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Hypersurfaces in space forms satisfying some generalized Einstein metric condition

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ABSTRACT

The difference tensor $C \cdot R - R \cdot C$ of Einstein manifolds, some quasi-Einstein manifolds and Roter type manifolds, of dimension $n \geq 4$, satisfy the following curvature condition: $(*)$ $C \cdot R - R \cdot C = Q(S, C) - (\kappa/(n-1))\, Q(g, C)$. We investigate hypersurfaces M in space forms N satisfying $(*)$. The main result states that if the tensor $C \cdot R - R \cdot C$ of a non-quasi-Einstein hypersurface M in N is a linear combination of the tensors $Q(g, C)$ and $Q(S, C)$ then $(*)$ holds on M. In the case when M is a quasi-Einstein hypersurface in N and some additional assumptions are satisfied then $(*)$ also holds on M.

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1. Pseudosymmetry type curvature conditions

Let (M, g), $n = \dim M \geq 3$, be a semi-Riemannian manifold. We denote by ∇, R, S, κ and C the Levi-Civita connection, the Riemann-Christoffel curvature tensor, the Ricci tensor, the scalar curvature and the Weyl conformal curvature tensor of (M, g), respectively.

It is well-known that if a semi-Riemannian manifold (M, g), $n \geq 3$, is locally symmetric then $\nabla R = 0$ on M (see, e.g., [66, Chapter 1.5]). This implies the following integrability condition $\mathcal{R}(X, Y) \cdot R = 0$, or briefly,

$$R \cdot R = 0. \tag{1.1}$$

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We refer to Sections 2 and 3 of the paper for precise definitions of the symbols used. Semi-Riemannian manifold satisfying (1.1) is called *semisymmetric* (see, e.g., [9, Chapter 8.5.3], [10, Chapter 20.7], [66, Chapter 1.6], [79,82]). Semisymmetric manifolds form a subclass of the class of pseudosymmetric manifolds. A semi-Riemannian manifold (M, g), $n \geq 3$, is said to be *pseudosymmetric* if the tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent at every point of M (see, e.g., [9, Chapter 8.5.3], [10, Chapter 20.7], [38, Chapter 6], [66, Chapter 12.4], [67, Chapter 7.3.1], [22,28,30,51,61,62,65,77,80–82]). This is equivalent to

$$R \cdot R = L_R\, Q(g, R) \tag{1.2}$$

on $\mathcal{U}_R = \{x \in M \mid R - (\kappa/(n-1)n)\, G \neq 0 \text{ at } x\}$, where L_R is some function on this set. According to [65], if the function L_R is constant then (M, g) is called a *pseudosymmetric manifold of constant type*. Examples of non-semisymmetric pseudosymmetric manifolds are presented among others in [21,45,51]. We also note that (1.2) implies

$$(a)\ R \cdot S = L_R\, Q(g, S) \quad \text{and} \quad (b)\ R \cdot C = L_R\, Q(g, C). \tag{1.3}$$

It is well-known that a semi-Riemannian manifold (M, g), $n \geq 3$, is said to be an *Einstein manifold* [7] if at every point of M its Ricci tensor S is proportional to the metric tensor g, i.e., on M we have

$$S = \frac{\kappa}{n}\, g. \tag{1.4}$$

According to [7, p. 432], (1.4) is called the *Einstein metric condition*. Einstein manifolds form a natural subclass of several classes of semi-Riemannian manifolds which are determined by curvature conditions imposed on their Ricci tensor [7, Table, pp. 432–433]. These conditions are named *generalized Einstein curvature conditions* [7, Chapter XVI].

Let $\mathcal{U}_S$ be the set of all points of a semi-Riemannian manifold (M, g), $n \geq 3$, at which S is not proportional to g, i.e., $\mathcal{U}_S = \{x \in M \mid S - (\kappa/n) g \neq 0 \text{ at } x\}$. Further, let $\mathcal{U}_C$ be the set of all points of a semi-Riemannian manifold (M, g), $n \geq 4$, at which $C \neq 0$. We note that $\mathcal{U}_S \cup \mathcal{U}_C = \mathcal{U}_R$ (see, e.g., [28]). The conditions (1.2), (1.3)(a) and (1.3)(b) are equivalent on the set $\mathcal{U}_S \cap \mathcal{U}_C$ of any 4-dimensional warped product manifold $\overline{M} \times_F \widetilde{N}$, (see, e.g., [34] and references therein).

A semi-Riemannian manifold (M, g), $n \geq 3$, is called *Ricci-pseudosymmetric* if the tensors $R \cdot S$ and $Q(g, S)$ are linearly dependent at every point of M (see, e.g., [9, Chapter 8.5.3], [22,30,82]). This is equivalent on $\mathcal{U}_S \subset M$ to

$$R \cdot S = L_S\, Q(g, S), \tag{1.5}$$

where L_S is some function on this set. According to [59], if the function L_S is constant then (M, g) is called a *Ricci-pseudosymmetric manifold of constant type*. Every warped product manifold $\overline{M} \times_F \widetilde{N}$ with an 1-dimensional $(\overline{M}, \overline{g})$ manifold and an $(n-1)$-dimensional Einstein semi-Riemannian manifold $(\widetilde{N}, \widetilde{g})$, $n \geq 3$, and a warping function F, is a Ricci-pseudosymmetric manifold (see, e.g., [9, Chapter 8.5.3], [14, Section 1], [34, Example 4.1]). A special subclass of Ricci-pseudosymmetric manifolds form Ricci-semisymmetric manifolds. A semi-Riemannian manifold (M, g), $n \geq 3$ is said to be *Ricci-semisymmetric* if

$$R \cdot S = 0 \tag{1.6}$$

on M (see, e.g., [9, Chapter 8.5.3], [30]). Ricci-semisymmetric manifolds were investigated by several authors, see, e.g., [1,9,26,27,66,68,70,71] and references therein. Ricci-semisymmetric manifolds (submanifolds) are also named *Ric-semisymmetric manifolds* (submanifolds), and in particular, *Ric-semisymmetric hypersurfaces* [66, Chapter 12.7], [68,70] or *Ryan hypersurfaces* [9, Chapter 8.5.3].

A semi-Riemannian manifold (M, g), $n \geq 4$, is said to be *Weyl-pseudosymmetric* if the tensors $R \cdot C$ and $Q(g, C)$ are linearly dependent at every point of M [22,28,30]. This is equivalent on $\mathcal{U}_C \subset M$ to

$$R \cdot C = L_1\, Q(g, C), \tag{1.7}$$

where L_1 is some function on this set. In particular, if the condition $R \cdot C = 0$ holds on $\mathcal{U}_C \subset M$ then (M, g), $n \geq 4$, is called *Weyl-semisymmetric* [10, Chapter 20.7], [30,38,80].

A semi-Riemannian manifold (M, g), $n \geq 4$, is said to have *pseudosymmetric Weyl tensor* if the tensors $C \cdot C$ and $Q(g, C)$ are linearly dependent at every point of M (see, e.g., [10, Chapter 20.7], [28,30,34]). This is equivalent on $\mathcal{U}_C \subset M$ to

$$C \cdot C = L_C\, Q(g, C), \tag{1.8}$$

where L_C is some function on this set. Every warped product manifold $\overline{M} \times_F \widetilde{N}$, with $\dim \overline{M} = \dim \widetilde{N} = 2$, satisfies (1.8) (see, e.g., [28,30,34] and references therein). Thus in particular, the Schwarzschild spacetime, the Kottler spacetime and the Reissner-Nordström spacetime satisfy (1.8). Recently, manifolds satisfying (1.8) were investigated among others in [28,34,44].

Warped product manifolds $\overline{M} \times_F \widetilde{N}$, of dimension ≥ 4, satisfying on $\mathcal{U}_C \subset \overline{M} \times_F \widetilde{N}$ the condition

$$R \cdot R - Q(S, R) = L\, Q(g, C), \tag{1.9}$$

where L is some function on this set, were studied among others in [20,34]. For instance, in [20] necessary and sufficient conditions for $\overline{M} \times_F \widetilde{N}$ to be a manifold satisfying (1.9) are given. Moreover, in that paper it was proved that any

4-dimensional warped product manifold $\overline{M}\times_F\widetilde{N}$, with an 1-dimensional base $(\overline{M},\overline{g})$, satisfies (1.9) [20, Theorem 4.1]. The warped product manifold $\overline{M}\times_F\widetilde{N}$, with 2-dimensional base $(\overline{M},\overline{g})$ and $(n-2)$-dimensional space of constant curvature $(\widetilde{N},\widetilde{g})$, $n\geq 4$, is a manifold satisfying (1.8) and (1.9) [34, Theorem 7.1(i)].

We refer to [14,22,28,30,32,34,38,44,46,77,80] for results on semi-Riemannian manifolds satisfying (1.2) and (1.5)–(1.9), as well as other conditions of this kind, named *pseudosymmetry type curvature conditions* or *pseudosymmetry type conditions*. It seems that (1.2) is the most important condition of that family of curvature conditions (see, e.g., [34]). We also can state that the Schwarzschild spacetime, the Kottler spacetime, the Reissner–Nordström spacetime, as well as the Friedmann-Lemaître-Robertson-Walker spacetimes are the "oldest" examples of pseudosymmetric warped product manifolds (see, e.g., [34,38,51,77]).

Some pseudosymmetry type conditions are also satisfied on Einstein manifolds. For instance, on every Einstein manifold the condition $(*)$ is satisfied (see Theorem 2.3), i.e.

$$C\cdot R-R\cdot C=Q(S,C)-\frac{\kappa}{n-1}\,Q(g,C). \tag{1.10}$$

The condition (1.10) is a generalized Einstein metric condition. We refer to [4,14,30–33,35,41–43,46] for results on semi-Riemannian manifolds, in particular, hypersurfaces in space forms, satisfying generalized Einstein metric conditions.

A semi-Riemannian manifold (M,g), $n\geq 3$, is said to be a *quasi-Einstein manifold* if

$$\operatorname{rank}(S-\alpha\,g)=1 \tag{1.11}$$

on $\mathcal{U}_S\subset M$, where α is some function on this set. In Section 2 (see Remark 2.5(i)-(iii)) we present some facts related to that class of manifolds. We mention that some quasi-Einstein warped product manifolds satisfy the following condition (see Remark 2.5(iii))

$$(n-2)(R\cdot C-C\cdot R)=Q(S,C)-L_S\,Q(g,C). \tag{1.12}$$

Quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations and the investigation on quasi-umbilical hypersurfaces of conformally flat spaces (see, e.g., [30,34] and references therein). Recently quasi-Einstein manifolds satisfying some pseudosymmetry type conditions were investigated among others in [4,14,28,32,44]. Quasi-Einstein hypersurfaces in semi-Riemannian spaces of constant curvature were studied among others in [29,40,43,59], see also [9, Chapter 6.2], [10, Chapter 19.5], [11, Chapter 4.6], [30,82] and references therein. We mention that there are different extensions of the class of quasi-Einstein manifolds. For instance we have the class of almost quasi-Einstein manifolds [12] and the class of 2-quasi-Einstein manifolds (see, e.g. [33,34]).

Investigations on semi-Riemannian manifolds (M,g), $n\geq 4$, satisfying (1.2) and (1.8) or (1.2) and (1.9) on $\mathcal{U}_S\cap\mathcal{U}_C\subset M$ lead to the following condition ([54, Theorem 3.2(ii)], [39, Lemma 4.1], see also [34, Section 1])

$$R=\frac{\phi}{2}\,S\wedge S+\mu\,g\wedge S+\frac{\eta}{2}\,g\wedge g, \tag{1.13}$$

where ϕ, μ and η are some functions on $\mathcal{U}_S\cap\mathcal{U}_C$. We note that if (1.13) is satisfied at a point of $\mathcal{U}_S\cap\mathcal{U}_C$ then at this point we have

$$\operatorname{rank}(S-\alpha\,g)>1,\quad \text{for any}\ \ \alpha\in\mathbb{R}. \tag{1.14}$$

A semi-Riemannian manifold (M,g), $n\geq 4$, satisfying (1.13) on $\mathcal{U}_S\cap\mathcal{U}_C\subset M$ is called a *Roter type manifold*, or a *Roter type space*, or a *Roter space* [24,35,36]. Roter type manifolds and in particular Roter type hypersurfaces (i.e. hypersurfaces satisfying (1.13)), in semi-Riemannian spaces of constant curvature were studied in: [24,28,32,35,40,45–47,49,58,63,64]. Roter type manifolds satisfy (1.10), as well as some other pseudosymmetry type conditions (see Theorem 2.4 and Remark 2.5(iv)–(vi)).

