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On the existence of stable bundles with prescribed Chern classes on Calabi–Yau threefolds

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ABSTRACT

We prove a case of the conjecture of Douglas, Reinbacher and Yau about the existence of stable vector bundles with prescribed Chern classes on a Calabi–Yau threefold. For this purpose we prove the existence of certain stable vector bundle extensions over elliptically fibered Calabi–Yau threefolds.

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1. Introduction

The present note is concerned with the question of existence of stable bundles V with prescribed Chern class $c_2(V)$ on a given Calabi–Yau threefold X. The 'DRY'-conjecture of Douglas, Reinbacher and Yau in [1] gives a sufficient condition for cohomology classes on X to be equal to the Chern classes of a stable sheaf V. A weak form of the conjecture asserts the existence of V with a suitable prescribed second Chern class c.

We will consider always only the case that V is actually a vector bundle and $c_1(V) = 0$. In [2] we showed that infinitely many classes on an X which is elliptically fibered over a base surface B exist for which the conjecture is true. In [3] we showed that rank 4 *polystable* vector bundles exist for such suitable cohomology classes c ("DRY-classes", cf. the definition below). In this note we prove the weak form of the conjecture by providing corresponding *stable* bundles V, for all ranks $\mathcal{N} \geq 4$, for B a Hirzebruch surface $\mathbf{F_g}$ or a del Pezzo surface $\mathbf{dP_k}$, in all but finitely many cases.

We recall the following definition (for use in this note) and the weak DRY-conjecture.

Definition 1.1. Let X be a Calabi–Yau threefold with $\pi_1(X) = 0$ and $c \in H^4(X, \mathbf{Z})$,

(i) c is called a *Chern class* if a vector bundle V on X exists with $c = c_2(V)$ which is μ-stable with respect to some ample class H',

(ii) c is called a *DRY class* if an ample class $H \in H^2(X, \mathbf{R})$ exists (and an integer $\mathcal{N}$) with

$$c = \mathcal{N}\left(H^2 + \frac{c_2(X)}{24}\right). \tag{1.1}$$

Conjecture 1.2. *On a Calabi–Yau threefold X with $\pi_1(X) = 0$ every DRY class $c \in H^4(X, \mathbf{Z})$ is a Chern class.*

Here it is understood that the integer $\mathcal{N}$ occurring in the two definitions is the same.

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In this paper we will construct a class of stable bundle extensions on a smooth elliptically fibered Calabi–Yau threefold $\pi: X \to B$ with section σ (we also denote by σ the image divisor in X and its cohomology class). We will consider B to be a surface with ample K_B^{-1} such as the Hirzebruch surface $\mathbf{F_g}$ with $g = 0, 1$ or the del Pezzo surface $\mathbf{dP_k}$ with $k = 0, \ldots, 7$ (note that $\mathbf{dP_1} \cong \mathbf{F_1}$). We will actually exclude $B = \mathbf{P^2}$ and $\mathbf{dP_8}$.

The main result of this note is (using the decomposition $H^4(X, \mathbf{Z}) \cong H^2(B, \mathbf{Z})\sigma \oplus H^4(B, \mathbf{Z})$; we usually identify $H^4(B, \mathbf{Z})$ with $\mathbf{Z}$).

Theorem 1.3. *Let the class $c = \phi\sigma + \omega$ be a DRY class and $B = \mathbf{F_0}$ or $\mathbf{dP_k}$ with $k = 1, \ldots, 7$. Then c is a Chern class*

(i) *for each $\mathcal{N} = 4, 5$ and $\phi \equiv c_1(B) (\mathrm{mod}\ 2)$, up to finitely many exceptions in c,*
(ii) *for each $\mathcal{N} \geq 6$*
  (a) *and $\mathcal{N}$ odd, up to finitely many exceptions in c (for $\mathcal{N}$ fixed)*
  (b) *and $\mathcal{N}$ even, without any exceptions in c*
    (i) *for $\mathcal{N} \equiv 2 (\mathrm{mod}\ 4)$ and $\mathcal{N} \neq 6$ for $\mathbf{dP_7}$,*
    (ii) *for $\mathcal{N} \equiv 0 (\mathrm{mod}\ 4)$ and $\phi \equiv c_1 (\mathrm{mod}\ 2)$.*

This note has two parts. In Section 2 we prove the existence of stable bundle extensions under certain conditions (Eqs. (2.6)–(2.8)) on the input data.

