f}p$, and $de^{-5/2\pi_1^* f}$ is the action of $e^{-5/2\pi_1^* f}$ by differential on TM.

For $M = \mathbb{H}$, the result was proved by Hitchin [17].

Assume that $M = \mathcal{U}(N)$ is a 4-dimensional hyperKähler manifold. Using the above theorem, we show that away from a singular set, the generalised Seiberg–Witten equations can be interpreted in terms of almost-complex geometry of the underlying 4-manifold, as equations for a compatible almost-complex structure and a real-valued function which is associated to a conformal factor. Recall that on a Riemannian 4-manifold (X, g_X), the compatible almost-complex structures on X are parameterized by sections of the twistor bundle $\mathcal{Z}$, which is a sphere bundle in Λ^+. Thus the almost-complex structures can be thought of as self-dual, 2-forms Ω with $|\Omega| = 1$. An almost-complex structure gives a splitting of Λ^+ into the direct sum of the trivial bundle spanned by Ω and its orthogonal complement $\overline{K}$, where K is a complex line bundle. Since $|\Omega| = 1$, its covariant derivative is a section of $T^*X \otimes_{\mathbb{R}} \overline{K}$. Using the almost-complex structure, we get the isomorphism

$$T^*X \otimes_{\mathbb{R}} \overline{K} \cong T^*X \otimes_{\mathbb{C}} K \oplus T^*X \otimes_{\mathbb{C}} \overline{K}.$$

Moreover, the wedge product gives a complex, bi-linear map

$$T^*X \times T^*X \longrightarrow \Lambda^2 T^*X = K.$$

using which, we can identify $TX \cong T^*X \otimes_{\mathbb{C}} \overline{K}$. Thus $\nabla\Omega$ has two components: the first component in $T^*X \otimes_{\mathbb{C}} K$ is the Nijenhuis tensor and the second one in TX is $d\Omega$. Let $\langle \cdot, \cdot \rangle$ denote the obvious $\overline{K}$-valued pairing between TX and $T^*X \otimes \overline{K}$.

Let $G = \mathrm{U}(1)$ and $M = \mathcal{U}(N)$ be 4-dimensional hyperKähler manifold, which is total space of a Swann bundle, equipped with a tri-Hamiltonian action of U(1) that commutes with the permuting Sp(1)-action. We will call such an action a permuting action of $\mathrm{U}(2) \cong \mathrm{Sp}(1) \times_{\pm} \mathrm{U}(1)$.

Theorem 1.2. *Fix a metric g_X on X and let $[g_X]$ be its conformal class. Assume that M is obtained as a quotient of a flat, quaternionic space and equipped with a residual permuting action of* U(2) *from the flat space. Then, there exists a bijective correspondence between the following:*

- *pairs consisting of a metric $g'_X \in [g_X]$ and a solution (u, A) to the generalised Seiberg–Witten equations, such that the image of u does not contain a fixed point of the* U(1) *action on M*
- *pairs consisting of a metric $g''_X \in [g_X]$ and a self-dual 2-form Ω satisfying*

$$(\nabla^*\nabla\Omega)^\perp + 2\,\langle d\Omega, N_\Omega\rangle = 0, \quad \frac{3}{2}\,|N_\Omega|^2 + \frac{1}{2}\,|d\Omega|^2 + \frac{1}{2}\,s_X(g''_X) < 0 \tag{3}$$

where $s_X(g''_X)$ denotes the scalar curvature with respect to the metric g''_X.

Theorem 1.2 was proved by Donaldson [18] for the usual Seiberg–Witten equations.

Notice that first equation in the second bullet of Theorem 1.2 is nothing but a perturbation of the Euler–Lagrange equation for the energy functional

$$\int_X |\nabla \Omega|^2 . \tag{4}$$

The functional was studied by Wood [19]. Critical points of the functional correspond to a choice of "optimal" almost-complex structures, amongst all possible almost-complex structures on X.

2. Preliminaries and definitions

2.1. HyperKähler manifolds

A $4n$-dimensional Riemannian manifold (M, g_M) is *hyperKähler* if it admits a triple of almost-complex structures $I_i \in \mathrm{End}(TM)$ $i = 1, 2, 3$, which are covariantly constant with respect to the Levi-Civita connection and satisfy quaternionic relations $I_i I_j = \delta_{ijk} I_k$.

Let $\mathrm{Sp}(1)$ denote the group of unit quaternions and $\mathfrak{sp}(1)$ denote its Lie algebra. The quaternionic structure on M induces a covariantly constant endomorphism of TM with values in $\mathfrak{sp}(1)^* = (\mathfrak{Im}(\mathbb{H}))^*$.

$$I \in \Gamma(M, \mathrm{End}(TM) \otimes \mathfrak{sp}(1)^*), \quad I_\xi := \xi_1 I_1 + \xi_2 I_2 + \xi_3 I_3, \quad \xi \in \mathfrak{sp}(1). \tag{5}$$

Observe that for every $\xi \in S^2 \subset \mathfrak{Im}(\mathbb{H})$, the endomorphism I_ξ is a complex structure. In other words, M has an entire family of Kähler structures parameterized by S^2. Define the 2-form

$$\omega \in \Lambda^2 M \otimes \mathfrak{sp}(1)^*, \quad \omega_\xi(\cdot, \cdot) = g_M(I_\xi(\cdot), \cdot).$$

If $\xi \in S^2$, then ω_ξ is just the Kähler 2-form associated to I_ξ.

Definition 1. An isometric action of $\mathrm{Sp}(1)$ on M is said to be *permuting* if the induced action on the 2-sphere of complex structures is the standard action of $\mathrm{SO}(3) = \mathrm{Sp}(1)/\pm 1$ on S^2:

$$dq\, I_\xi\, dq^{-1} = I_{q\xi\bar{q}}, \quad \text{for} \;\; q \in \mathrm{Sp}(1), \;\; \xi \in \mathfrak{sp}(1), \quad \|\xi\|^2 = 1.$$

Definition 2. An isometric action of a Lie group G on M is *tri-holomorphic* or *hyperKähler*, if it preserves the hyperKähler structure

$$\eta_* I_i = I_i \eta_* \;\; i = 1, 2, 3, \quad \eta \in G.$$

In particular, G fixes the 2-sphere of complex structures on M. The action is *tri-Hamiltonian* (or hyperHamiltonian) if it is Hamiltonian with respect to each ω_i. The three moment maps can be combined together to define a single, G-equivariant map *hyperKähler moment map* $\mu : M \longrightarrow \mathfrak{sp}(1)^* \otimes \mathfrak{g}^*$, which satisfies

$$d(\langle \mu, \xi_i \otimes \eta \rangle) = \iota_{K^M_\eta}\, \omega_i, \quad \eta \in \mathfrak{g}, \quad \xi_i \in \mathfrak{sp}(1) \;\; \text{is the basis}$$

and K^M_η denotes the fundamental vector-field due to the infinitesimal action of η.

Definition 3. A *hyperKähler potential* is a smooth function $f : M \longrightarrow \mathbb{R}^+$ which is simultaneously a Kähler potential for all the three complex structures I_1, I_2, I_3.

2.2. Target hyperKähler manifold

Suppose that M is a hyperKähler manifold with a permuting action of $\mathrm{Sp}(1)$ and a tri-Hamiltonian action of a compact Lie group G which commutes with the $\mathrm{Sp}(1)$-action. Let $\varepsilon \in G$ be a central element of order two. Let $\mathbb{Z}/2\mathbb{Z} \subset \mathrm{Sp}(1) \times G$ denote the normal subgroup of order two, generated by the element $(-1, \varepsilon)$. Assume that $\mathbb{Z}/2\mathbb{Z}$ acts trivially on M so that the action of $\mathrm{Sp}(1) \times G$ descends to an action of $\mathrm{Spin}^G(3) := \mathrm{Sp}(1) \times_{\mathbb{Z}/2\mathbb{Z}} G$. We will refer to this action as a permuting action of $\mathrm{Spin}^G(3)$. An action of $\mathrm{Spin}^G(4) := (\mathrm{Sp}(1)_+ \times \mathrm{Sp}(1)_-) \times_{\mathbb{Z}/2\mathbb{Z}} G$ is said to be permuting if the action is induced by a permuting action of $\mathrm{Sp}(1) \cong \mathrm{Spin}(3)$ via the homomorphism

$$\rho : \mathrm{Spin}^G(4) \longrightarrow \mathrm{Spin}^G(4)/\mathrm{Sp}(1)_- \cong \mathrm{Spin}^G(3).$$

Note that $\mathrm{Sp}(1)_-$ acts trivially on M.

2.3. $\mathrm{Spin}^G(4)$structure

From the definition of the group $\mathrm{Spin}^G(4)$, we have the following exact sequence

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathrm{Spin}^G(4) \xrightarrow{\gamma} \mathrm{SO}(4) \times (G/\{1,\epsilon\}) \longrightarrow 0. \tag{6}$$

For simplicity, put $\overline{G} = G/\{1,\epsilon\}$. Let $P_{\mathrm{SO}(4)}$ denote the frame-bundle of X and $P \to X$ be a principal $\overline{G}$-bundle over X. A $\mathrm{Spin}^G(4)$-structure over X is a principal $\mathrm{Spin}^G(4)$-bundle $\pi : Q \to X$, which is an equivariant double cover of the bundle $P_{\mathrm{SO}(4)} \times_X P$, with respect to the map γ as defined in (6). We refer to [12] for details.

