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Connections on decorated path space bundles

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ABSTRACT

For a principal bundle $P \to M$ equipped with a connection $\bar{A}$, we study an infinite dimensional bundle $\mathcal{P}^{\text{dec}}_{\bar{A}}P$ over the space of paths on M, with the points of $\mathcal{P}^{\text{dec}}_{\bar{A}}P$ being horizontal paths on P decorated with elements of a second structure group. We construct parallel transport processes on such bundles and study holonomy bundles in this setting.

1. Introduction

The focus of our study is parallel transport on bundles whose elements are paths decorated with elements of a second structure group. Geometry of this type can be studied in the language of category theory but in this work we focus exclusively on differential geometric aspects. However, we shall make remarks indicating the significance of certain notions in the category theoretic development.

We begin with a connection form $\bar{A}$ on a principal G-bundle $\pi : P \to M$, where G is a Lie group, and consider first the structure

$$\pi_{\bar{A}} : \mathcal{P}_{\bar{A}}P \to \mathcal{P}M : \overline{\gamma} \mapsto \pi \circ \overline{\gamma}, \tag{1.1}$$

where $\mathcal{P}M$ is the space of smooth paths on M and $\mathcal{P}_{\bar{A}}P$ the space of $\bar{A}$-horizontal smooth paths on P. Fig. 1 illustrates this structure.

The group G acts on the space $\mathcal{P}_{\bar{A}}P$ by right translations $\overline{\gamma} \mapsto \overline{\gamma}g$, and the structure (1.1) has the essential features of a principal G-bundle. Next we introduce a Lie group H and a semidirect product $H \rtimes_\alpha G$, which serves as a 'higher' structure group. Using these we construct a *decorated bundle*

$$\pi^d_{\bar{A}} : \mathcal{P}^{\text{dec}}_{\bar{A}}P = \mathcal{P}_{\bar{A}}P \times H \to \mathcal{P}M : (\overline{\gamma}, h) \mapsto \pi \circ \overline{\gamma}, \tag{1.2}$$

where we view each pair $(\overline{\gamma}, h)$ as an $\bar{A}$-horizontal path $\overline{\gamma}$ on P *decorated* with an element h drawn from the second structure group H. It is this structure, illustrated in Fig. 2, that is the ultimate focus of our work in this paper. The decorated bundle

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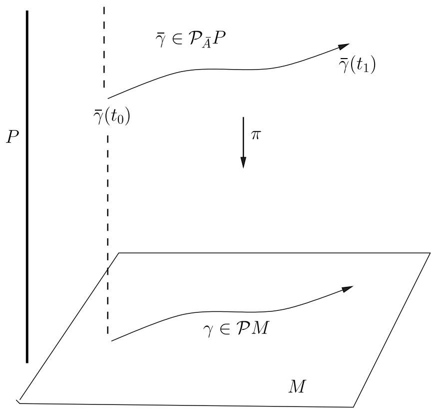


Fig. 1. Horizontal paths.

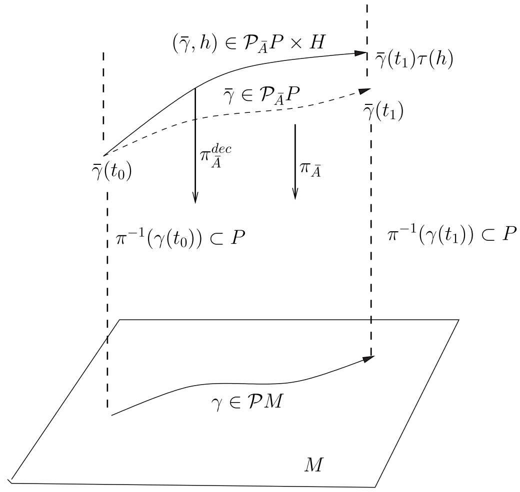


Fig. 2. Decorated paths.

arises as an example of a categorical principal bundle, as developed in [1]. Briefly put, the points of P are the objects of a category and the pairs $(\overline{\gamma}, h)$ are morphisms; the source of the morphism $(\overline{\gamma}, h)$ is the initial point $\overline{\gamma}_0$ of $\overline{\gamma}$ and the target is the point $\overline{\gamma}_1\tau(h)$, as shown in Fig. 2.

We prove results and explain how the structure (1.2) can be viewed as a principal $H \rtimes_\alpha G$-bundle. Parallel transport on this bundle takes a path on the base space $\mathcal{P}M$ of the form $[s_0, s_1] \to \mathcal{P}M : s \mapsto \Gamma_s$, and associates to it a path on the decorated bundle $\mathcal{P}_{\bar{A}}^{\mathrm{dec}}P$ of the form

$$[s_0, s_1] \to \mathcal{P}_{\bar{A}}^{\mathrm{dec}}P : s \mapsto (\hat{\Gamma}_s, h_s),$$

with a specified initial value $(\hat{\Gamma}_{s_0}, h_{s_0})$. This parallel transport process is obtained by using certain 1- and 2-forms on P with values in the Lie algebras $L(H)$ and $L(G)$. Given a suitable 1-form on P with values in the Lie algebra $L(H)$, we can associate, by a type of parallel transport process, a special element $h^*(\overline{\gamma}) \in H$ for each path $\overline{\gamma} \in \mathcal{P}_{\bar{A}}P$; this selects out an element $\left(\overline{\gamma}, h^*(\overline{\gamma})^{-1}\right) \in \mathcal{P}_{\bar{A}}^{\mathrm{dec}}P$ for each $\overline{\gamma} \in \mathcal{P}_{\bar{A}}P$. We then determine, in Section 7, conditions on the 1- and 2-forms that ensure that parallel transport of a point of $\mathcal{P}_{\bar{A}}^{\mathrm{dec}}P$ of the form $\left(\overline{\gamma}, h^*(\overline{\gamma})^{-1}\right)$ produces an element of the same type. This investigation is a study of the holonomy bundle for the decorated bundle (the holonomy bundle for a connection on a traditional finite dimensional principal bundle is a central object in the foundational work of Ambrose and Singer [2]).

The background motivation for our work arises from trying to construct a gauge theory for strings joining point particles. There is an active literature in this area, much of it focused on category theoretic aspects. In our recent works [3,1] we have developed a category theoretic framework centered on differential geometric notions such as parallel transport over spaces of decorated paths. In the present paper we establish a differential geometric development of the theory of connections over spaces of paths. For the category theoretic perspective we mention here the works of Abbaspour and Wagemann [4], Attal [5,6], Baez et al. [7,8], Barrett [9], Bartels [10], Parzygnat [11], Picken et al. [12–14], Soncini and Zucchini [15], Schreiber and Waldorf [16,17], and Wang [18,19].

1.1. Results and organization of material

All our constructions and results in this paper take as background a principal G-bundle $\pi : P \to M$, and a given set of connection forms and other forms on P. We denote by $\mathcal{P}_{\bar{A}}P$ the space of all paths on P that are horizontal with respect to a connection $\bar{A}$ on P. Here are the highlights of what we do in this paper:

- Section 2. We describe and study the bundle
$$\mathcal{P}_{\bar{A}}P \to \mathcal{P}M \tag{1.3}$$
and explain the sense in which this is a principal G-bundle. We also describe an explicit isomorphism between such bundles for different connections $\bar{A}$.
- Section 3. We construct a connection form ω on the bundle (1.3), determine horizontal lifts and parallel transport with respect to ω.
- Section 4. After reviewing the notion of a Lie crossed module, which involves two Lie groups G and H, with an action of G on H, we describe a *decorated bundle*
$$\mathcal{P}^{\rm dec}_{\bar{A}}P \to \mathcal{P}M. \tag{1.4}$$
A point on the decorated bundle is of the form
$$(\overline{\gamma}, h)$$
where $\overline{\gamma}$ is an $\bar{A}$-horizontal path on P and $h \in H$ is a 'decorating' element attached to $\overline{\gamma}$ (for example, h might arise by integration of an $L(H)$-valued 1-form along $\overline{\gamma}$). We study the decorated bundle and local trivializations for it.
- Sections 5 and 6. We construct a connection form Ω on the decorated bundle $\mathcal{P}^{\rm dec}_{\bar{A}}P$. Working out the splitting of a tangent vector to $\mathcal{P}^{\rm dec}_{\bar{A}}P$ into horizontal and vertical components, we determine horizontal lifts with respect to the connection Ω. Using this we determine explicitly the equations for parallel transport in the decorated path bundle.
- Section 7. Here we make an extensive examination of parallel transport of decorated paths. We consider a special type of decoration of a path $\overline{\gamma}$, where the decorating element in H arises by means of integration of an $L(H)$ valued 1-form along $\overline{\gamma}$. In Proposition 7.1 we find conditions under which the parallel transport of such a decorated path is itself decorated in the same manner.

The main objective of this paper is to study a differential geometric connection structure for the decorated bundles, with structure group $H \rtimes G$. The importance of such bundles arises from the fact that they provide a framework for categorical principal bundles, as explained in [1]. In Section 7, we consider a special type of subbundle of the decorated bundle with structure group be the subgroup G of $H \rtimes G$. We address the following question: when does a connection on the $H \rtimes G$ bundle reduce to a connection on that special subbundle. We obtain a curvature condition (7.10) that ensures that the connection reduces to the subbundle. The construction in Section 7 is motivated by the classical work of Ambrose and Singer [2] on the holonomy subbundle for a connection on a given principal G bundle.

2. A principal bundle of horizontal paths

We work with a principal G-bundle $\pi : P \to M$, where G is a Lie group, and a connection form $\bar{A}$ on this bundle. Our focus is on a pair of path spaces, one a space $\mathcal{P}M$ of paths on M and the other a space $\mathcal{P}_{\bar{A}}P$ of $\bar{A}$-horizontal paths on P. The projection map π induces a corresponding projection

$$\pi_{\bar{A}} : \mathcal{P}_{\bar{A}}P \to \mathcal{P}M : \overline{\gamma} \mapsto \pi \circ \overline{\gamma},$$

while the right action of G on the bundle space P induces a right action of G on $\mathcal{P}_{\bar{A}}P$ that preserves the fibers of $\pi_{\bar{A}}$. It is this structure, clearly analogous to a principal bundle, that we shall study. Specifying useful topologies and smooth structures on path spaces tend to be unrewarding tasks, and so we will keep the involvement of such structures to a minimum and make no attempt at formulating or using any general framework for them. However, it is important to note what exactly the elements of $\mathcal{P}M$ are. Unfortunately, even this requires a somewhat complex articulation of features that are intuitively quite clear.

2.1. The path spaces $\mathcal{P}M$ and $\mathcal{P}_{\bar{A}}P$

By a *parametrized path* on M we mean a C^∞ map $[t_0, t_1] \to M$, for some $t_0, t_1 \in \mathbb{R}$ with $t_0 < t_1$, that is constant near t_0 and near t_1. Thus the set of all such paths is

$$\bigcup_{t_0,t_1\in\mathbb{R},t_0<t_1} C_c^\infty\left([t_0, t_1]; M\right), \tag{2.1}$$

where the subscript c signifies the behavior near t_0 and t_1. If $\gamma_1 \in C_c^\infty[t_0, t_1]$ and $\gamma_2 \in C_c^\infty[t_1, t_2]$ then the composite $\gamma_2 \circ \gamma_1$ belongs to $C_c^\infty[t_0, t_2]$. It is often useful, at least for notational simplicity, to compose paths that are defined on the same parameter domain $[t_0, t_1]$. With this in mind, we introduce the quotient set $\mathcal{P}M$ obtained by identifying paths that differ by a time-translation reparametrization. Thus, $\gamma : [t_0, t_1] \to M$ is identified with $\gamma_{+a} : [t_0-a, t_1-a] \to M : t \mapsto \gamma(t+a)$ in $\mathcal{P}M$, for any $a \in \mathbb{R}$; this means that the parametrized paths γ and γ_{+a} correspond to the same element in $\mathcal{P}M$. We will usually not make a notational distinction between γ and its equivalence class $[\gamma]$ of such time-translation reparametrizations. We will then use the term 'path' to refer to an equivalence class such as this. Many important constructions are invariant under a far larger class of reparametrizations but at this stage we find it more convenient to keep the reparametrizations to a minimum.

We will not use a specific topology on $\mathcal{P}M$. Any topology of use in our context should (i) be Hausdorff, (ii) the initial and terminal points should be continuous functions of the path, and (iii) composition

$$(\gamma, \delta) \mapsto \delta \circ \gamma,$$

on the subset of $\mathcal{P}M \times \mathcal{P}M$ where defined, should be continuous.

Following the notational convention for $\mathcal{P}M$ we denoted by $\mathcal{P}_{\bar{A}}P$ the set of all $\bar{A}$-horizontal parametrized paths on P, where $\bar{A}$ is our given connection form. Thus, an element $\overline{\gamma} \in \mathcal{P}_{\bar{A}}P$ is represented by a C^∞ mapping $[t_0, t_1] \to P$, for some $t_0 < t_1$ in $\mathbb{R}$, constant near t_0 and t_1, such that

$$\bar{A}\big(\overline{\gamma}'(t)\big) = 0 \qquad \text{for all } t \in [t_0, t_1].$$

2.2. Local trivialization

We shall now construct a local trivialization of the path bundle $\pi : \mathcal{P}_{\bar{A}}P \to \mathcal{P}M$ using a local trivialization of the bundle $\pi : P \to M$. To this end consider an open set $U \subset M$ and a smooth diffeomorphism

$$\phi : U \times G \to \pi^{-1}(U) \tag{2.2}$$

that is G-equivariant in the sense that $\phi(u, gg') = \phi(u, g)g'$ for all $u \in U$ and $g, g' \in G$. Associated to U is the set U^0 of all paths that begin in U:

$$U^0 = \mathrm{ev}_0^{-1}(U)$$

in the base path space, where ev_0 gives the initial point, or source, of a path:

$$\mathrm{ev}_0 : \mathcal{P}M \to M : \gamma \mapsto \gamma_0 \stackrel{\text{def}}{=} \mathrm{ev}_0(\gamma) \stackrel{\text{def}}{=} \gamma(t_0).$$

We view U^0 as an open subset of $\mathcal{P}M$. We can construct a diffeomorphism between $\pi_{\bar{A}}^{-1}(U^0)$ and $U \times G$ by using the trivialization ϕ; to understand this let

$$\overline{\gamma}_p^{\bar{A}} \tag{2.3}$$

be the $\bar{A}$-horizontal path on P that starts at p and projects down to γ; thus

$$\bar{A}\Big(\{\overline{\gamma}_p^{\bar{A}}\}'(t)\Big) = 0 \qquad \text{for all } t \in [t_0, t_1],$$

the projection down to the base manifold is

$$\pi \circ \overline{\gamma}_p^{\bar{A}} = \gamma,$$

and the initial point is p:

$$\overline{\gamma}_p^{\bar{A}}(t_0) = p.$$

Then we define the map:

$$\phi^0 : U^0 \times G \to \pi_{\bar{A}}^{-1}(U^0) : (\gamma, g) \mapsto \phi^0(\gamma, g) \stackrel{\text{def}}{=} \overline{\gamma}_p^{\bar{A}}, \tag{2.4}$$

where $p = \phi(\gamma_0, g)$.

The mapping ϕ^0 is G-equivariant and is clearly bijective as well.

2.3. Transition functions

Let us now determine the transition function between trivializations ϕ^0 and ψ^0. For trivializations $\phi : U \times G \to \pi^{-1}(U)$ and $\psi : V \times G \to \pi^{-1}(V)$ we have the transition function

$$\theta_{\phi,\psi} : U \cap V \to G$$

given by

$$\psi(u, g) = \phi(u, g)\theta_{\phi,\psi}(u) \qquad \text{for all } (u, g) \in (U \cap V) \times G.$$

Then

$$\begin{aligned} \text{initial point of } \psi^0(\gamma, g) &= \psi\big(\gamma_0, g\big) \\ &= \phi\big(\gamma_0, g\big)\theta_{\phi,\psi}\big(\gamma_0\big) \\ &= \text{initial point of } \phi^0(\gamma, g)\theta_{\phi,\psi}\big(\gamma_0\big). \end{aligned} \tag{2.5}$$

Thus the transition function between ϕ^0 and ψ^0 is given by

$$\theta_{\phi^0,\psi^0} : U^0 \cap V^0 \to G : \gamma \mapsto \theta_{\phi,\psi}\big(\mathrm{ev}_0(\gamma)\big). \tag{2.6}$$

(Technically, we have not imposed a topology on the path space and so we do not have to verify continuity or smoothness of these transition functions.)

2.4. A pullback bundle

We can describe the bundle $\pi : \mathcal{P}_{\bar{A}}P \to \mathcal{P}M$ as a pullback of the bundle $\pi : P \to M$ under the evaluation map

$$\mathrm{ev}_0 : \mathcal{P}M \to M : \gamma \mapsto \gamma_0 = \mathrm{ev}_0(\gamma).$$

To this end let

$$\mathrm{ev}_0^*P := \{(\gamma, p) \in \mathcal{P}M \times P \mid \gamma_0 = \pi(p)\}. \tag{2.7}$$

Thus a point of ev_0^*P is specified by a point $p \in P$ along with a path γ on M that starts at $\pi(p)$.

The group G acts on this space by

$$(\gamma, p)g := (\gamma, pg).$$

The mapping

$$\pi_0 : \mathrm{ev}_0^*P \to \mathcal{P}M : (\gamma, p) \mapsto \gamma$$

is a surjective projection for which

$$\pi_0\big(\gamma g, p\big) = \pi_0(\gamma, p)$$

for all $p \in P$, $\gamma \in \mathcal{P}M$ and $g \in G$.