Let M, $n\geq 3$, be a connected hypersurface isometrically immersed in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, with signature $(s,n+1-s)$, $c=\frac{\widetilde{\kappa}}{n(n+1)}$, and the scalar curvature $\widetilde{\kappa}$. Let g be the metric tensor induced on M from the metric of the ambient space and let R, S, κ and C be the Riemann–Christoffel curvature tensor, Ricci tensor, the scalar curvature and the Weyl conformal curvature tensor of g, respectively. Further, let H and $\mathcal{A}$ be the second fundamental tensor and the shape operator of M, respectively. We have $H(X,Y)=g(\mathcal{A}X,Y)$, for any vector fields X, Y tangent to M. The $(0,2)$-tensors H^2 and H^3 are defined by $H^2(X,Y)=H(\mathcal{A}X,Y)$ and $H^3(X,Y)=H^2(\mathcal{A}X,Y)$, respectively.

Hypersurfaces in $N_s^{n+1}(c)$ satisfying pseudosymmetry type conditions were investigated in several papers, see, e.g., [76, Section 1] and references therein. We also refer to [33,36,76] for recent results related to this subject.

If M is a hypersurface in $N_s^{n+1}(c)$, $n\geq 3$, satisfying

$$H^2=\alpha\,H+\beta\,g, \tag{1.15}$$

for some functions α and β, then M is a pseudosymmetric manifold (see Theorem 3.1). Moreover, if some additional assumptions are satisfied then M is a Roter type manifold (see Theorem 3.5). In particular, every non-Einstein and non-conformally flat Clifford torus of dimension ≥ 5 is a Roter type manifold (see Example 3.6).

Let M be a hypersurface in $N_s^{n+1}(c)$, $n\geq 4$. We denote by $\mathcal{U}_H\subset M$ the set of all points at which the tensor H^2 is not a linear combination of the second fundamental tensor H and the metric tensor g of M. It is known that $\mathcal{U}_H\subset\mathcal{U}_S\cap\mathcal{U}_C\subset M$

([5, Proposition 2.1], [24, Section 1]). For instance, if M is the Cartan hypersurface then $\mathcal{U}_H = M$ ([9, Chapter 3.8.3], [10, Chapter 20.3]). In addition, if M is the Cartan hypersurface of dimension 6, 12, or 24 then it is a non-pseudosymmetric Ricci-pseudosymmetric manifold of constant type satisfying [53, Proposition 1(i), Theorem 1]

$$R \cdot S = \frac{\widetilde{\kappa}}{n(n+1)}\, Q(g, S), \tag{1.16}$$

provided that $n = 6, 12, 24$. The 3-dimensional Cartan hypersurface M is a pseudosymmetric manifold of constant type. In fact, we have on M $R \cdot R = (\widetilde{\kappa}/12)\, Q(g, R)$ [52, Section 5].

Evidently, (1.1) implies (1.6). The converse is not true. The problem of the equivalence of (1.1) and (1.6) on hypersurfaces, named *Ryan's problem* [66, Chapter 12.7], was investigated by several authors (see, e.g., [9, Chapter 8.5.3], [1,16,17,19,69] and references therein). This problem was stated as Problem P808 in [71] (cf., [66, Chapter 12.7]). We mention that (1.1) and (1.6) are equivalent on hypersurfaces in 5-dimensional semi-Riemannian spaces of constant curvature $N_s^5(c)$ [17]. For a presentation of results on the problem of the equivalence of semisymmetry, Ricci-semisymmetry and Weyl-semisymmetry, or, more generally, of pseudosymmetry, Ricci-pseudosymmetry and Weyl-pseudosymmetry on semi-Riemannian manifolds, and, in particular, on hypersurfaces in semi-Riemannian spaces, we refer to [30, Section 4].

The second fundamental tensor H of hypersurfaces M in $N_s^{n+1}(c)$, $n \geq 4$, realizing some pseudosymmetry type conditions on $\mathcal{U}_H \subset M$ satisfy also the following equation

$$H^3 = tr(H)\, H^2 + \psi\, H + \rho\, g, \tag{1.17}$$

where ψ and ρ are some functions on $\mathcal{U}_H$. We refer to [31,33,37,72–76] for results on hypersurfaces satisfying (1.17). We note that if M is a hypersurface in an 4-dimensional Riemannian space of constant curvature $N^4(c)$ then (1.17) holds $\mathcal{U}_H \subset M$.

If M is a hypersurface in $N_s^5(c)$ then (1.17) reduces on $\mathcal{U}_H \subset M$ to

$$H^3 = tr(H)\, H^2 + \psi\, H, \tag{1.18}$$

see [48, Proposition 2.1] and Proposition 4.3(i). If (1.18) is satisfied at every point of the set $\mathcal{U}_H$ of a hypersurface M in $N_s^{n+1}(c)$, $n \geq 3$, then (1.16) holds on this set (see Proposition 4.3(v)). Further, if $\operatorname{rank} H = 2$ at every point of the set $\mathcal{U}_H$ of a hypersurface M in $N_s^{n+1}(c)$, $n \geq 3$, then (1.18) holds on this set (see Proposition 4.3(ii), or [24, Lemma 2.1]). In addition, on this set we have (see Theorem 3.2(iii))

$$R \cdot R = \frac{\widetilde{\kappa}}{n(n+1)}\, Q(g, R). \tag{1.19}$$

If $\operatorname{rank} H = 2$ at every point of a hypersurface M in $N_s^{n+1}(c)$, $n \geq 3$, then M is called a *hypersurface with type number* 2 (see, e.g. [13]). Thus we see that if M in $N_s^{n+1}(c)$, $n \geq 3$, is a hypersurface with type number 2 then M is a pseudosymmetric manifold of constant type. We refer to [13] (see also [40, Section 5], [55]) for examples of submanifolds, and in particular, hypersurfaces in spaces of constant curvature with type number ≤ 2.

Evidently, (1.17) is a particular case of the equation

$$H^3 = \phi\, H^2 + \psi\, H + \rho\, g, \tag{1.20}$$

where ϕ, ψ and ρ are some functions on $\mathcal{U}_H$. Hypersurfaces M in $N_s^{n+1}(c)$, $n \geq 4$, satisfying (1.20) on $\mathcal{U}_H \subset M$ were investigated among others in [8,36,72,76], see also [9, Chapters 3.8.3 and 5.6] and references therein. If the tensor $R \cdot C$, $C \cdot R$ or $R \cdot C - R \cdot C$ is on $\mathcal{U}_H \subset M$ a linear combination of the tensor $R \cdot R$ and of a finite sum of the Tachibana tensors of the form $Q(A, B)$, where A is a symmetric $(0, 2)$-tensor and B a generalized curvature tensor, then the tensor H satisfies (1.17) on this set ([31, Corollary 4.1]). In Proposition 4.1 we present results on hypersurfaces M in $N_s^{n+1}(c)$, $n \geq 4$, satisfying (1.17) obtained in [73, Proposition 5.1]. Further, in that section we prove the following results. Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfying (1.17) on $\mathcal{U}_H \subset M$. We have: (i) (see Proposition 4.2(i)) The conditions

$$R \cdot S = Q(g, S^2) + \left(\varepsilon\psi - \frac{(2n-3)\widetilde{\kappa}}{n(n+1)}\right) Q(g, S), \tag{1.21}$$

$$R \cdot S^2 = Q(S, S^2) + \rho_1\, Q(g, S^2) + \rho_2\, Q(g, S), \tag{1.22}$$

$$S^3 = \left(-2\varepsilon\psi + \frac{3(n-1)\widetilde{\kappa}}{n(n+1)}\right) S^2 + \rho_2\, S + \rho_3\, g, \tag{1.23}$$

hold on $\mathcal{U}_H$, where ρ_1, ρ_2 and ρ_3 are defined by (4.11). (ii) (see Theorem 4.6) If on $\mathcal{U}_H$ the tensor $Q(S, R)$ is equal to the Tachibana tensor $Q(g, T)$, where T is a generalized curvature tensor, then any of the tensors: $R \cdot R$, $R \cdot C$, $C \cdot R$, $R \cdot C - C \cdot R$ and $C \cdot C$ is equal to some Tachibana tensor $Q(g, B)$, where B is a linear combination of the tensors R, $g \wedge g$, $g \wedge S$, $g \wedge S^2$ and $S \wedge S$. (iii) (see Proposition 4.7) The following conditions are satisfied on $\mathcal{U}_H$

$$C \cdot C = \frac{n-3}{n-2}\, R \cdot C + \frac{1}{n-2}\left(\frac{\kappa}{n-1} + \varepsilon\psi - \frac{(2n-3)\widetilde{\kappa}}{n(n+1)}\right) Q(g, C), \tag{1.24}$$

$$(n-2)C\cdot R+R\cdot C=(n-2)Q(S,C)+\left(\frac{\kappa}{n-1}+\varepsilon\psi-\frac{(n-1)^2\widetilde{\kappa}}{n(n+1)}\right)Q(g,C)$$
$$-\frac{1}{(n-2)}Q\left(g,\frac{n-2}{2}S\wedge S-\kappa\, g\wedge S+g\wedge S^2\right). \tag{1.25}$$

(iv) (see Theorem 4.8) If (1.11) and (1.18) are satisfied on $\mathcal{U}_H$ then (1.10) holds on $\mathcal{U}_H$ if and only if the following two conditions hold on this set

$$(a)\ \ \frac{\kappa}{n-1}=\frac{\widetilde{\kappa}}{n+1}\qquad\text{and}\qquad (b)\ \ Q\left(S-\frac{\kappa}{n}g,C\right)=0. \tag{1.26}$$

In Section 5 we consider hypersurfaces M in $N_s^{n+1}(c)$, $n\geq 4$, satisfying on $\mathcal{U}_H\subset M$

$$R\cdot C-C\cdot R=L_1\,Q(S,C)+L_2\,Q(g,C), \tag{1.27}$$

where L_1 and L_2 are some functions defined on this set. Theorem 5.1 states that if on $\mathcal{U}_H$ the conditions (1.11) and (1.27) are satisfied, for some functions α, L_1 and L_2, then on this set we have (1.18), for some function ψ, and

$$(n-2)(R\cdot C-C\cdot R)=Q(S,C)-\frac{\widetilde{\kappa}}{n(n+1)}Q(g,C). \tag{1.28}$$

Finally, Theorems 5.2 and 5.3 state that if on $\mathcal{U}_H$ the conditions (1.14) and (1.27) are satisfied, for some functions L_1 and L_2, then on this set we have (1.10) and $Q(S,R)=Q(g,T)$, i.e. (5.2), where T is a linear combination of the tensors R, $g\wedge g$, $g\wedge S$, $g\wedge S^2$ and $S\wedge S$. Moreover, any of the tensors: $R\cdot R$, $R\cdot C$, $C\cdot R$, $R\cdot C-C\cdot R$ and $C\cdot C$ is equal to some Tachibana tensor $Q(g,B)$, where B is a linear combination of the tensors R, $g\wedge g$, $g\wedge S$, $g\wedge S^2$ and $S\wedge S$.