In Section 3 we apply this to the weak DRY conjecture by showing that, for a class of cases described precisely below, a DRY class fulfills the assumptions of Section 2.

2. Stable bundle extensions

In this section we will construct a class of stable bundle extensions which will later, in Section 3, serve as the bundles which realize a given DRY class as Chern class.

We consider the following extension

$$0 \to \pi^*E \otimes \mathcal{O}_X(-nD) \to V \to W \otimes \mathcal{O}_X(rD) \to 0 \tag{2.1}$$

where E is a stable rank r vector bundle on B with $c_1(E) = 0$; these bundles exist on rational surfaces if [4]

$$c_2(E) \geq r + 2. \tag{2.2}$$

$D = \pi^*\alpha$ is a divisor in X with α a divisor in B and W a rank n spectral cover bundle with $c_1(W) = 0$.

Let us here recall what the latter notion means. For this let C be an irreducible surface in the linear system $|n\sigma + \pi^*\eta|$ (where we denote by η a divisor class in B and likewise its cohomology class) and $i: C \to X$ the immersion of C into X and let L be a rank one sheaf on C. We say W is a spectral cover bundle [5] of rank n if $W = \pi_{1*}(\pi_2^*(i_*L) \otimes \mathcal{P})$ where $\mathcal{P}$ is the Poincaré sheaf on the fiber product $X \times_B X$ and $\pi_{1,2}$ are the respective projections on the first and second factor. The condition $c_1(W) = 0$ leads to (here and in the sequel c_1 denotes $c_1(B)$ and $\lambda \in \frac{1}{2}\mathbf{Z}$) (cf. [5])

$$c_1(L) = n\left(\frac{1}{2} + \lambda\right)\sigma + \left(\frac{1}{2} - \lambda\right)\pi^*\eta + \left(\frac{1}{2} + n\lambda\right)\pi^*c_1. \tag{2.3}$$

Since $c_1(L)$ must be an integer class it follows that: if n is odd, then λ is strictly half-integral and for n even an integral λ requires $\eta \equiv c_1 (\mathrm{mod}\ 2)$ while a strictly half-integral λ requires c_1 even. Moreover note that, to assure that the linear system $|n\sigma + \pi^*\eta|$ contains an irreducible surface C it is sufficient to demand that the linear system $|\eta|$ is base-point free in B and that the divisor corresponding to the cohomology class $\eta - nc_1$ is effective [5].

Now let H_0 and H_B be fixed ample divisors in X and B, respectively. We have that π^*E and W are stable with respect to $H = \epsilon H_0 + \pi^*H_B$ for $\epsilon > 0$ chosen sufficiently small. For W this is due to Theorem 7.2 in [6] and for π^*E Theorem 3.1 in [7].

Two necessary conditions for V to be stable are:

(i) $DH^2 > 0$,
(ii) $Ext^1(W \otimes \mathcal{O}_X(rD), \pi^*E \otimes \mathcal{O}_X(-nD)) \neq 0$.

Here the first condition assures that $\pi^*E \otimes \mathcal{O}_X(-nD)$ is not a destabilizing subbundle of V and the second condition (which is equivalent to $H^1(X, \pi^*E \otimes W^* \otimes \mathcal{O}_X(-mD)) \neq 0$ with $m = r + n = \mathcal{N}$) assures that a non-split extension exists and shows that $W \otimes \mathcal{O}(X(rD))$ is not a destabilizing subbundle V.