Proposition 2.1. *The mapping*

$$\mu : \mathrm{ev}_0^*P \to \mathcal{P}_{\bar{A}}P : (\gamma, p) \mapsto \overline{\gamma}\bar{A}_p, \tag{2.8}$$

where $\overline{\gamma}_p^{\bar{A}}$ is the $\bar{A}$-horizontal lift of γ initiating at $p \in P$, is a G-equivariant bijection $\mu : \mathrm{ev}_0^*P \to \mathcal{P}_{\bar{A}}P$. The diagram

$$\begin{tikzcd} \mathrm{ev}_0^*P \arrow[r, "\mu"] \arrow[d, "\pi_0"] & \mathcal{P}_{\bar{A}}P \arrow[d, "\pi_{\bar{A}}"'] \\ \mathcal{P}M \arrow[r, "\mathrm{id}"'] & \mathcal{P}M \end{tikzcd} \tag{2.9}$$

commutes.

Let us note that $\pi_{\bar{A}} : \mathcal{P}_{\bar{A}}P \to \mathcal{P}M$ is a principal G-bundle (in the sense that the projection π is a surjection and the group G acts freely and transitively on each fiber of π); since the base and bundle spaces are both path spaces it might seem at first that the structure group is infinite dimensional but in fact it is just G because we are focusing on the $\bar{A}$-horizontal paths.

Proof. Consider any $(\gamma, p) \in \mathrm{ev}_0^*P$; then by definition $p \in \pi^{-1}(\gamma_0)$. Hence we can horizontally lift the path γ on M to P by $\bar{A}$ to obtain the $\bar{A}$-horizontal path $\overline{\gamma}_p^{\bar{A}}$ starting from $p \in P$ and that path is unique. On the other hand, any element $\overline{\gamma} \in \mathcal{P}_{\bar{A}}P$ is the image under μ of $\big(\gamma, \overline{\gamma}_0\big) \in \mathrm{ev}_0^*P$, where $\gamma = \pi_{\bar{A}}(\overline{\gamma}) \in \mathcal{P}M$. Thus

$$\mu : (\gamma, p) \mapsto \overline{\gamma}_p$$

is a bijection. Since horizontal lifts behave equivariantly under the action of G, the mapping μ is G-equivariant. The definition of μ also implies directly that the diagram (2.9) is commutative. $\square$

2.5. The tangent space $T_{\overline{\gamma}}\mathcal{P}_{\bar{A}}P$

We define a *tangent vector* $\tilde{v}$ at $\overline{\gamma} \in \mathcal{P}_{\bar{A}}P$ by means of the following description:

(i) $\tilde{v}$ is a C^∞ vector field along $\overline{\gamma}$,
(ii) $\tilde{v}$ satisfies the *tangency condition*

$$\frac{\partial \bar{A}(\tilde{v}(t))}{\partial t} = F^{\bar{A}}(\overline{\gamma}'(t), \tilde{v}(t)), \tag{2.10}$$

for all $t \in [t_0, t_1]$, where $[t_0, t_1]$ is the domain of $\overline{\gamma}$, and
(iii) $\tilde{v}$ is constant near t_0 and near t_1.

To be more precise two such vector fields are viewed as the same tangent vector if they differ by reparametrization by a translation as in the discussion following (2.1). We denote the set of all such vector fields by

$$T_{\overline{\gamma}}\mathcal{P}_{\bar{A}}P.$$

The linear nature of the differential equation (2.10) implies that this tangent space is indeed a vector space (closed under addition and scaling).

There is a unique tangent vector $\tilde{v} \in T_{\overline{\gamma}}\mathcal{P}_{\bar{A}}P$ with a specified projection vector field $v = \pi_*\tilde{v}$ and initial value $\tilde{v}(t_0)$. We review the proof from [3, Lemma 2.1]. Let $\tilde{v}(t)^h$ be the $\bar{A}$-horizontal vector in $T_{\overline{\gamma}(t)}P$ that projects by π_* to $v(t)$:

$$\bar{A}\big(\tilde{v}(t)^h\big) = 0 \quad \text{and} \quad \pi_*\tilde{v}(t)^h = v(t). \tag{2.11}$$

Now let

$$Z(t) = \bar{A}\big(\overline{v}(t_0)\big) + \int_{t_0}^{t} F^{\bar{A}}\big(\overline{\gamma}'(u), \tilde{v}(u)^h\big)\, du \in L(G),$$

and consider the vector field $\tilde{v}$ along $\overline{\gamma}$ specified by

$$\tilde{v}(t) = \tilde{v}(t)^h + \overline{\gamma}(t)Z(t) \tag{2.12}$$

for all $t \in [t_0, t_1]$. Then

$$\bar{A}\big(\tilde{v}(t)\big) = Z(t) = \bar{A}\big(\tilde{v}(t_0)\big) + \int_{t_0}^{t} F^{\bar{A}}\big(\overline{\gamma}'(u), \tilde{v}(u)\big)\, du. \tag{2.13}$$

Thus the differential equation (2.10) holds. Moreover, if $\tilde{v}$ is any vector field along $\overline{\gamma}$ projecting down to v and satisfying the differential equation (2.10) then the relation (2.13) holds and so $\tilde{v}(t)$ is given by (2.12). Because γ is constant near t_0 and t_1 we see that Z is constant near these endpoints. Similarly v and $\tilde{v}^h$ are also constants near the ends, and hence so is $\tilde{v}$.

Let us see how this is consistent with the pullback point of view in Proposition 2.1. If $\tilde{v}_0 \in T_pP$, where $p = \overline{\gamma}(t_0)$ and if v is a smooth vector field along $\gamma = \pi_{\bar{A}}(\overline{\gamma})$, viewed as a vector in $T_\gamma \mathcal{P}M$, with initial value $v(t_0) = d\pi|_p\tilde{v}_0$, then

$$d\mu\big|_{(\gamma,p)}(v, \tilde{v}_0) = \tilde{v}, \tag{2.14}$$

where the derivative $d\mu$ is taken in a formal but natural sense; more officially, we can take (2.14) as defining $d\mu$.

Working with the map μ that identifies $\mathcal{P}_{\bar{A}}P$ with the pullback bundle ev_0^*P it is possible to construct local trivializations of $\pi_{\bar{A}} : \mathcal{P}_{\bar{A}}P \to \mathcal{P}M$ from those of $\pi : P \to M$, and these coincide with the type given in (2.4).

2.6. Changing the base connection $\mathcal{P}_{\bar{A}}P$

We have been working with a fixed connection $\bar{A}$ on the principal G-bundle $\pi : P \to M$ and using this we have defined the path bundle $\mathcal{P}_{\bar{A}}P \to \mathcal{P}M$. Changing $\bar{A}$ to another connection $\bar{A}'$ produces a bundle $\mathcal{P}_{\bar{A}'}P \to \mathcal{P}M$. We show now that this is 'isomorphic' to $\mathcal{P}_{\bar{A}}P \to \mathcal{P}M$; this is, of course, quite natural from the point of view of Proposition 2.1. For the following result let us recall that the tangent space $T_{\overline{\gamma}}\mathcal{P}_{\bar{A}}P$ consists of all vector fields $\tilde{v}$ along $\overline{\gamma} \in \mathcal{P}_{\bar{A}}P$, constant near the initial and final points, that satisfy (2.10).

Let $\bar{A}$ and $\bar{A}'$ be connections on a principal G-bundle $\pi : P \to M$. Let $\mathcal{P}_{\bar{A}}P$ be the set of all paths on P that are $\bar{A}$-horizontal and $\mathcal{P}_{\bar{A}'}P$ the set of all $\bar{A}'$-horizontal paths. For each $\overline{\gamma} \in \mathcal{P}_{\bar{A}}P$ let $\mathcal{T}(\overline{\gamma})$ be the path on P that is $\bar{A}'$-horizontal, has the same initial point as $\overline{\gamma}$, and projects down to the same path $\pi \circ \overline{\gamma}$ on M as $\overline{\gamma}$. Thus we have a mapping

$$\mathcal{T} : \mathcal{P}_{\bar{A}}P \to \mathcal{P}_{\bar{A}'}P : \overline{\gamma} \mapsto \mathcal{T}(\overline{\gamma}). \tag{2.15}$$

For any vector field $\tilde{v}$ along $\overline{\gamma}$ that belongs to the tangent space $T_{\overline{\gamma}}\mathcal{P}_{\bar{A}}P$ let

$$\mathcal{T}_*\tilde{v} \in T_{\mathcal{T}(\overline{\gamma})}(\mathcal{P}_{\bar{A}'}P), \tag{2.16}$$

be the vector field along $\mathcal{T}(\overline{\gamma})$ whose initial value is $\tilde{v}(t_0)$ and whose projection by π_* is the vector field $\pi_* \circ \tilde{v}$ along the path $\gamma = \pi \circ \overline{\gamma} \in \mathcal{P}M$.

The mapping $\mathcal{T}_*$ is, in a natural intuitive sense, the derivative of the mapping $\mathcal{T}$. In more detail, suppose

$$\tilde{\Gamma} : [t_0, t_1] \times [s_0, s_1] \to P : (t, s) \mapsto \tilde{\Gamma}_s(t)$$

is a C^∞ map for which $\overline{\gamma} = \pi \circ \tilde{\Gamma}_{s_0}$ and

$$\tilde{v}(t) = \partial_s \tilde{\Gamma}_s\big|_{s=s_0}(t) \qquad \text{for all } t \in [t_0, t_1].$$

This displays the vector field $\tilde{v} \in T_{\overline{\gamma}}\mathcal{P}_{\tilde{A}}P$ as the 'tangent vector' to a path $s \mapsto \tilde{\Gamma}_s$ on $\mathcal{P}_{\tilde{A}}P$. Then the image of $\tilde{v}$ under the derivative of $\mathcal{T}$ at $\overline{\gamma}$ should be the tangent, at $s = s_0$, of the image path

$$s \mapsto \mathcal{T}(\tilde{\Gamma}_s).$$

This tangent is the vector field along $\mathcal{T}(\overline{\gamma})$ given by

$$t \mapsto w(t) \stackrel{\text{def}}{=} \partial_s \mathcal{T}(\tilde{\Gamma}_s)(t)\Big|_{s=s_0}.$$

Focusing on the initial 'time' $t = t_0$ we have

$$w(t_0) = \partial_s \tilde{\Gamma}_s(t_0)\Big|_{s=s_0}$$

because $\mathcal{T}(\tilde{\Gamma}_s)(t_0) = \tilde{\Gamma}_s(t_0)$ by definition of the mapping $\mathcal{T}$. Thus

$$w(t_0) = \tilde{v}(t_0).$$

Thus $w \in T_{\mathcal{T}(\overline{\gamma})}\mathcal{P}_{\tilde{A}'}P$, being uniquely determined by the initial value $w(t_0)$ and the projection $\pi_* \circ w = \pi_*\tilde{v} \in T_{\pi\circ\overline{\gamma}}M$, is exactly $\mathcal{T}_*(\tilde{v})$ as we have defined it above.

Proposition 2.2. *With A and A' connections on a principal G-bundle $\pi : P \to M$, and $\mathcal{T}$ and $\mathcal{T}$ as in (2.15) and (2.16), we have:*

(i) *the mapping $\mathcal{T} : \mathcal{P}_{\tilde{A}}P \to \mathcal{P}_{\tilde{A}'}P$ is a bijection;*
(ii) *for any point in $\mathcal{P}_{\tilde{A}}P$ given by a path $\overline{\gamma} : [t_0, t_1] \to P$ and any $\tilde{v} \in T_{\overline{\gamma}}\mathcal{P}_{\tilde{A}}P$, the vector*

$$\mathcal{T}_*(\tilde{v})(t) - \tilde{v}(t)g_{\overline{\gamma}}(t) \tag{2.17}$$

is vertical for all $t \in [t_0, t_1]$, where $g_{\overline{\gamma}}(t)$ is the element of G for which $\mathcal{T}(\overline{\gamma})(t) = \overline{\gamma}(t)g_{\overline{\gamma}}(t)$.

Proof. Horizontal lift of a path by a connection is uniquely determined by the initial point of the lifted path. Using this we see that $\overline{\gamma}$ is determined uniquely when $\mathcal{T}(\overline{\gamma})$ is known. Thus $\mathcal{T}$ is a bijection.

The right action mapping

$$R_g : P \to P : p \mapsto pg,$$

for any fixed $g \in G$, preserves fibers:

$$\pi \circ R_g(p) = \pi(p) \qquad \text{for all } p \in P.$$

Taking the derivative at p on any vector $v \in T_pP$ we then have

$$\pi_*|_{pg}\Big((R_g)_*|_p v\Big) = \pi_*|_p v \tag{2.18}$$

for all $v \in T_pP$ and all $p \in P$. Our notation vg means simply $(R_g)_*|_p v$:

$$vg \stackrel{\text{def}}{=} (R_g)_*|_p v.$$

Hence

$$\pi_*(vg) = \pi_*(v).$$

Next from the definition of $\mathcal{T}_*(\tilde{v})(t)$ we know that its projection by π_* is the vector $\pi_*(\tilde{v}(t)) \in T_{\pi\circ\overline{\gamma}(t)}M$. Thus the vectors $\mathcal{T}_*(\tilde{v})(t)$ and $\tilde{v}(t)g_{\overline{\gamma}}(t)$ in $T_{\mathcal{T}(\overline{\gamma})(t)}P$ both project down by π_* to the vector $\pi_*(\tilde{v}(t)) \in T_{\pi\circ\overline{\gamma}(t)}M$. Hence the difference $\mathcal{T}_*(\tilde{v})(t) - \tilde{v}(t)g_{\overline{\gamma}}(t)$ is a vertical vector. $\square$

2.7. Pullback of forms

We continue with the comparison of the horizontal path spaces $\mathcal{P}_{\bar{A}}P$ and $\mathcal{P}_{\bar{A}'}P$ using the mapping $\mathcal{T}$. We define pullbacks in the natural way: if $\tilde{D}$ is a k-form on $\mathcal{P}_{\bar{A}'}P$, with values in some vector space, then $\mathcal{T}^*\tilde{D}$ is the k-form on $\mathcal{P}_{\bar{A}}P$ given by

$$(\mathcal{T}^*\tilde{D})(\tilde{v}_1, \dots, \tilde{v}_k) = \tilde{D}(\mathcal{T}_*\tilde{v}_1, \dots, \mathcal{T}_*\tilde{v}_k). \tag{2.19}$$

For example, suppose B is 2-form on P with values in some vector space. Consider then the 2-form $\tilde{B}$ on $\mathcal{P}_{\bar{A}'}P$ given by

$$\tilde{B}_{\tilde{\gamma}}(\tilde{v}, \tilde{w}) = \int_{t_0}^{t_1} B_{\tilde{\gamma}(t)}\big(\tilde{v}(t), \tilde{w}(t)\big)\, dt \tag{2.20}$$

for all $\tilde{\gamma} \in \mathcal{P}_{\bar{A}'}P$ and $\tilde{v}, \tilde{w} \in T_{\tilde{\gamma}}\mathcal{P}_{\bar{A}'}P$. Then

$$\big(\mathcal{T}^*\tilde{B}\big)|_{\overline{\gamma}}(\overline{v}, \overline{w}) = \int_{t_0}^{t_1} B_{\mathcal{T}(\overline{\gamma})}\big((\mathcal{T}_*\overline{v})(t), (\mathcal{T}_*\overline{w})(t)\big)\, dt. \tag{2.21}$$

We can also pullback the 1-form on $\mathcal{P}_{\bar{A}'}P$ given by the Chen integral

$$B^{\mathrm{ch}}_{\tilde{\gamma}}(\tilde{v}) \stackrel{\text{def}}{=} \int_{t_0}^{t_1} B\big(\tilde{\gamma}'(t), \tilde{v}(t)\big)\, dt \tag{2.22}$$

to obtain the 1-form $\mathcal{T}^*B^{\mathrm{ch}}$ given by

$$(\mathcal{T}^*B^{\mathrm{ch}})_{\overline{\gamma}}(\overline{v}) \stackrel{\text{def}}{=} \int_{t_0}^{t_1} B\big(\overline{\gamma}'(t), \overline{v}(t)\big)\, dt. \tag{2.23}$$

(Chen integrals were introduced and developed in [20,21].) As another example, for a 1-form C on P, with values in some vector space, we have a 1-form $\tilde{C}_0$ on $\mathcal{P}_{\bar{A}'}P$ given by

$$\tilde{C}_0|_{\tilde{\gamma}}(\tilde{v}) = C_0|_{\tilde{\gamma}(t_0)}\big(\tilde{v}(t_0)\big)$$

and then the pullback $\mathcal{T}^*\tilde{C}_0$ is given by

$$\big(\mathcal{T}^*\tilde{C}_0\big)|_{\overline{\gamma}}(\overline{v}) = \tilde{C}_0|_{\mathcal{T}(\tilde{\gamma})(t_0)}\big((\mathcal{T}_*\overline{v})(t_0)\big), \tag{2.24}$$

for all $\overline{\gamma} \in \mathcal{P}_{\bar{A}}P$ and $\overline{v} \in T_{\overline{\gamma}}\mathcal{P}_{\bar{A}}P$.

3. A connection form on the space of horizontal paths

We continue working with a principal G-bundle

$$\pi : P \to M$$

equipped with a connection form $\bar{A}$, and the corresponding projection map

$$\pi_{\bar{A}} : \mathcal{P}_{\bar{A}}P \to \mathcal{P}M : \overline{\gamma} \mapsto \pi \circ \overline{\gamma},$$

where $\mathcal{P}M$ is the space of smooth paths on M and $\mathcal{P}_{\bar{A}}P$ the space of smooth horizontal paths in P. In the preceding section we have seen how $\pi_{\bar{A}} : \mathcal{P}_{\bar{A}}P \to \mathcal{P}M$ can be viewed, in a reasonable sense, as a principal G-bundle. Now we turn to a description of a 1-form ω on $\mathcal{P}_{\bar{A}}P$ (the sense in which this is a 1-form will become clear) that essentially provides a connection form on this path space bundle.

