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Noncommutative space from dynamical noncommutative geometries

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ABSTRACT

We present noncommutative topology as a basis for noncommutative geometry phrased completely in terms of partially ordered sets with operations. In this note we introduce a noncommutative space-time starting from a dynamical system of noncommutative topologies based on the notion of temporal points. At every moment a commutative topological space is constructed and it is shown to approximate the noncommutative space in sheaf theoretical terms; this so called moment space should be the space where observed phenomena should be described, the commutative shadow of the noncommutative space is to be thought of as the usual space-time.

0. Introduction

Recent developments in noncommutative geometry deal with different notions including: noncommutative manifolds [2,3], noncommutative (quantized) algebras, [4,5], or general quantization-deformations, [6,9]. In these theories the actual geometric objects are left as virtual objects and the results deal with noncommutative algebras or some categories thought of as rings of functions or categories of modules. In the approach of [1] a noncommutative topology is used and sheaf theory on a noncommutative topology replaces function theory to some extent. The noncommutativity of the topology and space is characterized by the lack of geometric points in the sense that there are not enough "points" in the space to identify the opens as sets of points, contrasting the set-theory based (commutative) geometry with its function theory and corresponding analysis. Yet, aiming at application in Physics, a suitable noncommutative model of space should allow explicit calculation and some level of geometric reasoning, making use of point coordinates in terms of real or complex numbers unavoidable.

In the model we propose, one of the examples being a candidate for noncommutative space–time, we start from a dynamical system of noncommutative topologies. All structural properties will depend only on a few intuitive axioms at the level of ordered structures. We start from a totally ordered set T and suitably connected noncommutative topological spaces $\Lambda_t, t \in T$. At each $t \in T$ we construct a new spectral space, called the moment space at t, which is in fact a commutative space. Then we view (pre-)sheaves on noncommutative topologies fitting together in what should be called dynamical geometry and connect them to sheaves on the moment spaces. The idea that those commutative moment spaces present us with a useful approximation of noncommutative space is reflected in one of the main results of this paper stating that the stalks at points of the moment space at $t \in T$ equal stalks at a "point" of some $\Lambda_{t'}, t' \in T$ in some prescribed T-interval containing t. The philosophy here is that there may not exist enough geometric points in the noncommutative geometry of Λ_t but it works in the moment space because the latter encodes dynamical information in some T-interval containing t. The construction is abstract and can be extended to further levels of generality and abstraction, however for possible application in physics only one case is of real interest . . . reality. To literally realize the noncommutative space we would think of T as a multidimensional irreversible time (with a suitable notion of dimension) of "dimension" big enough to make up for the missing geometric points in the noncommutative spaces $\Lambda_t, t \in T$, so that the T-size explains the difference

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in dimension between the noncommutative space or the moment spaces and the geometric dimension of its space of points that appears as the commutative shadow of the noncommutative space in the sense of [1]. The commutative shadow (defined as the lattice of idempotents in a noncommutative topology) may be thought of as the abstract space where mathematical reasoning will take place, that is to say: space–time with its own "abstract" time appearing as a reversible relativistic time. The moment space is the commutative space where the mathematics of observed and measured objects takes place and because observing and measuring takes time this moment space has to encode dynamical data. Allowing ourselves a slogan: observation creates space from time ! Thinking of a smallest possible observable in space, let us say a point standing still in space, it is in different $\Lambda_t, t \in T$ nevertheless, and therefore in the commutative shadow it appears as a string; on the other hand viewed in the moment space(s) it appears as a string of temporal points (not geometric points, see explanation in the paper) that is a string of opens in the commutative moment spaces to be thought of as a tube or higher-dimensional string, a brane perhaps? Obviously this suggests a link to M-theory that appears from the mathematical formalism, based on simple axioms solely dealing with partial ordered structures, describing the transfer between commutative shadows and the moment space. The difference between these two commutative worlds describing a noncommutative space (a next step of approximation of reality) is marked by the uncertainty principle whereby the noncommutative of measured objects in the moment space in fact reflects the "noncommutativity of reality" in the dynamic model (it makes sense to view it as a kind of quantum commutativity, in fact it is linked to the almost commutativity of the Weyl algebra with respect to the Bernstein filtration, see [1]).