2.4. Generalised Dirac operator

We define the space of *generalised spinors* to be the space of smooth, equivariant maps

$$\mathcal{S} := C^\infty(Q, M)^{\mathrm{Spin}^G(4)} \cong \Gamma(X, Q \times_{\mathrm{Spin}^G(4)} M).$$

The Levi-Civita connection φ on $P_{\mathrm{SO}(4)}$ and a connection a on the principal P together determine a unique connection on Q. Let $\mathscr{A}$ denote the space of all connections on Q, which are the lifts of the Levi-Civita connection. We define the covariant derivative of a spinor $u \in \mathcal{S}$, with respect to a connection $\mathsf{A} \in \mathscr{A}$ by[1]

$$D_{\mathsf{A}} : C^\infty(Q, M)^{\mathrm{Spin}^G(4)} \longrightarrow \mathrm{Hom}(TQ, TM)^{\mathrm{Spin}^G(4)}_{hor}, \quad D_{\mathsf{A}}u = du + K^M_{\mathsf{A}}|_u \tag{7}$$

where $K^M_{\mathsf{A}}|_u \colon TQ \to u^*TM$ is an equivariant bundle homomorphism defined by $K^M_{\mathsf{A}}|_u(v) = K^M_{\mathsf{A}(v)}|_{u(p)}$ for $v \in T_pQ$. Denote by $\pi_{\mathrm{SO}} : Q \longrightarrow P_{\mathrm{SO}(4)}$ the projection to the frame bundle. Then, alternatively, one can view the covariant derivative as

$$D_{\mathsf{A}} : C^\infty(Q, M)^{\mathrm{Spin}^G(4)} \longrightarrow C^\infty(Q, (\mathbb{R}^4)^* \otimes TM)^{\mathrm{Spin}^G(4)}, \quad \langle D_{\mathsf{A}}u(q), w\rangle = du(q)(\widetilde{w}) \tag{8}$$

where, $w \in \mathbb{R}^4$, $\widetilde{w}$ denotes the horizontal lift of $\pi_{\mathrm{SO}}(q)(w) \in T_{\pi(q)}X$.

Clifford multiplication

The second ingredient we need to define the Dirac operator is Clifford multiplication. From (5), we can construct an action of $\mathcal{C}l^0_4 \cong \mathcal{C}l_3$ on TM as

$$\mathbb{R}^3 \cong \mathfrak{Im}(\mathbb{H}) \longrightarrow \mathrm{End}(TM), \quad h \mapsto I_h.$$

The map extends to a $\mathrm{Spin}^G(4)$-equivariant map $\mathcal{C}l_3 \longrightarrow \mathrm{End}(TM)$. Thus TM is naturally a $\mathcal{C}l^0_4$ module. Now consider $W := \mathcal{C}l_4 \otimes_{\mathcal{C}l^0_4} E$, where $E = (TM, I_1)$. Since W is a $\mathcal{C}l^0_4$-module, we get a $\mathbb{Z}_2$-graded $\mathcal{C}l_4$-module

$$W = W^+ \oplus W^-, \quad W^+ = \mathcal{C}l^0_4 \otimes_{\mathcal{C}l^0_4} E, \quad W^- = \mathcal{C}l^1_4 \otimes_{\mathcal{C}l^0_4} E.$$

More precisely, W^+ is the $\mathrm{Spin}^G(4)$-equivariant bundle TM with an action induced by ρ, whereas W^- is the $\mathrm{Spin}^G(4)$-equivariant vector bundle TM equipped with the left-action:

$$[q_+, q_-, g] \cdot w_- = I_{q_-} I_{\bar{q}_+} dq_+ dg \; w_-.$$

Identify $\mathbb{R}^4$ with $\mathbb{H}$ by mapping the standard, oriented basis (e_1, e_2, e_3, e_4) of $\mathbb{R}^4$, to $(1, \bar{i}, \bar{j}, \bar{k})$. The $\mathrm{Spin}^G(4)$-action on $\mathbb{H}$ is given by $[q_+, q_-, g] \cdot h = q_- h \bar{q}_+$. Clifford multiplication is the $\mathrm{Spin}^G(4)$-equivariant map

$$\bullet : (\mathbb{R}^4)^* \cong \mathbb{H} \longrightarrow \mathrm{End}(W^+ \oplus W^-), \quad g_{\mathbb{R}^4}(h, \cdot) \longmapsto \begin{bmatrix} 0 & -I_{\bar{h}} \\ I_h & 0 \end{bmatrix}. \tag{9}$$

Since $h \bullet h = -g_{\mathbb{R}^4}(h, h) \cdot \mathrm{id}_{W^+ \oplus W^-}$, by universality property, the map $\bullet$ extends to a map of algebras $\bullet : \mathcal{C}l_4 \longrightarrow \mathrm{End}(W^+ \oplus W^-)$. Composing $\bullet$ with the covariant derivative, we get the *generalised Dirac operator*:

$$\mathcal{D}_{\mathsf{A}}u \in C^\infty(Q, u^*W^-)^{\mathrm{Spin}^G(4)}, \quad \mathcal{D}_{\mathsf{A}}u = \sum_{i=0}^{3} e_i \bullet D_{\mathsf{A}}u(\tilde{e}_i) \tag{10}$$

where the latter expression follows from Eq. (8).

[1] The subscript *hor* implies that $D_{\mathsf{A}}u$ vanishes on vertical vector fields.

Generalised Seiberg–Witten equations

Let μ be a hyperKähler moment map for the G-action on M and a be a connection on P. Then *generalised Seiberg–Witten equations* for a pair $(u, \mathsf{A}) \in \mathcal{S} \times \mathscr{A}$, in dimension four, are

$$\begin{cases} \mathcal{D}_{\mathsf{A}} u = 0 \\ F_a^+ - \Phi(\mu \circ u) = 0 \end{cases} \tag{11}$$

where $F_a^+ \in \mathrm{Map}(Q, \Lambda_+^2(\mathbb{R}^4)^*)^{\mathrm{Spin}^G(4)}$ is the self-dual part of the curvature of a and $\Phi : \mathfrak{sp}(1)^* \longrightarrow \Lambda_+^2(\mathbb{R}^4)^*$ is the isomorphism, mapping the basis elements $\xi_l \mapsto \beta_l,\ l = 1, 2, 3$, where

$$\beta_0 = dx_0 \wedge dx_1 + dx_2 \wedge dx_3,\ \beta_1 = dx_0 \wedge dx_2 + dx_3 \wedge dx_1,\ \beta_3 = dx_0 \wedge dx_3 + dx_1 \wedge dx_2. \tag{12}$$

We will suppress the isomorphism henceforth.

3. Conformal transformation of generalised Dirac operator

This section is divided into three parts. In the first part, Section 3.1, we study metric connections for metrics in the conformal class of g_X. Namely, given the Levi-Civita connection of g_X and a metric $g'_X \in [g_X]$, we explicitly construct the Levi-Civita connection for g'_X. In the second part, Section 3.2, we give a quick review of Swann's construction. In the third part, Section 3.3, we use the results from Section 3.1 to obtain a formula for conformal transformation of the generalised Dirac operator when the target hyperKähler manifold is obtained via Swann's construction. For details on ideas used in this section, we refer the interested reader to [20].

3.1. Metric connections on conformal bundle

Fix a metric g_X on X and let $[g_X]$ denote its conformal class. Let $\pi_1 : P_{\mathrm{CO}(4)} \longrightarrow X$ denote the bundle of all conformal frames on $(X, [g_X])$. A point $p \in P_{\mathrm{CO}(4)}$ is a CO(4)-equivariant, linear isomorphism $p : \mathbb{R}^4 \longrightarrow T_{\pi_1(p)}X$. Consider the canonical one-form $\theta : P_{\mathrm{CO}(4)} \longrightarrow \mathbb{R}^4$ defined as

$$\theta_p(v) = p^{-1}\left((\pi_1)_*(v)\right),\ \ p \in P_{\mathrm{CO}(4)},\ \ v \in T_p P_{\mathrm{CO}(4)}.$$

A metric on X is a section $g_X \in \Gamma(X, S^2(T^*X))$, which can viewed as an equivariant map in $C^\infty(P_{\mathrm{CO}(4)}, S^2(\mathbb{R}^4)^*)^{\mathrm{CO}(4)}$

$$\pi_1^* g_X(\cdot,\cdot) = g_{\mathbb{R}^4}\left(\theta_p(\cdot), \theta_p(\cdot)\right).$$

For a smooth, real-valued function f on X, consider the metric $g'_X = e^{2(\pi_1^* f)} g_X$ in the conformal class of g_X. The metrics g_X and g'_X determine two isomorphic SO(4) bundles:

$$P_{\mathrm{SO}(4)} = \{p \in P_{\mathrm{CO}(4)} \mid g_{\mathbb{R}^4}(\theta_p, \theta_p) = \pi_1^* g_X(\cdot,\cdot)\}$$
$$P'_{\mathrm{SO}(4)} = \{p \in P_{\mathrm{CO}(4)} \mid g_{\mathbb{R}^4}(\theta_p, \theta_p) = e^{2(\pi_1^* f)}\pi_1^* g_X(\cdot,\cdot)\}$$

where, $g_{\mathbb{R}^4}(\cdot,\cdot)$ is the standard metric on $\mathbb{R}^4$. Let φ be a connection on $P_{\mathrm{CO}(4)}$. Then $\varphi + \theta$ define a 1-form with values in $\mathfrak{co}(4) \oplus \mathbb{R}^4$. We can extend the bracket on the Lie algebra $\mathfrak{co}(4)$ to $\mathfrak{co}(4) \oplus \mathbb{R}^4$ as

$$[A, x] = -[x, A] = Ax,\ \ [x, y] = 0,\ \text{for } x, y \in \mathbb{R}^4 \text{ and } A \in \mathfrak{co}(4).$$

This defines an affine Lie algebra which is best identified with the frame bundle of $\mathbb{R}^4$. The failure of the 1-form $\varphi + \theta$ to conform with the associated Maurer–Cartan form is measured by

$$d(\varphi + \theta) + [\varphi + \theta, \varphi + \theta] = \mathcal{R}(\varphi) + T(\varphi)$$

where

$$\mathcal{R}(\varphi) = d\varphi + \frac{1}{2}[\varphi, \varphi], \quad T(\varphi) = d\theta + [\varphi, \theta].$$

Here the entities $\mathcal{R}$ and T are horizontal 2-forms on the conformal frame bundle, which are nothing but the curvature and the torsion tensors, respectively and the Lie bracket operations are carried out simultaneously with wedging of 1-forms.