3.1. The 2-form B_0 and a connection form A

Henceforth we will work with an $L(G)$-valued 2-form B_0 on P that has the following two special properties:

(i) B_0 is Ad-equivariant in the sense that

$$B_0|_{pg}(vg, wg) = \mathrm{Ad}(g^{-1})B_0(v, w),$$

for all $p \in P$ and all $v, w \in T_pP$;

(ii) B_0 is horizontal in the sense that

$$B_0(v, w) = 0$$

whenever v or w is a vertical vector, i.e. in $\ker \pi_*$.

Thus B_0 satisfies

$$\begin{aligned} B_0|_{pg}(vg, wg) &= \mathrm{Ad}(g^{-1})B_0(v, w), \qquad \forall v, w \in T_pP, g \in G, \\ B_0(v, w) &= 0, \qquad \text{if } v \text{ or } w \text{ is vertical} \end{aligned} \tag{3.1}$$

at all points $p \in P$.

As our final ingredient, let A be a connection form on the G-bundle $\pi : P \to M$.

3.2. The form $\omega^{(A,B_0)}$ on $\mathcal{P}_{\bar{A}}P$

We define an $L(G)$-valued 1-form $\omega^{(A,B_0)}$ on $\mathcal{P}_{\bar{A}}P$ by

$$\omega^{(A,B_0)}_{\overline{\gamma}}(\tilde{v}) := A\big(\tilde{v}(t_0)\big) + \int_{t_0}^{t_1} B_0\big(\tilde{v}(t), \overline{\gamma}'(t)\big)\, dt, \tag{3.2}$$

where $\tilde{v} \in T_{\overline{\gamma}}\mathcal{P}_{\bar{A}}P$.

3.3. The connection form ω on $\mathcal{P}_{\bar{A}}P$

In $\omega^{(A,B_0)}$ we have given a special role to the left endpoint $\overline{\gamma}(t_0)$; this could, however, be replaced by the right endpoint $\overline{\gamma}(t_1)$. In fact a slightly more general construction leads to a 1-form with dependence on both endpoints. To this end, let C_0^L and C_0^R be 1-forms on P, with values in $L(G)$, that vanish on vertical vectors and are equivariant:

$$C_0^{L,R}|_{pg}(vg) = \mathrm{Ad}(g^{-1})C_0^{L,R}|_p(v). \tag{3.3}$$

Consider then the $L(G)$-valued 1-form

$$\omega = \omega^{(A,B_0,C_0^L,C_0^R)}$$

on $\mathcal{P}_{\bar{A}}P$ given by

$$\omega_{\overline{\gamma}}(\tilde{v}) \overset{\text{def}}{=} A|_{\overline{\gamma}(t_0)}\big(\tilde{v}(t_0)\big) + C_0^R|_{\overline{\gamma}(t_1)}\big(\tilde{v}(t_1)\big) - C_0^L|_{\overline{\gamma}(t_0)}\big(\tilde{v}(t_0)\big) + \int_{t_0}^{t_1} B_0|_{\overline{\gamma}(t)}\big(\tilde{v}(t), \overline{\gamma}'(t)\big)\, dt. \tag{3.4}$$

Thus

$$\omega = \omega^{(A,B_0)} + \mathrm{ev}_1^* C_0^R - \mathrm{ev}_0^* C_0^L, \tag{3.5}$$

where ev_0 and ev_1 are the evaluations at the left and right endpoints respectively.

The additional terms in (3.5) allow us to include counterparts of $\omega^{(A,B_0)}$ that have a right endpoint term in place of the left endpoint term $A\big(\overline{v}(t_0)\big)$ by taking

$$C_0^L = C_0^R = A - \bar{A} \tag{3.6}$$

and replacing B_0 by $F^{\bar{A}} + B_0$ leads to the counterpart of $\omega^{(A,B_0)}$ involving the right endpoint in place of the left endpoint.

Proposition 3.1. *The* 1*-form ω on the principal G-bundle $\pi_{\bar{A}} : \mathcal{P}_{\bar{A}}P \to \mathcal{P}M$ has the following properties:*

(i)

$$\omega(\tilde{v}g) = \mathrm{Ad}(g^{-1})\omega(\tilde{v}) \tag{3.7}$$

for all $g \in G$ and all vector fields $\tilde{v} \in T_{\overline{\gamma}}\mathcal{P}_{\bar{A}}P$ and all $\overline{\gamma} \in \mathcal{P}_{\bar{A}}P$;

(ii) *If Y is any element of the Lie algebra $L(G)$ and $\widetilde{Y}$ is the vector field along $\overline{\gamma}$ given by $\widetilde{Y}(t) = \frac{d}{du}\big|_{u=0}\overline{\gamma}(t)\exp(uY)$, then*

$$\omega(\widetilde{Y}) = Y. \tag{3.8}$$

The property (3.8) can also be written as:

$$\omega_{\overline{\gamma}}(\overline{\gamma}Y) = Y \qquad \text{for all } Y \in L(G) \text{ and } \overline{\gamma} \in \mathcal{P}_{\bar{A}}P. \tag{3.9}$$

In [3, Proposition 2.2] we have established the preceding result with $C_0^R = 0$. The proof is simple, so we present a quick sketch here. The equivariance property (i) holds for each of the terms on the right in the definition of ω in (3.4) and so it holds for ω. Next, applying ω to the vector $\widetilde{Y} \in T_{\overline{\gamma}}\mathcal{P}_{\bar{A}}P$ all terms on the right in (3.4) are 0 except for the very first one which equals $A\big(\overline{\gamma}(t_0)\widetilde{Y}\big) = Y$ since A is a connection form on P; this establishes property (ii).

Properties (i) and (ii) are the essential properties of a connection form on a traditional principal bundle and so we will say that ω is a *connection* on $\pi : \mathcal{P}_{\bar{A}}P \to \mathcal{P}M$ even though we have not equipped the latter with a smooth structure. We will use this terminology henceforth for other forms that enjoy the properties (i) and (ii) in the relevant bundles.

3.4. Horizontal lifts of vectors using ω

Having constructed the connection form ω on the pathspace bundle, we now turn to a description of horizontal lifts with respect to ω. Consider a path $\gamma \in \mathcal{P}M$ and a tangent $v \in T_\gamma \mathcal{P}M$; this is just a smooth vector field along γ constant near the initial and terminal points. Our objective now is to show that for any $\overline{\gamma} \in \pi_{\bar{A}}^{-1}(\gamma)$, the connection ω provides a unique horizontal lift $\overline{v} \in T_{\overline{\gamma}}\mathcal{P}_{\bar{A}}P$; that is, $\overline{v}$ satisfies

$$\omega_{\overline{\gamma}}(\overline{v}) = 0 \quad \text{and} \quad \pi_{\bar{A}}(\overline{v}) = v. \tag{3.10}$$

We can choose a C^∞ vector field $\overline{v}$ along the path $\overline{\gamma}$ for which

$$d\pi_{\overline{\gamma}(t)}\big(\overline{v}_{\overline{\gamma}(t)}\big) = v_{\gamma(t)} \qquad \text{for all } t \in [t_0, t_1].$$

Now let Z_0 be the element of $L(G)$ given by

$$Z_0 = -\left[C_0^R|_{\overline{\gamma}(t_1)}\big(\overline{v}(t_1)\big) - C_0^L|_{\overline{\gamma}(t_0)}\big(\overline{v}(t_0)\big) + \int_{t_0}^{t_1} B_0|_{\overline{\gamma}(t)}\big(\overline{v}(t), \overline{\gamma}'(t)\big)\,dt\right]. \tag{3.11}$$

Let $\overline{v}_0 \in T_{\overline{\gamma}(t_0)}P$ for which

$$\overline{v}_0 = \overline{v}_0^h + \overline{\gamma}(t_0)Z_0, \tag{3.12}$$

where

$$\overline{v}_0^h \in T_{\overline{\gamma}(t_0)}P$$

is the unique A-horizontal vector that projects by π_* down to $v(t_0)$. Now let $\overline{v}$ be the vector field along $\overline{\gamma}$ that is in the tangent space $T_{\overline{\gamma}}\mathcal{P}_{\bar{A}}P$ and has initial value $\overline{v}_0$; this vector field is specified in (2.12) discussed earlier. Thus

$$A_{\overline{\gamma}(t_0)}\big(\overline{v}(t_0)\big) = A_{\overline{\gamma}(t_0)}(Z_0) \tag{3.13}$$

and so

$$A_{\overline{\gamma}(t_0)}\big(\overline{v}(t_0)\big) + C_0^R|_{\overline{\gamma}(t_1)}\big(\overline{v}(t_1)\big) - C_0^L|_{\overline{\gamma}(t_0)}\big(\overline{v}(t_0)\big) + \int_{t_0}^{t_1} B_0|_{\overline{\gamma}(t)}\big(\overline{v}(t), \overline{\gamma}'(t)\big)\,dt = 0. \tag{3.14}$$

This says precisely that

$$\omega_{\overline{\gamma}}(\overline{v}) = 0.$$

Thus we have shown existence of the ω-horizontal lift $\overline{v} \in T_{\overline{\gamma}}\mathcal{P}_{\bar{A}}P$ of the vector field $v \in T_\gamma\mathcal{P}M$. Uniqueness follows from the fact that the condition (3.14) implies that the initial value $\overline{v}(t_0)$ is given by $\overline{v}_0$ as in (3.12), and this uniquely specifies $\overline{v}$ as discussed in the context of (2.12).

3.5. Parallel transport of horizontal paths by ω

Let us now understand the process of parallel transport in the bundle $\mathcal{P}_{\bar{A}}P$ using the connection form ω; we recall that

$$\omega_{\overline{\gamma}}(\overline{v}) = A\big(\overline{v}(t_0)\big) + C_0^R\big(\overline{v}(t_1)\big) - C_0^L\big(\overline{v}(t_0)\big) + \int_{t_0}^{t_1} B_0\Big(\overline{v}(t), \overline{\gamma}'(t)\Big)\,dt, \tag{3.15}$$

for any $\overline{\gamma} \in \mathcal{P}_{\bar{A}}P$ and $\overline{v} \in T_{\overline{\gamma}}\mathcal{P}_{\bar{A}}P$. Consider a C^∞ map

$$[t_0, t_1] \times [s_0, s_1] \to M : (t, s) \mapsto \Gamma(t, s) = \Gamma_s(t),$$

forming a path $s \mapsto \Gamma_s$ on $\mathcal{P}M$, and consider an initial 'point' $\tilde{\Gamma}_{s_0} \in \mathcal{P}_{\bar{A}}P$ with

$$\pi \circ \tilde{\Gamma}_{s_0} = \Gamma_{s_0}.$$

Now let

$$\overline{\Gamma} : [t_0, t_1] \times [s_0, s_1] \to P : (t, s) \mapsto \overline{\Gamma}_s(t)$$

be the mapping specified by the requirements that each path

$$t \mapsto \overline{\Gamma}_s(t)$$

be $\bar{A}$-horizontal (hence $\overline{\Gamma}_s \in \mathcal{P}_{\bar{A}}P$), with $\pi \circ \overline{\Gamma}_s = \Gamma_s$, and the initial points $\overline{\Gamma}_s(t_0)$ trace out an A-horizontal path on P with initial point $\tilde{\Gamma}_{s_0}(t_0)$:

$$\begin{aligned} &s \mapsto \overline{\Gamma}_s(t_0) \in P \qquad \text{is an } A\text{-horizontal path,}\\ &\overline{\Gamma}_{s_0} = \tilde{\Gamma}_{s_0}. \end{aligned} \tag{3.16}$$

By the nature of the differential equation for parallel transport the path $s \mapsto \overline{\Gamma}_s(t_0)$ is C^∞ and then so is the mapping $\overline{\Gamma}$. Moreover, if Γ is constant in a 'rectangular' band of thickness $\epsilon > 0$ near the boundary of $[s_0, s_1] \times [t_0, t_1]$, then so is $\overline{\Gamma}$.

Now we would like to understand the nature of the path

$$s \mapsto \tilde{\Gamma}_s \in \mathcal{P}_{\bar{A}}P$$

that is the ω-horizontal lift of $s \mapsto \Gamma_s$. Since both $\tilde{\Gamma}_s$ and $\overline{\Gamma}_s$ are $\bar{A}$-horizontal and both project down to the same path $\Gamma_s \in \mathcal{P}M$ we can express $\tilde{\Gamma}_s$ as a rigid shift of $\overline{\Gamma}_s$:

$$\tilde{\Gamma}_s = \overline{\Gamma}_s a_s \tag{3.17}$$

for a unique $a_s \in G$, for each $s \in [s_0, s_1]$. The tangent vector to $\mathcal{P}_{\bar{A}}P$ for the derivative of $s \mapsto \tilde{\Gamma}_s$ is the vector field along $\tilde{\Gamma}_s$ given by

$$t \mapsto \partial_s \tilde{\Gamma}_s(t) = \partial_t \overline{\Gamma}_s(t) a_s + \overline{\Gamma}_s(t) a_s\, a_s^{-1} \dot{a}_s, \tag{3.18}$$

with the natural meaning for the notation used; for example, the second term on the right is the vector at the point $\overline{\Gamma}_s(t)a_s \in P$ arising from the vector $a_s^{-1}\dot{a}_s \in L(G)$. We note that the second term on the right is a vertical vector. We recall that B_0 vanishes on vertical vectors and A, being a connection form, maps a vertical vector of the form $pZ \in T_pP$ to $Z \in L(G)$. Applying ω to $\partial_t \tilde{\Gamma}_s(t)$ we then obtain

$$A\Big(\partial_t \overline{\Gamma}_s(t_0) a_s\Big) + a_s^{-1}\dot{a}_s + C_0^R\Big(\partial_t \overline{\Gamma}_s(t_1) a_s\Big) - C_0^L\Big(\partial_t \overline{\Gamma}_s(t_0) a_s\Big) + \int_{t_0}^{t_1} B_0\Big(\partial_s \overline{\Gamma}_s(t) a_s, \partial_t \overline{\Gamma}_s(t) a_s\Big)\, dt. \tag{3.19}$$

The condition that $s \mapsto \tilde{\Gamma}_s$ is ω-horizontal is that the above expression is 0 for $s \in [s_0, s_1]$. Using the equivariance properties of A, B_0, C_0^L and C_0^R, this condition is then equivalent to

$$\dot{a}_s a_s^{-1} = -A\Big(\partial_t \overline{\Gamma}_s(t_0)\Big) - C_0^R\Big(\partial_t \overline{\Gamma}_s(t_1)\Big) + C_0^L\Big(\partial_t \overline{\Gamma}_s(t_0)\Big) - \int_{t_0}^{t_1} B_0\Big(\partial_s \overline{\Gamma}_s(t), \partial_t \overline{\Gamma}_s(t)\Big)\, dt. \tag{3.20}$$

Now the definition of a_s given in (3.17), as the 'shift' that should be applied to $\overline{\Gamma}_s$ to yield $\tilde{\Gamma}_s$, shows that at $s = s_0$ the value is e because, by our definition of $s \mapsto \overline{\Gamma}_s$ the initial path $\overline{\Gamma}_{s_0}$ is the same as the given initial path $\tilde{\Gamma}_{s_0}$. Since the right hand side of (3.20) involves only C^∞ functions, there is a unique C^∞ solution path

$$[s_0, s_1] \to G : s \mapsto a_s.$$

Thus we have constructed the ω-horizontal lift

$$[s_0, s_1] \to \mathcal{P}_{\bar{A}}P : s \mapsto \tilde{\Gamma}_s = \overline{\Gamma}_s a_s \tag{3.21}$$

of the given path $s \mapsto \Gamma_s$ on $\mathcal{P}M$.

Since the ordinary differential equation (3.20) has a unique solution with given initial value $a_{s_0} = e$ it follows that *any C^∞ path $s \mapsto \Gamma_s$ on $\mathcal{P}M$ has a unique ω-horizontal lift to a path $s \mapsto \tilde{\Gamma}_s$ on $\mathcal{P}_{\bar{A}}P$ with given initial path $\tilde{\Gamma}_0$.*

Let us note that if $A = \bar{A}$ then the first term on the right in (3.20) is 0. No essential generality is achieved by working with an A different from $\bar{A}$ because their difference could be absorbed into C_0^L.

4. The decorated bundle

In this section we shall construct a 'decorated' principal bundle over a path space starting with a traditional principal bundle along with some additional data. This notion was introduced in our earlier work [1] where we developed it from a mainly category-theoretic point of view. In this section we shall explore this notion from a more differential geometric standpoint. Furthermore, we shall work out several formulas, such as for the derivatives of right actions on the relevant bundles, that will be of use later when we work with connection forms.

As we have remarked before, the motivation for constructing the decorated bundle comes from a physics context, where the decoration arises from a second structure group that describes a gauge theory where point particles are replaced by paths.

4.1. Lie crossed modules

A *Lie crossed module* (G, H, α, τ) is comprised of Lie groups G and H, and homomorphisms

$$\tau : H \to G \quad \text{and} \quad \alpha : G \to \mathrm{Aut}(H), \tag{4.1}$$

with τ being smooth and the map $G \times H \to H : (g, h) \mapsto \alpha(g)(h)$ also smooth, satisfying the Peiffer [23] relations:

$$\begin{aligned} \tau(\alpha(g)(h)) &= g\tau(h)g^{-1}, \qquad \forall g \in G, h \in H, \\ \alpha(\tau(h))(h') &= hh'h^{-1}, \qquad \forall h, h' \in H. \end{aligned} \tag{4.2}$$

For later use we note that the derivative of the first relation in (4.2) leads to

$$\tau\big[\alpha(g)X\big] = \mathrm{Ad}(g)\tau(X) \tag{4.3}$$

for all $g \in G$ and $X \in L(H)$, and we have used the following natural notation: in (4.3) τ means $\tau(X) = d\tau|_e X$, and $\alpha(g)X = d\alpha(g)|_e X$.