The noncommutative space in the dynamic model may be viewed as a spatial-temporal deformation of its commutative shadow and the construction of moment space as a dequantization i.e. a converse to quantization. Basic new ingredients are noncommutative topologies and mainly generalized Stone spaces. The latter generalize the idea of constructing the Stone space for the lattice of closed linear subspaces for a Hilbert space H. The quantization–dequantization technique should also allow tracing of certain physical aspects like: observables, spectral families, ... The final section is devoted to this and Γ-spectral families, for some totally ordered group Γ, are just separated filtrations on a noncommutative topology; for a Hilbert space H this yields a nice relation to (pseudo-)valuation theory via the appearance of pseudo-places on the projective Hilbert space $\mathbb{P}(H)$ of H.

There are many mathematical issues connected to the topic of the paper (we refer to [7] for more detail) most of those unfinished or at an initial stage of development. The bold idea behind this paper was to try to make noncommutative geometry real and to to propose a construction that would also allow concrete calculations in a commutative approximation; in any case author believes that the mathematics developed also has enough interest in its own right, while philosophy in the background of it may be thought provoking.

1. Preliminaries on noncommutative topology

We consider (partially ordered) Λ with 0 and 1. When Λ is equipped with operations $\wedge$ and $\vee$, we say that Λ is a **noncommutative topology** if the following axioms hold:

(i) for $x, y \in \Lambda$, $x \wedge y \leq x$ and $x \wedge y \leq y$ and $x \wedge 1 = 1 \wedge x = x$, $x \wedge 0 = 0 \wedge x = 0$, moreover $x \wedge \cdots \wedge x = 0$ if and only if $x = 0$.
(ii) For $x, y, z \in \Lambda$, $x \wedge y \wedge z = (x \wedge y) \wedge z = x \wedge (y \wedge z)$, and if $x \leq y$ then $z \wedge x \leq z \wedge y$, $x \wedge z \leq y \wedge z$.
(iii) Properties similar to (i) and (ii) with respect to $\vee$, in particular $x \vee x \vee \cdots \vee x = 1$ if and only if $x = 1$.
(iv) Let $\mathrm{id}_\wedge(\Lambda) = \{\lambda \in \Lambda, \lambda \wedge \lambda = \lambda\}$; for $x \in \mathrm{id}_\wedge(\Lambda)$ and $x \leq z$ we have: $x \vee (x \wedge z) \leq (x \vee x) \wedge z$, $x \vee (z \wedge x) \leq (x \vee z) \wedge x$.
(v) For $x \in \Lambda$ and $\lambda_1, \ldots, \lambda_n \in \Lambda$ such that $1 = \lambda_1 \vee \cdots \vee \lambda_n$, we have $x = (x \wedge \lambda_1) \vee \cdots \vee (x \wedge \lambda_n) = (\lambda_1 \wedge x) \vee \cdots \vee (\lambda_n \wedge x)$.

There are left (right) versions of this definition as introduced in [1], but we do not go into this here. In fact we restrict attention to the situation where $\vee$ is a commutative operation and Λ is $\vee$-complete i.e. for an arbitrary family $\mathcal{F}$ of elements in Λ, $\vee\mathcal{F}$ exists in Λ, where $\vee\mathcal{F}$ is characterized by the property: $\lambda \leq \vee\mathcal{F}$ for all $\lambda \in \mathcal{F}$ and if $\lambda \leq \mu$ for all $\lambda \in \mathcal{F}$ then $\vee\mathcal{F} \leq \mu$.