Suppose that φ is a connection on $P_{\mathrm{CO}(4)}$ satisfying

$$(d + \varphi)g_X = 0 \ \text{ and } \ (d + \varphi)\theta = 0. \tag{13}$$

Then φ is just the Levi-Civita connection for the metric g_X. Let φ' denote the Levi-Civita connection for the metric g'_X. The difference of the 2-connections is a horizontal 1-form on $P_{\mathrm{CO}(4)}$ and therefore can be written as contraction of θ with an equivariant function $\xi \in \mathrm{Hom}(\mathbb{R}^4, \mathfrak{co}(4)) \cong (\mathbb{R}^4)^* \otimes \mathfrak{co}(4)$. More precisely,

$$\left\langle \theta_p,\ \xi \right\rangle(Y) = \left\langle \theta_p(Y),\ \xi \right\rangle, \quad Y \in T_p P_{\mathrm{CO}(4)}.$$

Therefore we may write

$$\varphi' - \varphi = \langle \theta, \xi \rangle \quad \text{for some } \xi \in (\mathbb{R}^4)^* \otimes \mathfrak{co}(4). \tag{14}$$

Throughout, we will suppress the pairing with θ and simply write $\varphi' - \varphi = \xi$. Consider the covariant derivative of g'_X with respect to φ

$$(d + \varphi)(g'_X) = -e^{2(\pi_1^* f)}\, 2\,(\pi_1^* df)\, g_X. \tag{15}$$

The right hand side of the equation can be understood as follows. Define

$$f_i(p) = \pi_1^* df\,(\widetilde{p(e_i)}),$$

where, $e_i \in \mathbb{R}^4$ is the standard basis element of $\mathbb{R}^4$ and $\widetilde{p(e_i)}$ is the horizontal lift of $p(e_i)$ to $P_{\mathrm{CO}(4)}$ with respect to φ. We can write

$$\pi_1^* df(p) = \left\langle \sum_{i=1}^4 f_i(p)\, e^i,\ \theta_p \right\rangle, \qquad \sum_{i=1}^4 f_i(p)\, e^i \in (\mathbb{R}^4)^* \hookrightarrow (\mathbb{R}^4)^* \otimes \mathfrak{co}(4)$$

where e^i are the basis for $(\mathbb{R}^4)^*$. So the action of $\pi_1^* df$ is just the (left) action of $\sum_{i=1}^4 f_i\, e^i \in \mathrm{End}(\mathbb{R}^4)$.

Remark 1. The negative sign in Eq. (15) is due to the left action of $\mathrm{Aut}(\mathbb{R}^4) \curvearrowright S^2(\mathbb{R}^4)^*$, which is given by

$$S^2(\mathbb{R}^4)^* \ni g_X \longmapsto b \cdot g_X(\cdot, \cdot) := g_X(b^{-1}, b^{-1}),$$

where $b \in \mathrm{Aut}(\mathbb{R}^4)$.

It follows that $\varphi + \pi_1^* df$ is a metric connection for g'_X. But it has a non-zero torsion. Indeed

$$\left(d + \varphi + \pi_1^* df\right)\, \theta = \left\langle \sum_{i=1}^4 f_i\, e^i,\ \theta \right\rangle \wedge \theta. \tag{16}$$

Point-wise, the torsion tensor is a map

$$T(\varphi)(p) : \Lambda^2 D_p \cong \Lambda^2 \mathbb{R}^4 \xrightarrow{d\theta} \mathbb{R}^4.$$

For the connections φ and φ' on $P_{\mathrm{CO}(4)}$, the difference between their torsion tensors is

$$T(\varphi')_p\,(x \wedge y) - T(\varphi)_p\,(x \wedge y) = \frac{1}{2}(\xi_p(x)y - \xi_p(y)x), \quad x, y \in \mathbb{R}^4,$$

In terms of the CO(4)-equivariant homomorphism:

$$\delta : (\mathbb{R}^4)^* \otimes \mathfrak{co}(4) \hookrightarrow (\mathbb{R}^4)^* \otimes (\mathbb{R}^4)^* \otimes \mathbb{R}^4 \mapsto \Lambda^2(\mathbb{R}^4)^* \otimes \mathbb{R}^4 \cong \Lambda^2(\mathbb{R}^4)^* \otimes (\mathbb{R}^4)^*$$

where, the first map is the inclusion and the second one is the anti-symmeterization, we can write $T(\varphi')_p - T(\varphi)_p = -\delta\xi$. Therefore, it follows from (16) that

$$\left\langle \sum_{i=1}^4 f_i(p)\, e^i,\ \theta \right\rangle \wedge \theta = -\delta\left(\sum_{i=1}^4 f_i(p)\, e^i\right).$$

Identify $\mathfrak{so}(4) \cong \Lambda^2$ by associating the skew-symmetric endomorphism, to a pair of vectors $v, w \in \mathbb{R}^n$,

$$v \wedge w = \langle v, \cdot \rangle w - \langle w, \cdot \rangle v. \tag{17}$$

Lemma 3.1 (*[20], Prop. 2.1*)**.** *The restriction*

$$\delta|_{\mathfrak{so}(4)} \colon (\mathbb{R}^4)^* \otimes \Lambda^2(\mathbb{R}^4)^* \mapsto \Lambda^2(\mathbb{R}^4)^* \otimes (\mathbb{R}^4)^*$$

that maps the difference of two connections to the difference of their torsions is an isomorphism.

Proof. Let $a_{ijk} \in (\mathbb{R}^4)^* \otimes \Lambda^2(\mathbb{R}^4)^*$ denote the difference of Christoffel symbols of the two connections. Then, $\delta(a_{ijk}) = \frac{1}{2}(a_{ijk} - a_{jik})$. It is easily seen that if $a_{ijk} \in \ker(\delta)$, then $a_{ijk} = 0$ and hence $\delta|_{\mathfrak{so}(4)}$ is an isomorphism. □

Suppose that A is the Levi-Civita connection and B is a metric connection on $P_{\mathrm{CO}(4)}$. Then using the isomorphism $\delta|_{\mathfrak{so}(4)}$, we obtain the expression for A in terms of B. Let $B' = B - \alpha$ where $\alpha = \delta|_{\mathfrak{so}(4)}^{-1}(\delta(\xi))$. Then a straightforward computation shows that $T(B') = 0$. This is the strategy we are going to employ to express φ' in terms of φ and correction terms.

Pointwise, we can view $\sum_{i=1}^{4} f_i\, e^i$ as a 1-form with values in $(\mathbb{R}^4)^* \otimes \mathfrak{co}(4)$, by writing

$$\sum_{i=1}^{4} f_i\, e^i = \sum_{i,j} f_i\, e^i \otimes e^j \otimes e_j \in (\mathbb{R}^4)^* \otimes (\mathbb{R}^4)^* \otimes \mathbb{R}^4.$$

Using the isomorphism $\mathbb{R}^4 \cong (\mathbb{R}^4)^*$, we can write the right hand side as $\sum_{i,j} f_i\, e^i \otimes e^j \otimes e^j$. So,

$$\delta\left(\sum_{i,j} f_i\, e^i \otimes e^j \otimes e^j\right) = \frac{1}{2}\sum_{i,j} f_i\left(e^i \otimes e^j \otimes e^j - e^j \otimes e^i \otimes e^j\right)$$

and therefore

$$\delta|_{\mathfrak{so}(4)}^{-1}\left[\delta\left(\sum_{i,j} f_i\, e^i \otimes e^j \otimes e^j\right)\right] = \sum_{i,j} f_i\,(e^j \otimes e^j \otimes e^i - e^j \otimes e^i \otimes e^j) = -\sum_{i,j} f_i\, e^j \otimes (e^i \wedge e^j).$$

It is now easily verified that the torsion

$$T\left(\varphi + \pi_1^* df - \delta|_{\mathfrak{so}(4)}^{-1}\left(\delta\left(-\sum_{i=1}^{4} f_i\, e^i\right)\right)\right) = -\delta\left(\sum_{i=1}^{4} f_i\, e^i\right) - \delta\left(-\sum_{i=1}^{4} f_i\, e^i\right) = 0.$$

In conclusion, this is nothing but the Levi-Civita connection for the metric g'_X and therefore

$$\varphi' = \varphi + \pi_1^* df + \left\langle \sum_{i,j} f_i e^j \otimes (e^i \wedge e^j), \theta \right\rangle.$$

For simplicity, put $\alpha = \pi_1^* df + \langle \sum_{i,j} f_i e^j \otimes (e^i \wedge e^j), \theta \rangle$.

Proposition 3.2 ([21] *Prop. 6.2, Chap. I*)**.** *The adjoint representation induces the Lie algebra isomorphism* $\zeta : \mathfrak{spin}(n) \longrightarrow \mathfrak{so}(n)$ *is given by:*

$$\zeta(e_i e_j) = 2 e_i \wedge e_j,$$

where, $\{e_i e_j\}_{i<j}$ *are the basis elements of* $\mathfrak{spin}(n)$*. Consequently for* $v, w \in \mathbb{R}^n$*,*

$$\zeta^{-1}(v \wedge w) = \frac{1}{4}[v, w].$$

Under this isomorphism, α gets mapped to $\sum_{i=1}^{4} f_i\, e^i + \frac{1}{4}\sum_{i,j} f_i e^j \otimes (e^i e^j - e^j e^i)$. We denote this again by α.