We need only the semidirect product group $H \rtimes_\alpha G$. The map τ becomes significant in the category theoretic framework, where the Lie crossed module corresponds to a *categorical Lie group* $\mathbf{G}$, whose object set is G and whose morphisms are of the form (h, g), with source g and target $\tau(h)g$.

4.2. The semidirect product $H \rtimes_\alpha G$ and conjugation

Below in Section 4.3 we will construct a principal bundle whose structure group is the semidirect product $H \rtimes_\alpha G$; the product law in this group is given by

$$(h_1, g_1)(h_2, g_2) = \big(h_1\alpha(g_1)(h_2), g_1g_2\big). \tag{4.4}$$

Identifying H and G in the natural way as subgroups in $H \rtimes_\alpha G$ we have the commutation relation

$$hg = g\alpha(g^{-1})(h) \tag{4.5}$$

for all $h \in H$ and $g \in G$; to verify this note that the left side is, by definition, $(h, e)(e, g) = (h, g)$ and the right side is $(e, g)\big(\alpha(g^{-1})(h), e\big)$. The commutation relations can be used to reformulate some of our constructions below in a manner where the elements of G appear before the elements of H and for some relations this results in clearer expressions. For example, the commutation relation is also equivalent to

$$gh = \alpha(g)(h)g. \tag{4.6}$$

It is also very useful to note that after identifying G and H with subgroups of $H \rtimes_\alpha G$ (specifically, writing g for (e, g) and h for (h, e)), the commutation relation gives the following friendly form for the automorphism α:

$$\alpha(g)(h) = ghg^{-1}; \tag{4.7}$$

thus *the automorphism* $\alpha(g)$ *is simply conjugation* by g in $H \rtimes_\alpha G$ restricted to the subgroup $H \simeq H \times \{e\}$.

The identification of H and G with the corresponding subgroups of $H \rtimes_\alpha G$ often provides a great simplification of notation. An example of this simplification is seen in the derivative of the mapping $\alpha(g) : H \to H$ at $e \in H$, which can be obtained by restricting the operator

$$\mathrm{Ad}(g) : L(H \rtimes_\alpha G) \to L(H \rtimes_\alpha G) \tag{4.8}$$

to the subspace $L(H)$.

4.3. The decorated bundle

The total space of the bundle we will study is obtained by *decorating* P with elements of H:

$$\mathcal{P}^{\mathrm{dec}}_{\bar{A}}P := \mathcal{P}_{\bar{A}}P \times H. \tag{4.9}$$

The bundle projection is given by $(\overline{\gamma}, h) \mapsto \pi_{\bar{A}}(\overline{\gamma}) = \pi \circ \overline{\gamma}$. The group $H \rtimes_\alpha G$ acts on the right on the space $\mathcal{P}^{\mathrm{dec}}_{\bar{A}}P$ by

$$(\overline{\gamma}, h)(h_1, g_1) := \Big(\overline{\gamma}g_1, \alpha(g_1^{-1})(hh_1)\Big). \tag{4.10}$$

(This action has an important property that becomes clearer in the category theoretic framework: the categorical group arising from (G, H, α, τ) has a functorial right action on the category whose object set is P and whose morphisms are of the form $(\overline{\gamma}, h)$.) It will be notationally convenient to write $(\overline{\gamma}, h)$ as $\overline{\gamma}h$; then the action (with a dot, which we shall later omit, for visual clarity) reads

$$\overline{\gamma}h \cdot h_1g_1 = \overline{\gamma}g_1 \cdot \alpha(g_1^{-1})(hh_1), \tag{4.11}$$

an expression which has a formal consistency with the commutation relation (4.5). What we are denoting $\overline{\gamma}h$ here is what is denoted $(\overline{\gamma}, h^{-1})$ in [1]. We note that for any $\overline{\gamma} \in \mathcal{P}_{\bar{A}}P$, $g \in G$ and $h \in H$,

$$\overline{\gamma}g \text{ is an element of } \mathcal{P}_{\bar{A}}P;$$
$$\overline{\gamma}h \text{ is an element of } \mathcal{P}_{\bar{A}}P \times H.$$

The notation

$$\overline{\gamma}hg$$

might have two potentially different meanings:

$$(\overline{\gamma}e_H)\cdot(hg) = (\overline{\gamma}, e_H)(h,g) \text{ or } (\overline{\gamma}h)\cdot g = (\overline{\gamma}, h)(e_H, g),$$

where we have written e_H to stress that it is the identity in H. However, we can readily verify the following notational consistencies:

$$\begin{aligned}\overline{\gamma}e_H\cdot h &= \overline{\gamma}h\\ \overline{\gamma}e_H\cdot hg &= \overline{\gamma}g\cdot\alpha(g^{-1})(h) = \overline{\gamma}h\cdot g,\end{aligned} \tag{4.12}$$

where on the right sides h is, technically, (h,e) and g is (e,g) in $H\rtimes_\alpha G$.

Because of the relations (4.12), we can write $\overline{\gamma}e_H$ simply as $\overline{\gamma}$ if needed.

Proposition 4.1. *The mapping*

$$\mathcal{P}_{\bar{A}}^{\text{dec}}P\times(H\rtimes_\alpha G)\to\mathcal{P}_{\bar{A}}^{\text{dec}}P : (\overline{\gamma}h, h_1g_1)\mapsto\overline{\gamma}h\cdot h_1g_1 \tag{4.13}$$

is a right action. This action is free and is transitive on the fibers of $\pi_{\bar{A}}^{\text{dec}} : \mathcal{P}_{\bar{A}}^{\text{dec}}P\to\mathcal{P}M$.

Proof. We verify that (4.13) specifies a right action:

$$\begin{aligned}\overline{\gamma}h\cdot(h_1g_1h_2g_2) &= \overline{\gamma}h\cdot\big(h_1\alpha(g_1)(h_2)g_1g_2\big)\\ &= \overline{\gamma}g_1g_2\cdot\alpha(g_2^{-1}g_1^{-1})\big(hh_1\alpha(g_1)(h_2)\big)\\ &= \overline{\gamma}g_1g_2\cdot\alpha(g_2^{-1})\Big(\alpha(g_1^{-1})\big(hh_1\big)h_2\Big)\\ &= \Big(\overline{\gamma}g_1\cdot\alpha(g_1^{-1})(hh_1)\Big)\cdot h_2g_2\\ &= \big(\overline{\gamma}h\cdot h_1g_1\big)h_2g_2.\end{aligned} \tag{4.14}$$

Let us now verify that the action is free. Suppose

$$\text{a relation}\quad \overline{\gamma}h\cdot h_1g_1 = \overline{\gamma}h. \tag{4.15}$$

By (4.11), this means

$$\overline{\gamma}g_1\cdot\alpha(g_1^{-1})(hh_1) = \overline{\gamma}h,$$

which in turn is equivalent to

$$g_1 = e \qquad\text{and}\qquad hh_1 = h$$

with the latter relation being equivalent to $h_1 = e$. Thus the fixed point relation (4.15) implies that $h_1g_1 = e$.

Next suppose $\pi_{\bar{A}}^{\text{dec}}(\overline{\gamma}_1, h_1) = \pi_{\bar{A}}^{\text{dec}}(\overline{\gamma}_2, h_2)$. Then $\overline{\gamma}_1$ and $\overline{\gamma}_2$, both $\bar{A}$-horizontal paths in P, project down to the same path $\gamma\in\mathcal{P}M$, and so there is a $g\in G$ such that

$$\overline{\gamma}_2 = \overline{\gamma}_1 g.$$

Next we observe that

$$\begin{aligned}\overline{\gamma}_2h_2 &= \overline{\gamma}_1gh_2\\ &= \overline{\gamma}_1\alpha(g)(h_2)g \quad \text{(by the commutation relation (4.6))}\\ &= \overline{\gamma}_1h_1\cdot h_1^{-1}\alpha(g)(h_2)g.\end{aligned} \tag{4.16}$$

Thus $(\overline{\gamma}_2, h_2)$ is obtained by acting on $(\overline{\gamma}_1, h_1)$ with the element $h_1^{-1}\alpha(g)(h_2)g\in H\rtimes_\alpha G$. $\square$

As an illustration of the power of working within the semidirect product and using the notation $hg = (h,g)$ and $\overline{\gamma}hg = (\overline{\gamma}, h)(e_H, g)$, we realize that the computation (4.14) becomes completely natural using this notation:

$$\begin{aligned}\overline{\gamma}h\cdot(h_1g_1h_2g_2) &= \overline{\gamma}h\cdot\big(h_1\alpha(g_1)(h_2)g_1g_2\big)\\ &= \overline{\gamma}g_1g_2\cdot\alpha(g_2^{-1}g_1^{-1})\big(hh_1\alpha(g_1)(h_2)\big)\\ &= \overline{\gamma}g_1g_2\cdot\alpha(g_2^{-1})\Big(\alpha(g_1^{-1})\big(hh_1\big)h_2\Big)\\ &= \Big(\overline{\gamma}g_1\cdot\alpha(g_1^{-1})(hh_1)\Big)\cdot h_2g_2\\ &= \big(\overline{\gamma}h\cdot h_1g_1\big)h_2g_2.\end{aligned} \tag{4.17}$$

4.4. Local trivialization of the decorated bundle

We have seen that a local trivialization

$$\phi : U \times G \to \pi^{-1}(U)$$

of the original bundle $\pi : P \to M$ leads to a local trivialization of $(\pi_{\bar{A}}, \mathcal{P}_{\bar{A}}P, \mathcal{P}M)$ given in (2.4) by

$$\phi^0 : U^0 \times G \to \pi_{\bar{A}}^{-1}(U^0).$$

Let ϕ_0 be the inverse of ϕ^0; thus,

$$\phi_0 : \pi_{\bar{A}}^{-1}(U^0) \to U^0 \times G$$

is a G-equivariant bijection. Now we can construct a local trivialization for the $(H \rtimes_\alpha G)$-bundle $(\pi_{\rm dec}, \mathcal{P}_{\bar{A}}^{\rm dec}P, \mathcal{P}M)$ by means of the mapping:

$$\begin{aligned} \phi^{\rm dec} : U^0 \times (H \rtimes_\alpha G) &\to \pi_{\rm dec}^{-1}(U^0) \\ \big(\gamma, (h, g)\big) &\mapsto \big(\overline{\gamma}, \alpha(g^{-1})(h)\big) \end{aligned} \tag{4.18}$$

where $\overline{\gamma} = \phi^0(\gamma, g)$ is the $\bar{A}$-horizontal lift of γ starting at the point $\phi(\gamma_0, g)$, with γ_0 being the source (initial point) of γ. The inverse of this is:

$$\begin{aligned} \phi_{\rm dec} : \pi_{\rm dec}^{-1}(U^0) &\to U^0 \times (H \rtimes_\alpha G) \\ (\overline{\gamma}, h) &\mapsto \Big(\gamma, \big(\alpha(g)(h), g\big)\Big) \quad \text{where } g \text{ is specified by } (\gamma, g) = \phi_0(\overline{\gamma}). \end{aligned} \tag{4.19}$$

We can check that this is equivariant under the action of $H \rtimes_\alpha G$:

$$\begin{aligned} \phi^{\rm dec}\big(\gamma, hg\big) \cdot h_1 g_1 &= \big(\phi^0(\gamma, g), \alpha(g^{-1})(h)\big)h_1 g_1 \\ &= \Big(\phi^0(\gamma, g)g_1, \alpha\big(g_1^{-1}\big)(\alpha(g^{-1})(h)h_1\big)\Big) \\ &= \Big(\phi^0(\gamma, gg_1), \alpha(g_1^{-1}g^{-1})\big(h\alpha(g)(h_1)\big)\Big) \end{aligned} \tag{4.20}$$

which agrees with

$$\phi^{\rm dec}\big(\gamma, hgh_1g_1\big) = \Big(\phi^0\big(\gamma, gg_1\big), \alpha\big((gg_1)^{-1}\big)(h\alpha(g)(h_1))\Big) \tag{4.21}$$

Thus we have proved:

Proposition 4.2. *$(\pi_{\rm dec}, \mathcal{P}_{\bar{A}}^{\rm dec}P, \mathcal{P}M)$ is a principal $H \rtimes_\alpha G$-bundle with right-action given by (4.10) and local trivialization given by (4.18).*

Let us note that when working with bundles over spaces of paths we do not use a topology or an explicitly stated smooth structure. This is discussed further in Section 8.

4.5. The derivative of the right action

We turn now to some derivative computations that will be useful later, for example in Proposition 5.1, when we study a connection form Ω on the bundle of decorated paths. We view $\mathcal{P}_{\bar{A}}^{\rm dec}P = \mathcal{P}_{\bar{A}}P \times H$ as we would a product manifold. Thus we specify a tangent vector $\hat{v}$ at $(\overline{\gamma}, h) \in \mathcal{P}_{\bar{A}}^{\rm dec}P$ by

$$\hat{v} = \overline{v} + X,$$

where $\overline{v} \in T_{\overline{\gamma}}\mathcal{P}_{\bar{A}}P$ and $X \in T_hH$. (The tangent space $T_{\overline{\gamma}}\mathcal{P}_{\bar{A}}P$ is itself identifiable with $T_\gamma\mathcal{P}M \oplus L(G)$, where $\gamma = \pi \circ \overline{\gamma}$, by means of a local trivialization.) Thus we will take the tangent space $T_{(\overline{\gamma},h)}\mathcal{P}_{\bar{A}}^{\rm dec}P$ to be

$$T_{(\overline{\gamma},h)}\mathcal{P}_{\bar{A}}^{\rm dec}P = T_{\overline{\gamma}}\mathcal{P}_{\bar{A}}P \oplus T_hH. \tag{4.22}$$

Recalling from (4.10) the right action of $H \rtimes_\alpha G$ on $\mathcal{P}_{\bar{A}}^{\rm dec}P$ given by

$$(\overline{\gamma}, h)h_1g_1 = \big(\overline{\gamma}g_1, \alpha(g_1^{-1})(hh_1)\big), \tag{4.23}$$

we take, for fixed $(h_1, g_1) \in H \rtimes_\alpha G$, the 'differential' of the map

$$\mathcal{P}_{\bar{A}}^{\rm dec}P \to \mathcal{P}_{\bar{A}}^{\rm dec}P : (\overline{\gamma}, h) \mapsto (\overline{\gamma}, h)h_1g_1$$

to be given by

$$\begin{aligned} &\mathcal{R}_{(h_1,g_1)*} : T_{(\overline{\gamma},h)}\mathcal{P}_{\bar{A}}^{\rm dec}P \to T_{(\overline{\gamma},h)h_1g_1}\mathcal{P}_{\bar{A}}^{\rm dec}P \\ &\quad \mathcal{R}_{(h_1,g_1)*}(\overline{v} + X) := (\overline{v} + X)(h_1, g_1) = \overline{v}g_1 + g_1^{-1}(Xh_1)g_1, \end{aligned} \tag{4.24}$$

where $Xh_1 \in T_{hh_1}H$ is the image of $X \in T_hH$ under the derivative of the right-translation map $H \to H : x \mapsto xh_1$, and the last term on the right hand side is, more precisely, the derivative $d\alpha(g)|_{hh_1}$ applied to Xh_1.

4.6. Derivative of the orbit map

Next let us look at what should be taken to be the differential of the map

$$H \rtimes_\alpha G \to \mathcal{P}_{\bar{A}}^{\mathrm{dec}}P : (h_1, g_1) \mapsto (\overline{\gamma}, h)h_1g_1,$$

where $(\overline{\gamma}, h)$ is any fixed point in $\mathcal{P}_{\bar{A}}^{\mathrm{dec}}P$. (This will be useful when we study the connection form Ω in Proposition 5.1.) We use the realization of the tangent space to $H \rtimes_\alpha G$ as

$$T_{(h_1,g_1)}(H \rtimes_\alpha G) = T_{h_1}H \oplus T_{g_1}G,$$

and write a vector in this space in the form

$$h_1Y_1 + g_1Z_1 \stackrel{\mathrm{def}}{=} (h_1Y_1, g_1Z_1) \in T_{h_1}H \oplus T_{g_1}G,$$

where $Y_1 \in L(H)$ and $Z_1 \in L(G)$. Here, as always, xV means the derivative of the left-translation map $G \to G : y \mapsto xy$ by x on $V \in T_xG$ and Vx has an analogous meaning with respect to right translations. We will often use notation such as xV that makes it possible to see at a glance that we are speaking of a vector located at the point x.

