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Cohomologies, deformations and extensions of n-Hom-Lie algebras[☆]

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ABSTRACT

In this paper, first we give the cohomologies of an n-Hom-Lie algebra and introduce the notion of a derivation of an n-Hom-Lie algebra. We show that a derivation of an n-Hom-Lie algebra is a 1-cocycle with the coefficient in the adjoint representation. We also give the formula of the dual representation of a representation of an n-Hom-Lie algebra. Then, we study $(n-1)$-order deformation of an n-Hom-Lie algebra. We introduce the notion of a Hom-Nijenhuis operator, which could generate a trivial $(n-1)$-order deformation of an n-Hom-Lie algebra. Finally, we introduce the notion of a generalized derivation of an n-Hom-Lie algebra, by which we can construct a new n-Hom-Lie algebra, which is called the generalized derivation extension of an n-Hom-Lie algebra.

1. Introduction

The representation theory of an algebra is very important to this algebraic structure. Given a representation, one can obtain the corresponding cohomology, which could provide invariants. The cohomology plays important roles in the study of deformations and extension problems.

The notion of a Hom-Lie algebra was introduced by Hartwig, Larsson, and Silvestrov in [10] as part of a study of deformations of the Witt and the Virasoro algebras. In a Hom-Lie algebra, the Jacobi identity is twisted by a linear map, called the Hom-Jacobi identity. Some q-deformations of the Witt and the Virasoro algebras have the structure of a Hom-Lie algebra [10,11]. Because of close relation to discrete and deformed vector fields and differential calculus [10, 12,13], more people pay special attention to this algebraic structure. In particular, representations and cohomologies of Hom-Lie algebras are studied in [1,16,17]. On the other hand, the notion of an n-Hom-Lie algebra was introduced in [4], which is a generalization of an n-Lie algebra introduced in [9]. See the review article [5] for more information about n-Lie algebras. Then several aspects about n-Hom-Lie algebras are studied. For example, the cohomologies adapted to central extensions and deformations are studied in [2]; 2-cocycles that used to studied abelian extensions are studied in [6]; Construction of 3-Hom-Lie algebras from Hom-Lie algebras are studied in [3], and extensions of 3-Hom-Lie algebras are studied in [14]. However, the systematic study of the cohomology of an n-Hom-Lie algebra is still lost.

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The first purpose of this paper is to give a systematic study of cohomology of n-Hom-Lie algebras. We introduce the coboundary operator associated to a general representation of an n-Hom-Lie algebra and obtain the corresponding cohomology. As a byproduct, we introduce the notion of a derivation of an n-Hom-Lie algebra, which is different from existing ones. We show that a derivation is exactly a 1-cocycle with the coefficient in the adjoint representation, which generalizes the fact about derivations of a Lie algebra. We also studied $(n-1)$-order deformations of n-Hom-Lie algebras, which is inspired by the work in [15]. It turns out that for n-ary algebras, one should study $(n-1)$-order deformations, instead of 1-order deformations, to obtain invariants. The second purpose is to give the correct definition of a dual representation of a representation of an n-Hom-Lie algebra. In [14], the authors claimed that the usual definition of ad^* is still a representation without a proof. To solve this problem, in [6] the authors add a strong condition on a representation ρ of an n-Hom-Lie algebra to make ρ^* to be a representation, which generalizes the idea from [18]. Here we give a new formula to define the dual representation, which is quite natural. The last purpose is to give an approach to construct new n-Hom-Lie algebras. To do this, we introduce the notion of a generalized derivation of an n-Hom-Lie algebra, by which we can construct a new n-Hom-Lie algebra, which is called generalized derivation extension.

The paper is organized as follows. In Section 2, we recall some basic notions of Hom-Leibniz algebras and representations of n-Hom-Lie algebras. In Section 3, we study cohomologies, derivations and dual representations of n-Hom-Lie algebras. In Section 4, we studied $(n-1)$-order deformations of n-Hom-Lie algebras and introduce the notion of Hom-Nijenhuis operators which could generate trivial deformations. In Section 5, we introduce the notion of a generalized derivation of an n-Hom-Lie algebra, and construct a new n-Hom-Lie algebra, which is called generalized derivation extension.

In this paper, we work over an algebraically closed field $\mathbb{K}$ of characteristic 0 and all the vector spaces are over $\mathbb{K}$.

2. Preliminaries

In this section, we recall Hom-Leibniz algebras and representations of n-Hom-Lie algebras.

Definition 2.1. A (multiplicative) **Hom-Leibniz algebra** is a vector space $\mathfrak{g}$ together with a bracket operation $[\cdot,\cdot]_{\mathfrak{g}} : \mathfrak{g}\otimes\mathfrak{g} \longrightarrow \mathfrak{g}$ and an algebraic automorphism $\alpha : \mathfrak{g} \longrightarrow \mathfrak{g}$, such that for all $x, y, z \in \mathfrak{g}$, we have

$$[\alpha(x), [y, z]_{\mathfrak{g}}]_{\mathfrak{g}} = [[x, y]_{\mathfrak{g}}, \alpha(z)]_{\mathfrak{g}} + [\alpha(y), [x, z]_{\mathfrak{g}}]_{\mathfrak{g}}. \tag{1}$$

In particular, if the bracket operation $[\cdot,\cdot]_{\mathfrak{g}}$ is skew-symmetric, we obtain the definition of a Hom-Lie algebra.

Let V be a vector space, and $\beta \in GL(V)$. Define a skew-symmetric bilinear bracket operation $[\cdot,\cdot]_\beta : \wedge^2\mathfrak{gl}(V) \longrightarrow \mathfrak{gl}(V)$ by

$$[A, B]_\beta = \beta\circ A\circ\beta^{-1}\circ B\circ\beta^{-1} - \beta\circ B\circ\beta^{-1}\circ A\circ\beta^{-1}, \quad \forall A, B \in \mathfrak{gl}(V). \tag{2}$$

Denote by $\mathrm{Ad}_\beta : \mathfrak{gl}(V) \to \mathfrak{gl}(V)$ the adjoint action on $\mathfrak{gl}(V)$, i.e.

$$\mathrm{Ad}_\beta(A) = \beta\circ A\circ\beta^{-1}. \tag{3}$$

Proposition 2.2 (*[19, Proposition 4.1]*)**.** *With the above notations,* $(\mathfrak{gl}(V), [\cdot,\cdot]_\beta, \mathrm{Ad}_\beta)$ *is a regular Hom-Lie algebra.*

This Hom-Lie algebra plays an important role in the representation theory of Hom-Lie algebras. See [19] for more details.

Definition 2.3. An n-Hom-Lie algebra is a vector space $\mathfrak{g}$ equipped with a bracket operation $[\cdot,\ldots,\cdot]_{\mathfrak{g}} : \wedge^n\mathfrak{g} \longrightarrow \mathfrak{g}$ and an algebraic automorphism $\alpha : \mathfrak{g} \longrightarrow \mathfrak{g}$ such that for all $x_1,\ldots,x_{n-1}, y_1,\ldots,y_n \in \mathfrak{g}$, the following **Hom-Fundamental** identity holds:

$$\begin{aligned}\mathrm{HF}_{x_1,\ldots,x_{n-1},y_1,\ldots,y_n} \triangleq\ & [\alpha(x_1),\ldots,\alpha(x_{n-1}), [y_1,\ldots,y_n]_{\mathfrak{g}}]_{\mathfrak{g}}\\ & -\sum_{i=1}^{n}[\alpha(y_1),\ldots,\alpha(y_{i-1}), [x_1,\ldots,x_{n-1},y_i]_{\mathfrak{g}}, \alpha(y_{i+1}),\ldots,\alpha(y_n)]_{\mathfrak{g}}\\ =\ & 0.\end{aligned} \tag{4}$$

Any linear map $\alpha : \mathfrak{g} \longrightarrow \mathfrak{g}$ induces a linear map $\widetilde{\alpha} : \wedge^{n-1}\mathfrak{g} \longrightarrow \wedge^{n-1}\mathfrak{g}$ via

$$\widetilde{\alpha}(x_1\wedge\cdots\wedge x_{n-1}) = \alpha(x_1)\wedge\cdots\wedge\alpha(x_{n-1}). \tag{5}$$

Similar as the case of n-Lie algebras, elements in $\wedge^{n-1}\mathfrak{g}$ are called fundamental elements. On $\wedge^{n-1}\mathfrak{g}$, one can define a new bracket operation $[\cdot,\cdot]_F$ by

$$[X, Y]_F = \sum_{i=1}^{n-1}\alpha(y_1)\wedge\cdots\wedge\alpha(y_{i-1})\wedge[x_1,\ldots,x_{n-1},y_i]_{\mathfrak{g}}\wedge\alpha(y_{i+1})\wedge\cdots\wedge\alpha(y_{n-1}), \tag{6}$$

for all $X = x_1\wedge\cdots\wedge x_{n-1}$ and $Y = y_1\wedge\cdots\wedge y_{n-1}$. It is proved in [2] that $(\wedge^{n-1}\mathfrak{g}, [\cdot,\cdot]_F, \widetilde{\alpha})$ is a Hom-Leibniz algebra.