2. Preliminary results

Throughout this paper all manifolds are assumed to be connected paracompact manifolds of class C^∞. Let (M,g) be an n-dimensional semi-Riemannian manifold and let ∇ be its Levi-Civita connection and $\Xi(M)$ the Lie algebra of vector fields on M. We define on M the endomorphisms $X\wedge_A Y$ and $\mathcal{R}(X,Y)$ of $\Xi(M)$ by $(X\wedge_A Y)Z=A(Y,Z)X-A(X,Z)Y$ and

$$\mathcal{R}(X,Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z,$$

respectively, where A is a symmetric $(0,2)$-tensor on M and $X,Y,Z\in\Xi(M)$. The Ricci tensor S, the Ricci operator $\mathcal{S}$, the tensors S^2 and S^3 and the scalar curvature κ of (M,g) are defined by $S(X,Y)=\mathrm{tr}\{Z\to\mathcal{R}(Z,X)Y\}$, $g(\mathcal{S}X,Y)=S(X,Y)$, $S^2(X,Y)=S(\mathcal{S}X,Y)$, $S^3(X,Y)=S^2(\mathcal{S}X,Y)$ and $\kappa=\mathrm{tr}\,\mathcal{S}$, respectively. The endomorphism $\mathcal{C}(X,Y)$ of (M,g), $n\geq 3$, is defined by

$$\mathcal{C}(X,Y)Z=\mathcal{R}(X,Y)Z-\frac{1}{n-2}\left(X\wedge_g\mathcal{S}Y+\mathcal{S}X\wedge_g Y-\frac{\kappa}{n-1}X\wedge_g Y\right)Z.$$

The $(0,4)$-tensor G, the Riemann-Christoffel curvature tensor R and the Weyl conformal curvature tensor C of (M,g) are defined by $G(X_1,X_2,X_3,X_4)=g((X_1\wedge_g X_2)X_3,X_4)$,

$$R(X_1,X_2,X_3,X_4)=g(\mathcal{R}(X_1,X_2)X_3,X_4),\quad C(X_1,X_2,X_3,X_4)=g(\mathcal{C}(X_1,X_2)X_3,X_4),$$

respectively, where $X_1,X_2,X_3,X_4\in\Xi(M)$. Let $\mathcal{B}$ be a tensor field sending any $X,Y\in\Xi(M)$ to a skew-symmetric endomorphism $\mathcal{B}(X,Y)$ and let B be a $(0,4)$-tensor associated with $\mathcal{B}$ by

$$B(X_1,X_2,X_3,X_4)=g(\mathcal{B}(X_1,X_2)X_3,X_4). \tag{2.1}$$

The tensor B is said to be a *generalized curvature tensor* if the following conditions are satisfied

$$B(X_1,X_2,X_3,X_4)=B(X_3,X_4,X_1,X_2),$$
$$B(X_1,X_2,X_3,X_4)+B(X_3,X_1,X_2,X_4)+B(X_2,X_3,X_1,X_4)=0.$$

For $\mathcal{B}$ as above, let B be again defined by (2.1). We extend the endomorphism $\mathcal{B}(X,Y)$ to a derivation $\mathcal{B}(X,Y)\cdot$ of the algebra of tensor fields on M, assuming that it commutes with contractions and $\mathcal{B}(X,Y)\cdot f=0$, for any smooth function f on M. For a $(0,k)$-tensor field T, $k\geq 1$, we can define the $(0,k+2)$-tensor $B\cdot T$ by

$$(B\cdot T)(X_1,\dots,X_k,X,Y)=(\mathcal{B}(X,Y)\cdot T)(X_1,\dots,X_k)$$
$$=-T(\mathcal{B}(X,Y)X_1,X_2,\dots,X_k)-\cdots-T(X_1,\dots,X_{k-1},\mathcal{B}(X,Y)X_k).$$

If A is a symmetric $(0,2)$-tensor then we define the $(0,k+2)$-tensor $Q(A,T)$ by

$$Q(A,T)(X_1,\dots,X_k,X,Y)=(X\wedge_A Y\cdot T)(X_1,\dots,X_k)$$
$$=-T((X\wedge_A Y)X_1,X_2,\dots,X_k)-\cdots-T(X_1,\dots,X_{k-1},(X\wedge_A Y)X_k).$$

The tensor $Q(A,T)$ is called the *Tachibana tensor of the tensors* A and T, in short the Tachibana tensor (see, e.g., [28,32,34,37,38,44]). Thus, among other things, we have the $(0,6)$-tensors: $R\cdot R$, $R\cdot C$, $C\cdot R$, $C\cdot C$, $Q(g,R)$, $Q(S,R)$, $Q(g,C)$ and $Q(S,C)$, as well as the $(0,4)$-tensors: $R\cdot S$, $C\cdot S$, $Q(g,S)$, $Q(g,S^2)$ and $Q(S,S^2)$. For a symmetric $(0,2)$-tensor E and a $(0,k)$-tensor T, $k\geq 2$, we define their Kulkarni-Nomizu product $E\wedge T$ by (see, e.g., [28,37])

$$\begin{aligned}&(E\wedge T)(X_1,\dots,X_4,Y_3,\dots,Y_k)\\ &= E(X_1,X_4)T(X_2,X_3,Y_3,\dots,Y_k)+E(X_2,X_3)T(X_1,X_4,Y_3,\dots,Y_k)\\ &\quad -E(X_1,X_3)T(X_2,X_4,Y_3,\dots,Y_k)-E(X_2,X_4)T(X_1,X_3,Y_3,\dots,Y_k).\end{aligned}$$

On any semi-Riemannian manifold (M,g), $n\geq 3$, the following identities are satisfied (see, e.g., [34])

$$C = R-\frac{1}{n-2}\,g\wedge S+\frac{\kappa}{(n-2)(n-1)}\,G, \tag{2.2}$$

$$(a)\ \ Q(A,G)=-Q(g,g\wedge A),\qquad (b)\ \ Q(g,G)=0, \tag{2.3}$$

where $G=\frac{1}{2}\,g\wedge g$ and A is a symmetric $(0,2)$-tensor on M. Using (2.2) and (2.3)(a) we get

$$\begin{aligned}Q(S,C) &= Q\left(S,R-\frac{1}{n-2}\,g\wedge S+\frac{\kappa}{(n-2)(n-1)}\,G\right)\\ &= Q(S,R)-\frac{1}{n-2}\,Q(S,g\wedge S)+\frac{\kappa}{(n-2)(n-1)}\,Q(S,G)\\ &= Q(S,R)+\frac{1}{n-2}\,Q\left(g,\frac{1}{2}\,S\wedge S\right)-\frac{\kappa}{(n-2)(n-1)}\,Q(g,g\wedge S).\end{aligned} \tag{2.4}$$

Lemma 2.1 (*cf. [21, Lemma 1]*)**.** *(i) If A is a symmetric $(0,2)$-tensor and T a generalized curvature tensor at a point x of a semi-Riemannian manifold (M,g), $n\geq 3$, then the tensor $Q(A,T)$ satisfies at this point the identity*

$$\sum_{(X_1,X_2),(X_3,X_4),(X_5,X_6)} Q(A,T)(X_1,X_2,X_3,X_4,X_5,X_6)=0.$$

(ii) If T is a generalized curvature tensor at a point x of a semi-Riemannian manifold (M,g), $n\geq 3$, then the tensor $Q(g,T)$ vanishes at x if and only if $T=\frac{\kappa(T)}{(n-1)n}\,G$ at this point, where $\kappa(T)$ is the scalar curvature of T.

Proposition 2.2. *[34, Theorem 3.4(i)] On any semi-Riemannian manifold (M,g), $n\geq 4$, the following identity is satisfied*

$$R\cdot C+C\cdot R=R\cdot R+C\cdot C-\frac{1}{(n-2)^2}\,Q\left(g,-\frac{\kappa}{n-1}\,g\wedge S+g\wedge S^2\right). \tag{2.5}$$

Theorem 2.3. *Let (M,g), $n\geq 4$, be a semi-Riemannian Einstein manifold. (i) [25, Section 5] The condition (1.10) holds on M. (ii) [34, Theorem 3.1] If the condition (1.2) is satisfied on $\mathcal{U}_R\subset M$ then on this set we have*

$$\begin{aligned}R\cdot R &= Q(S,R)+\left(L_R-\frac{\kappa}{n}\right)Q(g,C),\\ C\cdot C &= \left(L_R-\frac{\kappa}{(n-1)n}\right)Q(g,C),\\ R\cdot C+C\cdot R &= Q(S,C)+\left(2L_R-\frac{\kappa}{n-1}\right)Q(g,C).\end{aligned}$$

Proof. (i) On any Einstein manifold (M,g), $n\geq 4$, the following identity is satisfied (see, e.g., [46, p. 107]) $R\cdot C-C\cdot R=(\kappa/((n-1)n))\,Q(g,C)$. This, by making use of $\kappa/((n-1)n)=\kappa/(n-1)-\kappa/n$ and (1.4) turns into (1.10), completing the proof. □

Theorem 2.4. *Let (M,g), $n\geq 4$, be a semi-Riemannian manifold satisfying (1.13) on $\mathcal{U}_S\cap\mathcal{U}_C\subset M$. (i) [30,58] The following equations are satisfied on $\mathcal{U}_S\cap\mathcal{U}_C$:*

$$\begin{aligned}&S^2=\alpha_1 S+\alpha_2 g,\quad \alpha_1=\kappa+((n-2)\mu-1)\phi^{-1},\quad \alpha_2=(\mu\kappa+(n-1)\eta)\phi^{-1},\\ &R\cdot C=L_R\,Q(g,C),\quad L_R=((n-2)(\mu^2-\phi\eta)-\mu)\phi^{-1},\\ &R\cdot R=L_R\,Q(g,R),\quad R\cdot S=L_R\,Q(g,S),\end{aligned}$$

$$\begin{aligned}&R\cdot R=Q(S,R)+L\,Q(g,C),\quad L=L_R+\mu\phi^{-1}=(n-2)(\mu^2-\phi\eta)\phi^{-1},\\ &C\cdot C=L_C\,Q(g,C),\quad L_C=L_R+\frac{1}{n-2}\left(\frac{\kappa}{n-1}-\alpha_1\right),\end{aligned}$$

$$C\cdot R = L_C\, Q(g,R), \quad C\cdot S = L_C\, Q(g,S),$$

$$R\cdot C - C\cdot R = \frac{1}{n-2}\,Q(S,R) + \left(\frac{(n-1)\mu-1}{(n-2)\phi} + \frac{\kappa}{n-1}\right) Q(g,R) + \frac{\mu((n-1)\mu-1)-(n-1)\phi\eta}{(n-2)\phi}\, Q(S,G),$$

$$R\cdot C - C\cdot R = \left(\frac{1}{\phi}\left(\mu - \frac{1}{n-2}\right) + \frac{\kappa}{n-1}\right) Q(g,R) + \left(\frac{\mu}{\phi}\left(\mu - \frac{1}{n-2}\right) - \eta\right) Q(S,G).$$

(ii) [34, Theorem 3.2 and Proposition 3.3] Moreover, we also have on $\mathcal{U}_S \cap \mathcal{U}_C$: (1.10) *and*

$$R\cdot C + C\cdot R = Q(S,C) + \left(L + L_C - \frac{1}{(n-2)\phi}\right) Q(g,C).$$

Remark 2.5. (i) (see [14,33,34] and references therein) It is known that every warped product manifold $\overline{M}\times_F \widetilde{N}$ with an 1-dimensional base manifold $(\overline{M},\overline{g})$ and a 2-dimensional manifold $(\widetilde{N},\widetilde{g})$ or an $(n-1)$-dimensional Einstein manifold $(\widetilde{N},\widetilde{g})$, $n\geq 4$, and a warping function F, is a quasi-Einstein manifold satisfying (1.5). (ii) It is easy to see that on the set $\mathcal{U}_S$ of any manifold (M,g) the condition (1.11) is equivalent to $(S-\alpha\, g)\wedge(S-\alpha\, g)=0$. This gives $(1/2)\,S\wedge S = \alpha\, g\wedge S - \alpha^2\, G$. From the last equation, by making use of (2.3)(b), we obtain $Q(g,(1/2)\,S\wedge S) = \alpha\, Q(g, g\wedge S)$. This and (2.4) yield

$$Q(S,R) = Q(S,C) - \frac{1}{n-2}\left(\alpha - \frac{\kappa}{n-1}\right) Q(g, g\wedge S). \tag{2.6}$$

(iii) (a) (see [34, Example 4.1] and references therein) The warped product manifold $\overline{M}\times_F \widetilde{N}$ with an 1-dimensional base manifold $(\overline{M},\overline{g})$, $\overline{g}_{11}=\pm 1$, and an $(n-1)$-dimensional Einstein fiber $(\widetilde{N},\widetilde{g})$, $n\geq 5$, which is not a space of constant curvature, and a warping function F, satisfies on $\mathcal{U}_S\cap\mathcal{U}_C \subset \overline{M}\times_F \widetilde{N}$ the conditions: (1.5), with some function L_S, (1.11), with $\alpha = (\kappa/(n-1)) - L_S$, and (1.12). In particular, if $\overline{g}_{11}=-1$ and $(\widetilde{N},\widetilde{g})$, $n\geq 3$, is a Riemannian manifold then $\overline{M}\times_F \widetilde{N}$ is a special generalized Robertson-Walker spacetime [2]. Generalized Robertson-Walker spacetimes satisfying curvature conditions of pseudosymmetry type were investigated among others in [4,14,34,49]. We also mention that Einstein generalized Robertson-Walker spacetimes were classified in [3]. (b) (see [1] and [33, Example 7.5(i)]) Let M be a hypersurface in an Euclidean space $\mathbb{E}^{n+1}$, $n=2p+1$, $p\geq 2$, having at every point three principal curvatures $\lambda_1=\lambda\neq 0$, $\lambda_2=-\lambda$ and $\lambda_3=0$, provided that the multiplicity of λ_1 and λ_2 is p. Evidently, we have $\mathcal{U}_H = M$. We can check that the following conditions are satisfied on $\mathcal{U}_H$: $H^3 = -(\kappa/(n-1))H$, (1.5), with $L_S=0$, (1.11), with $\alpha=\kappa/(n-1)$, $R\cdot C = Q(S,C)$, $C\cdot R = ((n-3)/(n-2))Q(S,C)$, and, $(n-2)(R\cdot C - C\cdot R) = Q(S,C)$, i.e. (1.12), with $L_S=0$. (iv) In the standard Schwarzschild coordinates $(t;r;\theta;\phi)$, and the physical units ($c=G=1$), the Reissner-Nordström-de Sitter ($\Lambda>0$), and the Reissner-Nordström-anti-de Sitter ($\Lambda<0$) metrics are given by the line element (see, e.g., [78])

$$ds^2 = -h(r)\,dt^2 + h(r)^{-1}\,dr^2 + r^2\,(d\theta^2 + \sin^2\theta\, d\phi^2), \quad h(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3}, \tag{2.7}$$

where M, Q and Λ are non-zero constants. (v) [25, Section 6] The metric (2.7) satisfies (1.13) with

$$\phi = \frac{3}{2}(Q^2 - Mr)r^4Q^{-4}, \quad \mu = \frac{1}{2}(Q^4 + 3Q^2\Lambda r^4 - 3\Lambda M r^5)Q^{-4},$$
$$\eta = \frac{1}{12}(3Q^6 + 4Q^4\Lambda r^4 - 3Q^4 Mr + 9Q^2\Lambda^2 r^8 - 9\Lambda^2 M r^9)r^{-4}Q^{-4}.$$