We also realize the tangent space $T_{(\overline{\gamma},h)}(\mathcal{P}_{\bar{A}}^{\mathrm{dec}}P)$ as

$$T_{(\overline{\gamma},h)}(\mathcal{P}_{\bar{A}}^{\mathrm{dec}}P) = T_{\overline{\gamma}}(\mathcal{P}_{\bar{A}}P) \oplus T_hH.$$

We will frequently need to use the derivative of the inversion map

$$j : G \to G : g \mapsto g^{-1},$$

and this is given by

$$dj|_g(W) = -g^{-1}Wg^{-1}, \tag{4.25}$$

for all tangent vectors $W \in T_gG$. In particular if $W = gZ$, where $Z \in L(G)$, then

$$dj|_g(gZ) = -Zg^{-1}. \tag{4.26}$$

As always we denote by

$$\overline{\gamma}g_1Z$$

the vertical vector field along $\overline{\gamma}g_1$ whose value at any parameter value t is

$$\overline{\gamma}(t)g_1Z = \frac{d}{ds}\Big|_{s=0}\overline{\gamma}(t)g_1\exp(sZ). \tag{4.27}$$

Let us write the right action of $H \rtimes_\alpha G$ on $\mathcal{P}_{\bar{A}}P \times H$, as we have done in (4.10), in the form

$$(\overline{\gamma}, h)h_1g_1 = (\overline{\gamma}g_1, g_1^{-1}hh_1g_1). \tag{4.28}$$

Holding $(\overline{\gamma}, h)$ fixed, the derivative of the orbit map

$$h_1g_1 \mapsto (\overline{\gamma}, h)h_1g_1$$

is given by

$$\begin{aligned} r_{(\overline{\gamma},h),(h_1,g_1)} : T_{(h_1,g_1)}(H \rtimes_\alpha G) &\to T_{(\overline{\gamma},h)h_1g_1}\mathcal{P}_{\bar{A}}^{\mathrm{dec}}P \\ h_1Y_1 + g_1Z_1 &\mapsto \overline{\gamma}g_1Z_1 + g_1^{-1}hh_1Y_1g_1 + \Big(g_1^{-1}hh_1g_1Z_1 - Z_1g_1^{-1}hh_1g_1\Big), \end{aligned} \tag{4.29}$$

where the last expression, comprised of two terms within $(\cdots)$, lies in $T_{hh_1}H$, by the reasoning used below in (4.31). Let us note here the distinction between the derivative $r_{(\overline{\gamma},h),(h_1,g_1)}$ and the derivative $\mathcal{R}_{(h,g)*}$.

Here and in most of our computations we identify the Lie algebras of H and G with the corresponding subalgebras inside $L(H \rtimes_\alpha G)$. Thus

$$L(H \rtimes_\alpha G) = L(H) \oplus L(G) \text{ as a direct sum of } \textit{vector spaces.} \tag{4.30}$$

Evaluating the derivative in (4.29) at the identity $(e, e) \in H \rtimes_\alpha G$ we obtain the linear map

$$\begin{aligned} r_{(\overline{\gamma},h)} : L(H \rtimes_\alpha G) &\to T_{(\overline{\gamma},h)}\mathcal{P}_{\bar{A}}^{\mathrm{dec}}P \\ Y_1 + Z_1 &\mapsto \overline{\gamma}Z_1 + hY_1 + hZ_1 - Z_1h \\ &= \overline{\gamma}Z_1 + h\Big(Y_1 + \big(1 - \mathrm{Ad}(h^{-1})\big)Z_1\Big), \end{aligned} \tag{4.31}$$

where $Y_1 \in L(H)$ and $Z_1 \in L(G)$. Note that $\mathrm{Ad}(h^{-1})Z_1$ is obtained by applying $\mathrm{Ad}(h^{-1})$ to Z_1, with everything taking place inside the Lie algebra $L(H \rtimes_\alpha G)$. The term $\big(1 - \mathrm{Ad}(h^{-1})\big)Z_1$ lies in $L(H)$, which can be seen by examining the derivative, at the identity in G, of the mapping

$$G \to H : g_1 \mapsto g_1hg_1^{-1} = \alpha(g_1)(h).$$

5. Connections on the decorated bundle

We continue with the framework established in the preceding sections. Thus $\bar{A}$ is a connection on a principal G-bundle

$$\pi : P \to M,$$

and (G, H, α, τ) is a Lie crossed module. We have then a connection $\omega^{(A,B_0)}$, constructed from a connection A on P and an $L(G)$-valued 2-form B_0 on P, on the horizontal path bundle

$$\mathcal{P}_{\bar{A}}P \to \mathcal{P}M,$$

which is also a principal G-bundle in the sense discussed before. We have constructed the decorated principal $H \rtimes_\alpha G$-bundle

$$\mathcal{P}_{\bar{A}}^{\mathrm{dec}}P = \mathcal{P}_{\bar{A}}P \times H \to \mathcal{P}M.$$

In this section we shall construct a connection on this decorated bundle by using the connection ω and two additional forms on P as ingredients.

5.1. The endpoint forms $C^{L,R}$ and a 2-form B

As noted before, the Lie algebra $L(H \rtimes_\alpha G)$ is the vector space direct sum $L(H) \oplus L(G)$ (the Lie algebra structure on $L(H \rtimes_\alpha G)$ is not, however, obtained as a direct sum of Lie algebras). Viewing G as a subgroup of $H \rtimes_\alpha G$ we have, for each $g \in G$, the operator $\mathrm{Ad}(g^{-1})$ on $L(H \rtimes_\alpha G)$. We shall work with an $L(H \rtimes_\alpha G)$-valued 2-form B on P with following properties:

$$\begin{aligned} &B(ug, vg) = \mathrm{Ad}(g^{-1})(B(u, v)) \qquad \forall u, v \in T_pP, g \in G, \\ &B(u, v) = 0, \qquad \text{if } u \text{ or } v \text{ is vertical.} \end{aligned} \tag{5.1}$$

Let us note here that in the first equation above, $\mathrm{Ad}(g)$ is acting on $L(H \rtimes_\alpha G)$ as in (4.8). Since this action maps $L(G)$ into itself and $L(H)$ also into itself, the equations in (5.1) mean that they hold separately for the components B_0 and B_1. We shall also use $L(H \rtimes_\alpha G)$-valued 1-forms C^L and C^R on P that have the following properties:

$$\begin{aligned} &C^L|_{pg}(vg) = \mathrm{Ad}(g^{-1})C^L|_p(v) \qquad \forall v \in T_pP, g \in G, \\ &C^L|_p(v) = 0, \qquad \text{if } v \in T_pP \text{ is any vertical vector,} \end{aligned} \tag{5.2}$$

for all $p \in P$ and the corresponding properties for C^R.

Let Σ be the Maurer–Cartan form on H:

$$\Sigma_h(X) = h^{-1}X, \qquad \forall h \in H,\ X \in T_hH,$$

where on the right the notation signifies the action of the derivative of the left-translation map $h_1 \mapsto h^{-1}h_1$.

5.2. The connection form Ω

As before we denote the evaluation map at the initial point by:

$$\mathrm{ev}_0 : \mathcal{P}_{\bar{A}}^{\mathrm{dec}}P \to P : (\overline{\gamma}, h) \mapsto \overline{\gamma}(t_0),$$

where the domain of $\overline{\gamma}$ is an interval $[t_0, t_1]$. Using a connection form A on P, along with the $L(H \rtimes_\alpha G)$-valued 2- and 1-forms

$$\begin{aligned} B &= B_0 + B_1 \\ C^{L,R} &= C_0^{L,R} + C_1^{L,R}, \end{aligned} \tag{5.3}$$

we define a 1-form Ω on $\mathcal{P}_{\bar{A}}^{\mathrm{dec}}P$, with values in $L(H \rtimes_\alpha G)$, as follows:

$$\Omega_{\overline{\gamma},h} \stackrel{\mathrm{def}}{=} \mathrm{Ad}(h^{-1})\left[\mathrm{ev}_0^*\big(A - C^L\big)|_{\overline{\gamma}(t_0)} + \mathrm{ev}_1^*(C^R)|_{\overline{\gamma}(t_1)} + \int_{\overline{\gamma}} B\right] + \Sigma_h, \tag{5.4}$$

where on the right we view Σ as a form on $\mathcal{P}_{\bar{A}}P \times H$ with the obvious pullback from the projection map onto H. Thus

$$\Omega_{\overline{\gamma},h}(\overline{v} + X) = \mathrm{Ad}(h^{-1})\left[\omega_{\overline{\gamma}}(\overline{v}) + C_1^R|_{\overline{\gamma}(t_1)}(\overline{v}(t_1)) - C_1^L|_{\overline{\gamma}(t_0)}(\overline{v}(t_0)) + \int_{t_0}^{t_1} B_1|_{\overline{\gamma}(t)}(\overline{v}(t), \overline{\gamma}'(t))dt + Xh^{-1}\right], \tag{5.5}$$

where $\overline{v} + X \in T_{(\overline{\gamma},h)}\mathcal{P}_{\bar{A}}^{\mathrm{dec}}P$, with $\overline{v}$ a vector field along the path $\overline{\gamma}$ and $X \in T_hH$, and the 1-form ω is as defined in (3.4):

$$\omega_{\overline{\gamma}}(\overline{v}) := A_{\overline{\gamma}(t_0)}(\overline{v}(t_0)) + C_0^R|_{\overline{\gamma}(t_1)}\big(v_{\overline{\gamma}(t_1)}\big) - C_0^L|_{\overline{\gamma}(t_0)}\big(v_{\overline{\gamma}(t_0)}\big) + \int_{t_0}^{t_1} B_0|_{\overline{\gamma}(t)}\big(\overline{v}(t), \overline{\gamma}'(t)\big)\, dt, \tag{5.6}$$

for every path $\overline{\gamma} : [t_0, t_1] \to P$ in $\mathcal{P}_{\bar{A}}P$.

Let us recall the right action

$$\mathcal{P}_{\bar{A}}^{\mathrm{dec}}P \times (H \rtimes_\alpha G) \to \mathcal{P}_{\bar{A}}^{\mathrm{dec}}P : \Big((\overline{\gamma}, h), (h_1, g_1)\Big) \mapsto (\overline{\gamma}g_1, g_1^{-1}hh_1g_1). \tag{5.7}$$

From this map we have the two 'directional derivatives':

$$\begin{aligned}&\mathcal{R}_{(h_1,g_1)*} : T_{\overline{\gamma},h}(\mathcal{P}_{\bar{A}}^{\mathrm{dec}}P) \to T_{(\overline{\gamma},h)h_1g_1}(\mathcal{P}_{\bar{A}}^{\mathrm{dec}}P)\\ &\qquad\qquad\text{and}\\ &r_{(\overline{\gamma},h)} : T_{(h_1,g_1)}(H \rtimes_\alpha G) \to T_{(\overline{\gamma},h)h_1g_1}(\mathcal{P}_{\bar{A}}^{\mathrm{dec}}P).\end{aligned} \tag{5.8}$$

Proposition 5.1. *The 1-form Ω is a connection on $(\pi_{\mathrm{dec}}, \mathcal{P}_{\bar{A}}^{\mathrm{dec}}P, \mathcal{P}M)$ in the sense that the following conditions hold.*

(i) *It is equivariant under the right-action of $H \rtimes_\alpha G$:*

$$\Omega_{(\overline{\gamma},h)h_1g_1}\mathcal{R}_{(h_1,g_1)*}(\overline{v}, X) = \mathrm{Ad}(h_1g_1)^{-1}\Omega_{(\overline{\gamma},h)}(\overline{v}, X) \tag{5.9}$$

for all $h, h_1 \in H$, $g_1 \in G$, $\overline{\gamma} \in \mathcal{P}_{\bar{A}}P$, $\overline{v} \in T_{\overline{\gamma}}\mathcal{P}_{\bar{A}}P$ and $X \in T_hH$;

(ii) *It returns the appropriate elements of $L(H \rtimes_\alpha G)$ when applied to vertical vectors:*

$$\Omega_{(\overline{\gamma},h)}r_{(\overline{\gamma},h)}(Y_1 + Z_1) = Y_1 + Z_1, \tag{5.10}$$

where $r_{(\overline{\gamma},h)}$ is the derivative of the right-action of $H \rtimes_\alpha G$ on $\mathcal{P}_{\bar{A}}P$ as in (4.29) and $(Y_1, Z_1) \in L(H) \oplus L(G)$.

Proof. For notational convenience we shall write $\overline{v}_0$ for $\overline{v}(t_0)$, $\overline{\gamma}_0$ for $\overline{\gamma}(t_0)$, and analogously for other paths and vector fields.

Working through the right-action we have

$$\begin{aligned}\Omega_{(\overline{\gamma},h)h_1g_1}\mathcal{R}_{(h_1,g_1)*}(\overline{v}, X) &= \Omega_{(\overline{\gamma}g_1, g_1^{-1}hh_1g_1)}(\overline{v}g_1 + g_1^{-1}Xh_1g_1) \quad \text{using (4.24)}\\ &= \mathrm{Ad}(g_1^{-1}h_1^{-1}h^{-1}g_1)\Big(\omega_{\overline{\gamma}}(\overline{v}g_1) + C_1^R|_{\overline{\gamma}_1g_1}(\overline{v}_0g_1) - C_1^L|_{\overline{\gamma}_0g_1}(\overline{\gamma}_0g_1)\Big)\\ &\quad + (g_1^{-1}hh_1g_1)^{-1}g_1^{-1}Xh_1g_1 + \mathrm{Ad}(g_1^{-1}h_1^{-1}h^{-1}g_1)\int_{t_0}^{t_1} B_1\big(\overline{v}(t)g_1, \overline{\gamma}'(t)g_1\big)\, dt.\end{aligned} \tag{5.11}$$

We work out the last term on the right separately:

$$\begin{aligned}\mathrm{Ad}(g_1^{-1}h_1^{-1}h^{-1}g_1)\int_{t_0}^{t_1} B_1\big(\overline{v}(t)g_1, \overline{\gamma}'(t)g_1\big)\, dt &= \mathrm{Ad}(g_1^{-1}h_1^{-1}h^{-1}g_1)\mathrm{Ad}(g_1^{-1})\int_{t_0}^{t_1} B_1\big(\overline{v}(t), \overline{\gamma}'(t)\big)\, dt\\ &\qquad \text{(using the equivariance property of } B_1 \text{ from (5.1))}\\ &= \mathrm{Ad}(g_1^{-1}h_1^{-1}h^{-1})\int_{t_0}^{t_1} D\big(\overline{v}(t), \overline{\gamma}'(t)\big)\, dt.\end{aligned} \tag{5.12}$$

Returning to our computation (5.11), we have:

$$\begin{aligned}\Omega_{(\overline{\gamma},h)h_1g_1}\mathcal{R}_{(h_1,g_1)*}(\overline{v}, X) &= \mathrm{Ad}(g_1^{-1}h_1^{-1}h^{-1}g_1)\mathrm{Ad}(g_1^{-1})\Big(\omega_{\overline{\gamma}}(\overline{v}) + C_1^R|_{\overline{\gamma}_1}(\overline{v}_0) - C_1^L|_{\overline{\gamma}_0}(\overline{v}_0)\Big)\\ &\quad + g_1^{-1}h_1^{-1}h^{-1}g_1\, g_1^{-1}Xh_1g_1 + \mathrm{Ad}(g_1^{-1}h_1^{-1}h^{-1})\int_{t_0}^{t_1} B_1\big(\overline{v}(t), \overline{\gamma}'(t)\big)\, dt.\end{aligned} \tag{5.13}$$

On the other hand

$$\begin{aligned}\mathrm{Ad}(h_1g_1)^{-1}\Omega_{(\overline{\gamma},h)}(\overline{v}, X) &= \mathrm{Ad}(h_1g_1)^{-1}\mathrm{Ad}(h^{-1})\Big(\omega_{\overline{\gamma}}(\overline{v}) + C_1^R|_{\overline{\gamma}_1}(\overline{v}_0) - C_1^L|_{\overline{\gamma}_0}(\overline{v}_0)\Big)\\ &\quad + \mathrm{Ad}(h_1g_1)^{-1}(h^{-1}X) + \mathrm{Ad}(h_1g_1)^{-1}\mathrm{Ad}(h^{-1})\int_{t_0}^{t_1} B_1\big(\overline{v}(t), \overline{\gamma}'(t)\big)\, dt.\end{aligned} \tag{5.14}$$

We see that this is equal to the expression on the right in (5.13). This proves property (i) for Ω.

Next we consider how the connection form acts on a 'vertical vector' in the bundle $\mathcal{P}_{\bar{A}}^{\mathrm{dec}}P$; such a vector is of the form

$$r_{(\overline{\gamma},h)}(Y_1 + Z_1) \in T_{(\overline{\gamma},h)}(\mathcal{P}_{\bar{A}}^{\mathrm{dec}}P),$$

where

$$Y_1 + Z_1 = (Y_1, Z_1) \in L(H) \oplus L(G),$$

is an arbitrary vector in $L(H \rtimes_\alpha G)$. Thus

$$\begin{aligned}\Omega_{(\overline{\gamma},h)} r_{(\overline{\gamma},h)}(Y_1+Z_1) &= \Omega_{(\overline{\gamma},h)}\Big(\overline{\gamma}Z_1 + hY_1 + \big(1-\mathrm{Ad}(h^{-1})\big)Z_1\Big) \\ &\quad \text{(using the expression for } r_{(\overline{\gamma},h)} \text{ obtained in (4.31))} \\ &= \mathrm{Ad}(h^{-1})\Big(\omega_{\overline{\gamma}}(\overline{\gamma}Z_1) + C_1^R|_{\overline{\gamma}_1}(\overline{\gamma}_1 Z_1) - C_1^L|_{\overline{\gamma}_0}(\overline{\gamma}_0 Z_1)\Big) \\ &\quad + \Big[Y_1 + \big(1-\mathrm{Ad}(h^{-1})\big)Z_1\Big] + \mathrm{Ad}(h^{-1})\int_{t_0}^{t_1} B_1\big(\overline{\gamma}(t)Z_1, \overline{\gamma}'(t)\big)\,dt. \end{aligned} \tag{5.15}$$

In the expression on the right, the terms with C and the last term, with B_1, are 0 because B_1, C_1^L and C_1^R vanish on vertical vectors (see (5.1) and (5.2)). The first term equals Z_1:

$$\omega_{\overline{\gamma}}(\overline{\gamma}Z_1) = Z_1, \tag{5.16}$$

as seen in (3.9).

Putting all this together we have

$$\begin{aligned}\Omega_{(\overline{\gamma},h)} r_{(\overline{\gamma},h)}(Y_1+Z_1) &= \mathrm{Ad}(h^{-1})Z_1 + \Big[Y_1 + \big(1-\mathrm{Ad}(h^{-1})\big)Z_1\Big] + 0 \\ &= Y_1 + Z_1. \end{aligned} \tag{5.17}$$

This proves property (ii). □

The conditions (i) and (ii) above imply that the form Ω splits each tangent space $T_{(\overline{\gamma},h)}\mathcal{P}_{\bar{A}}^{\mathrm{dec}}P$ into horizontal and vertical subspaces as explained in the following result.