Definition 2.4. A **morphism** of n-Hom-Lie algebras $f:(\mathfrak{g},[\cdot,\ldots,\cdot]_{\mathfrak{g}},\alpha)\to(\mathfrak{h},[\cdot,\ldots,\cdot]_{\mathfrak{h}},\gamma)$ is a linear map $f:\mathfrak{g}\to\mathfrak{h}$ such that

$$f[x_1,\ldots,x_n]_{\mathfrak{g}}=[f(x_1),\ldots,f(x_n)]_{\mathfrak{h}},\quad \forall x_1,\ldots,x_n\in\mathfrak{g}, \tag{7}$$

$$f\circ\alpha=\gamma\circ f. \tag{8}$$

Definition 2.5. A **representation** of an n-Hom-Lie algebra $(\mathfrak{g},[\cdot,\ldots,\cdot]_{\mathfrak{g}},\alpha)$ on a vector space V with respect to a linear automorphism $\beta\in GL(V)$ is a linear map $\rho:\wedge^{n-1}\mathfrak{g}\longrightarrow\mathfrak{gl}(V)$ such that for all $X,Y\in\wedge^{n-1}\mathfrak{g}$ and $x_1,\ldots,x_{n-2},y_1,\ldots,y_n\in\mathfrak{g}$, we have

(i) $\rho(\widetilde{\alpha}(X))\circ\beta=\beta\circ\rho(X)$;
(ii) $\rho(\widetilde{\alpha}(X))\circ\rho(Y)-\rho(\widetilde{\alpha}(Y))\circ\rho(X)=\rho([X,Y]_F)\circ\beta$;
(iii)

$$\begin{aligned}&\rho(\alpha(x_1),\ldots,\alpha(x_{n-2}),[y_1,\ldots,y_n]_{\mathfrak{g}})\circ\beta\\ =&\sum_{i=1}^{n}(-1)^{n-i}\rho(\alpha(y_1),\ldots,\hat{\alpha}(y_i),\ldots,\alpha(y_n))\circ\rho(x_1,\ldots,x_{n-2},y_i).\end{aligned}$$

We denote a representation by $(V;\rho,\beta)$.

Remark 2.6. Conditions (i) and (ii) in the above definition mean that $\rho:\wedge^{n-1}\mathfrak{g}\longrightarrow\mathfrak{gl}(\mathfrak{g})$ is a morphism of Hom-Leibniz algebras from $(\wedge^{n-1}\mathfrak{g},[\cdot,\cdot]_F,\widetilde{\alpha})$ to $(\mathfrak{gl}(V),[\cdot,\cdot]_\beta,\mathsf{Ad}_\beta)$.

Define $\mathsf{ad}:\wedge^{n-1}\mathfrak{g}\longrightarrow\mathfrak{gl}(\mathfrak{g})$ by

$$\mathsf{ad}_X y=[x_1,\ldots,x_{n-1},y]_{\mathfrak{g}},\quad \forall X=x_1\wedge\cdots\wedge x_{n-1}\in\wedge^{n-1}\mathfrak{g},\ y\in\mathfrak{g}. \tag{9}$$

Then $(\mathfrak{g};\mathsf{ad},\alpha)$ is a representation of the n-Hom-Lie algebra $(\mathfrak{g},[\cdot,\ldots,\cdot]_{\mathfrak{g}},\alpha)$ on $\mathfrak{g}$ with respect to α, which is called the **adjoint representation**.

The following result is straightforward.

Proposition 2.7. *Let $(V;\rho,\beta)$ be a representation of an n-Hom-Lie algebra $(\mathfrak{g},[\cdot,\ldots,\cdot]_{\mathfrak{g}},\alpha)$. Define a bracket operation $[\cdot,\ldots,\cdot]_\rho:\wedge^n(\mathfrak{g}\oplus V)\to\mathfrak{g}\oplus V$ by*

$$[x_1+u_1,\ldots,x_n+u_n]_\rho=[x_1,\ldots,x_n]_{\mathfrak{g}}+\sum_{i=1}^{n}(-1)^{n-i}\rho(x_1,\ldots,\hat{x}_i,\ldots,x_n)u_i,\quad \forall x_i\in\mathfrak{g},u_i\in V.$$

Define $\alpha+\beta:\mathfrak{g}\oplus V\to\mathfrak{g}\oplus V$ by

$$(\alpha+\beta)(x+u)=\alpha(x)+\beta(u).$$

*Then $(\mathfrak{g}\oplus V,[\cdot,\ldots,\cdot]_\rho,\alpha+\beta)$ is an n-Hom-Lie algebra, which we call the **semi-direct product** of the n-Hom-Lie algebra $(\mathfrak{g},[\cdot,\ldots,\cdot]_{\mathfrak{g}},\alpha)$ by the representation $(V;\rho,\beta)$. We denote this semidirect n-Hom-Lie algebra simply by $\mathfrak{g}\ltimes V$.*

3. Cohomologies, derivations and dual representations of *n*-Hom-Lie algebras

3.1. Cohomologies of n-Hom-Lie algebras

Let $(V;\rho,\beta)$ be a representation of an n-Hom-Lie algebra $(\mathfrak{g},[\cdot,\ldots,\cdot]_{\mathfrak{g}},\alpha)$. A p-cochain on $\mathfrak{g}$ with the coefficients in a representation $(V;\rho,\beta)$ is a multilinear map

$$f:\underbrace{\wedge^{n-1}\mathfrak{g}\otimes\cdots\otimes\wedge^{n-1}\mathfrak{g}}_{p-1}\wedge\mathfrak{g}\longrightarrow V.$$

Denote the space of p-cochains by $C^p(\mathfrak{g};V)$. Define the coboundary operator $\delta:C^p(\mathfrak{g};V)\to C^{p+1}(\mathfrak{g};V)$ by

$$\begin{aligned}&(\delta f)(X_1,\ldots,X_p,z)\\ =&\sum_{1\le i<k\le p}(-1)^i\beta f\Big(\widetilde{\alpha}^{-1}(X_1),\ldots,\hat{X}_i,\ldots,\widetilde{\alpha}^{-1}(X_{k-1}),[\widetilde{\alpha}^{-2}(X_i),\widetilde{\alpha}^{-2}(X_k)]_F,\\ &\widetilde{\alpha}^{-1}(X_{k+1}),\ldots,\widetilde{\alpha}^{-1}(X_p),\alpha^{-1}(z)\Big)\\ &+\sum_{i=1}^{p}(-1)^i\beta f(\widetilde{\alpha}^{-1}(X_1),\ldots,\hat{X}_i,\ldots,\widetilde{\alpha}^{-1}(X_p),[\widetilde{\alpha}^{-2}(X_i),\alpha^{-2}(z)]_{\mathfrak{g}})\end{aligned}$$

$$+\sum_{i=1}^{p}(-1)^{i+1}\rho(X_i)f(\widetilde{\alpha}^{-1}(X_1),\ldots,\hat{X}_i,\ldots,\widetilde{\alpha}^{-1}(X_p),\alpha^{-1}(z))$$
$$+\sum_{i=1}^{n-1}(-1)^{n+p-i+1}\rho(x_p^1,x_p^2,\ldots,\hat{x}_p^i,\ldots,x_p^{n-1},z)f(\widetilde{\alpha}^{-1}(X_1),\ldots,\widetilde{\alpha}^{-1}(X_{p-1}),\alpha^{-1}(x_p^i)),$$

for all $X_i=(x_i^1,x_i^2,\ldots,x_i^{n-1})\in\wedge^{n-1}\mathfrak{g}$ and $z\in\mathfrak{g}$.

Theorem 3.1. *With the above notations, $\delta^2=0$. Thus, we have a well-defined cohomological complex $\big(C^\bullet(\mathfrak{g};V)=\oplus_{p\geq 1}C^p(\mathfrak{g};V),\delta\big)$.*

Proof. It follows by straightforward computations. We omit details. ■

A p-cochain $f\in C^p(\mathfrak{g};V)$ is called a p-**cocycle** if $\delta(f)=0$. A p-cochain $f\in C^p(\mathfrak{g};V)$ is called a p-**coboundary** if $f=\delta(g)$ for some $g\in C^{p-1}(\mathfrak{g};V)$. Denote by $\mathcal{Z}^p(\mathfrak{g};V)$ and $\mathcal{B}^p(\mathfrak{g};V)$ the sets of p-cocycles and p-coboundaries respectively. We define the pth cohomology group $\mathcal{H}^p(\mathfrak{g};V)$ to be $\mathcal{Z}^p(\mathfrak{g};V)/\mathcal{B}^p(\mathfrak{g};V)$.

3.2. Derivations of n-Hom-Lie algebras

Generalizing the notion of a derivation of a Hom-Lie algebra given in [20], we give the notion of a derivation of an n-Hom-Lie algebra as follows.

Definition 3.2. A **derivation** of an n-Hom-Lie algebra $(\mathfrak{g},[\cdot,\ldots,\cdot]_\mathfrak{g},\alpha)$ is a linear map $D\in\mathfrak{gl}(\mathfrak{g})$, such that for all $x_1,\ldots,x_n\in\mathfrak{g}$, the following equality holds:

$$D[x_1,\ldots,x_n]_\mathfrak{g}=\sum_{i=1}^{n}[\alpha(x_1),\ldots,\alpha(x_{i-1}),(\mathrm{Ad}_{\alpha^{-1}}D)(x_i),\alpha(x_{i+1}),\ldots,\alpha(x_n)]_\mathfrak{g}. \tag{10}$$

Denote the set of derivations by $\mathrm{Der}(\mathfrak{g})$.