Theorem 2.1. *Assume* (i) *and* (ii) *are satisfied and $\alpha \cdot H_B = 0$. Then V as defined in (2.1) is stable with respect to $H = \epsilon H_0 + \pi^*H_B$ for sufficiently small $\epsilon > 0$.*

Proof. To prove stability of V consider the diagram of exact sequences

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \pi^*E/F_s \otimes \mathcal{O}_X(-nD) & \to & V/V'_{s+t} & \to & W/G_t \otimes \mathcal{O}_X(rD) & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \pi^*E \otimes \mathcal{O}_X(-nD) & \xrightarrow{i} & V & \xrightarrow{j} & W \otimes \mathcal{O}_X(rD) & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & F_s \otimes \mathcal{O}_X(-nD) & \to & V'_{s+t} & \to & G_t \otimes \mathcal{O}_X(rD) & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with $F_s \otimes \mathcal{O}_X(-nD) = i^{-1}V'_{s+t}$ and $G_t \otimes \mathcal{O}_X(nD) = j(V'_{s+t})$ of ranks $0 \le s \le r$ and $0 \le t \le n$ for a subsheaf V'_{s+t} of V.

Note that $s = 0$ or $t = 0$ implies $F_s = 0$ or $G_t = 0$, respectively. Moreover, note for $0 < s < r$ we have $c_1(F_s) = -A_1\sigma + \pi^*\kappa$ with $A_1 \ge 0$ and $\kappa \cdot H_B < 0$. To see this consider a subsheaf F_s of π^*E where we can assume that π^*E/F_s is torsion free. We have $0 \to F_s|_\sigma \to E$ and $c_1(F_s|_\sigma)H < 0$. Similarly we get for restriction to the fiber F that $0 \to F_s|_F \to \mathcal{O}_F^r$ thus $deg(F_s|_F) \le 0$ as $\mathcal{O}_F^r$ is semistable. Thus $A_1 \ge 0$ and $\kappa H_B < 0$.

For spectral cover bundles and $0 < t < n$ we have $c_1(G_t) = -A_2\sigma + \pi^*\beta$ with $A_2 > 0$ (Theorem 7.2, [6]).

We need to show for all subsheaves V'_{r+s} of V with $0 \le s \le r$ and $0 \le t \le n$ and $0 < s + t < n + r$ that $\mu(V'_{r+s}) < 0$. We can assume that the quotient V/V'_{r+s} is torsion free thus all subsheaves V'_{s+t} with $s = r$ need not be considered since π^*E/F_r is a torsion sheaf thus $0 \le s < r$.

(i) For subsheaves V'_{s+t} with $0 \le s < r$ and $0 < t < n$ the slope is given by

$$(s+t)\mu(V'_{s+t}) \le -H_B^2\sigma + O(\epsilon) < 0$$

where the latter inequality holds for ϵ sufficiently small. If $s = 0$ and $0 < t < n$ we find similarly $t\mu(V'_{0+t}) = -H_B^2\sigma + O(\epsilon)$ and $\mu(V'_{0+t}) < 0$ for ϵ sufficiently small.

(ii) For subsheaves V'_{s+t} with $0 < s < r$ and $t = 0$ we get

$$\mu(V'_s) = -nDH^2 + \mu(F_s)$$

but since $DH^2 > 0$ by assumption and $\mu(F_s) < 0$ by stability of π^*E we get $\mu(V'_s) < 0$.

So, after these cases (i) and (ii), we are left with the cases where $t = n$. Note that in all these cases W/G_n is a torsion sheaf. Let us write $c_1(W/G_n) = \bar{D}$; since W/G_n is a torsion sheaf one has $\bar{D} \cdot H^2 \ge 0$ and $\bar{D} \cdot H^2 = 0$ if and only if W/G_n is supported in codimension ≥ 2. Moreover, $\bar{D}$ is an effective divisor in X. To see this recall the following argument. From $G_n \to W$, we get the map $(\Lambda^n G_n)^{**} \to \Lambda^n W^{**}$ and that moreover $rk(\Lambda^n G_n) = rk(\Lambda^n W) = 1$ and $(\Lambda^n G_n)^{**}$ is a reflexive torsion free sheaf of rank one with $c_1(G_n) = c_1((\Lambda^n G_n)^{**})$. Furthermore, we have $(\Lambda^n W)^{**} = L$ with $c_1(L) = 0$. Thus $c_1(G_n) = -\bar{D}$ with $\bar{D}$ effective.