3.2. A review of Swann's construction

A quaternionic Kähler manifold is a $4n$ dimensional manifold, whose holonomy is contained in $\mathrm{Sp}(n)\mathrm{Sp}(1) := (\mathrm{Sp}(n) \times \mathrm{Sp}(1))/\pm 1$. Let N be a quaternionic Kähler manifold of positive scalar curvature and F be the $\mathrm{Sp}(n)\mathrm{Sp}(1)$ reduction of the frame bundle $P_{\mathrm{SO}(4n)}$ of N. Then $\mathcal{S}(N) := F/\mathrm{Sp}(n)$ is a principal SO(3)-bundle, which is the frame bundle of the 3-dimensional vector sub-bundle of skew symmetric endomorphisms of TN. The Sp(1)-action, by left multiplication, descends to an isometric action of SO(3) on $\mathbb{H}^*/\mathbb{Z}_2$. Swann bundle over N is the principal $\mathbb{H}^*/\mathbb{Z}_2$

$$\mathcal{U}(N) := \mathcal{S}(N) \times_{\mathrm{SO}(3)} (\mathbb{H}^*/Z_2) \longrightarrow N$$

Theorem 3.3 ([13])**.** *The manifold* $\mathcal{U}(N)$ *is a hyperKähler manifold with a free, permuting action of* SO(3) *and admits a hyperKähler potential given by* $\rho_0 = \frac{1}{2}r^2$*. The vector field* $\mathcal{X}_0 = -I_\xi K_\xi^M$ *is independent of* $\xi \in \mathfrak{sp}(1)$ *and* grad $\rho_0 = \mathcal{X}_0$*. Moreover, if a Lie group* G *acts on* N*, preserving the quaternionic Kähler structure, then the action can be lifted to a tri-Hamiltonian action of* G *on* $\mathcal{U}(N)$*.*

The Riemannian metric on the total space $\mathcal{U}(N)$ is given by $g_{\mathcal{U}(N)} = g_{\mathbb{H}^*/\mathbb{Z}_2} + r^2 g_N$ where r is the radial co-ordinate on $\mathbb{H}^*/\mathbb{Z}_2$ and $g_{\mathbb{H}^*/\mathbb{Z}_2}$ is the quotient metric obtained from $\mathbb{H}$. Alternatively, one can write

$$\mathcal{U}(N) = (0, \infty) \times \mathcal{S}(N)$$

with metric $g_{\mathcal{U}(N)} = dr^2 + r^2(g_N + g_{\mathbb{RP}^3})$, where $g_{\mathbb{RP}^3}$ is the quotient metric on $\mathbb{RP}^3$ derived from its double cover S^3. Thus, $\mathcal{U}(N)$ is a metric cone over $\mathcal{S}(N)$. The manifold $\mathcal{U}(N)$ is equipped with a natural left action of $\mathbb{H}^* \cong \mathbb{R}^+ \times \mathrm{Sp}(1)$

$$\big((\lambda, q)(r, s)\big) \longmapsto (\lambda \cdot r,\ q \cdot s). \tag{18}$$

3.3. Generalised Dirac operators for conformally related metrics

Henceforth, fix an $M = \mathcal{U}(N)$, for some quaternionic Kähler manifold N of positive scalar curvature and an action of G that preserves the quaternionic Kähler structure on N. By Theorem 3.3, the action lifts to a tri-Hamiltonian action of G on $\mathcal{U}(N)$. Therefore M carries a permuting action of $\mathrm{Spin}^G(4)$.

Define the conformal $\mathrm{Spin}^G(4)$ group $\mathrm{CSpin}^G(4) := \mathbb{R}^+ \times \mathrm{Spin}^G(4)$, which is a double cover of $\mathrm{CO}(4) \times G$

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathrm{CSpin}^G(4) \xrightarrow{\gamma} \mathrm{CO}(4) \times G \longrightarrow 0. \tag{19}$$

Definition 4. A $\mathrm{CSpin}^G(4)$-structure over X is a principal $\mathrm{CSpin}^G(4)$-bundle $\widetilde{\pi} : \widetilde{Q} \to X$, which is an equivariant double cover of bundle $P_{\mathrm{CO}(4)} \times_X P$, with respect to the map γ.

Let φ and φ' denote the Levi-Civita connections for metrics g_X and $g'_X \in [g_X]$ respectively. Fix a $\overline{G}$-connection A on P. Then A uniquely determines the connections A_φ and $A_{\varphi'}$, which are lifts of φ and φ' to $\widetilde{Q}$. Then, as shown in Section 3.1,

$$A_{\varphi'} - A_\varphi = \alpha \in C^\infty(\widetilde{Q}, (\mathbb{R}^4)^* \otimes \mathfrak{g})^{\mathrm{Spin}^G(4)}.$$

Consequently, the covariant derivative of u, with respect to $A_{\varphi'}$ is

$$D_{A_{\varphi'}} u = D_{A_\varphi} u + K^M_\alpha|_u \in C^\infty\big(\widetilde{Q}, (\mathbb{R}^4)^* \otimes u^*TM\big)^{\mathrm{CSpin}^G(4)}. \tag{20}$$

Recall that $\mathcal{U}(N)$ admits a hyperkähler potential ρ_0 and $\mathcal{X}_0 = \mathrm{grad}\,\rho_0$. For $\lambda \in \mathbb{R} \setminus \{0\}$,

$$\rho_0(e^\lambda x) = \frac{1}{2} g^M(\mathcal{X}_0|_{e^\lambda x}, \mathcal{X}_0|_{e^\lambda x}) = \frac{1}{2}\, e^{2\lambda} g^M(\mathcal{X}_0|_x, \mathcal{X}_0|_x) = e^{2\lambda}\rho_0(x).$$

Therefore

$$\frac{d}{dt}\rho_0(e^{2t\lambda}x)|_{t=0} = d\rho_0(\frac{d}{dt}(e^{2t\lambda}x)) = 2d\rho_0(K^{M,\mathbb{R}^+}_\lambda)|_x = g^M(\mathcal{X}_0|_x, K^{M,\mathbb{R}^+}_\lambda|_x).$$

On the other hand

$$\frac{d}{dt}\rho_0(e^{2t\lambda}x)|_{t=0} = \frac{d}{dt}(e^{2t\lambda})\rho_0(x) = 2\lambda\rho_0(x) = g_M(\mathcal{X}_0|_x, \mathcal{X}_0|_x),$$

which implies that $K^{M,\mathbb{R}^+}_\lambda = \lambda\mathcal{X}_0$.

We are now in a position to give the proof of Theorem 1.1. But first, we need the following Lemma:

Lemma 3.4. *For $f \in C^\infty(X, \mathbb{R})$, we have*

$$\mathcal{D}_A(e^{-\pi_1^* f} u) = de^{-\pi_1^* f}\, \mathcal{D}_A u - \pi_1^* df \bullet \mathcal{X}_0 \circ u, \tag{21}$$

where $de^{-\pi_1^* f}$ denotes the differential of the action of $e^{-\pi_1^* f}$ on TM.

Proof. Let $p \in \widetilde{Q}$ and $v \in T_p\widetilde{Q}$. Let $\gamma : [0, 1] \longrightarrow \widetilde{Q}$ be a curve in $\widetilde{Q}$ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. Evaluating the covariant derivative of $e^{-\pi_1^* f} u$ for v:

$$D_A(e^{-\pi_1^* f} u)(v) = d(e^{-\pi_1^* f} u)(v) + K^M_{A(v)}|_{e^{-\pi_1^* f(p)} u(p)}.$$

The first term of the above expression is

$$\begin{aligned} d(e^{-\pi_1^* f} u)(v) &= \frac{d}{dt}\big(e^{-\pi_1^* f} u\big)(\gamma(t))|_{t=0} \\ &= \frac{d}{dt}\big(e^{-\pi_1^* f(\gamma(t))} u(\gamma(t))\big)|_{t=0} \\ &= de^{-\pi_1^* f(p)} du(v) + K^M_{(-\pi_1^* df(v))}|_{u(p)} \\ &= de^{-\pi_1^* f(p)}\, du(v) - \Big\langle \sum_{i=1}^4 f_i\, e^i,\ \theta(v)\Big\rangle \mathcal{X}_0|_{u(p)} \end{aligned}$$

and the second term is

$$K^M_{A(v)}|_{e^{-\pi_1^* f(p)} u(p)} = de^{-\pi_1^* f(p)}\, K^M_{A(v)}|_{u(p)}.$$

In conclusion,

$$D_A(e^{-\pi_1^* f} u) = de^{-\pi_1^* f}\, D_A u - \Big\langle \sum_{i=1}^4 f_i\, e^i,\ \theta\Big\rangle \otimes \mathcal{X}_0 \circ u.$$

Applying Clifford multiplication, proves the statement of the Lemma. □

Proof of Theorem 1.1. With respect to the metric $e^{2\pi^* f} g_X$, the Clifford multiplication is given by $\bullet' = de^{-\pi_1^* f} \bullet$. Substituting for α in (20) and applying the Clifford multiplication we get:

$$\mathcal{D}_{A_{\varphi'}} u = de^{-\pi_1^* f} \left(\mathcal{D}_{A_\varphi} u + \pi_1^* df \bullet \mathcal{X}_0 \circ u + \frac{1}{4} \langle \sum_{i<j} f_i e^j ,\ \theta \rangle \bullet K^M_{(e^i e^j - e^j e^i)}|_u \right) \tag{22}$$

Note that in using the identification $(\mathbb{R}^4)^* \cong \mathbb{H}$, the element $(e^i e^j - e^j e^i)$ belongs to the Lie algebra $\mathfrak{sp}(1) \cong \mathfrak{Im}(\mathbb{H})$ and has norm 1. Now recall from Theorem 3.3 the vector field $\mathcal{X}_0 = -I_\xi K^M_\xi$ is independent of $\xi \in \mathfrak{sp}(1)$. In particular when $|\xi| = 1$, we get $I_\xi \mathcal{X}_0 = K^M_\xi$. Therefore,

$$K^M_{(e^i e^j - e^j e^i)}|_u = I_{(e^i e^j - e^j e^i)} \mathcal{X}_0 \circ u = (e^i e^j - e^j e^i) \bullet \mathcal{X}_0 \circ u.$$