Lemma 3.3. *For all $D\in\mathrm{Der}(\mathfrak{g})$, we have $\mathrm{Ad}_\alpha D\in\mathrm{Der}(\mathfrak{g})$.*

Proof. For all $x_1,\ldots,x_n\in\mathfrak{g}$, we have

$$\begin{aligned}\mathrm{Ad}_\alpha D[x_1,\ldots,x_n]_\mathfrak{g}&=\alpha D[\alpha^{-1}(x_1),\ldots,\alpha^{-1}(x_n)]_\mathfrak{g}\\&=\alpha\Big(\sum_{i=1}^{n}[x_1,\ldots,x_{i-1},(\mathrm{Ad}_{\alpha^{-1}}D)(\alpha^{-1}(x_i)),x_{i+1},\ldots,x_n]_\mathfrak{g}\Big)\\&=\sum_{i=1}^{n}[\alpha(x_1),\ldots,\alpha(x_{i-1}),(\mathrm{Ad}_{\alpha^{-1}}\mathrm{Ad}_\alpha D)(x_i),\alpha(x_{i+1}),\ldots,\alpha(x_n)]_\mathfrak{g},\end{aligned}$$

which implies that $\mathrm{Ad}_\alpha D$ is also a derivation. ■

Lemma 3.4. *For all $D,D'\in\mathrm{Der}(\mathfrak{g})$, we have $[D,D']_\alpha\in\mathrm{Der}(\mathfrak{g})$.*

Proof. For all $x_1,\ldots,x_n\in\mathfrak{g}$, by (2) and (10) we have

$$\begin{aligned}&[D,D']_\alpha([x_1,\ldots,x_n]_\mathfrak{g})\\&=(\alpha\circ D\circ\alpha^{-1}\circ D'\circ\alpha^{-1}-\alpha\circ D'\circ\alpha^{-1}\circ D\circ\alpha^{-1})[x_1,\ldots,x_n]_\mathfrak{g}\\&=(\alpha\circ D\circ\alpha^{-1})\sum_{i=1}^{n}[x_1,\ldots,x_{i-1},(\mathrm{Ad}_{\alpha^{-1}}D')\alpha^{-1}x_i,x_{i+1},\ldots,x_n]_\mathfrak{g}\\&\quad-(\alpha\circ D'\circ\alpha^{-1})\sum_{i=1}^{n}[x_1,\ldots,x_{i-1},(\mathrm{Ad}_{\alpha^{-1}}D)\alpha^{-1}x_i,x_{i+1},\ldots,x_n]_\mathfrak{g}\\&=(\alpha\circ D)\sum_{i=1}^{n}[\alpha^{-1}x_1,\ldots,\alpha^{-1}x_{i-1},\alpha^{-2}D'x_i,\alpha^{-1}x_{i+1},\ldots,\alpha^{-1}x_n]_\mathfrak{g}\\&\quad-(\alpha\circ D')\sum_{i=1}^{n}[\alpha^{-1}x_1,\ldots,\alpha^{-1}x_{i-1},\alpha^{-2}Dx_i,\alpha^{-1}x_{i+1},\ldots,\alpha^{-1}x_n]_\mathfrak{g}\end{aligned}$$

$$
\begin{aligned}
&= \alpha \sum_{i<j}[x_1, \ldots, x_{i-1}, \alpha^{-1}D'x_i, x_{i+1}, \ldots, x_{j-1}, (\mathsf{Ad}_{\alpha^{-1}}D)\alpha^{-1}x_j, x_{j+1}, \ldots, x_n]_{\mathfrak{g}} \\
&\quad + \alpha \sum_{i=1}^{n}[x_1, \ldots, x_{i-1}, (\mathsf{Ad}_{\alpha^{-1}}D)\alpha^{-2}D'x_i, x_{i+1}, \ldots, x_n]_{\mathfrak{g}} \\
&\quad + \alpha \sum_{i>j}[x_1, \ldots, x_{j-1}, (\mathsf{Ad}_{\alpha^{-1}}D)\alpha^{-1}x_j, x_{j+1}, \ldots, x_{i-1}, \alpha^{-1}D'x_i, x_{i+1}, \ldots, x_n]_{\mathfrak{g}} \\
&\quad - \alpha \sum_{i<j}[x_1, \ldots, x_{i-1}, \alpha^{-1}Dx_i, x_{i+1}, \ldots, x_{j-1}, (\mathsf{Ad}_{\alpha^{-1}}D')\alpha^{-1}x_j, x_{j+1}, \ldots, x_n]_{\mathfrak{g}} \\
&\quad - \alpha \sum_{i=1}^{n}[x_1, \ldots, x_{i-1}, (\mathsf{Ad}_{\alpha^{-1}}D')\alpha^{-2}Dx_i, x_{i+1}, \ldots, x_n]_{\mathfrak{g}} \\
&\quad - \alpha \sum_{i>j}[x_1, \ldots, x_{j-1}, (\mathsf{Ad}_{\alpha^{-1}}D')\alpha^{-1}x_j, x_{j+1}, \ldots, x_{i-1}, \alpha^{-1}Dx_i, x_{i+1}, \ldots, x_n]_{\mathfrak{g}} \\
&= \sum_{i=1}^{n}[\alpha(x_1), \ldots, \alpha(x_{i-1}), (\mathsf{Ad}_{\alpha^{-1}}[D, D']_\alpha)(x_i), \alpha(x_{i+1}), \ldots, \alpha(x_n)]_{\mathfrak{g}}.
\end{aligned}
$$

Therefore, we have $[D, D']_\alpha \in \mathsf{Der}(\mathfrak{g})$. ∎

Proposition 3.5. *With the above notations,* $(\mathsf{Der}(\mathfrak{g}), [\cdot,\cdot]_\alpha, \mathsf{Ad}_\alpha)$ *is a Hom-Lie algebra, which is a subalgebra of the Hom-Lie algebra* $(\mathfrak{gl}(\mathfrak{g}), [\cdot,\cdot]_\alpha, \mathsf{Ad}_\alpha)$ *given in Proposition 2.2.*

Proof. By Lemmas 3.3 and 3.4, $(\mathsf{Der}(\mathfrak{g}), [\cdot,\cdot]_\alpha, \mathsf{Ad}_\alpha)$ is a Hom-Lie subalgebra of the Hom-Lie algebra $(\mathfrak{gl}(\mathfrak{g}), [\cdot,\cdot]_\alpha, \mathsf{Ad}_\alpha)$. ∎

Proposition 3.6. *Let* $(\mathfrak{g}, [\cdot, \ldots, \cdot]_{\mathfrak{g}}, \alpha)$ *be an n-Hom-Lie algebra.* $f \in C^1(\mathfrak{g}; \mathfrak{g})$ *is a 1-cocycle with the coefficient in the adjoint representation if and only if f is a derivation on* $\mathfrak{g}$.

Proof. By straightforward computation, for $x_1, \ldots, x_n \in \mathfrak{g}$, we have

$$
\begin{aligned}
&\quad \delta(f)(x_1 \wedge \cdots \wedge x_{n-1}, x_n) \\
&= -\alpha f([\alpha^{-2}(x_1), \ldots, \alpha^{-2}(x_{n-1}), \alpha^{-2}(x_n)]_{\mathfrak{g}}) + [x_1, \ldots, x_{n-1}, f(\alpha^{-1}x_n)]_{\mathfrak{g}} \\
&\quad + \sum_{i=1}^{n-1}(-1)^{n-i}[x_1, \ldots, x_{i-1}, \hat{x}_i, x_{i+1}, \ldots, x_n, f(\alpha^{-1}(x_i))]_{\mathfrak{g}}.
\end{aligned}
$$

Thus, $\delta(f) = 0$ if and only if

$$
\begin{aligned}
&\quad f([\alpha^{-2}(x_1), \ldots, \alpha^{-2}(x_{n-1}), \alpha^{-2}(x_n)]_{\mathfrak{g}}) \\
&= \sum_{i=1}^{n}[\alpha^{-1}(x_1), \ldots, \alpha^{-1}(x_{i-1}), \alpha^{-1}(f(\alpha^{-1}(x_i))), \alpha^{-1}(x_{i+1}), \ldots, \alpha^{-1}(x_n)]_{\mathfrak{g}}.
\end{aligned}
$$

Thus, f is a derivation. The proof is finished. ∎

Remark 3.7. Recall that a derivation of a Lie algebra is exactly a 1-cocycle with the coefficient in the adjoint representation. Thus, the above proposition justifies our definition of a derivation of an n-Hom-Lie algebra.

For all $Y \in \wedge^{n-1}\mathfrak{g}$, ad_Y is a derivation of the n-Hom-Lie algebra $(\mathfrak{g}, [\cdot, \ldots, \cdot]_{\mathfrak{g}}, \alpha)$, which we call an **inner derivation**. This follows from

$$
\begin{aligned}
\mathsf{ad}_Y[x_1, \ldots, x_n]_{\mathfrak{g}} &= [y_1, \ldots, y_{n-1}, [x_1, \ldots, x_n]_{\mathfrak{g}}]_{\mathfrak{g}} \\
&= [\alpha(\alpha^{-1}y_1), \ldots, \alpha(\alpha^{-1}y_{n-1}), [x_1, \ldots, x_n]_{\mathfrak{g}}]_{\mathfrak{g}} \\
&= \sum_{i=1}^{n}[\alpha(x_1), \ldots, \alpha(x_{i-1}), [\alpha^{-1}y_1, \ldots, \alpha^{-1}y_{n-1}, x_i]_{\mathfrak{g}}, \alpha(x_{i+1}), \ldots, \alpha(x_n)]_{\mathfrak{g}} \\
&= \sum_{i=1}^{n}[\alpha(x_1), \ldots, \alpha(x_{i-1}), (\alpha^{-1} \circ \mathsf{ad}_Y \circ \alpha)(x_i), \alpha(x_{i+1}), \ldots, \alpha(x_n)]_{\mathfrak{g}}.
\end{aligned}
$$

Denote by $\mathsf{Inn}(\mathfrak{g})$ the set of inner derivations of the n-Hom-Lie algebra $(\mathfrak{g}, [\cdot, \ldots, \cdot]_{\mathfrak{g}}, \alpha)$, i.e.

$$
\mathsf{Inn}(\mathfrak{g}) = \{\mathsf{ad}_Y \mid Y \in \wedge^{n-1}\mathfrak{g}\}. \tag{11}
$$

Lemma 3.8. *Let $(\mathfrak{g}, [\cdot, \dots, \cdot]_{\mathfrak{g}}, \alpha)$ be an n-Hom-Lie algebra. For all $X = x_1 \wedge \cdots \wedge x_{n-1} \in \wedge^{n-1}\mathfrak{g}$ and $D \in \mathsf{Der}(\mathfrak{g})$, we have*

$$\mathsf{Ad}_\alpha \mathsf{ad}_X = \mathsf{ad}_{\widetilde{\alpha}(X)}, \quad [D, \mathsf{ad}_X]_\alpha = \mathsf{ad}_{\sum_{i=1}^{n-1}\alpha(x_1)\wedge\cdots\wedge D(x_i)\wedge\cdots\alpha(x_{n-1})}.$$

Therefore, $\mathsf{Inn}(\mathfrak{g})$ *is an ideal of the Hom-Lie algebra* $(\mathsf{Der}(\mathfrak{g}), [\cdot, \cdot]_\alpha, \mathsf{Ad}_\alpha)$.