5.3. Horizontal and vertical parts

We turn now to understand how the connection form Ω splits a vector $v \in T_{(\overline{\gamma},h)}\mathcal{P}_{\bar{A}}^{\mathrm{dec}}P$ into a horizontal and a vertical component.

Proposition 5.2. *At any $(\overline{\gamma}, h) \in \mathcal{P}_{\bar{A}}^{\mathrm{dec}}P$, Ω splits $T_{(\overline{\gamma},h)}\mathcal{P}_{\bar{A}}^{\mathrm{dec}}P$ into a direct sum:*

$$T_{(\overline{\gamma},h)}\mathcal{P}_{\bar{A}}^{\mathrm{dec}}P = \mathrm{H}_{(\overline{\gamma},h)}\mathcal{P}_{\bar{A}}^{\mathrm{dec}}P \oplus \mathrm{V}_{(\overline{\gamma},h)}\mathcal{P}_{\bar{A}}^{\mathrm{dec}}P, \tag{5.18}$$

where the 'horizontal subspace' $\mathrm{H}_{(\overline{\gamma},h)}\mathcal{P}_{\bar{A}}^{\mathrm{dec}}P$ is

$$\mathrm{H}_{(\overline{\gamma},h)}\mathcal{P}_{\bar{A}}^{\mathrm{dec}}P = \ker \Omega_{(\overline{\gamma},h)}, \tag{5.19}$$

and the 'vertical subspace' $\mathrm{V}_{(\overline{\gamma},h)}\mathcal{P}_{\bar{A}}^{\mathrm{dec}}P$ is the image of $r_{(\overline{\gamma},h)}$:

$$\mathrm{V}_{(\overline{\gamma},h)}\mathcal{P}_{\bar{A}}^{\mathrm{dec}}P = \{r_{(\overline{\gamma},h)}(Y_1+Z_1) : Y_1 \in L(H), Z_1 \in L(G)\}, \tag{5.20}$$

as noted in (4.31), with $r_{(\overline{\gamma},h)}$ being the right action of $H \rtimes_\alpha G$ on $\mathcal{P}_{\bar{A}}^{\mathrm{dec}}P$.

Proof. Let

$$\hat{v} = \overline{v} + X \in T_{(\overline{\gamma},h)}\mathcal{P}_{\bar{A}}^{\mathrm{dec}}P,$$

where $\overline{v} \in T_{\overline{\gamma}}\mathcal{P}_{\bar{A}}P$, $X \in T_hH$. Let $\overline{v}^{\mathrm{H}}$ and $\overline{v}^{\mathrm{V}}$ be, respectively, the horizontal and vertical components of $\overline{v}$ with respect to the connection $\omega = \omega^{(A,B_0)}$. Thus, in particular, $\omega_{\overline{\gamma}}(\overline{v}^{\mathrm{H}}) = 0$.

Let $\overline{\gamma}_1$ denote the right endpoint $\overline{\gamma}(t_1)$, and $\overline{\gamma}_0$ denote the left endpoint $\overline{\gamma}(t_0)$. We will now show that the horizontal and vertical components of $\hat{v} = \overline{v} + X$ are

$$\begin{aligned}\hat{v}^{\mathrm{H}} &= \overline{v}^{\mathrm{H}} - \left(C_1^R|_{\overline{\gamma}_1}\big(\overline{v}^{\mathrm{H}}(t_1)\big) - C_1^L|_{\overline{\gamma}_0}\big(\overline{v}^{\mathrm{H}}(t_0)\big) + \int_{t_0}^{t_1} B_1(\overline{v}^{\mathrm{H}}(t), \overline{\gamma}'(t))dt\right)h \\ \hat{v}^{\mathrm{V}} &= \overline{v}^{\mathrm{V}} + X + \left(C_1^R|_{\overline{\gamma}_1}\big(\overline{v}^{\mathrm{H}}(t_1)\big) - C_1^L|_{\overline{\gamma}_0}\big(\overline{v}^{\mathrm{H}}(t_0)\big) + \int_{t_0}^{t_1} B_1(\overline{v}^{\mathrm{H}}(t), \overline{\gamma}'(t))dt\right)h, \end{aligned} \tag{5.21}$$

respectively, where, as always, $Zh \in T_hH$ is the result of applying the derivative of the right translation map $H \to H : x \mapsto xh$ to $Z \in L(H)$.

Our objective now is to show that $\hat{v}^{\mathrm{H}}$ lies in the horizontal subspace, $\hat{v}^{\mathrm{V}}$ in the vertical subspace.

The relation (5.17) shows that when Ω is applied to a vertical vector, which, by definition, is of the form $v = r_{(\overline{\gamma},h)}(Y_1+Z_1)$ then the value obtained is $Y_1 + Z_1$; hence if this is 0 then v itself is 0. Thus the only vertical vector which is also horizontal is just the zero vector. Thus the sum in (5.18) is indeed a direct sum.

Inserting (5.21) in the expression for Ω given in (5.5) we get

$$\begin{aligned}\mathrm{Ad}(h)\Omega_{\overline{\gamma},h}(\hat{v}^{\mathrm{H}}) &= \omega_{\overline{\gamma}}(\overline{v}^{\mathrm{H}}) + C_1^R|_{\overline{\gamma}_1}\big(\overline{v}^{\mathrm{H}}(t_1)\big) - C_1^L|_{\overline{\gamma}_0}\big(\overline{v}^{\mathrm{H}}(t_0)\big) + \int_{t_0}^{t_1} B_1(\overline{v}^{\mathrm{H}}(t), \overline{\gamma}'(t))dt \\ &\quad - \left(C_1^R|_{\overline{\gamma}_1}\big(\overline{v}^{\mathrm{H}}(t_1)\big) - C_1^L|_{\overline{\gamma}_0}\big(\overline{v}^{\mathrm{H}}(t_0)\big) + \int_{t_0}^{t_1} B_1(\overline{v}^{\mathrm{H}}(t), \overline{\gamma}'(t))dt\right) \\ &= 0.\end{aligned} \tag{5.22}$$

The vector $\overline{v}^{\mathrm{V}}$, which is the ω-vertical part of $\overline{v}$, is given by

$$\overline{v}^{\mathrm{V}} = \overline{\gamma}Z, \tag{5.23}$$

where

$$Z = \omega_{\overline{\gamma}}(\overline{v})$$

(as we have discussed earlier in (3.8)). Applying $\Omega_{\overline{\gamma},h}$ to $\overline{v}^{\mathrm{V}}$ we have

$$\Omega_{\overline{\gamma},h}(\hat{v}^{\mathrm{V}}) = \mathrm{Ad}(h^{-1})\left[Z + Xh^{-1} + C_1^R|_{\overline{\gamma}_1}\big(\overline{v}^{\mathrm{H}}(t_1)\big) - C_1^L|_{\overline{\gamma}_0}\big(\overline{v}^{\mathrm{H}}(t_0)\big) + \int_{t_0}^{t_1} B_1(\overline{v}^{\mathrm{H}}(t), \overline{\gamma}'(t))dt\right]. \tag{5.24}$$

We can write the right hand side of (5.24) as a sum of a vector in $L(H)$ and a vector in $L(G)$ on using the observation, made earlier after (4.31), that $\big(1 - \mathrm{Ad}(h^{-1})\big)Z$ is in $L(H)$. Thus we have

$$\Omega_{\overline{\gamma},h}(\hat{v}^{\mathrm{V}}) = \Big(\mathrm{Ad}(h^{-1}) - 1\Big)Z + h^{-1}\left[X + \left(C_1^R|_{\overline{\gamma}_1}\big(\overline{v}^{\mathrm{H}}(t_1)\big) - C_1^L|_{\overline{\gamma}_0}\big(\overline{v}^{\mathrm{H}}(t_0)\big) + \int_{t_0}^{t_1} B_1(\overline{v}^{\mathrm{H}}(t), \overline{\gamma}'(t))dt\right)h\right] + Z. \tag{5.25}$$

Now we recall from (4.31) that the derivative of the orbit map

$$H \rtimes_\alpha G \to \mathcal{P}_{\bar{A}}^{\mathrm{dec}}P : (h_1, g_1) \mapsto (\overline{\gamma}, h)h_1g_1$$

at the identity element $(e, e) \in H \rtimes_\alpha G$ is given by

$$\begin{aligned} r_{(\overline{\gamma},h)} : L(H \rtimes_\alpha G) &\to T_{(\overline{\gamma},h)}\mathcal{P}_{\bar{A}}^{\mathrm{dec}}P \\ Y_1 + Z_1 &\mapsto \overline{\gamma}Z_1 + h\Big(Y_1 + \big(1 - \mathrm{Ad}(h^{-1})\big)Z_1\Big). \end{aligned} \tag{5.26}$$

Applying this to $\Omega_{\overline{\gamma},h}(\hat{v}^{\mathrm{V}})$ as given above we obtain

$$\begin{aligned} r_{(\overline{\gamma},h)}\Big(\Omega_{\overline{\gamma},h}(\hat{v}^{\mathrm{V}})\Big) &= \overline{\gamma}Z + h\Big(\mathrm{Ad}(h^{-1}) - 1\Big)Z \\ &\quad + \left[X + \left(C_1^R|_{\overline{\gamma}_1}\big(\overline{v}^{\mathrm{H}}(t_1)\big) - C_1^L|_{\overline{\gamma}_0}\big(\overline{v}^{\mathrm{H}}(t_0)\big) + \int_{t_0}^{t_1} B_1(\overline{v}^{\mathrm{H}}(t), \overline{\gamma}'(t))dt\right)h\right] + h\big(1 - \mathrm{Ad}(h^{-1})\big)Z \\ &= \overline{\gamma}Z + \left[X + \left(C_1^R|_{\overline{\gamma}_1}\big(\overline{v}^{\mathrm{H}}(t_1)\big) - C_1^L|_{\overline{\gamma}_0}\big(\overline{v}^{\mathrm{H}}(t_0)\big) + \int_{t_0}^{t_1} B_1(\overline{v}^{\mathrm{H}}(t), \overline{\gamma}'(t))dt\right)h\right], \end{aligned} \tag{5.27}$$

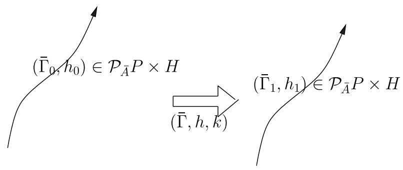
which we recognize to be $\hat{v}^{\mathrm{V}}$ as given in (5.21). Thus,

$$\hat{v}^{\mathrm{V}} = r_{(\overline{\gamma},h)}\Big(\Omega_{\overline{\gamma},h}(\hat{v}^{\mathrm{V}})\Big)$$

lies in the vertical subspace $\mathrm{V}_{(\overline{\gamma},h)}\mathcal{P}_{\bar{A}}^{\mathrm{dec}}P$. $\square$

6. Horizontal lifts of paths on decorated bundles

We work with the framework from the preceding sections, with a connection $\bar{A}$ on a principal G-bundle $\pi : P \to M$, and a Lie crossed module (G, H, α, τ); as noted before, we shall only use the semidirect product $H \rtimes_\alpha G$ and not τ at this stage. There is then a principal G-bundle $\mathcal{P}_{\bar{A}}P \to \mathcal{P}M$, where the elements of $\mathcal{P}_{\bar{A}}P$ are $\bar{A}$-horizontal paths on P. We have introduced a connection ω on $\mathcal{P}_{\bar{A}}P \to \mathcal{P}M$, and a connection Ω on the principal $H \rtimes_\alpha G$-bundle $\mathcal{P}_{\bar{A}}^{\mathrm{dec}}P \to \mathcal{P}M$. The elements of $\mathcal{P}_{\bar{A}}^{\mathrm{dec}}P$ are of the form $(\overline{\gamma}, h)$, where $\overline{\gamma} \in \mathcal{P}_{\bar{A}}P$ and $h \in H$. Our goal in this section is to determine parallel-transport, illustrated in Fig. 3, by the connection Ω (in the figure k encodes the parallel transport multiplier).



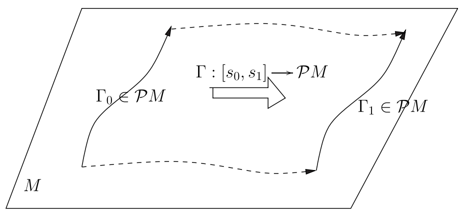


Fig. 3. Parallel transport of decorated paths.

6.1. Paths on the path space

We consider a path

$$\Gamma : [s_0, s_1] \to \mathcal{P}M : s \mapsto \Gamma_s$$

where each Γ_s is a C^∞ path $[t_0, t_1] \to M$, for some $t_0 < t_1$. We assume that Γ is smooth in the sense that

$$[t_0, t_1] \times [s_0, s_1] \to M : (t, s) \mapsto \Gamma(t, s) = \Gamma_s(t) \tag{6.1}$$

is C^∞. There are some additional technical requirements we impose in order to ensure that composition of paths of paths produces a path of paths of the same nature. To this end we assume that for the mapping Γ there exists an $\epsilon > 0$ such that for each fixed s the point

$$\Gamma_s(t)$$

remains constant when t is within distance ϵ of t_0 or t_1, and for each fixed $t \in [t_0, t_1]$ the point $\Gamma_s(t)$ remains constant when s is within distance ϵ of s_0 or s_1. Furthermore, we identify Γ with the mapping

$$\Gamma^{-v} : ([t_0, t_1] \times [s_0, s_1]) + v \to M : (t, s) \mapsto \Gamma\big((t, s) - v\big),$$

for any fixed $v \in \mathbb{R}^2$. More precisely, identification means that we form a quotient space $\mathcal{P}_2(M)$, where Γ and Γ^{-v} correspond to the same element.

6.2. The Ω-horizontal lift of a path on $\mathcal{P}M$

Our goal is to determine the Ω-horizontal lift of $s \mapsto \Gamma_s$, with a given initial point

$$(\tilde\Gamma_{s_0}, h_{s_0}) \in \mathcal{P}_{\bar A}^{\mathrm{dec}} P$$

where

$$\pi \circ \tilde\Gamma_{s_0} = \Gamma_{s_0}.$$

To this end let

$$[s_0, s_1] \to \mathcal{P}_{\bar A}P : s \mapsto \tilde\Gamma_s \tag{6.2}$$

be the ω-horizontal lift of the path $s \mapsto \Gamma_s$, with initial point $\tilde\Gamma_{s_0}$. (Recall that ω is a connection on $\mathcal{P}_{\bar A}P$ and in Section 3.5 we have shown the existence of ω-horizontal lifts.) Next let

$$s \mapsto h_s \tag{6.3}$$

be the solution of the differential equation

$$\dot h_s h_s^{-1} = -\left[C_1^R\big(\partial_s\tilde\Gamma_s(t_1)\big) - C_1^L\big(\partial_s\tilde\Gamma_s(t_0)\big) + \int_{t_0}^{t_1} B_1\Big(\partial_s\tilde\Gamma_s(t), \partial_t\tilde\Gamma_s(t)\Big)\,dt\right] \tag{6.4}$$

with an initial value $h_{s_0} = e \in H$.

We recall our assumptions that $C_1^{L,R}$ and B_1 take values in the Lie algebra $L(H) \subset L(H \rtimes_\alpha G)$. As a result, h_s lies in H.