(iii) For subsheaves V'_{s+t} with $0 < s < r$ and $t = n$ we get for ϵ sufficiently small

$$(s+n)\mu(V'_{s+n}) \le \epsilon\left(2H_0\pi^*(\kappa H_B) + \epsilon H_0^2(\pi^*\kappa + n(r-s)\pi^*\alpha)\right) < 0$$

since $\kappa H_B < 0$ and $H_0^2\lambda$ is bounded.

(iv) For subsheaves V'_{s+n} with $s = 0$ and $t = n$ we find

$$\mu(V'_{0+n}) = rDH^2 - \bar{D}H^2.$$

For $\bar{D}H^2 > 0$ we get (with $\bar{D} = y\sigma + \pi^*\bar{\beta}$)

$$\mu(V'_{0+n}) \le \epsilon\left(-2H_0\pi^*(\bar{\beta}H_B) + \epsilon r H_0^2\pi^*\alpha\right)$$

and for ϵ sufficiently small that $\mu(V'_{0+n}) < 0$.

It remains to prove that the case $\bar{D}H^2 = 0$ cannot happen. Note that V does not admit a destabilizing subsheaf $V'_{0+n} = G_n \otimes \mathcal{O}_X(rD)$ if the map

$$f\colon Ext^1(W \otimes \mathcal{O}_X(rD), \pi^*E \otimes \mathcal{O}_X(-nD)) \to Ext^1(G_n \otimes \mathcal{O}_X(rD), \pi^*E \otimes \mathcal{O}_X(-nD))$$

is injective [8, Lemma 2.3]. To see this in our case, we apply $Hom(\,, \pi^*E \otimes \mathcal{O}_X(-nD))$ to the exact sequence

$$0 \to G_n \otimes \mathcal{O}_X(rD) \to W \otimes \mathcal{O}_X(rD) \to W/G_n \otimes \mathcal{O}_X(rD) \to 0$$

to obtain (set $\mathcal{E} = \pi^*E \otimes \mathcal{O}_X(-nD)$ and $T = W/G_n$)

$$Ext^1(T \otimes \mathcal{O}_X(rD), \mathcal{E}) \to Ext^1(W \otimes \mathcal{O}_X(rD), \mathcal{E}) \to Ext^1(G_n \otimes \mathcal{O}_X(rD), \mathcal{E}).$$

Since T is supported in codimension ≥ 2, by Serre duality $Ext^1(T \otimes \mathcal{O}_X(rD), \mathcal{E}) = Ext^2(\mathcal{E}, T \otimes \mathcal{O}_X(rD) \otimes \mathcal{O}_X(K_X))^* = Ext^2(\mathcal{E}, T \otimes \mathcal{O}_X(rD)) = H^2(X, \mathcal{E}^* \otimes T \otimes \mathcal{O}_X(rD))^* = 0$ so that f is injective, and this finishes the proof. □

Finally let us analyze when the above extension can be chosen non-split. To simplify notation set $F = E \otimes \mathcal{O}_B(-m\alpha)$. The Hirzebruch–Riemann–Roch theorem on X gives

$$I_X = \sum_{i=0}^{3}(-1)^i \dim H^i(X, \pi^*F \otimes W^*) = \int_X ch(\pi^*F \otimes W^*)Td(X)$$
$$= r\big(-\lambda\eta + m\alpha\big)(\eta - nc_1). \tag{2.4}$$

The following lemma will be helpful for proving the next proposition.