Proof. For all $y \in \mathfrak{g}$, we have

$$\begin{aligned}(\mathsf{Ad}_\alpha \mathsf{ad}_X)(y) &= (\alpha \circ \mathsf{ad}_X \circ \alpha^{-1})(y) = \alpha[x_1, \dots, x_{n-1}, \alpha^{-1}(y)]_{\mathfrak{g}} = [\alpha(x_1), \dots, \alpha(x_{n-1}), y]_{\mathfrak{g}}\\ &= \mathsf{ad}_{\widetilde{\alpha}(X)}(y).\end{aligned}$$

By (10), we have

$$\begin{aligned}& [D, \mathsf{ad}_X]_\alpha(y)\\ =& (\alpha \circ D \circ \alpha^{-1} \circ \mathsf{ad}_X \circ \alpha^{-1})(y) - (\alpha \circ \mathsf{ad}_X \circ \alpha^{-1} \circ D \circ \alpha^{-1})(y)\\ =& \alpha(D[\alpha^{-1}(x_1), \dots, \alpha^{-1}(x_{n-1}), \alpha^{-2}(y)]_{\mathfrak{g}}) - \alpha[x_1, \dots, x_{n-1}, \alpha^{-1}D(\alpha^{-1}(y))]_{\mathfrak{g}}\\ =& \alpha\Big([x_1, \dots, x_{n-1}, (\mathsf{Ad}_{\alpha^{-1}}D)(\alpha^{-2}(y))]_{\mathfrak{g}} + \sum_{i=1}^{n-1}[x_1, \dots, (\mathsf{Ad}_{\alpha^{-1}}D)(\alpha^{-1}(x_i)), \dots, x_{n-1}, \alpha^{-1}(y)]_{\mathfrak{g}}\Big)\\ & - [\alpha(x_1), \dots, \alpha(x_{n-1}), D(\alpha^{-1}(y))]_{\mathfrak{g}}\\ =& \mathsf{ad}_{\sum_{i=1}^{n-1}\alpha(x_1)\wedge\cdots\wedge D(x_i)\wedge\cdots\alpha(x_{n-1})} y.\end{aligned}$$

The proof is finished. ■

3.3. Dual representations of n-Hom-Lie algebras

In this subsection, we study dual representation of an n-Hom-Lie algebra. See [8] for more details about dual representations of Hom-Lie algebras.

Let $(V; \rho, \beta)$ be a representation of an n-Hom-Lie algebra $(\mathfrak{g}, [\cdot, \dots, \cdot]_{\mathfrak{g}}, \alpha)$, where β is invertible, i.e. $\beta \in GL(V)$. Define $\rho^* : \wedge^{n-1}\mathfrak{g} \to \mathfrak{gl}(V^*)$ as usual by

$$\langle \rho^*(X)(\xi), u\rangle = -\langle \xi, \rho(X)(u)\rangle, \quad \forall X \in \wedge^{n-1}\mathfrak{g},\ u \in V,\ \xi \in V^*. \tag{12}$$

Then we define $\rho^\star : \wedge^{n-1}\mathfrak{g} \to \mathfrak{gl}(V^*)$ by

$$\rho^\star(X)(\xi) = \rho^*(\widetilde{\alpha}(X))((\beta^{-2})^*(\xi)). \tag{13}$$

Theorem 3.9. *Let $(V; \rho, \beta)$ be a representation of an n-Hom-Lie algebra $(\mathfrak{g}, [\cdot, \dots, \cdot]_{\mathfrak{g}}, \alpha)$, where β is invertible. Then $(V^*; \rho^\star, (\beta^{-1})^*)$ is also a representation of $(\mathfrak{g}, [\cdot, \dots, \cdot]_{\mathfrak{g}}, \alpha)$, which we call the* ***dual representation*** *of the representation $(V; \rho, \beta)$.*

Proof. By $\rho(\widetilde{\alpha}(X)) \circ \beta = \beta \circ \rho(X)$, we have

$$\begin{aligned}\langle \rho^\star(\widetilde{\alpha}(X))((\beta^{-1})^*(\xi)), u\rangle &= \langle \rho^*(\widetilde{\alpha}^2(X))((\beta^{-3})^*(\xi)), u\rangle = -\langle (\beta^{-3})^*(\xi), \rho(\widetilde{\alpha}^2(X))(u)\rangle\\ &= -\langle (\beta^{-2})^*(\xi), \beta^{-1}\rho(\widetilde{\alpha}^2(X))(u)\rangle = -\langle (\beta^{-2})^*(\xi), \rho(\widetilde{\alpha}(X))(\beta^{-1}(u))\rangle\\ &= \langle \rho^*(\widetilde{\alpha}(X))((\beta^{-2})^*(\xi)), \beta^{-1}(u)\rangle = \langle \rho^\star(X)(\xi), \beta^{-1}(u)\rangle\\ &= \langle (\beta^{-1})^*\rho^\star(X)(\xi), u\rangle,\end{aligned}$$

which implies that

$$\rho^\star(\widetilde{\alpha}(X)) \circ (\beta^{-1})^* = (\beta^{-1})^* \circ \rho^\star(X).$$

By straightforward computation, we have

$$\begin{aligned}\langle \rho^\star(\widetilde{\alpha}(X))\big(\rho^\star(Y)(\xi)\big), u\rangle &= \langle \rho^*(\widetilde{\alpha}^2(X))\big((\beta^{-2})^*\rho^*(\widetilde{\alpha}(Y))((\beta^{-2})^*(\xi))\big), u\rangle\\ &= -\langle (\beta^{-2})^*\rho^*(\widetilde{\alpha}(Y))((\beta^{-2})^*(\xi)), \rho(\widetilde{\alpha}^2(X))(u)\rangle\\ &= -\langle \rho^*(\widetilde{\alpha}(Y))((\beta^{-2})^*(\xi)), \beta^{-2}\rho(\widetilde{\alpha}^2(X))(u)\rangle\\ &= \langle (\beta^{-2})^*(\xi), \rho(\widetilde{\alpha}(Y))\big(\beta^{-2}\rho(\widetilde{\alpha}^2(X))(u)\big)\rangle\\ &= \langle (\beta^{-2})^*(\xi), \rho(\widetilde{\alpha}(Y))\big(\rho(X)(\beta^{-2}(u))\big)\rangle.\end{aligned}$$

Therefore, we have

$$\langle\big(\rho^\star(\widetilde{\alpha}(X)) \circ \rho^\star(Y) - \rho^\star(\widetilde{\alpha}(Y)) \circ \rho^\star(X)\big)(\xi), u\rangle$$

$$
\begin{aligned}
&= \langle(\beta^{-2})^*(\xi), \rho(\widetilde{\alpha}(Y))\big(\rho(X)(\beta^{-2}(u))\big) - \rho(\widetilde{\alpha}(X))\big(\rho(Y)(\beta^{-2}(u))\big)\rangle \\
&= \langle(\beta^{-2})^*(\xi), -\big(\rho(\widetilde{\alpha}(X)) \circ \rho(Y) - \rho(\widetilde{\alpha}(Y)) \circ \rho(X)\big)(\beta^{-2}(u))\rangle \\
&= \langle(\beta^{-2})^*(\xi), -\sum_{i=1}^{n-1} \rho(\alpha(y_1), \dots, \alpha(y_{i-1}), \mathrm{ad}_X(y_i), \alpha(y_{i+1}), \cdots, \alpha(y_{n-1}))(\beta^{-1}(u))\rangle \\
&= -\sum_{i=1}^{n-1} \langle(\beta^{-3})^*(\xi), \beta\rho(\alpha(y_1), \dots, \alpha(y_{i-1}), \mathrm{ad}_X(y_i), \alpha(y_{i+1}), \cdots, \alpha(y_{n-1}))(\beta^{-1}(u))\rangle \\
&= -\sum_{i=1}^{n-1} \langle(\beta^{-3})^*(\xi), \rho(\alpha^2(y_1), \dots, \alpha^2(y_{i-1}), \alpha(\mathrm{ad}_X(y_i)), \alpha^2(y_{i+1}), \cdots, \alpha^2(y_{n-1}))(u)\rangle \\
&= \sum_{i=1}^{n-1} \langle\rho^*(\alpha^2(y_1), \dots, \alpha^2(y_{i-1}), \alpha(\mathrm{ad}_X(y_i)), \alpha^2(y_{i+1}), \cdots, \alpha^2(y_{n-1}))((\beta^{-3})^*(\xi)), u\rangle \\
&= \langle\sum_{i=1}^{n-1} \rho^\star(\alpha(y_1), \dots, \alpha(y_{i-1}), \mathrm{ad}_X(y_i), \alpha(y_{i+1}), \dots, \alpha(y_{n-1}))((\beta^{-1})^*(\xi)), u\rangle,
\end{aligned}
$$

which implies that

$$
\rho^\star(\widetilde{\alpha}(X)) \circ \rho^\star(Y) - \rho^\star(\widetilde{\alpha}(Y)) \circ \rho^\star(X) = \sum_{i=1}^{n-1} \rho^\star([X, Y]_F) \circ (\beta^{-1})^*.
$$