Substituting this in (22), we get

$$\begin{aligned}
\mathcal{D}_{A_{\varphi'}} u &= de^{-\pi_1^* f} \left(\mathcal{D}_{A_\varphi} u + \pi_1^* df \bullet \mathcal{X}_0 \circ u + \frac{1}{4} \langle \sum_{i<j} f_i e^j ,\ \theta \rangle \bullet (e^i e^j - e^j e^i) \bullet \mathcal{X}_0 \circ u \right) \\
&= de^{-\pi_1^* f} \Big(\mathcal{D}_{A_\varphi} u + \pi_1^* df \bullet \mathcal{X}_0 \circ u + \frac{1}{4} \langle 4 \sum_i f_i e^i - 2 \sum_{i,j} f_i e^j \delta_{i,j} + 4 \sum_i f_i e^i , \theta \rangle \bullet \mathcal{X}_0 \circ u \Big) \\
&= de^{-\pi_1^* f} \Big(\mathcal{D}_{A_\varphi} u + \pi_1^* df \bullet \mathcal{X}_0 \circ u + \frac{3}{2} \pi_1^* df \bullet \mathcal{X}_0 \circ u \Big).
\end{aligned}$$

Now observe that

$$\begin{aligned}
\mathcal{D}_{A_{\varphi'}} (e^{-\pi_1^* f} u) &= de^{-\pi_1^* f} \left(de^{-\pi_1^* f}\, \mathcal{D}_{A_\varphi} u + \frac{3}{2} de^{-\pi_1^* f}\, \pi_1^* df \bullet \mathcal{X}_0 \circ u \right) \\
&= de^{-\pi_1^* f} \left(de^{-\frac{5}{2}\pi_1^* f}\, \mathcal{D}_{A_\varphi} (e^{\frac{3}{2}\pi_1^* f} u) \right).
\end{aligned}$$

Thus, in conclusion

$$\mathcal{D}_{A_{\varphi'}} (\mathscr{B} u) = \mathscr{B} \left(de^{-5/2\pi_1^* f}\, \mathcal{D}_{A_\varphi} (e^{3/2\pi_1^* f} u) \right). \quad \Box \tag{23}$$

4. Almost Hermitian geometry and generalised Seiberg–Witten

In this section, we give the proof of Theorem 1.2. Let the target hyperKähler manifold M be as in Section 3.3, but with $G = \mathrm{U}(1)$, so that M now carries a permuting action of $\mathrm{Spin}^c(4)$. Moreover, let $\dim M = 4$. Fix a $\mathrm{Spin}^c(4)$-structure $Q \to X$. In this section we restrict our attention to those $\mathcal{U}(N)$ which can be obtained by a hyperKähler reduction of a flat, quaternionic space. Examples include nilpotent co-adjoint, orbits of complex semi-simple Lie groups, the moduli spaces of instantons on 4-manifolds, etc. We describe this set-up below.

Let V be a finite-dimensional, Hermitian vector space and $\mathsf{H} := V \oplus V^*$. Then H is a flat-hyperKähler manifold. Identifying H with $\mathbb{H}^n$, for some n, it is easy to see that H carries a natural permuting action of $\mathrm{Sp}(1)$ given by multiplication by conjugate on the right. Consider the left action of $\mathrm{U}(1)$ on H

$$z \cdot (v, w) = (z \cdot v, z^{-1} \cdot w). \tag{24}$$

The action is tri-Hamiltonian, with a moment map

$$\mu_{\mathbb{R}}(v, w) = \frac{1}{2}(\|v\|^2 - \|w\|^2), \quad \mu_{\mathbb{C}}(v, w) = \langle v, w \rangle \tag{25}$$

Therefore, H admits a permuting action of $\mathrm{U}(2)$. Suppose that another compact Lie group $G \subset U(n) \hookrightarrow Sp(n)$ has a tri-Hamiltonian action on H that commutes with the $\mathrm{U}(2)$-action. Assume zero is a regular value of the G-moment map $\mu_{\mathfrak{g}} : \mathsf{H} \to \mathfrak{sp}(1)^* \otimes \mathfrak{g}^*$. Then, $\mathrm{U}(2)$ preserves the zero level set of $\mu_{\mathfrak{g}}$ and therefore descends to a permuting action on the quotient $M := \mu_{\mathfrak{g}}^{-1}(0)/G$. Put $\widehat{G} := \mathrm{Spin}^c(4) \times G$.

Remark 2. More generally, we can consider $\mathsf{H} = \sum_{i=1}^k V_i \oplus V_i^*$, where each V_i is a complex representation of $\mathrm{U}(2) \times G$, equipped with the tri-holomorphic action of $\mathrm{U}(1)$ by (weighted) left multiplication, so that it may happen that $\mathrm{U}(1)$ acts non-trivially on the first $\{V_l\}_{l=1}^m,\ 1 < m < k$ and trivially on the rest. However, we require that the image of the spinor be devoid of fixed points of the $\mathrm{U}(1)$-action. Therefore, we stick to the case where $\mathsf{H} = V \oplus V^*$ and $\mathrm{U}(1) \hookrightarrow \mathrm{Sp}(n) \curvearrowright \mathsf{H}$.

4.1. Modified Seiberg–Witten equations

By assumption $\mu_{\mathfrak{g}}^{-1}(0)/G = M$. Let $P := \mu_{\mathfrak{g}}^{-1}(0)$ denote the Spin^c-equivariant principal G-bundle over M.

$$\begin{tikzcd}
\widehat{Q} \arrow[r, "\widehat{u}"] \arrow[d, "\pi_1"'] & P \subset \mathsf{H} \arrow[d, "\pi_2"] \\
Q \arrow[r, "u"] \arrow[d, "\pi"'] & M \\
X &
\end{tikzcd}$$

Consider a $\widehat{G}$-bundle $\widehat{Q} \to X$, as in the diagram. Given a smooth, equivariant map $\widehat{u} : \widehat{Q} \longrightarrow \mathsf{H}$, such that $\mu_{\mathfrak{g}} \circ \widehat{u} = 0$, define $u : Q \to M$ by $u(q) = \pi_2(\widehat{u}(p))$, $q \in Q$, $p \in \pi_1^{-1}(q)$. Clearly then, u is a $\mathrm{Spin}^c(4)$-equivariant map and the diagram commutes. On the other hand, given a smooth spinor $u : Q \longrightarrow M$, it defines a principal $\widehat{G}$-bundle over X, via pull-back of P and canonically defines $\widehat{u}$, making the diagram commutative. In summary,

Lemma 4.1. *There is a bijective correspondence between*

$$\{u \in C^\infty(Q, M)^{\mathrm{Spin}^c}\} \Longleftrightarrow \{\widehat{u} \in C^\infty(\widehat{Q}, \mathsf{H})^{\widehat{G}} \mid \mu_{\mathfrak{g}} \circ \widehat{u} = 0\}.$$

Fix a connection A on Q. This is uniquely determined by the Levi-Civita connection on X and a connection b on the determinant bundle $P_{\mathrm{U}(1)}$. The bundle $P \to M$ is a Riemannian submersion and therefore carries a canonical connection a. This is defined as follows. For $p \in P$, let $K_\eta^{P,G}|_p$ denote the fundamental vector field at p due to $\eta \in \mathfrak{g}$. For $v \in T_pP$, define $\mathsf{a}_p(v) \in \mathfrak{g}$ be the unique element such that

$$K_{\mathsf{a}}^{P,G}|_p(v) = K_{\mathsf{a}(v)}^{P,G}|_p = -\,\mathrm{proj}^{\mathrm{im}\,K^{P,G}}(v)$$

where $\mathrm{proj}^{\mathrm{im}\,K^{P,G}}$ denotes the orthogonal projection to the vertical sub-bundle, which is nothing but the image of the map

$$K^{P,G} : \mathfrak{g} \to TP, \quad \eta \longmapsto K^{P,G}(\eta)|_p = K_\eta^{P,G}|_p.$$

The pull-back of this connection by $\widehat{u}$, along with the connection A on Q, uniquely determine a connection $\widehat{\mathsf{A}}$ on $\widehat{Q}$ (see [4])

$$\widehat{\mathsf{A}} = \pi^*\mathsf{A} \oplus \widehat{\mathsf{A}}_{\mathfrak{g}} \in \Lambda^1\left(\widehat{Q}, \widehat{\mathfrak{g}}\right)^{\widehat{G}}, \quad \widehat{\mathsf{A}}_{\mathfrak{g}} = \widehat{u}^*\mathsf{a} - \langle \pi_1^*\mathsf{A}, \iota_{\mathfrak{spin}^c}\widehat{u}^*\mathsf{a}\rangle. \tag{26}$$

We can define a twisted Dirac operator $\mathcal{D}_{\widehat{\mathsf{A}}}$ acting on maps $\widehat{u}$.

Proposition 4.2. *Then, there is a bijective correspondence between*

$$\{(\widehat{u}, \widehat{\mathsf{A}}) \mid \mathcal{D}_{\widehat{\mathsf{A}}}\widehat{u} = 0, \ \ \mu_{\mathfrak{g}} \circ \widehat{u} = 0\} \ \ \textit{and} \ \ \{(u, \mathsf{A}) \mid \mathcal{D}_{\mathsf{A}}u = 0\}. \tag{27}$$

Whenever $\mathcal{D}_{\widehat{\mathsf{A}}}\widehat{u} = 0$, $\mu_{\mathfrak{g}} \circ \widehat{u} = 0$ and $\mathrm{proj}_{\mathfrak{g}}\widehat{\mathsf{A}} = \widehat{\mathsf{A}}_{\mathfrak{g}}$ as in (26) and therefore, $\widehat{\mathsf{A}}$ is uniquely determined by a $\mathrm{U}(1)$-connection a on $P_{\mathrm{U}(1)}$.