Lemma 2.2. *Let W be a spectral cover bundle over X. Then*

(i) $\pi_*W = \pi_*W^* = 0$,
(ii) *the sheaves R^1p_*W and $R^1\pi_*W^*$ are supported on $\mathcal{A} = C \cap \sigma$.*

Proof. (i) For a given spectral cover bundle W one has $\pi_*W = 0$. At a generic point $b \in B$ one has the stalk $(\pi_*W)_b = H^0(F, W|_F) = \bigoplus_{i=1}^{n} H^0(F, \mathcal{O}_F(q_i - p))$ where $p = \sigma F$ is the zero element in the group law on the fiber F over $b \in B$ and q_i are the points at which the spectral cover of V_n intersects F. Now $\mathcal{O}(q_i - p)$ is generically a non-trivial bundle of degree zero which over an elliptic curve admits no global sections. Thus $H^0(F, \mathcal{O}_F(q_i - p)) = 0$ for all i and so $(\pi_*W)|_b = 0$. However, since W is torsion free, π_*W is also torsion free. Thus $(\pi_*W)|_b = 0$ for generic $b \in B$ gives $\pi_*W = 0$ everywhere. As W^* is again a spectral cover bundle one has also $\pi_*W^* = 0$.

(ii) For generic points $b \in B$ where p is distinct from the n points p_i we have $(R^1\pi_*W)_b = 0$. However, at points in B at which p is equal to one of the points p_i we find that $(R^1\pi_*W)_b \neq 0$. The locus of such points forms a curve $C \cap \sigma$ in B. □

Proposition 2.3. $H^3(X, \pi^*F \otimes W^*) = 0$.

Proof. The Leray spectral sequence applied to $\pi : X \to B$ leads to (using the fact that $\pi_*W^* = 0$)

$$H^3(X, \pi^*F \otimes W^*) \cong H^2(B, R^1\pi_*W^* \otimes F). \tag{2.5}$$

As $R^1\pi_*W^*$ is a torsion sheaf on B supported on the curve of class $\mathcal{A} = C \cap \sigma$ we get $H^2(B, R^1\pi_*W^* \otimes F) = 0$ and conclude. □

Thus imposing $I_X < 0$ one gets a non-split extension (2.1) as $H^1(X, \pi^*F \otimes W^*) \neq 0$.

In summary, we get the following list of conditions (where $\alpha \neq 0$ because of (2.8)):

$$[\lambda\eta - m\alpha]\,(\eta - nc_1) > 0 \tag{2.6}$$
$$\alpha H_B = 0 \quad (\Longrightarrow \alpha \neq \pm\text{effective}) \tag{2.7}$$
$$\pi^*\alpha H_0^2 > 0 \tag{2.8}$$

(recall that $m = r + n$). To solve (2.8) we note

Lemma 2.4. *Let X be an elliptically fibered Calabi–Yau threefold with section σ, then $H_0 = x\sigma + \pi^*\rho$ is ample if and only if $x > 0$ and $\rho - xc_1$ is ample in B.*

The proof of this Lemma is given in Appendix A in [2]. Thus condition (2.8) becomes

$$(2\rho - xc_1)\alpha > 0. \tag{2.9}$$

In the following, (2.9) will not be considered as a condition on α but rather as a condition on x and ρ (for each respective α). To show that it is possible to solve this for x and ρ note that it is enough to show that an ample class h exists with $h\alpha > 0$: on the one hand the expression in brackets in (2.9) $h := \rho + (\rho - xc_1)$ is such a class, and on the other hand any h can be written in such a way.[1] Now the existence of a nonzero α which is neither effective nor anti-effective, cf. Eq. (2.6), presupposes that $h^{1,1}(B) \geq 2$. Let $\alpha = \beta - \gamma$ be a decomposition with β and γ effective, then it is possible to choose in the open ample cone an element h with $h\beta = 2$ and $h\gamma = 1$.