If we set $\Lambda=0$ in (2.7) then we obtain the line element of the Reissner-Nordström spacetime, see, e.g., [60, Section 9.2] and references therein. It seems that the Reissner-Nordström spacetime is the "oldest" example of the Roter type warped product manifold. (vi) Some comments on pseudosymmetric manifolds (also called Deszcz symmetric spaces), as well as Roter spaces, are given in [15, Section 1]: "*From a geometric point of view, the Deszcz symmetric spaces may well be considered to be the simplest Riemannian manifolds next to the real space forms.*" and "*From an algebraic point of view, Roter spaces may well be considered to be the simplest Riemannian manifolds next to the real space forms.*" For further comments we refer to [82].

3. Hypersurfaces in space forms

Let M, $n\geq 3$, be a connected hypersurface isometrically immersed in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, with signature $(s, n+1-s)$. The Gauss equation of M in $N_s^{n+1}(c)$ reads

$$R = \frac{\varepsilon}{2}\,H\wedge H + \frac{\widetilde{\kappa}}{n(n+1)}\,G, \quad G = \frac{1}{2}\,g\wedge g, \quad \varepsilon = \pm 1. \tag{3.1}$$

From (3.1), by suitable contractions, we get

$$S = \varepsilon\,(\mathrm{tr}(H)H - H^2) + \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\,g, \quad \kappa = \varepsilon\,((\mathrm{tr}(H))^2 - \mathrm{tr}(H^2)) + \frac{(n-1)\widetilde{\kappa}}{n+1}. \tag{3.2}$$

It is known that (1.9) holds on M. Precisely, we have on M (see, e.g., [36, eq. (14)], [50, Proposition 3.1])

$$R\cdot R = Q(S,R) - \frac{(n-2)\widetilde{\kappa}}{n(n+1)}\,Q(g,C). \tag{3.3}$$

It is easy to see that (3.3), by making use of (2.2) and (2.3), turns into

$$R\cdot R = Q(S,R) - \frac{(n-2)\widetilde{\kappa}}{n(n+1)}\,Q(g,R) - \frac{\widetilde{\kappa}}{n(n+1)}\,Q(S,G). \tag{3.4}$$

We present now some results on pseudosymmetric hypersurfaces in $N_s^{n+1}(c)$, $n \geq 3$.

Theorem 3.1. *[52, Lemma 1, Theorem 1] If M is a hypersurface in $N_s^{n+1}(c)$, $n \geq 3$, satisfying (1.15) on $\mathcal{U}_R \subset M$, for some functions α and β, then on this set we have*

$$R\cdot R = \left(\frac{\widetilde{\kappa}}{n(n+1)} - \varepsilon\beta\right) Q(g,R).$$

Theorem 3.2. *(i) [23, Remark 3.2] If M is a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfying (1.2) on $\mathcal{U}_R \subset M$, then the following condition is satisfied on $\mathcal{U}_R$*

$$Q\left(S - \left(L_R + \frac{(n-2)\widetilde{\kappa}}{n(n+1)}\right)g,\, R - \frac{\widetilde{\kappa}}{n(n+1)}\,G\right) = 0. \tag{3.5}$$

(ii) [23, Lemma 1, Theorem 1] A hypersurface M in $N_s^{n+1}(c)$, $n \geq 4$, is pseudosymmetric if and only if at every point of M one of the following conditions is satisfied: (1.15) or $\mathrm{rank}\,H = 2$. (iii) [18, Theorem 4.2] If $\mathrm{rank}\,H = 2$ at a point of a hypersurface M in $N_s^{n+1}(c)$, $n \geq 4$, then (1.19) holds at this point. (iv) If $\mathrm{rank}\,H = 2$ at a point of $\mathcal{U}_H \subset M$ of a hypersurface M in $N_s^{n+1}(c)$, $n \geq 4$, then at this point we have

$$Q\left(S - \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\,g,\, R - \frac{\widetilde{\kappa}}{n(n+1)}\,G\right) = 0. \tag{3.6}$$

(v) (cf. [49, Section 4, eq. (62)], [64, Remark 5.2]) If $\mathrm{rank}\,H = 2$ and $\mathrm{rank}\left(S - \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\,g\right) > 1$ at a point of $\mathcal{U}_H \subset M$ of a hypersurface M in $N_s^{n+1}(c)$, $n \geq 4$, then at this point we have

$$R - \frac{\widetilde{\kappa}}{n(n+1)}\,G = \frac{\phi}{2}\left(S - \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\,g\right) \wedge \left(S - \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\,g\right), \quad \phi \in \mathbb{R}. \tag{3.7}$$

Proof. (iv) The condition (3.6) follows immediately from (iii) and (3.5). (v) The condition (3.7) is an immediate consequence of (iv) and [28, Proposition 2.4] (see also [40, Lemma 3.1]). □

Note that from (3.2) it follows that (1.15) and $\mathrm{tr}(H) - \alpha \neq 0$ are satisfied at every point of $(\mathcal{U}_S \cap \mathcal{U}_C) \setminus \mathcal{U}_H$. We have

Theorem 3.3. *[57, Proposition 3.3] If M is a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfying (1.15) on $(\mathcal{U}_S \cap \mathcal{U}_C)\setminus\mathcal{U}_H$, where α and β are some functions on this set, then (1.13) holds on $(\mathcal{U}_S \cap \mathcal{U}_C)\setminus\mathcal{U}_H$, where the functions ϕ, μ, η are defined by $\phi = \varepsilon(\mathrm{tr}(H) - \alpha)^{-2}$ and*

$$\mu = -\phi\left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} - \varepsilon\beta\right), \qquad \eta = \phi\left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} - \varepsilon\beta\right)^2 + \frac{\widetilde{\kappa}}{n(n+1)}.$$

Theorem 3.4. *If (1.15) is satisfied at every point of a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, then (1.10) holds on M.*

Proof. Evidently, (1.10) is satisfied at all points of M at which $C = 0$. From Theorem 3.1(i) it follows that our assertion is also true at all points of M at which $S = (\kappa/n)g$. Finally, in view of Theorem 2.4(ii) and Theorem 3.3, (1.10) holds on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$. Our theorem is thus proved. □

As an immediate consequence of the last result we have the following theorem.

Theorem 3.5. *If M is a hypersurface in a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 4$, having at every point at most two distinct principal curvatures then (1.10) holds on M.*

Example 3.6 (*cf. [57, Proposition 3.4, Example 3.2, Corollary 3.1]*). Let $M = S^p(\sqrt{(p/n)}) \times S^{n-p}(\sqrt{((n-p)/n)})$ be the Clifford torus in the $(n+1)$-dimensional unit sphere $S^{n+1}(1)$, $n \geq 4$. Thus we have $c = \widetilde{\kappa}/(n(n+1)) = 1$. In addition, we assume that $n \neq 2p$ and $2 \leq p \leq n-2$. Thus at every point of M the tensors $S-(\kappa/n)g$ and C are non-zero. Therefore $\mathcal{U}_S \cap \mathcal{U}_C = M$. As it was shown in [57, Proposition 3.4], (1.13) is satisfied on M. Precisely, we have on M

$$R = \frac{p(n-p)}{2(n-2p)^2}(S-(n-2)g) \wedge (S-(n-2)g) + \frac{1}{2}g \wedge g.$$

This, in view of Theorem 3.2(ii), implies (1.10). Further, it is known that every Clifford torus is a semisymmetric manifold. Thus $R \cdot R = 0$ and, in a consequence, $R \cdot C = 0$ on M. Now (1.10) reduces to $C \cdot R = Q(S, C) - (\kappa/(n-1))Q(g, C)$ on M.

Remark 3.7. Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$. (i) If (1.4) holds at a point $x \in M$ then at this point (3.2) turns into

$$H^2 = \operatorname{tr}(H)H + \frac{(n-1)\varepsilon}{n}\left(\frac{\widetilde{\kappa}}{n+1} - \frac{\kappa}{n-1}\right)g.$$

(ii) The Weyl conformal curvature tensor C of M vanishes at a point $x \in M$ if and only if at x we have $\operatorname{rank}(H-\alpha_1 g) \leq 1$, for some $\alpha_1 \in \mathbb{R}$ [50, Theorem 4.1]. But the last condition, in view of [5, Lemma 2.2], implies (3.2), for some $\alpha, \beta \in \mathbb{R}$.

4. The condition $H^3 = \operatorname{tr}(H)H^2 + \psi H + \rho g$

In this section we consider hypersurfaces M in $N_s^{n+1}(c)$, $n \geq 4$, satisfying on $\mathcal{U}_H \subset M$ curvature conditions of the kind: the tensor $R \cdot C$, $C \cdot R$ or $R \cdot C - R \cdot C$ is a linear combination of the tensor $R \cdot R$ and of a finite sum of the Tachibana tensors of the form $Q(A, B)$, where A is a symmetric $(0,2)$-tensor and B a generalized curvature tensor. As it was mentioned in Section 1, if such condition is satisfied on $\mathcal{U}_H$ then (1.17) holds on this set.

Proposition 4.1. *[73, Proposition 5.1, eq. (29)] If M is a hypersurface in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 4$, satisfying (1.17) on $\mathcal{U}_H \subset M$, for some functions ψ and ρ on $\mathcal{U}_H$, then on $\mathcal{U}_H$ we have*

$$R \cdot C = Q(S, R) - \frac{(n-2)\widetilde{\kappa}}{n(n+1)}Q(g, R) + \alpha_2 Q(S, G) + \frac{\rho}{n-2}Q(H, G), \tag{4.1}$$

$$C \cdot R = \frac{n-3}{n-2}Q(S, R) + \alpha_1 Q(g, R) + \alpha_2 Q(S, G), \tag{4.2}$$

$$\begin{aligned}(n-2)(R \cdot C - C \cdot R) = {} & Q(S, R) + \rho Q(H, G) \\ & + \left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} - \frac{\kappa}{n-1} - \varepsilon\psi\right)Q(g, R),\end{aligned} \tag{4.3}$$

$$\begin{aligned}(n-2)C \cdot C = {} & (n-3)Q(S, R) + (n-2)\alpha_1 Q(g, R) \\ & + (\alpha_1 - \alpha_2)Q(S, G) + \frac{n-3}{n-2}\rho Q(H, G),\end{aligned} \tag{4.4}$$

$$R \cdot S = \frac{\widetilde{\kappa}}{n(n+1)}Q(g, S) + \rho Q(g, H), \tag{4.5}$$

$$\alpha_1 = \frac{1}{n-2}\left(\frac{\kappa}{n-1} + \varepsilon\psi - \frac{(n^2-3n+3)\widetilde{\kappa}}{n(n+1)}\right), \tag{4.6}$$

$$\alpha_2 = -\frac{(n-3)\widetilde{\kappa}}{(n-2)n(n+1)}. \tag{4.7}$$

Proposition 4.2. *Let M be a hypersurface in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 4$, satisfying (1.17) on $\mathcal{U}_H \subset M$, for some functions ψ and ρ on $\mathcal{U}_H$. (i) The following conditions are satisfied on $\mathcal{U}_H$*

$$Q(\rho H - \alpha_3 S - S^2, G) = 0, \tag{4.8}$$

$$\rho H = S^2 + \alpha_3 S + \frac{\lambda}{n}g, \quad \lambda = \rho \operatorname{tr}(H) - \kappa\alpha_3 - \operatorname{tr}(S^2), \tag{4.9}$$

$$\alpha_3 = (n-2)^2\left(\frac{1}{n-2}(\alpha_1 - \alpha_2) - 2\alpha_2 - \frac{\widetilde{\kappa}}{n(n+1)}\right) - \frac{\kappa}{n-1} = \varepsilon\psi - \frac{2(n-1)\widetilde{\kappa}}{n(n+1)}, \tag{4.10}$$

where α_1 and α_2 are defined by (4.6) and (4.7), respectively. Moreover, (1.21), (1.22) and (1.23) hold on $\mathcal{U}_H$, where ρ_1, ρ_2 and ρ_3 are defined by

$$\rho_1 = -\frac{(n-2)\widetilde{\kappa}}{n(n+1)} - \alpha_3, \qquad \rho_2 = -\frac{\lambda}{n} - \left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} + \alpha_3\right)\alpha_3,$$
$$\rho_3 = \frac{1}{n}\left(\operatorname{tr}(S^3) + \left(2\varepsilon\psi - \frac{3(n-1)\widetilde{\kappa}}{n(n+1)}\right)\operatorname{tr}(S^2) - \kappa\rho_2\right), \tag{4.11}$$

respectively. (ii) If at a point $x \in \mathcal{U}_H$ we have $S^2 = \beta_1 S + \beta_2 g$, for some $\beta_1,\ \beta_2 \in \mathbb{R}$, then $\rho = 0$, $\beta_1 = \alpha_3$ and $\beta_2 = -(\lambda/n)$ at this point.