Finally, by straightforward computations, we can obtain

$$
\begin{aligned}
&\rho^\star(\alpha(x_1), \dots, \alpha(x_{n-2}), [y_1, \dots, y_n]_{\mathfrak{g}}) \circ (\beta^{-1})^* \\
=& \sum_{i=1}^{n} (-1)^{n-i} \rho^\star(\alpha(y_1), \dots, \hat{y}_i, \dots, \alpha(y_n)) \circ \rho^\star(x_1, \dots, x_{n-2}, y_i).
\end{aligned}
$$

Therefore, $(V^*; \rho^\star, (\beta^{-1})^*)$ is a representation of $(\mathfrak{g}, [\cdot, \dots, \cdot]_{\mathfrak{g}}, \alpha)$. ■

Remark 3.10. By straightforward computation, one can deduce that the usual ρ^* defined by (12) is not a representation anymore. In particular, for the adjoint representation ad, the dual map ad^* is not a representation in general. On the other hand, to solve this problem, the authors in [6] add some strong conditions to make ρ^* still being a representation. Here, we see that there is a natural definition of $\rho^\star$ such that it is a representation. This makes our definition of $\rho^\star$ nontrivial. Furthermore, the definition of $\rho^\star$ is a generalization of the one given in [8] for Hom-Lie algebras.

4. $(n-1)$-order deformations, Hom-Nijenhuis operators and Hom-$\mathcal{O}$-operators of an n-Hom-Lie algebra

4.1. $(n-1)$-order deformations of an n-Hom-Lie algebra

Let $(\mathfrak{g}, [\cdot, \dots, \cdot]_{\mathfrak{g}}, \alpha)$ be an n-Hom-Lie algebra. For convenience, we denote by $\omega_0(x_1, \dots, x_n) = [x_1, \dots, x_n]_{\mathfrak{g}}$. Let $\omega_i : \wedge^n \mathfrak{g} \longrightarrow \mathfrak{g}$, $1 \le i \le n-1$ be skew-symmetric multilinear maps. Consider a λ-parametrized family of linear operations:

$$
\omega_\lambda(x_1, \dots, x_n) = \sum_{i=0}^{n-1} \lambda^i \omega_i(x_1, \dots, x_n). \tag{14}
$$

Here $\lambda \in \mathbb{K}$, where $\mathbb{K}$ is the base field. If all $(\mathfrak{g}, \omega_\lambda, \alpha)$ are n-Hom-Lie algebras, we say that $\omega_1, \dots, \omega_{n-1}$ generate **an** $(n-1)$-**order deformation** of the n-Hom-Lie algebra $(\mathfrak{g}, \omega_0, \alpha)$. We also denote by $[x_1, \dots, x_n]_\lambda = \omega_\lambda(x_1, \dots, x_n)$.

Proposition 4.1. *With the above notations, $\omega_1, \dots, \omega_{n-1}$ generate an $(n-1)$-order deformation of the n-Hom-Lie algebra $(\mathfrak{g}, \omega_0, \alpha)$ if and only if for all $i, j = 1, 2, \dots, n-1$ and $k = 1, 2, \dots, 2n-2$ the following conditions are satisfied:*

$$
\omega_i \circ \alpha^{\otimes n} = \alpha \circ \omega_i, \tag{15}
$$

$$
\sum_{i+j=k} \omega_i \circ \omega_j = 0. \tag{16}
$$

Here $\omega_i \circ \omega_j : \wedge^{n-1}\mathfrak{g} \otimes \wedge^{n-1}\mathfrak{g} \wedge \mathfrak{g} \longrightarrow \mathfrak{g}$ is defined by

$$
(\omega_i \circ \omega_j)(X, Y, z) = \omega_i(\omega_j(X, \cdot) * Y, \alpha(z)) - \omega_i(\widetilde{\alpha}(X), \omega_j(Y, z)) + \omega_i(\widetilde{\alpha}(Y), \omega_j(X, z)),
$$

where $\omega_j(X,\cdot)*Y\in\wedge^{n-1}\mathfrak{g}$ *is given by*

$$\omega_j(X,\cdot)*Y=\sum_{l=1}^{n-1}\alpha(y_1)\wedge\cdots\wedge\alpha(y_{l-1})\wedge\omega_j(X,y_l)\wedge\alpha(y_{l+1})\wedge\cdots\wedge\alpha(y_{n-1}). \tag{17}$$

Proof. $(\mathfrak{g},\omega_\lambda,\alpha)$ are n-Hom-Lie algebra structures if and only if

$$\omega_\lambda\circ\alpha^{\otimes n}=\alpha\circ\omega_\lambda, \tag{18}$$

$$\omega_\lambda(\widetilde{\alpha}(X),\omega_\lambda(Y,z))=\omega_\lambda(\omega_\lambda(X,\cdot)*Y,\alpha(z))+\omega_\lambda(\widetilde{\alpha}(Y),\omega_\lambda(X,z)). \tag{19}$$

By (18), we have

$$\omega_i\circ\alpha^{\otimes n}=\alpha\circ\omega_i.$$

Expanding the equations in (19) and collecting coefficients of λ^k, we see that (19) is equivalent to the system of equations

$$\sum_{i+j=k}\omega_i(\widetilde{\alpha}(X),\omega_j(Y,z))=\sum_{i+j=k}\omega_i(\omega_j(X,\cdot)*Y,\alpha(z))+\sum_{i+j=k}\omega_i(\widetilde{\alpha}(Y),\omega_j(X,z)).$$

Thus, we have

$$\sum_{i+j=k}\omega_i(\omega_j(X,\cdot)*Y,\alpha(z))-\omega_i(\widetilde{\alpha}(X),\omega_j(Y,z))+\omega_i(\widetilde{\alpha}(Y),\omega_j(X,z)),$$

which finishes the proof. ■

Corollary 4.2. *If* $\omega_1,\dots,\omega_{n-1}$ *generate an* $(n-1)$*-order deformation of the n-Hom-Lie algebra* $(\mathfrak{g},\omega_0,\alpha)$*, then* ω_1 *is a* 2*-cocycle of the n-Hom-Lie algebra* $(\mathfrak{g},\omega_0,\alpha)$ *with the coefficients in the adjoint representation.*

Proof. By (16), let k=1, we deduce that

$$\omega_0\circ\omega_1+\omega_1\circ\omega_0=0,$$

which is equivalent to that ω_1 is a 2-cocycle. We omit details. ■

Corollary 4.3. *If* $\omega_1,\dots,\omega_{n-1}$ *generate an* $(n-1)$*-order deformation of the n-Hom-Lie algebra* $(\mathfrak{g},\omega_0,\alpha)$*, then* $(\mathfrak{g},\omega_{n-1},\alpha)$ *is an n-Hom-Lie algebra.*

Proof. By (15), let $i=n-1$, we deduce that

$$\omega_{n-1}\circ\alpha^{\otimes n}=\alpha\circ\omega_{n-1},$$

and by (16), let $k=2n-2$, we deduce that

$$\omega_{n-1}\circ\omega_{n-1}=0,$$

which is equivalent to that $(\mathfrak{g},\omega_{n-1},\alpha)$ is an n-Hom-Lie algebra. ■

4.2. Hom-Nijenhuis operators and Hom-$\mathcal{O}$-operators of an n-Hom-Lie algebra

In this subsection, we study trivial $(n-1)$-order deformations of an n-Hom-Lie algebra and introduce the notion of a Hom-Nijenhuis operator of an n-Hom-Lie algebra, which could generate a trivial deformation. Then we give the relation between Hom-$\mathcal{O}$-operators and Hom-Nijenhuis operators.

Definition 4.4. An $(n-1)$-order deformation is said to be **trivial** if there exists a linear map $N:\mathfrak{g}\longrightarrow\mathfrak{g}$ such that for all λ, $T_\lambda=\alpha+\lambda N$ satisfies

$$T_\lambda\circ\alpha=\alpha\circ T_\lambda, \tag{20}$$

$$T_\lambda[x_1,\dots,x_n]_\lambda=[T_\lambda x_1,\dots,T_\lambda x_n]_{\mathfrak{g}},\quad\forall x_1,\dots,x_n\in\mathfrak{g}. \tag{21}$$

Eq. (20) equals

$$N\circ\alpha=\alpha\circ N. \tag{22}$$

The left hand side of Eq. (21) equals

$$\alpha[x_1,\dots,x_n]_{\mathfrak g}+\lambda\big(\alpha\omega_1(x_1,\dots,x_n)+N[x_1,\dots,x_n]_{\mathfrak g}\big)$$
$$+\sum_{i=2}^{n-1}\lambda^i\big(\alpha\omega_i(x_1,\dots,x_n)+N\omega_{i-1}(x_1,\dots,x_n)\big)+\lambda^nN\omega_{n-1}(x_1,\dots,x_n).$$

The right hand side of Eq. (21) equals

$$[\alpha(x_1),\dots,\alpha(x_n)]_{\mathfrak g}+\lambda\sum_{i=1}^{n}[\alpha(x_1),\dots,Nx_i,\dots,\alpha(x_n)]_{\mathfrak g}$$
$$+\lambda^2\sum_{l_1<l_2}[\alpha(x_1),\dots,Nx_{l_1},\dots,Nx_{l_2},\dots,\alpha(x_n)]_{\mathfrak g}+\cdots+\lambda^n[Nx_1,\dots,Nx_n]_{\mathfrak g}.$$

Therefore, by Eq. (21), we have

$$\alpha\omega_1(x_1,\dots,x_n)+N[x_1,\dots,x_n]_{\mathfrak g}=\sum_{i=1}^{n}[\alpha(x_1),\dots,Nx_i,\dots,\alpha(x_n)]_{\mathfrak g},\tag{23}$$
$$N\omega_{n-1}(x_1,\dots,x_n)=[Nx_1,\dots,Nx_n]_{\mathfrak g},\tag{24}$$

and

$$\begin{aligned}&\alpha\omega_i(x_1,\dots,x_n)+N\omega_{i-1}(x_1,\dots,x_n)\\=&\sum_{l_1<l_2\cdots<l_i}[\alpha(x_1),\dots,Nx_{l_1},\dots,Nx_{l_2},\dots,Nx_{l_i},\dots,\alpha(x_n)]_{\mathfrak g},\end{aligned}\tag{25}$$

for all $2\le i\le n-1$.