Finally let us give the expressions for the second Chern classes of V and W

$$c_2(V) = c_2(W) + c_2(\pi^*E) - \frac{rn(r+n)}{2}\pi^*\alpha^2, \tag{2.10}$$

$$c_2(W) = \pi^*\eta \cdot \sigma - \frac{n^3 - n}{24}\pi^*c_1^2 + \left(\lambda^2 - \frac{1}{4}\right)\frac{n}{2}\pi^*\eta \cdot (\pi^*\eta - n\pi^*c_1). \tag{2.11}$$

[1] One has $h = (A + xc_1) + A = 2[A + \frac{x}{2}c_1]$ with the ample class $A := \rho - xc_1$; conversely one can, if an ample h is given, solve this for an ample class A as $\frac{1}{2}h$ will be also ample and thus also $A = \frac{1}{2}h - \frac{x}{2}c_1$ for $x > 0$ sufficiently small (the ample cone is an open set).

We also note that for the special case $n = r$ a simplified version of the construction is possible where the twisting is as follows

$$0 \to \pi^*E \otimes \mathcal{O}_X(-D) \to V' \to W \otimes \mathcal{O}_X(D) \to 0 \tag{2.12}$$

and the stability proof of V is completely analogous. In this case we get

$$c_2(V') = c_2(W) + c_2(\pi^*E) - n\pi^*\alpha^2. \tag{2.13}$$

3. Proof of main result

The aim of this section is to prove Theorem 1.3. For this let us first recall a result characterizing DRY classes on elliptically fibered Calabi–Yau threefolds obtained in [2].

Theorem 3.1. *A class $c = \phi\sigma + \omega \in H^4(X, \mathbf{Z})$ is a DRY class if and only if the following condition is fulfilled: there exists some $b \in \mathbf{R}^{>0}$ and $\omega \in H^4(B, \mathbf{Z}) \cong \mathbf{Z}$ such that $\phi - \mathcal{N}(\frac{1}{2} + b)c_1$ is ample and $\frac{1}{\mathcal{N}}\omega > \omega_0(\phi; b) := R + \frac{c_1^2}{4}(b + \frac{q}{b})$.*

Here we use the following abbreviations (cf. [2]): $R := \frac{1}{2\mathcal{N}}\phi c_1 + \frac{1}{6}c_1^2 + \frac{1}{2}$, $q := \frac{(\phi - \frac{\mathcal{N}}{2}c_1)^2}{\mathcal{N}^2 c_1^2}$. Note that under the hypothesis that $\phi - \frac{\mathcal{N}}{2}c_1 = A + b\mathcal{N}c_1$ with an ample class A on B (and with $c_1 \neq 0$ effective) one has $(\phi - \frac{\mathcal{N}}{2}c_1)^2 > b^2\mathcal{N}^2c_1^2$, thus one has $b < \sqrt{q}$. On the other hand ω_0 reaches its minimum as a function of b for $b = \sqrt{q}$, giving that $\omega_0(\phi) > R + \frac{c_1^2}{2}\sqrt{q} = R + \frac{1}{2\mathcal{N}^2}\sqrt{c_1^2}\sqrt{(\phi - \frac{\mathcal{N}}{2}c_1)^2}$.

Important for us will be especially the following (recall that we assume that c_1 is ample).

Corollary 3.2. *For a DRY class $c = \phi\sigma + \omega$ one has that*

(i) *$\phi - \frac{\mathcal{N}}{2}c_1$ is ample,*

(ii) $\omega > \mathcal{N}\left(R + \frac{1}{2\mathcal{N}^2}\sqrt{c_1^2}\sqrt{(\phi - \frac{\mathcal{N}}{2}c_1)^2}\right)$.

Our goal is to show that, for a certain class of cases the conditions for a DRY class c imply indeed the conditions under which the stable extension bundle V of the previous section exist (the latter will have the prescribed class c as $c_2(V)$).