Proof. (i) The identities (2.5) and (3.4), together with (4.1), (4.2) and (4.4), give

$$\begin{aligned}
&\left(1+\frac{n-3}{n-2}\right)Q(S,R) + \alpha_1 Q(g,R) + 2\alpha_2 Q(S,G) - \frac{(n-2)\widetilde{\kappa}}{n(n+1)}Q(g,R) + \frac{\rho}{n-2}Q(H,G)\\
&= \left(1+\frac{n-3}{n-2}\right)Q(S,R) + \alpha_1 Q(g,R) + \frac{1}{n-2}\left((\alpha_1-\alpha_2)Q(S,G) + \frac{(n-3)\rho}{n-2}Q(H,G)\right)\\
&\quad -\frac{(n-2)\widetilde{\kappa}}{n(n+1)}Q(g,R) - \frac{\widetilde{\kappa}}{n(n+1)}Q(S,G) - \frac{1}{(n-2)^2}Q\left(g, -\frac{\kappa}{n-1}g\wedge S + g\wedge S^2\right),
\end{aligned}$$

$$\begin{aligned}
\frac{\rho}{(n-2)^2}Q(H,G) &= \left(\frac{1}{n-2}(\alpha_1-\alpha_2) - 2\alpha_2 - \frac{\widetilde{\kappa}}{n(n+1)}\right)Q(S,G)\\
&\quad - \frac{1}{(n-2)^2}Q\left(g, -\frac{\kappa}{n-1}g\wedge S + g\wedge S^2\right),
\end{aligned}$$

$$\begin{aligned}
\rho\, Q(H,G) &= (n-2)^2\left(\frac{1}{n-2}(\alpha_1-\alpha_2) - 2\alpha_2 - \frac{\widetilde{\kappa}}{n(n+1)}\right)Q(S,G)\\
&\quad - Q\left(g, -\frac{\kappa}{n-1}g\wedge S + g\wedge S^2\right).
\end{aligned}$$

This, by making use of (2.3) and (4.10), yields $\rho\, Q(H,G) = \alpha_3 Q(S,G) - Q(g, g\wedge S^2)$ and $\rho\, Q(H,G) = \alpha_3 Q(S,G) + Q(S^2,G)$ and (4.8). From (4.8), by a suitable contraction, we get $Q(\rho H - \alpha_3 S - S^2, g) = 0$ and, in consequence, (4.9). Now we prove that (1.21) and (1.22) hold on $\mathcal{U}_H$. From (4.9) we get

$$\rho\, Q(g,H) = Q(g,S^2) + \alpha_3 Q(g,S), \tag{4.12}$$
$$\rho\, Q(H,S) = -Q(S,S^2) + \frac{\lambda}{n}Q(g,S), \tag{4.13}$$
$$R\cdot S^2 = \rho\,(R\cdot H) - \alpha_3\,(R\cdot S). \tag{4.14}$$

The conditions (4.5), (4.10) and (4.12) yield immediately (1.21). Further, (4.14), by (4.5) and (4.12), turns into

$$R\cdot S^2 = \rho\,(R\cdot H) - \left(\frac{\widetilde{\kappa}\alpha_3}{n(n+1)} + \alpha_3^2\right)Q(g,S) - \alpha_3 Q(g,S^2). \tag{4.15}$$

From the Gauss equation (3.1) of M in $N_s^{n+1}(c)$ we get

$$\begin{aligned}
g^{rs}(H_{hr}R_{sijk} + H_{ir}R_{shjk}) &= \varepsilon\,(H_{hk}^2H_{ij} - H_{hj}^2H_{ik} + H_{ik}^2H_{hj} - H_{ij}^2H_{hk})\\
&\quad + \frac{\widetilde{\kappa}}{n(n+1)}(H_{hk}g_{ij} - H_{hj}g_{ik} + H_{ik}g_{hj} - H_{ij}g_{hk}),\\
R\cdot H &= \varepsilon\, Q(H,H^2) + \frac{\widetilde{\kappa}}{n(n+1)}Q(g,H).
\end{aligned}$$

This, together with

$$\varepsilon\, Q(H,H^2) = Q(H,S) - \frac{\widetilde{\kappa}}{n(n+1)}Q(g,H),$$

which is an immediate consequence of (3.2), turns into

$$R\cdot H = -Q(H,S) - \frac{(n-2)\widetilde{\kappa}}{n(n+1)}Q(g,H).$$

Applying in this (4.12) and (4.13) we obtain

$$\rho\,(R\cdot H)=Q(S,S^2)-\frac{(n-2)\widetilde{\kappa}}{n(n+1)}\,Q(g,S^2)-\left(\frac{\lambda}{n}+\frac{(n-2)\widetilde{\kappa}\alpha_3}{n(n+1)}\right)Q(g,S),$$

which together with (4.15) yields (1.22). We prove now that (1.23) holds on $\mathcal{U}_H$. From (1.21) it follows that

$$\begin{aligned}S_h^rR_{rijk}+S_i^rR_{rhjk}\;=\;&\rho_4\,(g_{hj}S_{ik}+g_{ij}S_{hk}-g_{hk}S_{ij}-g_{ik}S_{hj})\\&+g_{hj}S_{ik}^2+g_{ij}S_{hk}^2-g_{hk}S_{ij}^2-g_{ik}S_{hj}^2,\end{aligned}\tag{4.16}$$

where $S_h^r=S_{hk}g^{kr}$ and $\rho_4=\varepsilon\psi-((2n-3)\widetilde{\kappa})/(n(n+1))$. Transvecting (4.16) with S_l^h we get

$$\begin{aligned}S_l^hS_h^rR_{rijk}+S_l^hS_i^rR_{rhjk}\;=\;&\rho_4\,(S_{lj}S_{ik}-S_{lk}S_{ij}+g_{ij}S_{lk}^2-g_{ik}S_{lj}^2)\\&+S_{lj}S_{ik}^2-S_{lk}S_{ij}^2+g_{ij}S_{hk}^3-g_{ik}S_{lj}^3,\end{aligned}$$

which by symmetrization in i,l leads to $R\cdot S^2=Q(S,S^2)+Q(g,S^3)+\rho_4\,Q(g,S^2)$. This and (1.22) yield $Q(g,S^3+(\rho_4-\rho_1)S^2-\rho_2\,S)=0$, which in view of [50, Lemma 2.4(i)] implies (1.23), completing the proof of (i). (ii) From (4.9), by our assumption and (3.2), we obtain

$$\rho\,H=(\beta_1-\alpha_3)\,\varepsilon\,(\mathrm{tr}\,(H)\,H-H^2)+\left((\beta_1-\alpha_3)\,\frac{(n-1)\widetilde{\kappa}}{n(n+1)}+\beta_2+\frac{\lambda}{n}\right)g,$$

which completes the proof. □

Proposition 4.3. *Let M be a hypersurface in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n\geq 4$.*
(i) [48, Proposition 2.1] If $n=4$ then (1.17) reduces on $\mathcal{U}_H\subset M$ to (1.18). (ii) [72, Theorem 3.2(iv), Proposition 3.1] If $\mathrm{rank}\,H=2$ *at every point of $\mathcal{U}_H\subset M$ then (1.18) is satisfied on this set with*

$$\psi=\frac{1}{2}\,(\mathrm{tr}\,(H^2)-(\mathrm{tr}\,(H))^2)=\frac{(n-1)\varepsilon}{2}\left(\frac{\widetilde{\kappa}}{n+1}-\frac{\kappa}{n-1}\right).\tag{4.17}$$

(iii) If at every point of $\mathcal{U}_H\subset M$ the conditions: $\mathrm{rank}\,H=2$ *and* $\mathrm{rank}\left(S-\frac{(n-1)\widetilde{\kappa}}{n(n+1)}\,g\right)>1$ *are satisfied then (1.13) holds on this set with the functions ϕ, μ and η defined by*

$$\frac{2}{(n-1)\phi}=\frac{\widetilde{\kappa}}{n+1}-\frac{\kappa}{n-1},\tag{4.18}$$

$$\mu=-\frac{(n-1)\widetilde{\kappa}}{n(n+1)}\,\phi,\quad \eta=\frac{\widetilde{\kappa}}{n(n+1)}\left(\frac{(n-1)^2\widetilde{\kappa}}{n(n+1)}\,\phi+1\right),\tag{4.19}$$

respectively. Moreover, the following conditions are satisfied on $\mathcal{U}_H$: (1.19) and

$$R\cdot C=\frac{\widetilde{\kappa}}{n(n+1)}\,Q(g,C),\tag{4.20}$$

$$C\cdot R=\frac{n-3}{(n-2)(n-1)\phi}\,Q(g,R),\tag{4.21}$$

$$C\cdot C=\frac{n-3}{(n-2)(n-1)\phi}\,Q(g,C),\tag{4.22}$$

(cf., [72, Theorem 3.2(ii), Proposition 4.3]). (iv) (cf., [6, Theorem 3.1]) On $\mathcal{U}_H\subset M$ (1.19) and (4.20) are equivalent. (v) [18, Proposition 3.2] If (1.18) is satisfied on $\mathcal{U}_H\subset M$ then (1.16) holds on this set.