Let $(\mathfrak g,[\cdot,\dots,\cdot]_{\mathfrak g},\alpha)$ be an n-Hom-Lie algebra, and $N:\mathfrak g\longrightarrow\mathfrak g$ a linear map. Define an n-ary bracket $[\cdot,\dots,\cdot]_N^1:\wedge^n\mathfrak g\longrightarrow\mathfrak g$ by

$$[x_1,\dots,x_n]_N^1=\sum_{i=1}^{n}[x_1,\dots,N\alpha^{-1}(x_i),\dots,x_n]_{\mathfrak g}-N[\alpha^{-1}(x_1),\dots,\alpha^{-1}(x_n)]_{\mathfrak g}.\tag{26}$$

Then we define n-ary brackets $[\cdot,\dots,\cdot]_N^i:\wedge^n\mathfrak g\longrightarrow\mathfrak g,(2\le i\le n-1)$ via induction by

$$\begin{aligned}[x_1,\dots,x_n]_N^i=&\sum_{l_1<l_2\cdots<l_i}[x_1,\dots,N\alpha^{-1}(x_{l_1}),\dots,N\alpha^{-1}(x_{l_2}),\dots,N\alpha^{-1}(x_{l_i}),\dots,x_n]_{\mathfrak g}\\&-N[\alpha^{-1}(x_1),\dots,\alpha^{-1}(x_n)]_N^{i-1}.\end{aligned}\tag{27}$$

Definition 4.5. Let $(\mathfrak g,[\cdot,\dots,\cdot]_{\mathfrak g},\alpha)$ be an n-Hom-Lie algebra. A linear map $N:\mathfrak g\longrightarrow\mathfrak g$ is called a **Hom-Nijenhuis operator** if

$$N\circ\alpha=\alpha\circ N,\tag{28}$$
$$[Nx_1,\dots,Nx_n]_{\mathfrak g}=N([x_1,\dots,x_n]_N^{n-1}),\quad\forall x_1,\dots,x_n\in\mathfrak g.\tag{29}$$

Remark 4.6. When $n=2$, (29) reduces to

$$[Nx,Ny]_{\mathfrak g}=N[x,y]_N^1=N([x,N\alpha^{-1}(y)]_{\mathfrak g}+[N\alpha^{-1}(x),y]_{\mathfrak g}-N[\alpha^{-1}(x),\alpha^{-1}(y)]_{\mathfrak g}),$$

which is the same as the condition given in [7, Proposition 6.2]. Thus, the notion of a Hom-Nijenhuis operator of an n-Hom-Lie algebra is a natural generalization of the Hom-Nijenhuis operator of a Hom-Lie algebra given in [7].

Theorem 4.7. *Let N be a Hom-Nijenhuis operator of an n-Hom-Lie algebra $(\mathfrak g,[\cdot,\dots,\cdot]_{\mathfrak g},\alpha)$. Then an $(n-1)$-order deformation can be obtained by putting*

$$\omega_i(x_1,x_2,\dots,x_n)=[x_1,x_2,\dots,x_n]_N^i,\quad 1\le i\le n-1.\tag{30}$$

Moreover, this $(n-1)$-order deformation is trivial.

Lemma 4.8. *Let $(\mathfrak g,[\cdot,\dots,\cdot]_{\mathfrak g},\alpha)$ be an n-Hom-Lie algebra and $\mathfrak h$ a vector space with a linear automorphism $\gamma\in GL(\mathfrak h)$. If there exists an linear isomorphism $f:\mathfrak h\longrightarrow\mathfrak g$, such that $\alpha\circ f=f\circ\gamma$. We define an n-ary bracket $[\cdot,\dots,\cdot]'$ on $\mathfrak h$ by*

$$[u_1,u_2,\dots,u_n]'=f^{-1}[f(u_1),f(u_2),\dots,f(u_n)]_{\mathfrak g},\quad\forall\,u_i\in\mathfrak h,$$

then $(\mathfrak h,[\cdot,\dots,\cdot]',\gamma)$ is an n-Hom-Lie algebra.

Proof. It follows from straightforward computations. ■

The proof of Theorem 4.7. It is obvious that for a Hom-Nijenhuis operator N, the maps $\omega_1, \dots, \omega_{n-1}$ given by Eq. (30) satisfy Eqs. (23)–(25). Therefore, for any λ, T_λ satisfies

$$\begin{aligned} T_\lambda \circ \alpha &= \alpha \circ T_\lambda, \\ T_\lambda[x_1, x_2, \dots, x_n]_\lambda &= [T_\lambda x_1, T_\lambda x_2, \dots, T_\lambda x_n]_{\mathfrak{g}}, \quad \forall x_1, \dots, x_n \in \mathfrak{g}. \end{aligned}$$

For λ sufficiently small, we see that T_λ is an isomorphism between vector spaces. Thus, we have

$$[x_1, x_2, \dots, x_n]_\lambda = T_\lambda^{-1}[T_\lambda x_1, T_\lambda x_2, \dots, T_\lambda x_n]_{\mathfrak{g}}.$$

By Lemma 4.8, we deduce that $(\mathfrak{g}, [\cdot, \dots, \cdot]_\lambda, \alpha)$ is an n-Hom-Lie algebra, for λ sufficiently small. Thus, $\omega_1, \dots, \omega_{n-1}$ given by Eq. (30) satisfy the conditions (15)–(16) in Proposition 4.1. Therefore, $(\mathfrak{g}, [\cdot, \dots, \cdot]_\lambda, \alpha)$ is an n-Hom-Lie algebra for all λ, which means that $\omega_1, \dots, \omega_{n-1}$ given by Eq. (30) generate an $(n-1)$-order deformation. It is obvious that this $(n-1)$-order deformation is trivial. ■

Corollary 4.9. *Let N be a Hom-Nijenhuis operator of an n-Hom-Lie algebra $(\mathfrak{g}, [\cdot, \dots, \cdot]_{\mathfrak{g}}, \alpha)$. Then $(\mathfrak{g}, [\cdot, \dots, \cdot]_N^{n-1}, \alpha)$ is an n-Hom-Lie algebra, and N is a homomorphism from $(\mathfrak{g}, [\cdot, \dots, \cdot]_N^{n-1}, \alpha)$ to $(\mathfrak{g}, [\cdot, \dots, \cdot]_{\mathfrak{g}}, \alpha)$.*

In the sequel, we introduce the notion of a Hom-$\mathcal{O}$-operator associated to a representation of an n-Hom-Lie algebra and give the relation between Hom-$\mathcal{O}$-operators and Hom-Nijenhuis operators.

Definition 4.10. Let $(\mathfrak{g}, [\cdot, \dots, \cdot]_{\mathfrak{g}}, \alpha)$ be an n-Hom-Lie algebra and $(V; \rho, \beta)$ a representation. A linear map $T : V \to \mathfrak{g}$ is called a **Hom-$\mathcal{O}$-operator** if for all $v_1, \dots, v_n \in V$,

$$T \circ \beta = \alpha \circ T, \tag{31}$$

$$[Tv_1, \dots, Tv_n]_{\mathfrak{g}} = T\Big(\sum_{i=1}^{n}(-1)^{n-i}\rho\big(T\beta^{-1}(v_1), \dots, \widehat{T\beta^{-1}(v_i)}, \dots, T\beta^{-1}(v_n)\big)(v_i)\Big). \tag{32}$$

Proposition 4.11. *Let $(\mathfrak{g}, [\cdot, \dots, \cdot]_{\mathfrak{g}}, \alpha)$ be an n-Hom-Lie algebra and $(V; \rho, \beta)$ a representation. A linear map $T : V \to \mathfrak{g}$ is a Hom-$\mathcal{O}$-operator if and only if*

$$\bar{T} = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix} : \mathfrak{g} \oplus V \to \mathfrak{g} \oplus V$$

is a Hom-Nijenhuis operator acting on the semidirect product n-Hom-Lie algebra $\mathfrak{g} \ltimes V$.