Proof. For $h \in P$ such that $\mu_{\mathfrak{g}}(h) = 0$, define $\mathcal{H}_h := \ker d\mu_{\mathfrak{g}}(h) \cap (\mathrm{Im}\,K^{P,G})^\perp$. This is just the horizontal subspace over h with respect to the canonical connection a.

We will prove the proposition in two steps. In what follows, we shall denote the G and $Spin^c$-components of $\widehat{\mathsf{A}}$ by $\widehat{\mathsf{A}}_{\mathfrak{g}}$ and A respectively.

Step 1: In the first step we will prove that $I_\xi D_{\widehat{\mathsf{A}}}\widehat{u}(v) \in \mathcal{H}_{\widehat{u}}$ for every $\xi \in \mathfrak{sp}(1)$ and $v \in \mathcal{H}_{\widehat{\mathsf{A}}} \subset T\widehat{Q}$. Indeed, if $\mu_{\mathfrak{g}} \circ \widehat{u} = 0$, then $d\widehat{u}(v) \in \ker d\mu_{\mathfrak{g}}(\widehat{u}(p))$. Also, $K_{\widehat{\mathsf{A}}_{\mathfrak{g}}}^{P,G}|_{\widehat{u}} \in \ker d\mu_{\mathfrak{g}}(\widehat{u}(p))$ and $K_{\widehat{\mathsf{A}}}^{P,\mathrm{Spin}^c}|_{\widehat{u}} \in \ker d\mu_{\mathfrak{g}}(\widehat{u}(p))$. Therefore, $D_{\widehat{\mathsf{A}}}\widehat{u}(v) \in \ker d\mu_{\mathfrak{g}}(\widehat{u}(p))$. Consequently

$$0 = \langle d\mu_{\mathfrak{g}}(D_{\widehat{\mathsf{A}}}\widehat{u}(v)), \xi \otimes \eta\rangle = \langle I_\xi K_\eta^{P,G}|_{\widehat{u}(p)}, D_{\widehat{\mathsf{A}}}\widehat{u}(v)\rangle = -\langle K_\eta^{P,G}|_{\widehat{u}(p)}, I_\xi D_{\widehat{\mathsf{A}}}\widehat{u}(v)\rangle$$

for $\xi \in \mathfrak{sp}(1)$, $\eta \in \mathfrak{g}$ and so $I_\xi D_{\widehat{\mathsf{A}}}\widehat{u}(v) \in (\operatorname{Im} K^{P,G})^\perp$ for all $\xi \in \mathfrak{sp}(1)$. For $\xi' \in \mathfrak{sp}(1)$,

$$\langle d\mu_{\mathfrak{g}}(I_\xi D_{\widehat{\mathsf{A}}}\widehat{u}(v)), \xi' \otimes \eta \rangle = \langle d\mu_{\mathfrak{g}}(D_{\widehat{\mathsf{A}}}\widehat{u}(v)), [\xi, \xi'] \otimes \eta \rangle = 0$$

which implies $I_\xi D_{\widehat{\mathsf{A}}}\widehat{u}(v) \in \ker d\mu_G(\widehat{u}(p))$ for all $\xi \in \mathfrak{sp}(1)$. Thus, $I_\xi D_{\widehat{\mathsf{A}}}\widehat{u}(v) \in \mathcal{H}_{\widehat{u}}$.

Step 2: In this step, we prove the equivalence (27). If $\mathcal{D}_{\widehat{\mathsf{A}}}\widehat{u} = 0$, then from (10), we have

$$0 = D_{\widehat{\mathsf{A}}}\widehat{u}(\tilde{e}_0) - \sum_{i=1}^{3} I_i D_{\widehat{\mathsf{A}}}\widehat{u}(\tilde{e}_i)$$

From Step 1, $D_{\widehat{\mathsf{A}}}\widehat{u}(\tilde{e}_0) \in \mathcal{H}_{\widehat{u}}$. It follows that $D_{\widehat{\mathsf{A}}}\widehat{u}(\tilde{e}_i) \in \mathcal{H}_{\widehat{u}}$ for all $i = 1, 2, 3$. Consequently, for any $v \in \mathcal{H}_{\widehat{\mathsf{A}}}$, $\operatorname{proj}^{\operatorname{Im} K^{P,G}} D_{\widehat{\mathsf{A}}}\widehat{u}(v) = 0$ and we get $K^{P,G}_{\widehat{\mathsf{A}}_{\mathfrak{g}}(v)} = -\operatorname{proj}^{\operatorname{Im} K^{P,G}} d\widehat{u}(v)$. In other words, the $\mathfrak{g}$-connection component of $\widehat{\mathsf{A}}$ is just the pull-back of the canonical connection on P. Since the diagram commutes, $d\pi_2(D_{\widehat{\mathsf{A}}}\widehat{u}) = D_{\mathsf{A}}u$. Also, as $D_{\widehat{\mathsf{A}}}\widehat{u}(\tilde{e}_i) \in \mathcal{H}_{\widehat{u}}$ for all $i = 0, 1, 2, 3$, we have $\iota^* I_i = \pi_2^* \tilde{I}_i$ and so,

$$0 = d\pi_2(\mathcal{D}_{\widehat{\mathsf{A}}}\widehat{u}) = d\pi_2 \left(D_{\widehat{\mathsf{A}}}\widehat{u}(\tilde{e}_0) - \sum_{i=1}^{3} \iota^* I_i \, D_{\widehat{\mathsf{A}}}\widehat{u}(\tilde{e}_i) \right) = \mathcal{D}_{\mathsf{A}} u$$

Thus, $\mathcal{D}_{\widehat{\mathsf{A}}}\widehat{u} = 0$ implies $\mathcal{D}_{\mathsf{A}} u = 0$. On the other hand if $K^{P,G}_{\widehat{\mathsf{A}}_{\mathfrak{g}(v)}} = -\operatorname{proj}^{\operatorname{Im} K^{P,G}} d\widehat{u}(v)$ then $D_{\widehat{\mathsf{A}}}\widehat{u} \in \mathcal{H}_{\widehat{u}}$ and so $d\pi_2(\mathcal{D}_{\widehat{\mathsf{A}}}\widehat{u}) = \mathcal{D}_{\mathsf{A}} u$. Therefore, if $\mathcal{D}_{\mathsf{A}} u = 0$, it implies that $\mathcal{D}_{\widehat{\mathsf{A}}}\widehat{u} \in \operatorname{Im} K^{P,G}$. But since,

$$\mathcal{D}_{\widehat{\mathsf{A}}}\widehat{u} = D_{\widehat{\mathsf{A}}}\widehat{u}(\tilde{e}_0) - \sum_{i=1}^{3} \pi_2^* \tilde{I}_i \, D_{\widehat{\mathsf{A}}}\widehat{u}(\tilde{e}_i) \in \mathcal{H}_{\widehat{u}}$$

it follows that $\mathcal{D}_{\widehat{\mathsf{A}}}\widehat{u} \in (\operatorname{Im} K^{P,G})^\perp$ and so $\mathcal{D}_{\widehat{\mathsf{A}}}\widehat{u} = 0$. This proves the statement. □

With this observation, it is now easy to construct a "lift" of the equations as follows.

Proposition 4.3. *Fix a connection a on $P_{\mathrm{U}(1)}$. There is a bijective correspondence between the following systems of equations*

$$\begin{cases} \mathcal{D}_{\widehat{\mathsf{A}}}\widehat{u} = 0 \\ F_{\mathsf{b}}^{+} - \mu \circ \widehat{u} = 0 \\ \mu_{\mathfrak{g}} \circ \widehat{u} = 0 \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{D}_{\mathsf{A}} u = 0 \\ F_{\mathsf{b}}^{+} - \mu \circ u = 0 \end{cases} \tag{28}$$

where $\mu : \mathsf{H} \to i\mathbb{R}$ denotes the moment map for $\mathrm{U}(1)$*-action on* H.

Since the tri-Hamiltonian action of $\mathrm{U}(1)$ descends to M, we denote the $\mathrm{U}(1)$-moment map by μ itself. The above correspondence was independently obtained by Pidstrygach [22] and also by Haydys [23] (Prop. 4.5 and Thm. 4.6).

4.2. Almost-complex geometry and generalised Seiberg–Witten

In this subsection, we give a proof of Theorem 1.2. It exploits the equivalence (28) and Theorem 1.1. Firstly, note that the generalised Seiberg–Witten are not conformally invariant. On the other hand, from Theorem 1.1, we know that the space of harmonic, generalised spinors is conformally invariant. It follows that there is bijective correspondence between the solutions $(\widehat{u}', \widehat{\mathsf{A}}')$ of the system (28) with respect to the metric $g'_X \in [g_X]$, such that image of $\widehat{u}$ does not contain a fixed point of the $\mathrm{U}(1)$-action on H, and the triples $(g''_X, \widehat{u}'', \widehat{\mathsf{A}}'')$ such that $\left|\mu \circ \widehat{u}''\right| = 1$ and $(\widehat{u}'', \widehat{\mathsf{A}}'')$ satisfy the equations

$$\begin{cases} \mathcal{D}_{\widehat{\mathsf{A}}''}\widehat{u}'' = 0 \\ F_{\mathsf{b}}^{+} - \lambda\mu \circ \widehat{u}'' = 0 \\ \mu_{\mathfrak{g}} \circ \widehat{u}'' = 0 \end{cases} \tag{29}$$

where λ is a strictly positive function given by $\lambda = |\mu \circ u|^{-1}$. To see the correspondence, choose $g''_X = \left|\mu \circ \widehat{u}'\right|^{-4/3} g'_X$. Then $u'' = \left|\mu \circ \widehat{u}'\right|^{-1/2} u'$. By virtue of Theorem 1.1, u'' is harmonic and the third equation of (28) remains invariant under the conformal scaling. Moreover, $\lambda\mu \circ \widehat{u}'' = \mu \circ \widehat{u}'$. The said correspondence follows from the map $(u', \mathsf{A}') \mapsto (u'', \mathsf{A}')$.