Proof. (iii) From Theorem 3.2(v) it follows that (4.19) holds on $\mathcal{U}_H$, where ϕ is some function on this set. We prove now that ϕ satisfies (4.18). From Theorem 2.4(i) and (4.19) it follows that

$$S^2=\left(\kappa-\frac{(n-2)(n-1)\widetilde{\kappa}}{n(n+1)}-\phi^{-1}\right)S-\frac{(n-1)\widetilde{\kappa}}{n(n+1)}\left(\kappa-\frac{(n-1)^2\widetilde{\kappa}}{n(n+1)}-\phi^{-1}\right)g.\tag{4.23}$$

But on the other hand, (4.9), by (4.10) making use of and (4.17), yields

$$S^2=\frac{1}{2}\left(\kappa-\frac{(n-1)(n-4)\widetilde{\kappa}}{n(n+1)}\right)S-\frac{\lambda}{n}\,g.\tag{4.24}$$

Now using (4.23) and (4.24) we get (4.18). Further, using (4.18), (4.19) and suitable formulas given in Theorem 2.4(i) we can check that (1.19) and (4.20)–(4.22) hold good, which completes the proof of (iii). □

Theorem 4.4 ([37, Theorem 5.1, Theorem 5.2]). *Let M be a hypersurface in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 4$. (i) If a generalized curvature tensor B_1 satisfies*

$$R \cdot R = Q(g, B_1) \tag{4.25}$$

on $\mathcal{U}_H \subset M$, then on this set we have

$$\begin{aligned}(n-1)B_1 = &\left(\kappa + \varepsilon\psi - \frac{(n-1)^2\widetilde{\kappa}}{n(n+1)}\right) R - \frac{1}{2} S \wedge S + g \wedge S^2 \\ &+ \left(\varepsilon\psi - \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\right) g \wedge S + \lambda G,\end{aligned} \tag{4.26}$$

where λ is some function on $\mathcal{U}_H$. (ii) If a generalized curvature tensor B_2 satisfies

$$R \cdot C = Q(g, B_2) \tag{4.27}$$

on $\mathcal{U}_H \subset M$, then on this set we have

$$\begin{aligned}(n-1)B_2 = &\left(\kappa + \varepsilon\psi - \frac{(n-1)^2\widetilde{\kappa}}{n(n+1)}\right) R - \frac{1}{n-2} g \wedge S^2 \\ &- \frac{1}{2} S \wedge S - \frac{1}{n-2}\left(\varepsilon\psi - \frac{(n-1)^2\widetilde{\kappa}}{n(n+1)}\right) g \wedge S + \lambda G,\end{aligned} \tag{4.28}$$

where λ is some function on $\mathcal{U}_H$. (iii) If a generalized curvature tensor B_3 satisfies

$$C \cdot R = Q(g, B_3) \tag{4.29}$$

on $\mathcal{U}_H \subset M$, then on this set we have

$$\begin{aligned}B_3 = &\left(\frac{\kappa}{n-1} + \frac{2\varepsilon\psi}{n-1} - \frac{\widetilde{\kappa}}{n+1}\right) R + \lambda G \\ &+ \frac{n-3}{(n-2)(n-1)}\left(\left(\varepsilon\psi - \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\right) g \wedge S - \frac{1}{2} S \wedge S + g \wedge S^2\right),\end{aligned} \tag{4.30}$$

where λ is a function on $\mathcal{U}_H$. (iv) If a generalized curvature tensor B_4 satisfies

$$R \cdot C - C \cdot R = Q(g, B_4) \tag{4.31}$$

on $\mathcal{U}_H \subset M$, then on this set we have

$$\begin{aligned}B_4 = &\left(-\frac{\varepsilon\psi}{n-1} + \frac{\widetilde{\kappa}}{n(n+1)}\right) R + \left(-\frac{\varepsilon\psi}{n-1} + \frac{2\widetilde{\kappa}}{n(n+1)}\right) g \wedge S \\ &- \frac{1}{n-1} g \wedge S^2 - \frac{1}{2(n-2)(n-1)} S \wedge S + \lambda G,\end{aligned} \tag{4.32}$$

where λ is a function on $\mathcal{U}_H$.

Theorem 4.5. *Let M be a hypersurface in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 4$, satisfying (1.17) on $\mathcal{U}_H \subset M$, for some functions ψ and ρ on $\mathcal{U}_H$. If the tensor $C \cdot C$ and a generalized curvature tensor B satisfy*

$$C \cdot C = Q(g, B) \tag{4.33}$$

on $\mathcal{U}_H$, then on this set we have

$$\begin{aligned}B = &\left(\frac{\kappa}{n-1} + \frac{2\varepsilon\psi}{n-1} - \frac{\widetilde{\kappa}}{n+1}\right) C + \lambda G \\ &- \frac{n-3}{(n-2)^2(n-1)}\left(\frac{n-2}{2} S \wedge S - \kappa\, g \wedge S + g \wedge S^2\right),\end{aligned} \tag{4.34}$$

where λ is some function on $\mathcal{U}_H$.

Proof. From (2.3)(a), (4.4) and (4.33) it follows that the tensor $Q(S, R)$ is a linear combination of the tensors of the form $Q(g, g \wedge A)$, where A is a symmetric $(0, 2)$-tensor, and the tensors $Q(g, R)$ and $Q(g, B)$. Now, from (3.4), (4.1) and (4.2) it follows that the tensors $R \cdot R$, $R \cdot C$ and $C \cdot R$ satisfy (4.25), (4.27) and (4.29), respectively, where B_1, B_2 and B_3 are some generalized curvature tensors. In view of Theorem 4.4, the tensors B_1, B_2 and B_3 satisfy (4.26), (4.28) and (4.30), respectively. Further, (4.25), (4.27) and (4.29) together with (2.5) yield

$$C \cdot C = \frac{1}{(n-2)^2} Q\left(g, -\frac{\kappa}{n-1} g \wedge S + g \wedge S^2\right) + Q(g, -B_1 + B_2 + B_3).$$

This by making use of (4.26), (4.28) and (4.30) leads to (4.33), with

$$B = \left(\frac{\kappa}{n-1} + \frac{2\varepsilon\psi}{n-1} - \frac{\widetilde{\kappa}}{n+1}\right) R - \frac{n-3}{2(n-2)(n-1)}\, S \wedge S$$
$$-\frac{1}{n-2}\left(\frac{\kappa}{(n-2)(n-1)} + \frac{2\varepsilon\psi}{n-1} - \frac{\widetilde{\kappa}}{n+1}\right) g \wedge S - \frac{n-3}{(n-2)^2(n-1)}\, g \wedge S^2 + \lambda_2\, G, \tag{4.35}$$

where λ_2 is some function on $\mathcal{U}_H$. Now (4.35), by (2.2), gives (4.34), completing the proof. □

From Proposition 4.1 and Theorems 4.3 and 4.4 it follows the following result.

Theorem 4.6. *Let M be a hypersurface in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 4$, satisfying (1.17) on $\mathcal{U}_H \subset M$, for some functions ψ and ρ on $\mathcal{U}_H$. If the tensor $Q(S,R)$ is equal to the Tachibana tensor $Q(g,T)$, where T is a generalized curvature tensor, then the tensors $R\cdot R$, $R\cdot C$, $C\cdot R$, $R\cdot C - C\cdot R$ and $C\cdot C$ satisfy (4.25), (4.27), (4.29), (4.31) and (4.33), with the tensors B_1, B_2, B_3, B_4 and B, defined by (4.26), (4.28), (4.30), (4.32) and (4.34), respectively.*

Proposition 4.7. *Let M be a hypersurface isometrically immersed in $N_s^{n+1}(c)$, $n \geq 4$.*
(i) [34, Theorem 3.7] The following identity is satisfied on M

$$R\cdot C + C\cdot R = Q(S,C) - \frac{(n-2)\widetilde{\kappa}}{n(n+1)}\, Q(g,C) + C\cdot C$$
$$-\frac{1}{(n-2)^2}\, Q\left(g, \frac{n-2}{2}\, S\wedge S - \kappa\, g\wedge S + g\wedge S^2\right). \tag{4.36}$$

(ii) If (1.17) is satisfied on $\mathcal{U}_H \subset M$, for some functions ψ and ρ, then (1.24) and (1.25) hold on this set.

Proof. (ii) From (4.1), (4.4) and (4.7) we get

$$\begin{aligned}
&C\cdot C - \frac{n-3}{n-2}\, R\cdot C \\
&= \frac{n-3}{n-2}\, Q(S,R) + \alpha_1\, Q(g,R) + \frac{1}{n-2}\left((\alpha_1-\alpha_2)\, Q(S,G) + \frac{n-3}{n-2}\rho\, Q(H,G)\right) \\
&\quad -\frac{n-3}{n-2}\, Q(S,R) + \frac{n-3}{n-2}\,\frac{(n-2)\widetilde{\kappa}}{n(n+1)}\, Q(g,R) - \frac{n-3}{n-2}\,\alpha_2\, Q(S,G) - \frac{n-3}{n-2}\,\frac{\rho}{n-2}\, Q(H,G) \\
&= \left(\alpha_1 + \frac{(n-3)\widetilde{\kappa}}{n(n+1)}\right) Q(g,R) + \left(\frac{\alpha_1-\alpha_2}{n-2} - \frac{n-3}{n-2}\alpha_2\right) Q(S,G) \\
&= \left(\alpha_1 + \frac{(n-3)\widetilde{\kappa}}{n(n+1)}\right) Q(g,R) + \left(\frac{\alpha_1}{n-2} - \frac{n-3+1}{n-2}\alpha_2\right) Q(S,G) \\
&= \alpha_1\, Q(g,R) + \frac{\alpha_1}{n-2}\, Q(S,G) + \frac{(n-3)\widetilde{\kappa}}{n(n+1)}\, Q(g,R) - \alpha_2\, Q(S,G) \\
&= \alpha_1\, Q(g,C) + \frac{(n-3)\widetilde{\kappa}}{n(n+1)}\, Q(g,R) + \frac{(n-3)\widetilde{\kappa}}{(n-2)n(n+1)}\, Q(S,G) \\
&= \alpha_1\, Q(g,C) + \frac{(n-3)\widetilde{\kappa}}{n(n+1)}\left(Q(g,R) - \frac{1}{n-2}\, Q(g,g\wedge S)\right) = \left(\alpha_1 + \frac{(n-3)\widetilde{\kappa}}{n(n+1)}\right) Q(g,C) \\
&= \left(\frac{1}{n-2}\left(\frac{\kappa}{n-1} + \varepsilon\psi - \frac{(n^2-3n+3)\widetilde{\kappa}}{n(n+1)}\right) + \frac{(n-3)\widetilde{\kappa}}{n(n+1)}\right) Q(g,C) \\
&= \frac{1}{n-2}\left(\frac{\kappa}{n-1} + \varepsilon\psi - \frac{(2n-3)\widetilde{\kappa}}{n(n+1)}\right) Q(g,C),
\end{aligned}$$

i.e. (1.24). Now (4.36) and (1.24) lead to (1.25). Our proposition is thus proved. □

We finish this section with the following result on quasi-Einstein hypersurfaces M in $N_s^{n+1}(c)$, $n \geq 4$, satisfying (1.18) on $\mathcal{U}_H \subset M$, i.e. (1.17), with $\rho = 0$, Precisely, we have

Theorem 4.8. *Let M be a hypersurface isometrically immersed in $N_s^{n+1}(c)$, $n \geq 4$, and let (1.11) and (1.18) be satisfied on $\mathcal{U}_H \subset M$. (i) The following conditions are satisfied on $\mathcal{U}_H$*

$$\alpha = \frac{\kappa}{n-1} - \frac{\widetilde{\kappa}}{n(n+1)}, \tag{4.37}$$
$$R\cdot C - C\cdot R = \frac{\kappa}{n-1}\, Q(g,C) - Q(S,C) + A, \tag{4.38}$$

where the function α is defined by (1.11) and the $(0,6)$-tensor A is defined by

$$A = Q\left(S - \frac{1}{n-1}\left(\frac{(n-2)\kappa}{n-1} + \frac{\widetilde{\kappa}}{n(n+1)}\right)g, C\right). \tag{4.39}$$

(ii) The condition (1.10) is satisfied on $\mathcal{U}_H$ if and only if (1.26) holds on this set.

Proof. (i) If (1.18) is satisfied on $\mathcal{U}_H$ then (1.16) holds on this set (see, e.g., Proposition 4.3(v)). Now (1.16), in view of [14, Theorem 2.3], implies

$$\sum_{(X_1,X_2),(X_3,X_4),(X_5,X_6)} (R\cdot C - C\cdot R)(X_1,X_2,X_3,X_4,X_5,X_6) = 0. \tag{4.40}$$

Further, (1.11) and (4.40), in view of [43, Proposition 2.1], yield (4.37) and

$$(n-2)(R\cdot C - C\cdot R) = Q(S,R) - \frac{\widetilde{\kappa}}{n(n+1)}Q(g,R). \tag{4.41}$$

The condition (4.41), by (4.37), (2.2) and (2.6), turns into

$$R\cdot C - C\cdot R = \frac{1}{n-2}Q(S,C) - \frac{\widetilde{\kappa}}{(n-2)n(n+1)}Q(g,C). \tag{4.42}$$

Now from (4.42) we easily get (4.38). (ii) From (4.38) it follows immediately that (1.10) holds on $\mathcal{U}_H$ if and only if $A=0$ on this set. Furthermore, we note that if $\operatorname{rank}(S-\alpha g)=1$ and $\operatorname{rank}(S-\beta g)=1$ on $\mathcal{U}_S\subset M$, for some functions α and β, respectively, then $\alpha=\beta$ at every point of $\mathcal{U}_S$ [41, Section 3]. Let now $A=0$ holds on $\mathcal{U}_H$. From this, in view of [28, Proposition 2.4] (see also [39, Lemma 3.4], or, [40, Lemma 3.1], or [41, Lemma 2.2]), (1.11), (4.37) and the mentioned above remark we easily deduce that

$$\frac{\kappa}{n-1} - \frac{\widetilde{\kappa}}{n(n+1)} = \frac{1}{n-1}\left(\frac{(n-2)\kappa}{n-1} + \frac{\widetilde{\kappa}}{n(n+1)}\right) \tag{4.43}$$

on $\mathcal{U}_H$. From (4.43) we immediately get (1.26)(a). Now $A=0$, by (1.26)(a) and (4.43), turns into (1.26)(b). Conversely, if (1.26) holds on $\mathcal{U}_H$ then we can check that $A=0$ on $\mathcal{U}_H$. □