Proof. First it is obvious that $\bar{T} \circ (\alpha + \beta) = (\alpha + \beta) \circ \bar{T}$ if and only if $T \circ \beta = \alpha \circ T$. Then for all $x_1, \dots, x_n \in \mathfrak{g}$, $v_1, \dots, v_n \in V$, we have

$$[\bar{T}(x_1 + v_1), \bar{T}(x_2 + v_2), \dots, \bar{T}(x_n + v_n)]_\rho = [Tv_1, Tv_2, \dots, Tv_n]_{\mathfrak{g}}.$$

On the other hand, since $\bar{T}^2 = 0$, we have

$$\begin{aligned} &\bar{T}[x_1 + v_1, x_2 + v_2, \dots, x_n + v_n]_{\bar{T}}^{n-1} \\ = &\bar{T}\Big(\sum_{l_1 < l_2 \dots < l_{n-1}} [\cdots, \bar{T}(\alpha^{-1}(x_{l_1}) + \beta^{-1}(v_{l_1})), \dots, \bar{T}(\alpha^{-1}(x_{l_{n-1}}) + \beta^{-1}(v_{l_{n-1}})), \dots]_\rho\Big) \\ = &\bar{T}\sum_{i=1}^{n}[T\beta^{-1}(v_1), \dots, T\beta^{-1}(v_{i-1}), x_i + v_i, T\beta^{-1}(v_{i+1}), \dots, T\beta^{-1}(v_n)]_\rho \\ = &T\sum_{i=1}^{n}(-1)^{n-i}\rho(T\beta^{-1}(v_1), \dots, T\beta^{-1}(v_{i-1}), \widehat{T\beta^{-1}(v_i)}, T\beta^{-1}(v_{i+1}), \dots, T\beta^{-1}(v_n))(v_i), \end{aligned}$$

which implies that $\bar{T}$ is a Hom-Nijenhuis operator if and only if T is a Hom-$\mathcal{O}$-operator. ■

5. Generalized derivation extensions of n-Hom-Lie algebras

In this section, we give a new approach to construct n-Hom-Lie algebras, which is called generalized derivation extensions of n-Hom-Lie algebras. First we give the notion of a generalized derivation of an n-Hom-Lie algebra.

Definition 5.1. Let $(\mathfrak{g}, [\cdot, \dots, \cdot]_{\mathfrak{g}}, \alpha)$ be an n-Hom-Lie algebra. A linear map $D : \wedge^{n-1}\mathfrak{g} \to \mathfrak{g}$ is called a **generalized derivation**, if for all $x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1} \in \mathfrak{g}$, the following conditions are satisfied:

(i) $\alpha \circ D = D \circ \widetilde{\alpha}$;

(ii)

$$
\begin{aligned}
&[\alpha(x_1), \dots, \alpha(x_{n-1}), D(y_1, \dots, y_{n-1})]_{\mathfrak{g}} - [\alpha(y_1), \dots, \alpha(y_{n-1}), D(x_1, \dots, x_{n-1})]_{\mathfrak{g}} \\
= &\sum_{i=1}^{n-1} D(\alpha(y_1), \dots, \alpha(y_{i-1}), [x_1, \dots, x_{n-1}, y_i]_{\mathfrak{g}}, \alpha(y_{i+1}), \dots, \alpha(y_{n-1}));
\end{aligned}
$$

(iii)

$$
\begin{aligned}
&D(\alpha(x_1), \dots, \alpha(x_{n-2}), [y_1, \dots, y_n]_{\mathfrak{g}}) \\
= &\sum_{i=1}^{n} [\alpha(y_1), \dots, \alpha(y_{i-1}), D(x_1, \dots, x_{n-2}, y_i), \alpha(y_{i+1}), \dots, \alpha(y_n)]_{\mathfrak{g}};
\end{aligned}
$$

(iv)

$$
\begin{aligned}
&D(\alpha(x_1), \dots, \alpha(x_{n-2}), D(y_1, \dots, y_{n-1})) \\
= &\sum_{i=1}^{n-1} D(\alpha(y_1), \dots, \alpha(y_{i-1}), D(x_1, \dots, x_{n-2}, y_i), \alpha(y_{i+1}), \dots, \alpha(y_{n-1})).
\end{aligned}
$$

Remark 5.2. Let $D : \wedge^{n-1}\mathfrak{g} \to \mathfrak{g}$ be a generalized derivation on $(\mathfrak{g}, [\cdot, \dots, \cdot]_{\mathfrak{g}}, \alpha)$. By Conditions (i) and (iv) in Definition 5.1, D defines an $(n-1)$-Hom-Lie algebra structure on the vector space $\mathfrak{g}$.

For all $x \in \mathfrak{g}$, satisfying $\alpha(x) = x$, define $\mathfrak{ad}_x : \wedge^{n-1}\mathfrak{g} \to \mathfrak{g}$ by

$$
\mathfrak{ad}_x(y_1, \dots, y_{n-1}) = [x, y_1, \dots, y_{n-1}]_{\mathfrak{g}}.
$$

Then we have

Lemma 5.3. *For all $x \in \mathfrak{g}$, satisfying $\alpha(x) = x$, $\mathfrak{ad}_x$ is a generalized derivation on the n-Hom-Lie algebra $(\mathfrak{g}, [\cdot, \dots, \cdot]_{\mathfrak{g}}, \alpha)$, which is called an inner generalized derivation.*

Proof. By $\alpha(x) = x$ we have

$$
\begin{aligned}
(\alpha \circ \mathfrak{ad}_x)(y_1, \dots, y_{n-1}) &= [\alpha(x), \alpha(y_1), \dots, \alpha(y_{n-1})]_{\mathfrak{g}} \\
&= [x, \alpha(y_1), \dots, \alpha(y_{n-1})]_{\mathfrak{g}} \\
&= (\mathfrak{ad}_x \circ \widetilde{\alpha})(y_1, \dots, y_{n-1}).
\end{aligned}
$$

Thus, Condition (i) in Definition 5.1 holds.

By $\alpha(x) = x$ and the Hom Fundamental identity, we have

$$
\begin{aligned}
&[\alpha(x_1), \dots, \alpha(x_{n-1}), \mathfrak{ad}_x(y_1, \dots, y_{n-1})]_{\mathfrak{g}} - [\alpha(y_1), \dots, \alpha(y_{n-1}), \mathfrak{ad}_x(x_1, \dots, x_{n-1})]_{\mathfrak{g}} \\
= &[\alpha(x_1), \dots, \alpha(x_{n-1}), [x, y_1, \dots, y_{n-1}]_{\mathfrak{g}}]_{\mathfrak{g}} - [\alpha(y_1), \dots, \alpha(y_{n-1}), [x, x_1, \dots, x_{n-1}]_{\mathfrak{g}}]_{\mathfrak{g}} \\
= &\sum_{i=1}^{n-1} [\alpha(x), \alpha(y_1), \dots, \alpha(y_{i-1}), [x_1, \dots, x_{n-1}, y_i]_{\mathfrak{g}}, \alpha(y_{i+1}), \dots, \alpha(y_{n-1})] \\
= &\sum_{i=1}^{n-1} \mathfrak{ad}_x(\alpha(y_1), \dots, \alpha(y_{i-1}), [x_1, \dots, x_{n-1}, y_i]_{\mathfrak{g}}, \alpha(y_{i+1}), \dots, \alpha(y_{n-1})).
\end{aligned}
$$

Thus, Condition (ii) in Definition 5.1 holds.

Similarly, we have

$$
\begin{aligned}
&\mathfrak{ad}_x(\alpha(x_1), \dots, \alpha(x_{n-2}), [y_1, \dots, y_n]_{\mathfrak{g}}) \\
= &[\alpha(x), \alpha(x_1), \dots, \alpha(x_{n-2}), [y_1, \dots, y_n]_{\mathfrak{g}}]_{\mathfrak{g}} \\
= &\sum_{i=1}^{n} [\alpha(y_1), \dots, \alpha(y_{i-1}), [x, x_1, \dots, x_{n-2}, y_i]_{\mathfrak{g}}, \alpha(y_{i+1}), \dots, \alpha(y_n)]_{\mathfrak{g}} \\
= &\sum_{i=1}^{n} [\alpha(y_1), \dots, \alpha(y_{i-1}), \mathfrak{ad}_x(x_1, \dots, x_{n-2}, y_i), \alpha(y_{i+1}), \dots, \alpha(y_n)]_{\mathfrak{g}},
\end{aligned}
$$

and

$$
\mathfrak{ad}_x(\alpha(x_1), \dots, \alpha(x_{n-2}), \mathfrak{ad}_x(y_1, \dots, y_{n-1}))
$$

$$
\begin{aligned}
&= [\alpha(x), \alpha(x_1), \ldots, \alpha(x_{n-2}), [x, y_1, \ldots, y_{n-1}]_{\mathfrak{g}}]_{\mathfrak{g}} \\
&= \sum_{i=1}^{n-1} [x, \alpha(y_1), \ldots, \alpha(y_{i-1}), [x, x_1, \ldots, x_{n-2}, y_i]_{\mathfrak{g}}, \alpha(y_{i+1}), \ldots, \alpha(y_n)]_{\mathfrak{g}} \\
&= \sum_{i=1}^{n-1} \mathfrak{ad}_x(\alpha(y_1), \ldots, \alpha(y_{i-1}), \mathfrak{ad}_x(x_1, \ldots, x_{n-2}, y_i), \alpha(y_{i+1}), \ldots, \alpha(y_n)),
\end{aligned}
$$

which implies that Conditions (iii) and (iv) in Definition 5.1 hold. The proof is finished. ■

For any linear map $D : \wedge^{n-1}\mathfrak{g} \to \mathfrak{g}$, denote by $\mathbb{K}D$ the 1-dimensional vector space generated by D. On the direct sum $\mathfrak{g} \oplus \mathbb{K}D$, define a totally skew-symmetric linear map $[\cdot, \ldots, \cdot]_D : \wedge^n(\mathfrak{g} \oplus \mathbb{K}D) \to \mathfrak{g} \oplus \mathbb{K}D$ by

$$
[x_1 + k_1 D, \ldots, x_n + k_n D]_D = [x_1, \ldots, x_n]_{\mathfrak{g}} + \sum_{i=1}^{n} (-1)^{i-1} k_i D(x_1, \ldots, \hat{x}_i, \ldots, x_n).
$$