Before we come to the general discussion of all cases with $\mathcal{N} \geq 4$ we have to distinguish two cases. This is caused by a restriction in the relation between the ranks n and r of the spectral bundle and the pullback bundle, respectively. From the Corollary we know given a DRY class c, that $\eta - \frac{n+r}{2}c_1$ is effective (we will put $\eta := \phi$); on the other hand for the spectral construction we need that $\eta - nc_1$ is effective. Thus one is led to the condition

$$r \geq n. \tag{3.1}$$

3.3. *The cases $\mathcal{N} = 4, 5$.* Recall that $m = r + n = \mathcal{N}$. In this case one has that $n = 2$, that is that n is even and so one gets the condition $\eta \equiv c_1 (\mathrm{mod}\ 2)$; thus we can only cover the cases with $\phi \equiv c_1 (\mathrm{mod}\ 2)$ (on the other hand this will be no problem, of course, if $\mathcal{N} = n + r \geq 6$ as one can then choose $n = 3$).

3.4. *The cases $\mathcal{N} \geq 6$.* Let us therefore assume that we are in the case $\mathcal{N} \geq 6$, and thus n can be chosen odd (actually $n = 3$). We can then use a strictly half-integral λ; we will have to choose $\lambda = \pm 1/2$ to kill the corresponding contribution in $c_2(V)$ (which would lift the socle of realizable ω). Consider now the non-split-condition (2.6) (where H_B has to be chosen orthogonal to α). As $\eta(\eta - nc_1) > 0$ we will just choose (for each η) $\lambda = +1/2$ and a suitable α such that $\alpha \cdot (\eta - nc_1) \leq 0$. Now let us consider two cases.

3.4.1. *The case of B being a Hirzebruch surface $\mathbf{F_g}$.* Because of our assumption that c_1 is ample we have actually $g = 0$ or 1. Let us take first $\alpha = (-1, 1)$ such that $-\alpha^2 = g + 2$ and $H_B := (e, f) = e(1, g + 1)$ (which is ample as $f > ge$). The evaluation $\alpha \cdot (\eta - nc_1) = (g + 1)a - b - gn$ shows that one chooses this α if the latter expression is ≤ 0; if it is > 0 one chooses just $-\alpha$ instead of α.

3.4.2. *The case of B being a del Pezzo surface $\mathbf{dP_k}$.* Let us now also investigate the situation for the del Pezzo surfaces $\mathbf{dP_k}$ with $k = 1, \ldots, 7$ and $c_1 = 3l - \sum_k E_k$ (where l is of course the class of the pull-back of the line). Note that on $\mathbf{P^2}$ the condition (2.7) would amount, with $\alpha = al$, to $a = 0$, giving a contradiction. The case of $\mathbf{dP_1} \cong \mathbf{F_1}$ is already settled. Now in the case of $2 \leq k \leq 7$ let $\alpha = kl - 3\sum_{i=1}^{k} E_i$ such that $-\alpha^2 = k(9 - k)$ and $H_B = c_1$. Then one finds with $\eta = al + \sum b_i E_i$ that $\alpha \cdot (\eta - nc_1) = ka + 3\sum b_i$: if this expression is ≤ 0 the choice of α was already successful; otherwise one just takes $-\alpha$.

3.5. *The realizability of DRY classes by Chern classes of stable bundles.* Let us now look at the conditions under which we can realize a given DRY class $c = \phi\sigma + \omega$ as $c_2(V)$ for the extension bundle V we have constructed. In the σ term one just takes $\eta := \phi$ where under our assumed condition $r \geq n$ the spectral requirement that $\eta - nc_1$ is effective follows from the DRY property that $\phi - \frac{n+r}{2}c_1$ is effective.

Let us check that $|\eta|$ is base point free as required in the spectral cover construction. For B given by the Hirzebruch surface $\mathbf{F_g}$ the linear system $|\eta|$ is base point free if $\eta \cdot b \geq 0$ (where b is the class of the base $\mathbf{P}^1$), and on B given by the del Pezzo surface $\mathbf{dP_k}$ ($2 \leq k \leq 7$) if $\eta \cdot E \geq 0$ for all curves E with $E^2 = -1$ and $E \cdot c_1 = 1$ (such curves generate the effective cone). Now note that because of Corollary 3.2 and Eq. (3.1) one has that $\eta - nc_1$ is ample such that $\eta \cdot \beta > 0$ for all irreducible curves β (note that we did assume c_1 is ample).