We note that

$$\dot{h}_s h_s^{-1} = -\left[C_1^R\left(\partial_s\tilde{\Gamma}_s(t_1)\right) - C_1^L\left(\partial_s\tilde{\Gamma}_s(t_0)\right) + \int_{t_0}^{t_1} B_1\left(\partial_s\tilde{\Gamma}_s(t), \partial_t\tilde{\Gamma}_s(t)\right) dt\right]. \tag{6.5}$$

Let us recall from (5.5) the connection form Ω on $\mathcal{P}_{\bar{A}}^{\mathrm{dec}}P$ given by:

$$\Omega_{\overline{\gamma},h}(\overline{v}+X) = \mathrm{Ad}(h^{-1})\left[\omega_{\overline{\gamma}}(\overline{v}) + C_1^R\left(\partial_s\tilde{\Gamma}_s(t_1)\right) - C_1^L\left(\partial_s\tilde{\Gamma}_s(t_0)\right) + \int_{t_0}^{t_1} B_1|_{\overline{\gamma}(t_0)}(\overline{v}(t), \overline{\gamma}'(t))dt + Xh^{-1}\right], \tag{6.6}$$

where $\overline{v}+X \in T_{(\overline{\gamma},h)}\mathcal{P}_{\bar{A}}^{\mathrm{dec}}P$, with $\overline{v}$ a vector field along the path $\overline{\gamma} : [t_0, t_1] \to P$ belong to $\mathcal{P}_{\bar{A}}P$ and $X \in T_hH$, the 1-form ω is as defined in (3.2):

$$\omega_{\overline{\gamma}}(\overline{v}) := A_{\overline{\gamma}(t_0)}(\overline{v}(t_0)) + C_0^R(\overline{v}(t_1)) - C_0^L(\overline{v}(t_0)) + \int_{t_0}^{t_1} B_0|_{\overline{\gamma}(t_0)}\left(\overline{v}(t), \overline{\gamma}'(t)\right) dt. \tag{6.7}$$

Proposition 6.1. *Suppose (G, H, α, τ) is a Lie crossed module and $\bar{A}$ is a connection on a principal G-bundle $\pi : P \to M$. We have then as above the bundle $\mathcal{P}_{\bar{A}}P \to \mathcal{P}M$ of $\bar{A}$-horizontal paths on M over the space $\mathcal{P}M$ of paths on M, and the decorated bundle*

$$\mathcal{P}_{\bar{A}}^{\mathrm{dec}}P = \mathcal{P}_{\bar{A}}P \times H \to \mathcal{P}M,$$

equipped with a connection form Ω given above in (6.6), involving the forms $C_1^{R,L}$ and B_1 that take values in $L(H)$. Then the path

$$[s_0, s_1] \to \mathcal{P}_{\bar{A}}^{\mathrm{dec}}P : s \mapsto (\tilde{\Gamma}_s, h_s) \tag{6.8}$$

is Ω-horizontal if and only if $s \mapsto \tilde{\Gamma}_s$ is an ω-horizontal path on $\mathcal{P}_{\bar{A}}P$ and $s \mapsto h_s$ satisfies the differential equation

$$\dot{h}_s h_s^{-1} = -C_1^R\left(\partial_s\tilde{\Gamma}_s(t_1)\right) + C_1^L\left(\partial_s\tilde{\Gamma}_s(t_0)\right) - \int_{t_0}^{t_1} B_1\left(\partial_s\tilde{\Gamma}_s(t), \partial_t\tilde{\Gamma}_s(t)\right) dt. \tag{6.9}$$

Proof. Evaluating Ω on the tangent vector (field)

$$\partial_s(\tilde{\Gamma}_s, h_s) = \left(\partial_s\tilde{\Gamma}_s, \dot{h}_s\right) \in T_{(\tilde{\Gamma}_s,h_s)}\mathcal{P}_{\bar{A}}^{\mathrm{dec}}P,$$

we have

$$\begin{aligned}\Omega_{(\tilde{\Gamma}_s,h_s)}\partial_s(\tilde{\Gamma}_s, h_s) = {} & \mathrm{Ad}(h_s^{-1})\omega\left(\partial_s\tilde{\Gamma}_s\right) + \mathrm{Ad}(h_s^{-1})\Big[\dot{h}_s h_s^{-1} \\ & + C_1^R\left(\partial_s\tilde{\Gamma}_s(t_1)\right) - C_1^L\left(\partial_s\tilde{\Gamma}_s(t_0)\right) + \int_{t_0}^{t_1} B_1\left(\partial_s\tilde{\Gamma}_s(t), \partial_t\tilde{\Gamma}_s(t)\right) dt\Big].\end{aligned} \tag{6.10}$$

Here, on the right, the first term is in $L(G)$ and the second term is in $L(H)$. The entire expression is 0 if and only if each of these terms is 0. This is equivalent to $s \mapsto \tilde{\Gamma}_s$ being ω-horizontal and $s \mapsto h_s$ satisfying the differential equation (6.9). □

7. Curvature conditions for reduction to holonomy bundle

We continue to work in the framework of the decorated bundle $\mathcal{P}_{\bar{A}}^{\mathrm{dec}}P$. Let C_1 be an $L(H)$-valued 1-form on P that is equivariant and vanishes on vertical vectors. Then we can associate to each $\overline{\gamma} \in \mathcal{P}_{\bar{A}}P$ a special decoration $h^*(\overline{\gamma})$ that is given by

$$h^*(\overline{\gamma}) = h_{\overline{\gamma}}(t_1), \tag{7.1}$$

where $[t_0, t_1] \to H : t \mapsto h_{\tilde{\gamma}}(t)$ is the solution of

$$h'_{\overline{\gamma}}(t)h_{\overline{\gamma}}(t)^{-1} = -C_1\left(\overline{\gamma}'(t)\right), \tag{7.2}$$

with initial value $h_{\overline{\gamma}}(t_0) = e$.

Then we have a sub-bundle $\overline{\mathcal{P}}_{\bar{A}}^{\mathrm{dec}}P$ of $\mathcal{P}_{\bar{A}}^{\mathrm{dec}}P$ specified by:

$$\overline{\mathcal{P}}_{\bar{A}}^{\mathrm{dec}}P := \{\left(\overline{\gamma}, h^*(\overline{\gamma})^{-1}\right) \mid \overline{\gamma} \in \mathcal{P}_{\bar{A}}P\} \subset \mathcal{P}_{\bar{A}}^{\mathrm{dec}}P. \tag{7.3}$$

More precisely,

$$\overline{\mathcal{P}}_{\bar{A}}^{\text{dec}}P \to \mathcal{P}M : \left(\overline{\gamma}, h^*(\overline{\gamma})^{-1}\right) \mapsto \pi \circ \overline{\gamma} \tag{7.4}$$

is a principal G-bundle. Henceforth, in this section we take the connection A to be the same as the connection $\bar{A}$:

$$A = \bar{A}.$$

Our goal in this section is to determine a type of connection $\hat{\Omega}$ on $\mathcal{P}_A^{\text{dec}}P$ that reduces to a connection on the sub-bundle $\overline{\mathcal{P}}_A^{\text{dec}}P$. Section 7.4 serves as a technical appendix to this section and presents some of the background computations for the proof of the main result Proposition 7.1. Later in Section 7.5 we present a description of the notion of a holonomy bundle and a result of Ambrose and Singer [2], that form a motivational background for our investigations.

7.1. Statement of the result

Let us first summarize the notation and framework.

Let A be a connection form on a principal G-bundle $\pi : P \to M$, and suppose (G, H, α, τ) is a Lie crossed module. (As noted before, we will use only the semidirect products here, and not the target map τ, which is useful in the categorical framework.) Let C be an $L(H \rtimes_\alpha G)$-valued 1-form on P that vanishes on vertical vectors and satisfies the equivariance

$$C|_{pg}(vg) = \text{Ad}(g^{-1})C|_p(v) \tag{7.5}$$

for all $p \in P$, $v \in T_pP$ and $g \in H$. Here, as usual, on the right we take $\text{Ad}(g^{-1})$ to be an operator on $L(H \rtimes_\alpha G)$. We decompose C into its component in $L(H)$ and the component in $L(G)$:

$$C = C_0 + C_1, \tag{7.6}$$

where C_0 takes values in $L(G)$ and C_1 in $L(H)$.

Let B be an $L(H \rtimes_\alpha G)$-valued 1-form on P that is equivariant analogously to C and vanishes when contracted on any vertical vector; we write

$$B = B_0 + B_1, \tag{7.7}$$

where B_0 is the $L(G)$-component of B and B_1 is the $L(H)$-component.

Let $\mathcal{P}_AP$ be, as before, the path space of A-horizontal paths $\overline{\gamma}$ on P, and $\mathcal{P}M$ the path space for M. On the decorated bundle

$$\mathcal{P}_A^{\text{dec}}P \stackrel{\text{def}}{=} \mathcal{P}_AP \times H \to \mathcal{P}M$$

consider the connection form $\hat{\Omega}$ given by

$$\begin{aligned}\hat{\Omega}_{\overline{\gamma},h}(\overline{v} + X) = \text{Ad}(h^{-1})\Big[&A_{\overline{\gamma}(t_0)}\big(\overline{v}(t_0)\big) - \text{Ad}\big(g_{\overline{\gamma}(t_1)}^{-1}\big)C|_{\overline{\gamma}(t_1)}\big(\overline{v}(t_1)\big) + C|_{\overline{\gamma}(t_0)}\big(\overline{v}(t_0)\big) \\ &+ \int_{t_0}^{t_1} \text{Ad}\big(g_{\overline{\gamma}}(t)^{-1}\big)B_{\overline{\gamma}(t)}\big(\overline{v}(t), \overline{\gamma}'(t)\big)\,dt + Xh^{-1}\Big]\end{aligned} \tag{7.8}$$

for all $\overline{\gamma} \in \mathcal{P}_AP$, $h \in H$, $\overline{v} \in T_{\overline{\gamma}}\mathcal{P}_{\bar{A}}P$ and $X \in T_hH$. (Let us note that in relation to our previous notation, $C^R = C^L = -C$.) In (7.8) the element $g_{\overline{\gamma}}(t) \in G$ is given by

$$g_{\overline{\gamma}}(t) = \tau\big(h_{\overline{\gamma}}(t)\big), \tag{7.9}$$

where $h_{\overline{\gamma}}(t) \in H$ is as given by the differential equation (7.2).

The connection $\hat{\Omega}$ can be obtained as a pullback $\mathcal{T}^*\Omega$, where Ω is the connection given in (5.4) on the decorated bundle and $\mathcal{T}$ is the change of base connection map discussed in Section 2.6 from the connection A to the connection $A + \tau C_1$; however, we leave this as a remark and omit verification.

Suppose that B_1 and C_1 are related by

$$B_1 = dC_1 + \frac{1}{2}[C_1, C_1]. \tag{7.10}$$

Consider a path $[s_0, s_1] \to \mathcal{P}M : s \mapsto \Gamma_s$ given by a C^∞ map

$$[t_0, t_1] \times [s_0, s_1] \to M : (t, s) \mapsto \Gamma_s(t) = \Gamma(t, s)$$

and let $\overline{\gamma} \in \mathcal{P}_AP$ be an $\bar{A}$-horizontal lift of the initial path Γ_{s_0}. Let $h^*(\overline{\gamma})$ be the value $h_{s_0}(t_1)$ where $[t_0, t_1] \to H : t \mapsto h_s(t)$ is the solution of the equation

$$\begin{aligned} h_s'(t) + C_1\big(\overline{\gamma}'(t)\big)h_s(t) = 0 \quad &\text{for all } t \in [t_0, t_1]; \\ h_s(t_0) = e. &\end{aligned} \tag{7.11}$$

We can conjugate by any $g \in G$ to see that the path

$$[t_0, t_1] \to G : t \mapsto h_{s;g}(t) \stackrel{\text{def}}{=} g^{-1}h_s(t)g$$

satisfies

$$\begin{aligned} h'_{s;g}(t) + g^{-1}C_1\big(\overline{\gamma}'(t)\big)g\, h_{s;g}(t) = 0 \quad &\text{for all } t \in [t_0, t_1]; \\ h_{s;g}(t_0) = e. \end{aligned} \tag{7.12}$$

By the equivariance of C, and hence of the $L(H)$-component C_1, given by (7.5), the first equation in (7.12) is equivalent to

$$h'_{s;g}(t) + C_1\big((\overline{\gamma}g)'(t)\big)\, h_{s;g}(t) = 0. \tag{7.13}$$

Thus, by uniqueness of the solution of such differential equations,

$$h_{s;g}(t) = g^{-1}h_s(t)g \qquad \text{for all } (t, s) \in [t_0, t_1] \times [s_0, s_1]. \tag{7.14}$$

Let

$$\begin{aligned} h^*(\overline{\gamma}) &= h_{\overline{\gamma}}(t_1) \\ g^*(\overline{\gamma}) &= \tau\big(h^*(\overline{\gamma})\big). \end{aligned} \tag{7.15}$$

We can now state the main result of this section.

Proposition 7.1. *With notation and framework as above, suppose the relation (7.10) holds. Then parallel transport by the connection $\hat{\Omega}$ carries elements of the form $\big(\overline{\gamma}, h^*(\overline{\gamma})^{-1}\big)$ to elements of the same form.*

The remainder of this section is devoted to an understanding and proof of this result.

7.2. Horizontal lifts by $\hat{\omega}$ and $\hat{\Omega}$

Consider a path $s \mapsto \Gamma_s$ on $\mathcal{P}M$ specified by a C^∞ map

$$\Gamma : [t_0, t_1] \times [s_0, s_1] \to M : (t, s) \mapsto \Gamma(t, s) = \Gamma_s(t)$$

and an A-horizontal lift

$$\overline{\Gamma}_0 : [t_0, t_1] \to P : t \mapsto \overline{\Gamma}_0(t)$$

of the initial path Γ_{s_0} on M. Now let

$$[t_0, t_1] \times [s_0, s_1] \to P : (t, s) \mapsto \hat{\Gamma}_s(t)$$

be the $\hat{\omega}$-horizontal lift of the path $s \mapsto \Gamma_s$, with initial value being the given path $\overline{\Gamma}_0$:

$$\hat{\Gamma}_{s_0} = \overline{\Gamma}_0.$$

We follow the strategy used to study ω-horizontal lifts, as in (3.17). We compare $\hat{\Gamma}$ to another path

$$\overline{\Gamma} : [s_0, s_1] \to \mathcal{P}_A P : s \mapsto \overline{\Gamma}_g$$

that is constructible in terms of just the connection for A and the initial path $\overline{\Gamma}_0$. We define

$$\overline{\Gamma} : [t_0, t_1] \times [s_0, s_1] \to P : (t, s) \mapsto \overline{\Gamma}_s(t)$$

to be the C^∞ map for which (i) each path $\overline{\Gamma}_s$ is $\bar{A}$-horizontal, (ii) the initial points $s \mapsto \overline{\Gamma}_s(t_0)$ constitute a path

$$[t_0, t_1] \to P : t \mapsto \overline{\Gamma}_{s_0}(t)$$

that is horizontal with respect to the connection for A, and (iii) the initial path $\overline{\Gamma}_{s_0}$ is the given initial path $\overline{\Gamma}_0 \in \mathcal{P}_A P$.

Then by (3.20) (applied to the connection form $\hat{\omega}$) it follows that the path $\hat{\Gamma}_s$ is obtained from $\overline{\Gamma}_s$ by translation with an element $a_s \in G$:

$$\hat{\Gamma}_s = \overline{\Gamma}_s a_s,$$

where $s \mapsto a_s$ satisfies the differential equation

$$\dot{a}_s a_s^{-1} = \mathrm{Ad}\big(g^*(\overline{\Gamma}_s(t_1))^{-1}\big)C_0\Big(\partial_t\overline{\Gamma}_s(t_1)\Big) - C_0\Big(\partial_t\overline{\Gamma}_s(t_0)\Big) - \int_{t_0}^{t_1} \mathrm{Ad}\big(g^*(\overline{\Gamma}_s(t))^{-1}\big)B_0\Big(\partial_s\overline{\Gamma}_s(t), \partial_t\overline{\Gamma}_s(t)\Big)\, dt \tag{7.16}$$

with initial value $a_{s_0} = e$. (In (3.20) there is a first term on the right that is absent here because $A = \bar{A}$ in this context.) Note that this differential equation is for a path on the group G.

Next, by Proposition 6.1 applied to $\hat{\Omega}$, the path

$$[s_0, s_1] \to \mathcal{P}_A^{\mathrm{dec}} P : s \mapsto (\hat{\Gamma}_s, x_s)$$

is $\hat{\Omega}$-horizontal if and only if the path $s \mapsto x_s \in H$ satisfies the differential equation

$$\dot{x}_s x_s^{-1} = \mathrm{Ad}\big(g^*(\hat{\Gamma}_s(t_1))^{-1}\big)C_1\big(\partial_s \hat{\Gamma}_s(t_1)\big) - C_1\big(\partial_s \hat{\Gamma}_s(t_0)\big) - \int_{t_0}^{t_1} \mathrm{Ad}\big(g^*(\hat{\Gamma}_s(t))^{-1}\big)B_1\Big(\partial_s \hat{\Gamma}_s(t), \partial_t \hat{\Gamma}_s(t)\Big)\, dt. \tag{7.17}$$

This differential equation, for the decoration element, is for a path on the group H. We can also verify (7.17) directly by focusing on the $L(H)$-component of the expression for $\hat{\Omega}$ given in (7.8) applied to the vector $(\partial_s \hat{\Gamma}_s, \dot{h}_s)$, the result being equal to 0.

From the second relation between α and τ that have noted in (4.2) we have

$$\mathrm{Ad}\big(\tau(h)\big)X = \alpha\big(\tau(h)\big)X = \mathrm{Ad}(h)X \tag{7.18}$$

for all $h \in H$ and $X \in L(H)$. From this we see that in the right side of (7.17) we can replace each g^* by an h^*, and so

$$\dot{x}_s x_s^{-1} = C_1\big(\partial_s \hat{\Gamma}_s(t_0)\big) - \mathrm{Ad}\big(h^*(\hat{\Gamma}_s(t_1))^{-1}\big)C_1\big(\partial_s \hat{\Gamma}_s(t_1)\big) - \int_{t_0}^{t_1} \mathrm{Ad}\big(h^*(\hat{\Gamma}_s(t))^{-1}\big)B_1\Big(\partial_s \hat{\Gamma}_s(t), \partial_t \hat{\Gamma}_s(t)\Big)\, dt. \tag{7.19}$$

7.3. *Comparison with variation of parallel transport*

We continue with the framework as above. For fixed $s \in [s_0, s_1]$, consider the path

$$[t_0, t_1] \to H : t \mapsto h_s(t)$$

that satisfies

$$\begin{aligned} h_s'(t) + C_1\big(\hat{\Gamma}_s'(t)\big)h_s(t) &= 0 \\ h_s(t_0) &= e. \end{aligned} \tag{7.20}$$

We use the notation

$$h^*(\hat{\Gamma}_s) = h_s(t_1). \tag{7.21}$$

Our objective is to show that the path

$$[s_0, s_1] \to \overline{\mathcal{P}}_A^{\mathrm{dec}} P : s \mapsto \Big(\hat{\Gamma}_s, h^*(\hat{\Gamma}_s)^{-1}\Big) \tag{7.22}$$

is $\hat{\Omega}$-horizontal. Let

$$y_s(t) \stackrel{\mathrm{def}}{=} h_s(t)^{-1}. \tag{7.23}$$

Thus, we need to show that

$$[s_0, s_1] \to \overline{\mathcal{P}}_A^{\mathrm{dec}} P : s \mapsto \big(\hat{\Gamma}_s, y_s(t_1)\big)$$

is $\hat{\Omega}$-horizontal. Expressing Eq. (7.20) in terms of y_s we have:

$$\begin{aligned} y_s(t)^{-1}y_s'(t) &= h_s(t)\big(-h_s(t)^{-1}h_s'(t)h_s(t)^{-1}\big) \\ &= -h_s'(t)h_s(t)^{-1} \\ &= C_1\big(\hat{\Gamma}_s'(t)\big). \end{aligned} \tag{7.24}$$

Then, as we show below in (7.60),

$$\begin{aligned} \dot{y}_s(t_1)y_s(t)^{-1} - \dot{y}_s(t_0)y_s(t_0)^{-1} &= -\int_{t_0}^{t_1} y_s(u)\left(d\hat{C}_1 + \frac{1}{2}[\hat{C}_1, \hat{C}_1]\right)(\partial_t, \partial_s)y_s(u)^{-1}du \\ &\quad + y_s(t_1)C_1\big(\partial_s \hat{\Gamma}_s(t_1)\big)y_s(t_1)^{-1} - y_s(t_0)C_1\big(\partial_s \hat{\Gamma}_s(t_0)\big)y_s(t_0)^{-1}, \end{aligned} \tag{7.25}$$

where

$$\hat{C}_1 = \hat{\Gamma}^* C_1. \tag{7.26}$$

Since $y_s(t_0)$ is held fixed at e we have then, on using (7.25),

$$\dot{y}_s(t_1)y_s(t)^{-1} = -\int_{t_0}^{t_1} y_s(u)\left(d\hat{C}_1 + \frac{1}{2}[\hat{C}_1, \hat{C}_1]\right)(\partial_t, \partial_s)y_s(u)^{-1}du + y_s(t_1)C_1(\partial_s \hat{\Gamma}_s(t_1)\big)y_s(t_1)^{-1} - C_1\big(\partial_s \hat{\Gamma}_s(t_0)\big). \tag{7.27}$$

Comparing with the equation for $\hat{\Omega}$-parallel transport (7.17) we see that the two equations agree if

$$B_1 = dC_1 + \frac{1}{2}[C_1, C_1]. \tag{7.28}$$

Thus the path (7.22) is $\hat{\Omega}$-horizontal, and the proof of Proposition 7.1 is complete.