Define a linear map $\alpha_D : \mathfrak{g} \oplus \mathbb{K}D \to \mathfrak{g} \oplus \mathbb{K}D$ by $\alpha_D(x + kD) = \alpha(x) + kD$, i.e.

$$
\alpha_D = \begin{pmatrix} \alpha & 0 \\ 0 & \mathrm{Id} \end{pmatrix}.
$$

Theorem 5.4. *Let $(\mathfrak{g}, [\cdot, \ldots, \cdot]_{\mathfrak{g}}, \alpha)$ be an n-Hom-Lie algebra and $D : \wedge^{n-1}\mathfrak{g} \to \mathfrak{g}$ a linear map. Then $(\mathfrak{g} \oplus \mathbb{K}D, [\cdot, \ldots, \cdot]_D, \alpha_D)$ is an n-Hom-Lie algebra if and only if D is generalized derivation on $\mathfrak{g}$. We call $(\mathfrak{g} \oplus \mathbb{K}D, [\cdot, \ldots, \cdot]_D, \alpha_D)$ the generalized derivation extension of $\mathfrak{g}$ by the generalized derivation D.*

Proof. For all $x_1, \ldots, x_n \in \mathfrak{g}$, $k_1, \ldots, k_n \in \mathbb{K}$, we have

$$
\begin{aligned}
\alpha_D[x_1 + k_1 D, \ldots, x_n + k_n D]_D &= \alpha_D([x_1, \ldots, x_n]_{\mathfrak{g}} + \sum_{i=1}^{n} (-1)^{i-1} k_i D(x_1, \ldots, \hat{x}_i, \ldots, x_n)) \\
&= \alpha([x_1, \ldots, x_n]_{\mathfrak{g}}) + \sum_{i=1}^{n} (-1)^{i-1} k_i \alpha(D(x_1, \ldots, \hat{x}_i, \ldots, x_n)),
\end{aligned}
$$

and

$$
\begin{aligned}
&[\alpha_D(x_1 + k_1 D), \ldots, \alpha_D(x_n + k_n D)]_D \\
&= [\alpha(x_1) + k_1 D, \ldots, \alpha(x_n) + k_n D]_D \\
&= [\alpha(x_1), \ldots, \alpha(x_n)]_{\mathfrak{g}} + \sum_{i=1}^{n} (-1)^{i-1} k_i D(\alpha(x_1), \ldots, \hat{\alpha}(x_i), \ldots, \alpha(x_n)).
\end{aligned}
$$

Since α is an algebra morphism, α_D is an algebra morphism if and only if

$$
\alpha \circ D = D \circ \widetilde{\alpha}.
$$

By the definition of the bracket $[\cdot, \ldots, \cdot]_D$, we deduce that $(\mathfrak{g} \oplus \mathbb{K}D, [\cdot, \ldots, \cdot]_D, \alpha_D)$ satisfy the Hom Fundamental identity if and only if

$$
\mathrm{HF}_{x_1, \ldots, x_{n-2}, x_{n-1}, D, y_1, \ldots, y_{n-1}} = 0, \tag{33}
$$

$$
\mathrm{HF}_{D, x_1, \ldots, x_{n-2}, y_1, \ldots, y_{n-1}, y_n} = 0, \tag{34}
$$

$$
\mathrm{HF}_{D, x_1, \ldots, x_{n-2}, D, y_1, \ldots, y_{n-1}} = 0. \tag{35}
$$

By straightforward computation, we have

$$
\begin{aligned}
[\alpha_D(x_1), \ldots, \alpha_D(x_{n-1}), [D, y_1, \ldots, y_{n-1}]_D]_D &= [\alpha(x_1), \ldots, \alpha(x_{n-1}), D(y_1, \ldots, y_{n-1})]_D \\
&= [\alpha(x_1), \ldots, \alpha(x_{n-1}), D(y_1, \ldots, y_{n-1})]_{\mathfrak{g}},
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{i=1}^{n-1} [\alpha_D(D), \alpha_D(y_1), \ldots, \alpha_D(y_{i-1}), [x_1, \ldots, x_{n-1}, y_i]_D, \alpha_D(y_{i+1}), \ldots, \alpha_D(y_{n-1})]_D \\
&+ [[x_1, \ldots, x_{n-1}, D]_D, \alpha_D(y_1), \ldots, \alpha_D(y_{n-1})]_D \\
&= \sum_{i=1}^{n-1} [D, \alpha(y_1), \ldots, \alpha(y_{i-1}), [x_1, \ldots, x_{n-1}, y_i]_{\mathfrak{g}}, \alpha(y_{i+1}), \ldots, \alpha(y_{n-1})]_D
\end{aligned}
$$

$$
\begin{aligned}
&+[(-1)^{n-1}D(x_1,\ldots,x_{n-1}),\alpha(y_1),\ldots,\alpha(y_{n-1})]_D\\
=&\sum_{i=1}^{n-1}D(\alpha(y_1),\ldots,\alpha(y_{i-1}),[x_1,\ldots,x_{n-1},y_i]_{\mathfrak{g}},\alpha(y_{i+1}),\ldots,\alpha(y_{n-1}))\\
&+[\alpha(y_1),\ldots,\alpha(y_{n-1}),D(x_1,\ldots,x_{n-1})]_{\mathfrak{g}}.
\end{aligned}
$$

Thus, (33) is equivalent to Condition (ii) in Definition 5.1.

Similarly, we have

$$
\begin{aligned}
[\alpha_D(D),\alpha_D(x_1),\ldots,\alpha_D(x_{n-2}),[y_1,\ldots,y_n]_D]_D &= [D,\alpha(x_1),\ldots,\alpha(x_{n-2}),[y_1,\ldots,y_n]_{\mathfrak{g}}]_D\\
&= D(\alpha(x_1),\ldots,\alpha(x_{n-2}),[y_1,\ldots,y_n]_{\mathfrak{g}}),
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{i=1}^{n}[\alpha_D(y_1),\ldots,\alpha_D(y_{i-1}),[D,x_1,\ldots,x_{n-2},y_i]_D,\alpha_D(y_{i+1}),\ldots,\alpha_D(y_n)]_D\\
=&\sum_{i=1}^{n}[\alpha(y_1),\ldots,\alpha(y_{i-1}),D(x_1,\ldots,x_{n-2},y_i),\alpha(y_{i+1}),\ldots,\alpha(y_n)]_D\\
=&\sum_{i=1}^{n}[\alpha(y_1),\ldots,\alpha(y_{i-1}),D(x_1,\ldots,x_{n-2},y_i),\alpha(y_{i+1}),\ldots,\alpha(y_n)]_{\mathfrak{g}}.
\end{aligned}
$$

Thus, (34) is equivalent to Condition (iii) in Definition 5.1.

Finally, we have

$$
\begin{aligned}
&[\alpha_D(D),\alpha_D(x_1),\ldots,\alpha_D(x_{n-2}),[D,y_1,\ldots,y_{n-1}]_D]_D\\
=&[D,\alpha(x_1),\ldots,\alpha(x_{n-2}),D(y_1,\ldots,y_{n-1})]_D\\
=&D(\alpha(x_1),\ldots,\alpha(x_{n-2}),D(y_1,\ldots,y_{n-1})),
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{i=1}^{n-1}[\alpha_D(D),\alpha_D(y_1),\ldots,\alpha_D(y_{i-1}),[D,x_1,\ldots,x_{n-2},y_i]_D,\alpha_D(y_{i+1}),\ldots,\alpha_D(y_{n-1})]_D\\
=&\sum_{i=1}^{n-1}[D,\alpha(y_1),\ldots,\alpha(y_{i-1}),D(x_1,\ldots,x_{n-2},y_i),\alpha(y_{i+1}),\ldots,\alpha(y_{n-1})]_D\\
=&\sum_{i=1}^{n-1}D(\alpha(y_1),\ldots,\alpha(y_{i-1}),D(x_1,\ldots,x_{n-2},y_i),\alpha(y_{i+1}),\ldots,\alpha(y_{n-1})).
\end{aligned}
$$

Thus, (35) is equivalent to Condition (iv) in Definition 5.1. Therefore, $(\mathfrak{g}\oplus\mathbb{K}D,[\cdot,\ldots,\cdot]_D,\alpha_D)$ is an n-Hom-Lie algebra if and only if D is a generalized derivation on $\mathfrak{g}$. The proof is finished. ■

Proposition 5.5. *Let D^2 and D^1 be two generalized derivations on an n-Hom-Lie algebra $(\mathfrak{g},[\cdot,\ldots,\cdot]_{\mathfrak{g}},\alpha)$. If there exists $x\in\mathfrak{g}$ such that $\alpha(x)=x$ and $D^1=D^2+\mathfrak{ad}_x$, then the corresponding generalized derivation extensions $(\mathfrak{g}\oplus\mathbb{K}D^2,[\cdot,\ldots,\cdot]_{D^2},\alpha_{D^2})$ and $(\mathfrak{g}\oplus\mathbb{K}D^1,[\cdot,\ldots,\cdot]_{D^1},\alpha_{D^1})$ are isomorphic.*

Proof. Define $\bar{x}:\mathbb{K}D^1\longrightarrow\mathfrak{g}$ by

$$
\bar{x}(kD^1)=kx,\quad \forall k\in\mathbb{K}.
$$

Then $\begin{pmatrix}\mathrm{Id}_{\mathfrak{g}} & \bar{x}\\ 0 & 1\end{pmatrix}$ is an isomorphism from the n-Hom-Lie algebra $(\mathfrak{g}\oplus\mathbb{K}D^1,[\cdot,\ldots,\cdot]_{D^1},\alpha_{D^1})$ to $(\mathfrak{g}\oplus\mathbb{K}D^2,[\cdot,\ldots,\cdot]_{D^2},\alpha_{D^2})$. We leave details to readers. ■

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