Example 4.9. (i) As it was mentioned in Proposition 4.3(i), for every hypersurface M in $N_s^5(c)$ (1.17) reduces on $\mathcal{U}_H\subset M$ to (1.18). (ii) Let M be a hypersurface in $N_s^{n+1}(c)$, $n\geq 4$, satisfying (1.18) on $\mathcal{U}_H\subset M$ and let $Q(S-(\kappa/n)g,C)=0$ (i.e. (1.26)(b)) at a point $x\in\mathcal{U}_H$. From the last equation, in view of [28, Proposition 2.4] (also see [32, Proposition 2.1], or, [39, Lemma 3.4], or, [41, Lemma 2.2]) we have at x: (a) $\operatorname{rank}(S-(\kappa/n)g)=1$ and

$$\omega(X_1)C(X_2,X_3,X_4,X_5) + \omega(X_2)C(X_3,X_1,X_4,X_5) + \omega(X_3)C(X_1,X_2,X_4,X_5) = 1,$$

where ω is some 1-form at x and $X_1,X_2,\dots,X_5$ are vectors tangent to M at x, or, (b) $\operatorname{rank}(S-(\kappa/n)g)>1$ and the tensors C and $(S-(\kappa/n)g)\wedge(S-(\kappa/n)g)$ are linearly dependent. From this we get (1.13). An example of a quasi-Einstein hypersurface M in $N_s^{n+1}(c)$, $n\geq 4$, satisfying on $\mathcal{U}_H\subset M$ the conditions: (1.11), with $\alpha=\kappa/n$, (1.18) and (1.26) is given in [40, Section 5]. (iii) An example of a non-quasi-Einstein hypersurface M in an Euclidean space $\mathbb{E}^{n+1}$, $n\geq 5$, satisfying (1.17) on $\mathcal{U}_H=M$, with non-zero functions ψ and ρ, is given in [76, Example 5.1(iii)]. Precisely, on $\mathcal{U}_H$ we have: $\rho=(\kappa\operatorname{tr}(H))/(n-1)\neq 0$, $\psi=-(\kappa/(n-1))$, $\widetilde{\kappa}=0$,

$$H^3 = \operatorname{tr}(H)H^2 - \frac{\kappa}{n-1}H + \frac{\kappa\operatorname{tr}(H)}{n-1}g, \quad S^3 = \frac{2\kappa}{n-1}S^2 + \left(\frac{\operatorname{tr}(S^3)}{\kappa} - \frac{2\operatorname{tr}(S^2)}{n-1}\right)S,$$

$$R\cdot C = Q(S,R) + \frac{1}{n-2}Q\left(S^2 - \frac{\kappa}{n-1}S, G\right), \tag{4.44}$$

$$C\cdot R = \frac{n-3}{n-2}Q(S,R), \tag{4.45}$$

$$C\cdot C = \frac{n-3}{n-2}\left(Q(S,R) + \frac{1}{n-2}Q\left(S^2 - \frac{\kappa}{n-1}S, G\right)\right). \tag{4.46}$$

From (4.44)–(4.46) it follows immediately that

$$\begin{aligned}(n-2)(R\cdot C - C\cdot R) &= Q(S,R) + Q\left(S^2 - \frac{\kappa}{n-1}S, G\right),\\ C\cdot C &= C\cdot R + \frac{n-3}{(n-2)^2}Q\left(S^2 - \frac{\kappa}{n-1}S, G\right),\\ C\cdot C &= \frac{n-3}{n-2}R\cdot C,\end{aligned} \tag{4.47}$$

Using (4.47), $\psi = -(\kappa/(n-1))$, $\varepsilon = 1$ and $\widetilde{\kappa} = 0$ we can check that (1.24) holds on M. Furthermore, using (2.4), (4.44) and (4.45) we obtain

$$\begin{aligned} C\cdot R + \frac{1}{n-2}R\cdot C &= Q(S,R) - \frac{1}{(n-2)^2}Q\left(g, g\wedge\left(S^2 - \frac{\kappa}{n-1}S\right)\right)\\ &= Q(S,C) - \frac{1}{n-2}Q\left(g, \frac{1}{2}S\wedge S\right) + \frac{\kappa}{(n-2)(n-1)}Q(g, g\wedge S)\\ &\quad - \frac{1}{(n-2)^2}Q\left(g, g\wedge\left(S^2 - \frac{\kappa}{n-1}S\right)\right)\\ &= Q(S,C) - \frac{1}{(n-2)^2}Q\left(g, \frac{n-2}{2}S\wedge S - \kappa\, g\wedge S + g\wedge S^2\right) \end{aligned}$$

and in consequence (1.25).

5. The condition $R\cdot C - C\cdot R = L_1 Q(S,C) + L_2 Q(g,C)$

Let M be a hypersurface in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 4$, satisfying (1.27) on $\mathcal{U}_H \subset M$, for some functions L_1 and L_2 on $\mathcal{U}_H$. We note that in view of [31, Corollary 4.1] (1.17) holds on $\mathcal{U}_H$, and, in a consequence, (4.3) is satisfied on this set. First we consider quasi-Einstein hypersurfaces satisfying (1.27).

Theorem 5.1. *If M is a hypersurface in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 4$, satisfying on $\mathcal{U}_H \subset M$ the conditions (1.11) and (1.27), for some functions α, L_1 and L_2, then on this set we have (1.18), for some function ψ, and (1.28), (4.37), (4.38), (4.39). Moreover, (1.10) is satisfied on $\mathcal{U}_H$ if and only if (1.26) holds on this set.*

Proof. From (1.27), by an application of Lemma 2.1(i), we get (4.40). Now (1.11) and (4.40), in view of [43, Proposition 2.1], yield (1.16), (4.37) and

$$(n-2)(R\cdot C - C\cdot R) = Q(S,R) - \frac{\widetilde{\kappa}}{n(n+1)}Q(g,R). \tag{5.1}$$

From [18, Proposition 3.2, Theorem 3.1] it follows that on $\mathcal{U}_H$ (1.16) is equivalent to (1.18). Next, applying to (5.1), the conditions (2.2), (2.3)(b) and (2.6), we get (1.28). Now Theorem 4.8 completes the proof. □

In [43, Section 4, Example 4.1] an example of a quasi-Einstein non-pseudosymmetric Ricci-pseudosymmetric warped product $\overline{M}\times_F \widetilde{M}$, $\dim\overline{M} = 1$, $\dim\widetilde{N} = n-1 \geq 4$, with $\kappa/(n-1) \neq \widetilde{\kappa}/(n+1)$, which can be locally realized as a hypersurface M isometrically immersed in some semi-Riemannian space of constant sectional curvature $N_s^{n+1}(c)$, $n \geq 5$, was constructed. The manifold $(\widetilde{N}, \widetilde{g})$ used in that construction is an Einstein manifold. Moreover, the condition (1.16) holds on M [43, eq. (4.7)]. Thus in view of [18, Proposition 3.1(iii)] (1.18) holds on that hypersurface. Finally, in view of Theorem 5.1, Eq. (1.28) is also satisfied on M.

Now we consider non-quasi-Einstein hypersurfaces satisfying (1.27). Precisely, we consider (1.27) at points of $\mathcal{U}_H$ at which (1.14) holds. We have

Theorem 5.2. *Let M be a hypersurface in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 4$, satisfying on $\mathcal{U}_H \subset M$ Eq. (1.27). If at a point $x \in \mathcal{U}_H$ the condition (1.14) is satisfied then at this point we have: (1.10) and*

$$\begin{aligned} (n-1)\,Q(S,R) &= Q\left(g, \left(\varepsilon\psi + \kappa - \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\right)R\right.\\ &\quad \left. + \left(\varepsilon\psi - \frac{2(n-1)\widetilde{\kappa}}{n(n+1)}\right)g\wedge S + g\wedge S^2 - \frac{1}{2}S\wedge S\right). \end{aligned} \tag{5.2}$$

Proof. The conditions (4.3) and (1.27), by (2.2), (2.3)(a), (2.4) and (4.8), turn into

$$\begin{aligned} (n-2)(R\cdot C - C\cdot R) &= Q(S,R) - \alpha_3\, Q(g, g\wedge S) - Q(g, g\wedge S^2)\\ &\quad + \left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} - \frac{\kappa}{n-1} - \varepsilon\psi\right)Q(g,R), \end{aligned} \tag{5.3}$$

$$\begin{aligned} (n-2)(R\cdot C - C\cdot R) &= (n-2)L_1\, Q(S,R) + L_1\, Q\left(g, \frac{1}{2}S\wedge S\right)\\ &\quad + (n-2)L_2\, Q(g,R) - \left(L_2 + \frac{\kappa L_1}{n-1}\right)Q(g, g\wedge S), \end{aligned} \tag{5.4}$$

respectively.

I. The case: $L_1 = 1/(n-2)$ at a point $x \in \mathcal{U}_H$. From (5.3) and (5.4) we obtain

$$\left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} - \frac{\kappa}{n-1} - \varepsilon\psi - (n-2)L_2\right) Q(g,R) - Q(g, g\wedge S^2)$$
$$-\frac{1}{n-2}\, Q(g, \frac{1}{2}\, S\wedge S) + \left(L_2 + \frac{\kappa}{(n-2)(n-1)} - \alpha_3\right) Q(g, g\wedge S) = 0. \tag{5.5}$$

I(*a*). The subcase:

$$L_2 = \frac{1}{n-2}\left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} - \frac{\kappa}{n-1} - \varepsilon\psi\right)$$

at a point $x \in \mathcal{U}_H$. Now (5.5) reduces to

$$\frac{1}{n-2}\, Q(g, \frac{1}{2}\, S\wedge S) + Q(g, g\wedge S^2) + \left(\alpha_3 - L_2 - \frac{\kappa}{(n-2)(n-1)}\right) Q(g, g\wedge S) = 0,$$

which yields

$$\frac{1}{2(n-2)}\, S\wedge S + g\wedge S^2 + \left(\alpha_3 - L_2 - \frac{\kappa}{(n-2)(n-1)}\right) g\wedge S + \lambda_1 G = 0, \quad \lambda_1 \in \mathbb{R},$$

and, by a suitable contraction, leads to

$$\frac{(n-3)(n-1)}{n-2}\, S^2 + \left(\alpha_3 - L_2 + \frac{\kappa}{(n-2)(n-1)}\right) S + \lambda_2 g = 0, \quad \lambda_2 \in \mathbb{R}.$$

The last two equations yield $(1/2)S\wedge S = \beta_1 g\wedge S + \beta_2 G$, $\beta_1,\ \beta_2 \in \mathbb{R}$. From this, in view of [56, Lemma 3.1], we obtain (1.11), with $\alpha = \beta_1$, which contradicts (1.14). Thus we see that the case *I*(*a*) cannot occur at *x*.

I(*b*). The subcase:

$$L_2 \neq \frac{1}{n-2}\left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} - \frac{\kappa}{n-1} - \varepsilon\psi\right)$$

at a point $x \in \mathcal{U}_H$. Now (5.5) turns into

$$Q(g,R) = \alpha_4\, Q(g, g\wedge S^2) + \frac{\alpha_4}{n-2}\, Q\left(g, \frac{1}{2}\, S\wedge S\right) + \alpha_5\, Q(g, g\wedge S), \tag{5.6}$$
$$\alpha_4 = \left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} - \frac{\kappa}{n-1} - \varepsilon\psi - (n-2)L_2\right)^{-1},$$
$$\alpha_5 = \alpha_4\left(\alpha_3 - L_2 - \frac{\kappa}{(n-2)(n-1)}\right).$$

From (5.6) we get

$$R = \alpha_4\, g\wedge S^2 + \frac{\alpha_4}{2(n-2)}\, S\wedge S + \alpha_5\, g\wedge S + \lambda_4 G, \quad \lambda_4 \in \mathbb{R}. \tag{5.7}$$

This, by a suitable contraction, yields

$$\frac{(n-3)(n-1)}{n-2}\, S^2 = \alpha_6\, S + \lambda_5\, g, \quad \alpha_6, \lambda_5 \in \mathbb{R}. \tag{5.8}$$

From (5.7) and (5.8) we get $R = (\alpha_4/(2(n-2)))S\wedge S + \alpha_7\, g\wedge S + \lambda_6 G$, $a_7, \lambda_6 \in \mathbb{R}$, which, in view of Theorem 2.4, gives (1.2) and (1.10). Now $L_1 = 1/(n-2)$, (1.10) and (1.27) lead to

$$Q(S,C) = \frac{n-2}{n-1}\left(\frac{\kappa}{n-1} - L_2\right) Q(g,C),$$
$$R\cdot C - C\cdot R = \frac{1}{n-1}\left(\frac{\kappa}{n-1} + (n-2)L_2\right) Q(g,C).$$

The last condition, in view of [42, Theorem 4.1], yields

$$C\cdot R = 0, \quad R\cdot C = \frac{1}{n-1}\left(\frac{\kappa}{n-1} + (n-2)L_2\right) Q(g,C). \tag{5.9}$$

As it was stated above, (1.2) holds at *x*. Since $x \in \mathcal{U}_H$, from Proposition 3.2(ii) it follows that $\operatorname{rank} H = 2$ and (1.19) are satisfied at this point. Further, from Proposition 4.3(ii) we have (4.20) and (4.21). Now (4.21), (5.9) and $Q(g,R) \neq 0$ imply $\widetilde{\kappa}/(n+1) = \kappa/(n-1)$, which contradicts (4.18). Thus we see that the case *I*(*b*) cannot occur at *x*.