Suppose we are given a triple $(g''_X, \widehat{u}, \widehat{\mathsf{A}})$ satisfying (29) and $|\mu \circ \widehat{u}| = 1$. Then $\Omega = \Phi(\mu \circ \widehat{u})$ is a non-degenerate, self-dual 2-form on X, where $\Phi : \mathfrak{sp}(1)^* \longrightarrow \Lambda^2_+(\mathbb{R}^4)^*$ is the isomorphism, and defines an almost-complex structure on X.

Lemma 4.4. *Suppose that the target hyperKähler manifold M is 4-dimensional. Let A_0 be a fiducial connection on Q and u be a spinor such that the range of u does not contain a fixed point of the* $\mathrm{U}(1)$*-action on M. Then there exists a unique 1-form a_0 on X such that $\mathcal{D}_{\mathsf{A}} u = 0$, where $\mathsf{A} = \mathsf{A}_0 + \mathbf{i}\mathsf{a}_0$.*

Proof. Observe that $\mathcal{D}_{\mathsf{A}}u = \mathcal{D}_{\mathsf{A}_0}u + \sum_{i=0}^{3} e^i \bullet K^M_{\mathbf{i}\mathsf{a}_0(\widetilde{e}_i)}|_u$. At a point $q \in Q$,

$$K^M_{\mathbf{i}\mathsf{a}_0\,(\widetilde{e}_i(q))}\,|_{u(q)} = \frac{d}{dt}\exp\left(\,\mathbf{i}\,t\,\mathsf{a}_0(\widetilde{e}_i(q))\right)u(q)|_{t=0} = \,(\mathsf{a}_0(\widetilde{e}_i(q)))\,K^M_{\mathbf{i}}|_{u(q)}.$$

Therefore

$$\begin{aligned}\mathcal{D}_{\mathsf{A}}u(q) &= \mathcal{D}_{\mathsf{A}_0}u(q) + \sum_{i=0}^{3}\left(\mathsf{a}_0(\widetilde{e}_i(q))e^i\right)\bullet K^M_{\mathbf{i}}|_{u(q)}\\ &= \mathcal{D}_{\mathsf{A}_0}u(q) + \mathsf{a}_0(q)\bullet K^M_{\mathbf{i}}|_{u(q)}.\end{aligned}$$

Suppose that $\mathcal{D}_{\mathsf{A}}u = 0$. Then, we need to solve the equation

$$-\mathcal{D}_{\mathsf{A}_0}u = \mathsf{a}_0 \bullet K^M_{\mathbf{i}}|_u.$$

Point-wise, we can choose identification of $T_{u(q)}M$ and $\mathbb{R}^4$ with quaternions, such that the Clifford multiplication is just the usual quaternionic multiplication. Since the image of u does not contain a fixed point of the $\mathrm{U}(1)$ action on M, $K^M_{\mathbf{i}}|_u$ is a non-vanishing, equivariant section of $u^*TM \to Q$. The statement of the Lemma follows. □

In essence, this translates to saying that given a non-vanishing spinor $\widehat{u}$ such that $\mu_{\mathfrak{g}} \circ \widehat{u} = 0$, then there exists a unique 1-form a_0 on X such that $\mathcal{D}_{\widehat{A}}\widehat{u} = 0$. Therefore, the connection $\widehat{A}$ is entirely determined by $\widehat{u}$ and hence by the almost complex structure $\Omega = \Phi(\mu \circ \widehat{u})$.

Let $B : \mathsf{H} \times \mathsf{H} \longrightarrow \mathfrak{sp}(1)$ denote the symmetric (real) bi-linear form associated to the $\mathrm{U}(1)$-moment map and $\widetilde{B}$ denote the induced map on $(T^*X \otimes \mathsf{H}) \times (T^*X \otimes \mathsf{H})$, obtained using contraction furnished by the Riemannian metric on X. Then, $\Omega = B(\widehat{u}, \widehat{u})$ and so

$$\nabla^*\nabla\Omega = 2\left(B(D^*_{\widehat{\mathsf{A}}}D_{\widehat{\mathsf{A}}}\widehat{u},\,\widehat{u}) - \widetilde{B}(D_{\widehat{\mathsf{A}}}\widehat{u},\,D_{\widehat{\mathsf{A}}}\widehat{u})\right)$$

Applying the Weitzenböck formula

$$\mathcal{D}^*_{\widehat{\mathsf{A}}}\mathcal{D}_{\widehat{\mathsf{A}}}\widehat{u} = D^*_{\widehat{\mathsf{A}}}D_{\widehat{\mathsf{A}}}\widehat{u} + \frac{s_X(g''_X)}{4}\widehat{u} + F^+_{\mathsf{b}}\bullet\widehat{u} + F^+_{\widehat{\mathsf{A}}_{\mathfrak{g}}}\bullet\widehat{u} \tag{30}$$

gives

$$\nabla^*\nabla\Omega = -\frac{s_X(g''_X)}{2}\Omega - B(F^+_{\widehat{\mathsf{A}}_{\mathfrak{g}}}\bullet\widehat{u},\widehat{u}) - B(F^+_{\mathsf{b}}\bullet\widehat{u},\widehat{u}) - 2\widetilde{B}(D_{\widehat{\mathsf{A}}}\widehat{u},D_{\widehat{\mathsf{A}}}\widehat{u})$$

We claim that the term $B(F^+_{\widehat{\mathsf{A}}_{\mathfrak{g}}}\bullet\widehat{u},\widehat{u})$ vanishes. This follows from the following Lemma:

Lemma 4.5. *Assume that $\mu_{\mathfrak{g}}(h) = 0$ and let $\xi \in \mathfrak{sp}(1)$ and $\eta \in \mathfrak{g}$. Then*

$$B(\widehat{u},\ \eta\,\widehat{u}\,\overline{\xi}) = 0$$

Proof. This follows from the fact that the $\mathrm{U}(1)$-moment map is G-invariant. For $\eta \in \mathfrak{g}$, computing $\frac{d}{dt}B\left(u,\ \exp(t\eta)\,u\,\overline{\xi}\right)|_{t=0}$ proves the statement of the Lemma. □

It follows that $B(F^+_{\widehat{\mathsf{A}}_{\mathfrak{g}}}\bullet\widehat{u},\widehat{u}) = 0$. Therefore,

$$\nabla^*\nabla\Omega = -\left(\frac{s_X(g''_X)}{2} + \lambda\right)\Omega - 2\widetilde{B}(D_{\widehat{\mathsf{A}}}\widehat{u},D_{\widehat{\mathsf{A}}}\widehat{u}) \tag{31}$$

We are now in a position to give the proof of Theorem 1.2. The arguments of the proof are essentially the same as those of Donaldson's [18]. Nonetheless, for the sake of completeness, we present them here once again.

Proof of Theorem 1.2. Observe that since $|\Omega| = 1$,

$$0 = \Delta\,|\Omega| = 2\left\langle\nabla^*\nabla\Omega,\,\Omega\right\rangle - 2\,|\nabla\Omega|^2\,.$$

Using (31), we get

$$2\lambda = -s_X(g''_X) - 2\,|\nabla\Omega|^2 - 2\left\langle\widetilde{B}(D_{\widehat{\mathsf{A}}}\widehat{u},D_{\widehat{\mathsf{A}}}\widehat{u}),\,\Omega\right\rangle.$$

Therefore, re-arranging, we have

$$|\nabla\Omega|^2 + \frac{1}{2}\,s_X(g''_X) + \left\langle\widetilde{B}(D_{\widehat{\mathsf{A}}}\widehat{u},D_{\widehat{\mathsf{A}}}\widehat{u}),\,\Omega\right\rangle < 0. \tag{32}$$

Also, from (31) we have that $(\nabla^*\nabla\Omega)^{\perp_\Omega} + \widetilde{B}(D_{\widehat{\mathsf{A}}}\widehat{u},D_{\widehat{\mathsf{A}}}\widehat{u})^{\perp_\Omega} = 0$. Thus comparing with the identities (3) of Theorem 1.2, to complete our proof, we merely need to show that

$$\widetilde{B}(D_{\widehat{\mathsf{A}}}\widehat{u},D_{\widehat{\mathsf{A}}}\widehat{u})^{\perp_\Omega} = 2\,\langle d\Omega, N_\Omega\rangle\,,\quad \left\langle\widetilde{B}(D_{\widehat{\mathsf{A}}}\widehat{u},D_{\widehat{\mathsf{A}}}\widehat{u}),\,\Omega\right\rangle = \frac{1}{4}\left(|N_\Omega|^2 - |d\Omega|^2\right). \tag{33}$$

The key issue here is to identify the map $\widetilde{B}$ on kernel of the Clifford multiplication. In order to do this, it suffices to restrict to the standard model when $X = \mathbb{R}^4$ and the connection $\widehat{\mathrm{A}}$ is trivial. This is because at any point $x \in X$, there exists a trivialisation in which the connection matrix $\widehat{\mathrm{A}}$ vanishes at the point x.

Since $\widehat{u} \in \ker \mu_{\mathfrak{g}}$, the derivative $D\widehat{u} \in \mathcal{H}_{\widehat{u}} \subset \ker d\mu_{\mathfrak{g}}$. At every point $p \in \ker \mu_{\mathfrak{g}}$, the horizontal subspace $\mathcal{H}_p$ can be identified with $T_{\pi_2(p)}M$. Since M is 4-dimensional, $\mathcal{H}_p$ is 4-dimensional and so $\mathcal{H}_p \cong \mathbb{H}$.