7.4. Variation of differential equations

In this subsection we work through the details of the computation that leads to the equation (7.27) which was central to the proof of Proposition 7.1.

Consider the differential equation

$$b(t)^{-1}b'(t) = C\big(\overline{\gamma}'(t)\big) \tag{7.29}$$

for $t \in [0, 1]$, where $\overline{\gamma} \in \mathcal{P}_A P$. We shall determine how fast the terminal point $b(1)$ changes when we change the path $\overline{\gamma}$.

Our strategy is to consider the family of differential equations

$$b_s(t)^{-1}b_s'(t) = C\big(\hat{\Gamma}_s'(t)\big) \tag{7.30}$$

where $s \in [0, 1]$ and $t \in [0, 1]$, and the prime is derivative with respect to t. Here

$$\hat{\Gamma} : [0, 1] \times [0, 1] \to P : (t, s) \mapsto \hat{\Gamma}_s(t) \tag{7.31}$$

is a smooth map. We think of s as a variational parameter and our goal is to compute how fast $b_s(1)$ changes with s.

Let us denote the right hand side by $E_s(t) \in L(G)$:

$$E_s(t) = C\big(\hat{\Gamma}_s'(t)\big). \tag{7.32}$$

Thus our differential equation is

$$b_s(t)^{-1}b_s'(t) = E_s(t). \tag{7.33}$$

Now let

$$D_s(t) = \dot{b}_s(t)b_s(t)^{-1}, \tag{7.34}$$

where

$$\dot{b}_s(t) = \partial_s b_s(t) \tag{7.35}$$

is the derivative which contains the information we are ultimately seeking. Our goal is to compute $D_s(t)$.

As always, we will denote the s-derivative by a dot over the letter:

$$\dot{x}_s(t) = \partial_s x_s(t). \tag{7.36}$$

Our strategy is to compute $D_s'(t) = \partial_t D_s(t)$ and then obtain $D_s(t)$ by integrating:

$$D_s(t) = \int_0^t D_s'(u)\,du + D_s(0).$$

So now let us compute the derivative $D_s'(t)$. From (7.34) we have

$$D_s'(t) = -\dot{b}_s(t)b_s(t)^{-1}b_s'(t)b_s(t)^{-1} + \big(\partial_s\partial_t b_s(t)\big)b_s(t)^{-1}. \tag{7.37}$$

Now we are going to work out $\partial_s E_s(t)$, but first let us recall what $E_s(t)$ is:

$$E_s(t) = b_s(t)^{-1}b_s'(t). \tag{7.38}$$

It is important that we have $b_s(t)^{-1}$ on the left for $E_s(t)$ and on the right for $D_s(t)$. Returning to the calculation, we have:

$$\dot{E}_s(t) = b_s(t)^{-1}\partial_s\partial_t b_s(t) - b_s(t)^{-1}\dot{b}_s(t)b_s(t)^{-1}b_s'(t). \tag{7.39}$$

Comparing with $D_s'(t)$ we see that it is useful to conjugate $\dot{E}_s(t)$ by $b_s(t)$:

$$b_s(t)\dot{E}_s(t)b_s(t)^{-1} = \big(\partial_{st}^2 b_s(t)\big)b_s(t)^{-1} - \dot{b}_s(t)b_s(t)^{-1}b_s'(t)b_s(t)^{-1}, \tag{7.40}$$

which is exactly $D_s'(t)$! Thus:

$$D_s'(t) = b_s(t)\dot{E}_s(t)b_s(t)^{-1}. \tag{7.41}$$

Integrating, we obtain

$$D_s(t) = D_s(0) + \int_0^t b_s(u)\dot{E}_s(u)b_s(u)^{-1}\,du. \tag{7.42}$$

This is in itself a nice formula for $D_s(t) = \dot{b}_s(t)b_s(t)^{-1}$, the rate of change of $b_s(t)$ when s is varied.

Let us formally summarize what we have proved so far as a self-contained result.

Proposition 7.2. *Let H be a Lie group and suppose*

$$[t_0, t_1] \times [s_0, s_1] \to H : (t, s) \mapsto b_s(t)$$

is a C^∞ function. Let

$$E : [t_0, t_1] \times [s_0, s_1] \to L(H) : (t, s) \mapsto E_s(t)$$

be the function given by

$$E_s(t) = b_s(t)^{-1} b'_s(t) \tag{7.43}$$

for all $(t, s) \in [t_0, t_1] \times [s_0, s_1]$. Then

$$\dot{b}_s(t) b_s(t)^{-1} = \dot{b}_s(t_0) b_s(t_0)^{-1} + \int_{t_0}^{t} \mathrm{Ad}\big(b_s(u)\big) \dot{E}_s(u)\, du \tag{7.44}$$

for all $(t, s) \in [t_0, t_1] \times [s_0, s_1]$, with a dot over a letter denoting the derivative with respect to s.

Results of this type are sometimes called 'non-abelian Stokes formulas'.

Now let us return to the geometric context, with notation as before. Thus, as in (7.32), $E_s(t)$ is given by:

$$E_s(t) = C\Big(\partial_t \hat{\Gamma}_s(t)\Big). \tag{7.45}$$

We can write this as

$$E_s(t) = \hat{C}(\partial_t), \tag{7.46}$$

where $\hat{C}$ is the pull back

$$\hat{C} = \hat{\Gamma}^* C, \tag{7.47}$$

which is a 1-form on $[0, 1] \times [0, 1]$. Let us now use the formula for the exterior differential of a 1-form:

$$d\hat{C}(v, w) = v[\hat{C}(w)] - w[\hat{C}(v)] - \hat{C}([v, w]), \tag{7.48}$$

for any smooth vector fields v and w. Then

$$d\hat{C}(\partial_t, \partial_s) = \partial_t\Big(\hat{C}(\partial_s)\Big) - \partial_s\Big(\hat{C}(\partial_t)\Big) - \hat{C}\Big([\partial_t, \partial_s]\Big). \tag{7.49}$$

The Lie bracket of the coordinate vector fields ∂_t and ∂_s appearing on the right is 0. So we have

$$\dot{E}_s(t) = \partial_s\Big[\hat{C}(\partial_t)\Big] = \partial_t\Big[\hat{C}(\partial_s)\Big] - d\hat{C}(\partial_t, \partial_s). \tag{7.50}$$

To keep the notation simple let us write

$$F_s(t) = \hat{C}_{(t,s)}(\partial_s). \tag{7.51}$$

Then

$$\dot{E}_s(t) = -d\hat{C}(\partial_t, \partial_s) + \partial_t F_s(t). \tag{7.52}$$

Looking back at $D'_s(t)$ as given in (7.41) we compute

$$D'_s(t) = b_s(t)\Big(-d\hat{C}(\partial_t, \partial_s) + F'_s(t)\Big) b_s(t)^{-1}. \tag{7.53}$$

We focus for now on the second term and compute:

$$\begin{aligned} b_s(t) F'_s(t) b_s(t)^{-1} &= \partial_t\Big(b_s(t) F_s(t) b_s(t)^{-1}\Big) - b'_s(t) F_s(t) b_s(t)^{-1} - b_s(t) F_s(t) \partial_t\big(b_s(t)^{-1}\big) \\ &= \partial_t\Big(b_s(t) F_s(t) b_s(t)^{-1}\Big) - b_s(t) E_s(t) F_s(t) b_s(t)^{-1} + b_s(t) F_s(t) E_s(t) b_s(t)^{-1} \\ &= \partial_t\Big(b_s(t) F_s(t) b_s(t)^{-1}\Big) - b_s(t) [E_s(t), F_s(t)] b_s(t)^{-1}. \end{aligned} \tag{7.54}$$

Let us analyze the Lie bracket term

$$[E_s(t), F_s(t)] = [\hat{C}(\partial_t), \hat{C}(\partial_s)]. \tag{7.55}$$

The 2-form $[\hat{C}, \hat{C}]$ is defined by

$$[\hat{C}, \hat{C}](v, w) = [\hat{C}(v), \hat{C}(w)] - [\hat{C}(w), \hat{C}(v)] = 2[\hat{C}(v), \hat{C}(w)]. \tag{7.56}$$

There is also the related notation

$$(\hat{C} \wedge \hat{C})(v, w) = [\hat{C}(v), \hat{C}(w)], \tag{7.57}$$

which is directly meaningful if $\hat{C}$ takes values in a matrix Lie algebra.

Hence

$$[E_s(t), F_s(t)] = \frac{1}{2}[\hat{C}, \hat{C}](\partial_t, \partial_s). \tag{7.58}$$

Now glancing back a few steps at (7.53) we see that

$$D_s'(t) = -b_s(t)d\hat{C}(\partial_t, \partial_s)b_s(t)^{-1} + \partial_t\Big(b_s(t)F_s(t)b_s(t)^{-1}\Big) - b_s(t)\frac{1}{2}[\hat{C}, \hat{C}](\partial_t, \partial_s)b_s(t)^{-1}. \tag{7.59}$$

Integrating, and recalling from (7.34) that $D_s(t)$ is $\dot{b}_s(t)b_s(t)^{-1}$, we have

$$\begin{aligned}\dot{b}_s(t)b_s(t)^{-1} - \dot{b}_s(0)b_s(0)^{-1} &= -\int_0^t b_s(u)\Big(d\hat{C} + \frac{1}{2}[\hat{C}, \hat{C}]\Big)(\partial_t, \partial_s)b_s(u)^{-1}du \\ &\qquad + b_s(t)F_s(t)b_s(t)^{-1} - b_s(0)F_s(0)b_s(0)^{-1},\end{aligned} \tag{7.60}$$

wherein, as before, $F_s(t) = \hat{C}_{(t,s)}(\partial_s)$.

7.5. The holonomy bundle

For a principal G-bundle $\pi : P \to M$ equipped with a connection A we denote by $P_A(u)$ the set of all terminal points of A-horizontal paths that initiate at any given point $u \in P$. The *holonomy group* $H_A(u)$ consists of all $g \in G$ for which $ug \in P_A(u)$. If $\overline{\gamma}_u^p$ is an A-horizontal path on P initiating at u and terminating at $p \in P_A(u)$ then $\overline{\gamma}_u^p g$ is also A-horizontal, initiating at the point ug and terminating at pg; if $g \in H_A(u)$ then we can choose an A-horizontal path $\overline{\gamma}_u^{ug}$ from u to ug, and the composite $(\overline{\gamma}_u^p g) \circ \overline{\gamma}_u^{ug}$ is an A-horizontal path from u to the point pg. Thus $P_A(u)$ is mapped into itself by the right action of the holonomy group $H_A(u) \subset G$. In this way the structure

$$\pi : P_A(u) \to M : p \mapsto \pi(p) \tag{7.61}$$

is a principal $H_A(u)$-bundle over M. The connection A reduces to a connection on this bundle. A celebrated result of Ambrose and Singer [2] relates the Lie algebra of the holonomy group to the Lie subalgebra of $L(G)$ spanned by elements $F^A(v, w)$, where F^A is the curvature of A and v and w run over all vectors in T_pP with p running over the holonomy bundle $P_A(u)$. (Since composition of paths is crucial in these discussions, such as even to see that $H_A(u)$ is a subgroup, the definition of the holonomy bundle should involve a family of paths that is closed under composition; in fact we may use just the type of paths we have been using, C^∞ and constant near the initial and final times.) In our context, for the connection $\hat{\Omega}$ on the bundle $\mathcal{P}_A^{\rm dec}P$, Proposition 7.1 says that the holonomy subbundle of any point in $\overline{\mathcal{P}}_A^{\rm dec}P$ is contained inside this subbundle.

8. Differential calculus on path spaces

We have avoided putting a manifold structure on the spaces of paths with which we have worked. Such a structure is not logically needed for any of our constructions and is useful only as an idea. It is standard practice in the theory of stochastic processes (which is concerned with integration on path spaces) to work primarily with notions of differentiation and integration defined in the specific context of path or function spaces rather than on any abstract infinite dimensional manifold. Although an abstract theory of such integration was constructed (Kuo [22]) it has been found to be more useful to define geometric, differential and measure theoretic notions directly on path spaces. Let us then summarize here the differential notions we need for our work.

Consider a set X whose points are paths on a given manifold M. In this context we require that the paths be C^∞, and there might be additional restrictions placed.

By a *tangent vector* v to X at a point in X given by a path $\gamma : [t_0, t_1] \to M$ we mean a C^∞ vector field $v : [t_0, t_1] \to TM$ along γ that is constant near t_0 and near t_1. For example, there is the special vector $\gamma' \in T_\gamma X$ which is just the tangent vector field along γ (the tangent vector field along γ is zero near the initial and final times). We denote the set of all vectors tangent to X at γ by $T_\gamma X$ and call this the *tangent space* to X at γ. This is clearly a vector space under pointwise addition and scaling.

If v is a C^∞ vector field on an open subset of M and $\gamma \in \mathcal{P}(M)$ lies entirely in U then we obtain a vector field v_γ along γ given by

$$v_\gamma(t) = v\big(\gamma(t)\big) \quad \text{for all } t \in [t_0, t_1].$$

Then v_γ is C^∞ and constant near t_0 and t_1, and hence is a vector in the tangent space $T_\gamma \mathcal{P}(M)$.

A *k-form* Θ on X is an assignment to each $\gamma \in X$ an alternating multilinear mapping

$$(T_\gamma X)^k \to \mathbb{R} : (v_1, \ldots, v_k) \mapsto \Theta_\gamma(v_1, \ldots, v_k).$$

A typical example of interest is a k-form $I(\theta)$ that arises from a k-form θ on M by the specification:

$$I(\theta)_\gamma(v_1, \ldots, v_k) = \int_{t_0}^{t_1} \theta_{\gamma(t)}\big(v_1(t), \ldots, v_k(t)\big)\, dt. \tag{8.1}$$

Many forms on X of interest to us have some additional features: for example, they are invariant under a class of reparametrizations of the paths. Moreover, many of the forms we use vanish when contracted on the tangent vector field. As an example consider, with $I(\theta)$ as above, the $(k-1)$-form on X given by:

$$(i_{\gamma'} I(\theta)_\gamma)(v_1, \ldots, v_{k-1}) = \int_{t_0}^{t_1} \theta_{\gamma(t)}\big(\gamma'(t), v_1(t), \ldots, v_{k-1}(t)\big)\, dt. \tag{8.2}$$

This form vanishes when one of the vectors v_j happens to be γ'. The form $i_{\gamma'} I(\theta)_\gamma$ is the *Chen integral*

$$\int_\gamma \theta \stackrel{\text{def}}{=} i_{\gamma'} I(\theta)_\gamma. \tag{8.3}$$

Intuitively we think of X as a bundle over a quotient space $[X]$ after quotienting by a group of reparametrizations. Of interest then are forms on X that vanish along the orbital directions and are invariant under translations (reparametrizations) by the action of the structure group; thus these correspond to forms on $[X]$ pulled back up to the space X.

Now let

$$\Gamma : [t_0, t_1] \times [s_0, s_1] \to M : (t, s) \to \Gamma_s(t)$$

be a C^∞ map which is stationary near the boundary in the following sense: there is an $\epsilon > 0$ such that for each fixed s the point $\Gamma_s(t)$ is the same when t is at distance $< \epsilon$ from $\{t_0, t_1\}$, and for each fixed t the point $\Gamma_s(t)$ is the same when s is at distance $< \epsilon$ from $\{s_0, s_1\}$. Thus each Γ_s is in $\mathcal{P}(M)$ as defined in (2.1). Then there is for each $s \in [s_0, s_1]$ the tangent vector $\dot{\Gamma}_s \in T_{\Gamma_s}\mathcal{P}(M)$ given by

$$\dot{\Gamma}_s(t) = \partial_s \Gamma_s(t) \qquad \text{for all } t \in [t_0, t_1]. \tag{8.4}$$

Other differential geometric notions such as bundles and connections over spaces of paths can be defined by natural extension of the usual definitions on finite dimensional spaces.

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