II. The case: $L_1 \neq 1/(n-2)$ at a point $x \in \mathcal{U}_H$. From (5.3) and (5.4) it follows that the tensor $Q(S,R)$ is a linear combination of the tensors of the form $Q(g,T)$, where T is a generalized curvature tensor. Thus, in view of Theorem 4.6, the tensors $R\cdot R$, $R\cdot C$, $C\cdot R$, $R\cdot C - C\cdot R$ and $C\cdot C$ satisfy (4.25), (4.27), (4.29), (4.31) and (4.33), with the tensors B_1, B_2, B_3, B_4 and B, defined by (4.26), (4.28), (4.30), (4.32) and (4.34), respectively. Now, (4.31), (4.32) and (5.4) yield

$$\begin{aligned}
&(n-2)\left(-\frac{\varepsilon\psi}{n-1}+\frac{\widetilde{\kappa}}{n(n+1)}-L_2\right)Q(g,R)\\
&+\left((n-2)\left(-\frac{\varepsilon\psi}{n-1}+\frac{2\widetilde{\kappa}}{n(n+1)}\right)+L_2+\frac{\kappa L_1}{n-1}\right)Q(g,g\wedge S)\\
&-\frac{n-2}{n-1}Q(g,g\wedge S^2)-\left(\frac{1}{n-1}+L_1\right)Q\left(g,\frac{1}{2}S\wedge S\right)=(n-2)L_1\,Q(S,R)
\end{aligned}$$

and

$$\begin{aligned}
(n-2)L_1\,Q(S,R) = Q\Bigg(&g,(n-2)\left(-\frac{\varepsilon\psi}{n-1}+\frac{\widetilde{\kappa}}{n(n+1)}-L_2\right)R\\
&+\left((n-2)\left(-\frac{\varepsilon\psi}{n-1}+\frac{2\widetilde{\kappa}}{n(n+1)}\right)+L_2+\frac{\kappa L_1}{n-1}\right)g\wedge S\\
&-\frac{n-2}{n-1}g\wedge S^2-\left(\frac{1}{n-1}+L_1\right)\frac{1}{2}S\wedge S\Bigg).
\end{aligned}\tag{5.10}$$

But on the other hand, (2.2), (3.3) and (4.26) yield

$$\begin{aligned}
&(n-2)L_1\,Q(S,R)\\
&=Q\left(g,\frac{(n-2)L_1}{n-1}\left(\left(\kappa+\varepsilon\psi-\frac{(n-1)^2\widetilde{\kappa}}{n(n+1)}\right)R-\frac{1}{2}S\wedge S+g\wedge S^2+\left(\varepsilon\psi-\frac{(n-1)\widetilde{\kappa}}{n(n+1)}\right)g\wedge S\right)\right)\\
&\quad+\frac{(n-2)^2\widetilde{\kappa}L_1}{n(n+1)}Q(g,R)-\frac{(n-2)\widetilde{\kappa}L_1}{n(n+1)}Q(g,g\wedge S)
\end{aligned}$$

and

$$\begin{aligned}
(n-2)L_1\,Q(S,R) = Q\Bigg(&g,\frac{(n-2)L_1}{n-1}\left(\left(\kappa+\varepsilon\psi-\frac{(n-1)\widetilde{\kappa}}{n(n+1)}\right)R-\frac{1}{2}S\wedge S+g\wedge S^2\right.\\
&\left.+\left(\varepsilon\psi-\frac{2(n-1)\widetilde{\kappa}}{n(n+1)}\right)g\wedge S\right)\Bigg).
\end{aligned}\tag{5.11}$$

Comparing now the right hand sides of (5.10) and (5.11) we obtain

$$\begin{aligned}
&\left(-\frac{\varepsilon\psi}{n-1}+\frac{\widetilde{\kappa}}{n(n+1)}-L_2-\frac{L_1}{n-1}\left(\kappa+\varepsilon\psi-\frac{(n-1)\widetilde{\kappa}}{n(n+1)}\right)\right)R\\
&+\left(-\frac{\varepsilon\psi}{n-1}+\frac{2\widetilde{\kappa}}{n(n+1)}+\frac{L_2}{n-2}+\frac{L_1}{n-1}\left(\frac{\kappa}{n-2}-\varepsilon\psi+\frac{2(n-1)\widetilde{\kappa}}{n(n+1)}\right)\right)g\wedge S\\
&-\frac{1}{n-1}(1+L_1)g\wedge S^2-\frac{1}{(n-2)(n-1)}(1+L_1)\frac{1}{2}S\wedge S-\frac{\lambda_1}{n-2}G=0.
\end{aligned}\tag{5.12}$$

The case *II*(*a*): $L_1\neq -1$ at a point $x\in\mathcal{U}_H$. From (5.12), by a suitable contraction, we get $S^2=\beta_1 S+\beta_2 g$, for some $\beta_1,\ \beta_2\in\mathbb{R}$. From Proposition 4.3(ii) it follows that $\rho=0$, $\beta_2=-(\lambda/n)$ and $\beta_1=\alpha_3$, where ρ, λ and α_3 are defined by (1.17), (4.9) and (4.10), respectively. Now (5.12) turns into

$$\begin{aligned}
&\left(-\frac{\varepsilon\psi}{n-1}+\frac{\widetilde{\kappa}}{n(n+1)}-L_2-\frac{L_1}{n-1}\left(\kappa+\varepsilon\psi-\frac{(n-1)\widetilde{\kappa}}{n(n+1)}\right)\right)R\\
&=\frac{1}{(n-2)(n-1)}(1+L_1)\frac{1}{2}S\wedge S+\beta_3 g\wedge S+\beta_4 G,\quad \beta_3,\beta_4\in\mathbb{R}.
\end{aligned}\tag{5.13}$$

We note that if

$$-\frac{\varepsilon\psi}{n-1}+\frac{\widetilde{\kappa}}{n(n+1)}-L_2-\frac{L_1}{n-1}\left(\kappa+\varepsilon\psi-\frac{(n-1)\widetilde{\kappa}}{n(n+1)}\right)=0$$

at a given point then (5.13) reduces to $(1/2)S\wedge S=\beta_5 g\wedge S+\beta_6 G$, $\beta_5,\beta_6\in\mathbb{R}$. From this, in view of [56, Lemma 3.1], we get (1.11), with $\alpha=\beta_5$, which contradicts (1.14). Thus, we see that at this point we must necessarily have

$$-\frac{\varepsilon\psi}{n-1}+\frac{\widetilde{\kappa}}{n(n+1)}-L_2-\frac{L_1}{n-1}\left(\kappa+\varepsilon\psi-\frac{(n-1)\widetilde{\kappa}}{n(n+1)}\right)\neq 0.$$

Now (5.13) turns into (1.13). Thus, in view of Theorem 2.4(ii), (1.10) holds at x. The conditions (1.10) and (1.27) give

$$Q((L_1+1)S-(L_2-\frac{\kappa}{n-1})g,C)=0.$$

From this, in view of [28, Proposition 2.4] (also see [32, Proposition 2.1], or, [39, Lemma 3.4], or, [41, Lemma 2.2]) and (1.14), we get

$$C=\frac{\lambda}{2}\left((L_1+1)S-\left(L_2-\frac{\kappa}{n-1}\right)g\right)\wedge\left((L_1+1)S-\left(L_2-\frac{\kappa}{n-1}\right)g\right),\quad \lambda\in\mathbb{R}.$$

This, by a suitable contraction, yields

$$\begin{aligned}S^2&=(\kappa-(n-2)(L_2-\frac{\kappa}{n-1})(L_1+1)^{-1})S\\&-\left(\left(L_2-\frac{\kappa}{n-1}\right)^2+\left(\kappa(L_1+1)-n\left(L_2-\frac{\kappa}{n-1}\right)\right)\left(L_2-\frac{\kappa}{n-1}\right)\right)(L_1+1)^{-2}g.\end{aligned}\tag{5.14}$$

But on the other hand, in view Theorem 2.4(i), we have

$$S^2=(\kappa+((n-2)\mu-1)\phi^{-1})S+(\mu\kappa+(n-1)\eta)\phi^{-1}g.\tag{5.15}$$

From (5.14) and (5.15) we get

$$L_2-\frac{\kappa}{n-1}=\frac{1}{n-2}(1-(n-2)\mu)\phi^{-1}(L_1+1),\tag{5.16}$$

$$\begin{aligned}&\left((n-1)\left(L_2-\frac{\kappa}{n-1}\right)-\kappa(L_1+1)\right)\left(L_2-\frac{\kappa}{n-1}\right)\\&=(\mu\kappa+(n-1)\eta)\phi^{-1}(L_1+1)^2.\end{aligned}\tag{5.17}$$

Now (5.16) and (5.17) yield

$$\begin{aligned}&(n-1)(1-(n-2)\mu)^2-(n-2)\kappa(1-(n-2)\mu)\phi=(n-2)^2(\mu\kappa+(n-1)\eta)\phi,\\&\kappa=-(n-2)(n-1)\eta+\frac{n-1}{n-2}(1-(n-2)\mu)^2\phi^{-1}.\end{aligned}$$

The last condition, by making use of Proposition 4.3(iii), leads to $\widetilde{\kappa}/(n+1)=\kappa/(n-1)$, a contradiction with (4.18). Thus we see that the case $II(a)$ cannot occur at x.

The case $II(b)$: $L_1=-1$ at a point $x\in\mathcal{U}_H$. Thus (5.12) reduces to

$$\left(L_2-\frac{\kappa}{n-1}\right)\left(R-\frac{1}{n-2}g\wedge S\right)+\lambda_2 G=0,\quad \lambda_2\in\mathbb{R},$$

which yields

$$\left(L_2-\frac{\kappa}{n-1}\right)\left(R-\frac{1}{n-2}g\wedge S+\frac{\kappa}{(n-2)(n-1)}G\right)=\lambda_3 G,\quad \lambda_3\in\mathbb{R}.$$

This, by (2.2), turns into $(L_2-\kappa/(n-1))C=\lambda_3 G$, which implies $(L_2-\kappa/(n-1))C=0$ and, in a consequence, $L_2=\kappa/(n-1)$. Now (5.10) reduces to (5.2). Our theorem is thus proved. □

As a consequence of [31, Corollary 4.1] (see also Section 4) and Theorems 4.6 and 5.2 we get

Theorem 5.3. *Let M be a hypersurface in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n\geq 4$, and let (1.14) and (1.27) be satisfied at every point of the set $\mathcal{U}_H\subset M$. Then (1.10) and (5.2) hold on $\mathcal{U}_H$. Moreover, the tensors $R\cdot R$, $R\cdot C$, $C\cdot R$, $R\cdot C-C\cdot R$ and $C\cdot C$ satisfy (4.25), (4.27), (4.29), (4.31) and (4.33), with the tensors B_1, B_2, B_3, B_4 and B, defined by (4.26), (4.28), (4.30), (4.32) and (4.34), respectively.*

Finally, in view of Theorems 2.3(i), 2.4(ii), 3.3, 3.4 and 5.2, we obtain

Theorem 5.4. *If at every point of a non-quasi-Einstein hypersurface M in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n\geq 4$, the difference tensor $R\cdot C-C\cdot R$ is a linear combination of the Tachibana tensors $Q(g,C)$ and $Q(S,C)$, then (1.10) holds on M.*

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