Let (x_0, x_1, x_2, x_3) be the standard co-ordinates on $\mathbb{R}^4$. Let $s_1, s_2, \dots \cdot s_{2n}$ denote the complex basis for the spinors and write $\widehat{u}$ as

$$\widehat{u} : \mathbb{R}^4 \longrightarrow \mathrm{H}, \quad \widehat{u} = \sum_{i=1}^{n} f_i\, s_i + \sum_{i=n+1}^{2n} g_{i-n}\, s_i \quad \text{where} \quad f_i, g_i \in C^\infty(\mathbb{R}^4, \mathbb{C}).$$

By Step 2 of Proposition 4.2, $D\widehat{u} \in \mathcal{H}_{\widehat{u}}$, which means that without loss of generality, at the origin, we can assume that

$$(f_i)_{x_j} = (g_i)_{x_j} = 0 \quad \text{for} \quad i = 2, 3, \dots, n \quad \text{and} \quad j = 0, 1, 2, 3.$$

Consequently, in the decomposition (33), the only contributing terms are the 1-jets of f_1, g_1 at the origin. Therefore, without loss of generality, we can assume that at the origin, $f_i, g_i = 0$ for $i = 2, 3, \dots \cdot n$. Let $f_0 = f_1(0)$ and $g_0 = g_1(0)$. Then, at the origin $u = f_0\, s_1 + g_0\, s_2$. Moreover, since $|\Omega| = 1$, $|f_0|^2 + |g_0|^2 = 1$ and

$$B(\widehat{u}, \widehat{u}) = \left(\frac{|f_0|^2 - |g_0|^2}{2} \right) \beta_0 \; + \; \mathrm{Re}\, \langle f_0, g_0 \rangle \; \beta_1 \; + \; \mathrm{Im}\, \langle f_0, g_0 \rangle \; \beta_2$$

where β_i are the basis of self-dual 2-forms on $\mathbb{R}^4$, given as in (12). The group Spin(4) acts on the base $\mathbb{R}^4$ and also transitively on unit positive spinors. In particular, for a suitable choice of an element in Spin(4), we may further assume that at the origin, $f_0 = 1$ and $g_0 = 0$. In particular, $\Omega = \frac{1}{2}\beta_0$ at the origin. Thus Ω defines the standard complex structure $\frac{1}{2}\, \beta_0$ on $\mathbb{R}^4$. This allows us to use the complex co-ordinates

$$z = x_0 + ix_1, \quad w = x_2 + ix_3.$$

From the Dirac equation we have

$$f_{1\,\bar{z}} = g_{1\,w}, \quad f_{1\,\overline{w}} = -g_{1\,z}. \tag{34}$$

Moreover, since $f_1 = 1$ at the origin, the derivatives of f_1 at the origin are purely imaginary. Therefore, at the origin,

$$f_{1\,z} = -\overline{f_{1\,\bar{z}}} \quad \text{and} \quad f_{1\,w} = -\overline{f_{1\,\overline{w}}}. \tag{35}$$

Now, the component of $\widetilde{B}(D\widehat{u}, D\widehat{u})$ along $\frac{1}{2}\, \beta_0$ is

$$\frac{1}{4} \sum_{l=0}^{3} \left| \frac{\partial f_1}{\partial x_l} \right|^2 - \left| \frac{\partial g_1}{\partial x_l} \right|^2 = \frac{1}{16} \left(|f_{1\,z}|^2 + |f_{1\,\bar{z}}|^2 + |f_{1\,w}|^2 + |f_{1\,\overline{w}}|^2 - |g_{1\,z}|^2 - |g_{1\,\bar{z}}|^2 - |g_{1\,w}|^2 - |g_{1\,\overline{w}}|^2 \right).$$

Using the identities (34) and (35), we get

$$\left\langle \widetilde{B}(D\widehat{u}, D\widehat{u}),\; \frac{1}{2}\beta_0 \right\rangle = \frac{1}{16} \left(|g_{1\,z}|^2 + |g_{1\,w}|^2 \right) - \frac{1}{16} \left(|g_{1\,\bar{z}}|^2 + |g_{1\,\overline{w}}|^2 \right). \tag{36}$$

The space orthogonal to $\frac{1}{2}\, \beta_0$ is spanned by $\beta_c = d\bar{z} \cdot d\overline{w}$ and therefore the component of $B(D\widehat{u}, D\widehat{u})$ orthogonal to $\frac{1}{2}\, \beta_0$ is

$$\begin{aligned} (B(D\widehat{u}, D\widehat{u}))^{\perp \beta_0} &= \sum_{l=0}^{3} \left[\left(\frac{\partial f_1}{\partial x_l} \right)^{\dagger} \frac{\partial g_1}{\partial x_l} \right] \beta_c \\ &= \frac{1}{4} \left(f_{1\,z}\, \overline{g_{1\,z}} + f_{1\,\bar{z}}\, \overline{g_{1\,\bar{z}}} + f_{1\,w}\, \overline{g_{1\,w}} + f_{1\,\overline{w}}\, \overline{g_{1\,\overline{w}}} \right) \beta_c = \frac{1}{4} \left(g_{1\,z}\, \overline{g_{1\,\overline{w}}} + g_{1\,w}\, \overline{g_{1\,\bar{z}}} \right) \beta_c \end{aligned}$$

where, once again, we have used the identities (34) and (35) in the penultimate step. Now Ω is a section of the twistor bundle and therefore its covariant derivative at the origin is given by the derivative of $f_1\, \overline{g}_1$ which is nothing but the derivative of g_1. The holomorphic part $(g_{1\,z}, g_{1\,w})$ corresponds to the Nijenhuis tensor N_Ω whereas the anti-holomorphic component $(g_{1\,\bar{z}}, g_{1\,\overline{w}})$ corresponds to $d\Omega$, due to the vanishing of the rest of the partial derivatives.

Recall that there is a natural $\overline{K}$-valued pairing between TX and $T^*X \otimes \overline{K}$. Applying this to $d\Omega$ and N_Ω, the pairing corresponds to $(g_{1\,z}\, \overline{g_{1\,\overline{w}}} + g_{1\,w}\, \overline{g_{1\,\bar{z}}})\; \beta_c$. Therefore,

$$(B(D\widehat{u}, D\widehat{u}))^{\perp \Omega_0} = \frac{1}{4} \times 4\, \langle d\Omega, N_\Omega \rangle = \langle d\Omega, N_\Omega \rangle \tag{37}$$

$$\left\langle \widetilde{B}(D\widehat{u}, D\widehat{u}),\; \frac{1}{2}\Omega_0 \right\rangle = \frac{1}{16} \times 4 \left(|N_\Omega|^2 - |d\Omega|^2 \right) = \frac{1}{4} \left(|N_\Omega|^2 - |d\Omega|^2 \right) \tag{38}$$

Substituting in Eq. (31), we have

$$\nabla^*\nabla\Omega = -\left(\frac{s_X(g''_X)}{2} + \lambda\right)\Omega + \frac{1}{2}\left(|d\Omega|^2 - |N_\Omega|^2\right)\Omega - 2\,\langle d\Omega, N_\Omega\rangle \tag{39}$$

Also, observe that $|\nabla\Omega|^2 = |d\Omega|^2 + |N_\Omega|^2$. The statement of the theorem follows from Eq. (39) and Eq. (32). □

5. Some remarks

For the usual Seiberg–Witten equations, Donaldson remarks that for a fixed metric, the Seiberg–Witten equations are in bijective correspondence with solutions to the following equations

$$\begin{aligned}\nabla^*\nabla\Omega = &-\left(\frac{s}{2} + |\Omega|^2\right)\Omega - 2\langle d\Omega + *d\,|\Omega|\,, N_\Omega\rangle + \frac{1}{2}\left(\frac{|d\Omega|^2}{|\Omega|^2} - |N_\Omega|^2\right)\Omega \\ &+ \frac{1}{2}\left(|d\,|\Omega|\,|^2 + 2\langle d\,|\Omega|\,, *d\Omega\rangle\right)\frac{\Omega}{|\Omega|^2}\end{aligned} \tag{40}$$

Many examples of hyperKähler manifolds with requisite properties can be obtained via hyperKähler reduction of flat space. Using Proposition 4.3 and applying Donaldson's arguments, one can show that the Abelian, generalised Seiberg–Witten equations, for a 4-dimensional target hyperKähler manifold, can be expressed as (40).

Note that the specification of an almost-complex structure I compatible with Ω imposes a topological constraint on X. Namely, in terms of the Euler characteristic χ and the signature τ of X,

$$c_1^2(L) = 2\,\chi + 3\,\tau$$

where L is the line-bundle associated to the determinant bundle $P_{\mathrm{U}(1)}$. For the usual Seiberg–Witten equations, this is precisely the condition under which the expected dimension of the moduli space is zero. Therefore Theorem 1.2, in combination with Donaldson's result [18] delivers a potential candidate to get a compact moduli space.

The arguments in the latter half of the article can be extended for target hyperKähler manifolds of higher dimensions, using similar techniques. However, in this case, one obtains a map from the moduli space of generalised Seiberg–Witten to the usual Seiberg–Witten equations, which may be neither injective nor surjective.

Acknowledgments

The author wishes to thank Prof. Clifford Taubes for pointing out a crucial error in the earlier version of the article. A major part of this article is based on the author's doctoral dissertation, during which he was financially supported by the Deutsche Forschungsgemeinschaft (DFG), as a part of RTG 1493 "Mathematical Structures in Modern Quantum Physics" at the University of Göttingen. The author wishes to thank his advisor Prof. V. Ya. Pidstrygach for his unwavering support and constant encouragement. The author would also like to thank the anonymous referee for many helpful remarks and suggestions, especially on the second half of the article.

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