Now let us compare the ω terms, i.e. the lower bound for ω_{DRY} which follows from the DRY condition with the lower bound of the ω term in the expression $c_2(V)$ which comes from a stable bundle. Here one finds (with $n = 3$, $\lambda = 1/2$; note that we had a different choice for α for $\mathbf{F_1}$ and $\mathbf{dP_1}$, respectively)

$$\begin{aligned}\omega_{DRY} &> \frac{1}{2}\eta c_1 + \frac{m}{6}c_1^2 + \frac{m}{2} + \frac{1}{2m}\sqrt{c_1^2}\sqrt{\left(\eta - \frac{m}{2}c_1\right)^2} \\ &> \frac{1}{2}\eta c_1 + \frac{m}{6}c_1^2 + \frac{m}{2} = \begin{cases} \frac{1}{2}\eta c_1 + \frac{11}{6}\,m & \text{for } B = \mathbf{F_g} \\ \frac{1}{2}\eta c_1 + \frac{12-k}{6}m & \text{for } B = \mathbf{dP_k} \end{cases}\end{aligned} \tag{3.2}$$

$$\begin{aligned}\omega_V &= -\frac{n^3-n}{24}c_1^2 + r + 2 - \frac{nr(n+r)}{2}\alpha^2 \\ &= \begin{cases} -\frac{n^3-n}{3} + m - 1 + \frac{3}{2}(m-3)m(g+2) & \text{for } B = \mathbf{F_g} \\ -(9-k)\frac{n^3-n}{24} + m - 1 + \frac{3}{2}(m-3)mk(9-k) & \text{for } B = \mathbf{dP_k}. \end{cases}\end{aligned} \tag{3.3}$$

Note that the bound for ω_V is not only a necessary condition for an integer being the ω_V of a stable bundle V (of the type we constructed above) but even a sufficient one according to Eq. (2.2). Thus one finds that, as the bound for ω_{DRY} depends on η while the bound for ω_V does not, only for finitely many choices of η the corresponding bound for ω_{DRY} does not lie above the bound for ω_V (keeping m fixed). This finishes the proof of part (i) and part (a) of (ii) of Theorem 1.3.

For part (b1) of (ii) note that for $\mathcal{N} = m$ being $\equiv 2 (\mathrm{mod}\ 4)$ one can take $n = r$ (being odd) and use (still with $\lambda = 1/2$) the modified construction (2.12). In this case one gets

$$\omega_V = -\frac{n^3-n}{24}c_1^2 + \frac{m}{2} + 2 - \alpha^2 = \begin{cases} \frac{m}{2} + g + 4 - \frac{n^3-n}{3} & \text{for } B = \mathbf{F_g} \\ \frac{m}{2} + 10 - \frac{n^3-n}{24}(9-k) & \text{for } B = \mathbf{dP_k} \end{cases} \tag{3.4}$$

where this time for dP_k we choose $\alpha = l - 3E_1$ (which is also orthogonal to c_1) with $-\alpha^2 = 8$. To finish the proof for $B = \mathbf{F_g}$ one just has to note that $\frac{11}{6}m > \frac{m}{2} + g + 4 - \frac{n^3-n}{3}$ (recall $n = m/2$). For $B = \mathbf{dP_k}$ note that, using $\eta c_1 > \frac{m}{2}(9-k)$ because $\eta - nc_1$ is ample, one has $\omega_{DRY} > [\frac{5}{12}(9-k) + \frac{1}{2}]m$. For $k = 1, \ldots, 6$ the lower bound for ω_{DRY} is greater than the lower bound for ω_V because $\frac{5}{12}(9-k)m > 10 - \frac{n^3-n}{24}(9-k)$. For $k = 7$ only a small $m = 6$ is excluded.

For part (b2) of (ii) where one takes $n = r$ with n even and $\lambda = 1/2$ one has the condition $\phi \equiv c_1 (\mathrm{mod}\ 2)$ and one notes that $\frac{11}{6}m > \frac{1}{2}m - 4$.

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