In case only (v) is dropped we call Λ a **skew topology**; this is sometimes interesting e.g. the lattice $L(H)$ of a Hilbert space H is a skew topology and condition (v) does not hold! Here $L(H)$ is the lattice of closed linear subspaces of H with respect to intersection and closure of the sum. The definition of a noncommutative topology allows $\lambda \vee \lambda \neq \lambda$, $\lambda \wedge \lambda \neq \lambda$, in fact this is the main aspect pf noncommutativity. Let us write $\mathrm{id}_\wedge(\Lambda) = \{\lambda \in \Lambda, \lambda \wedge \lambda = \lambda\}$, $\mathrm{id}_\vee(\Lambda) = \{\lambda \in \Lambda, \lambda \vee \lambda = \lambda\}$. The restriction to abelian $\vee$ (most interesting examples are like this) entails that $\mathrm{id}_\wedge(\Lambda)$ is closed under the operation $\vee$. Moreover on $\mathrm{id}_\wedge(\Lambda)$ we introduce a new operation $\underset{\cdot}{\wedge}$ defined by $\sigma \underset{\cdot}{\wedge} \tau = \vee\{\gamma \in \mathrm{id}_\wedge(\Lambda), \gamma \leq \sigma \wedge \tau\}$ for $\sigma, \tau \in \mathrm{id}_\wedge(\Lambda)$. Let us write $SL(\Lambda)$ for the set $\mathrm{id}_\wedge(\Lambda)$ with the operations $\underset{\cdot}{\wedge}$ and $\vee$; then $SL(\Lambda)$ is easily checked to be again a skew topology (with $\vee$ commutative and being $\vee$-complete). We refer to $SL(\Lambda)$ as the **commutative shadow** of Λ.

1.1. Lemma (cf. [1] or [7] 2.2.3. and 2.2.5)

If Λ is a skew topology $\vee$-complete with respect to a commutative operation $\vee$, then $SL(\Lambda)$ is a lattice satisfying the modular inequality.

A subset $X \subset \Lambda$ is **directed** if for every $x, y \in X$ there is a $z \in X$ such that $z \leq x$, $z \leq y$; we say that X is a **filter** in Λ if it is directed and for $x \in X$, $x \leq y$ yields $y \in X$. Two directed sets A and B in Λ are equivalent if they define the same filter $\overline{A} = \overline{B}$ where for any directed set A we put $\overline{A} = \{\lambda \in \Lambda$, there is an $a \in A$ such that $a \leq \lambda\}$. Let $\mathcal{D}(\Lambda)$ be the set of directed subsets in Λ and we write $A \sim B$ when the directed subsets A and B are equivalent, we write $[A]$ for the equivalence class of A and let $C(\Lambda)$ be the set of classes of directed subsets of Λ. We introduce a partial order in $C(\Lambda)$ by putting $A \leq B$ if $\overline{B} \subset \overline{A}$ and write $[A] \leq [B]$ for the partial order on $\mathcal{D}(\Lambda)$ induced by the foregoing. For A and B in $\mathcal{D}(\Lambda)$ define $A\dot{\wedge}B = \{a \wedge b, a \in A, b \in B\}$, $A\dot{\vee}B = \{a \vee b, a \in A, b \in B\}$ and: $[A] \wedge [B] = [A\dot{\wedge}B]$, $[A] \vee [B] = [A\dot{\vee}B]$.

The lattice $L(H)$ mentioned before is its own commutative shadow, even though it is not distributive (hence the failure of condition v. as above it means that the geometry of $L(H)$ is in some sense still commutative and only by looking at words in the projection operators P_U for $U \in L(H)$ does one arrive at a noncommutative situation (a linear basis for the algebra of bounded linear operators $\mathcal{L}(H)$). The lattice of torsion theories for a ring or in fact for any abelian category is the commutative shadow of a noncommutative topology obtained by looking at the left exact preradicals, see [7] for full detail. An interesting specific example of a skew topology is obtained by taking the poset of reflexive relations $\mathcal{R}_r(A)$ on some set A, that is the subsets of $A \times A$ containing the diagonal $\Delta(A)$ in $A \times A$. For $\vee$ we take the intersection of such subsets and for $\wedge$ the composition, that is to say for relations ρ_1, ρ_2 on A we let $\rho_1 \wedge \rho_2$ be given by the subset of $A \times A$ consisting of (a, a'') for which there exists an $a' \in A$ such that $(a, a') \in \rho_1$ and $(a', a'') \in \rho_2$. The commutative shadow of this is given by the reflexive transitive relations, $R_{r,t}(A)$.

1.2. Lemma

If Λ is a skew topology, resp. a noncommutative topology, then so is $C(\Lambda)$ with respect to the partial order and operations $\wedge$ and $\vee$ as defined above. The canonical map $\Lambda \to C(\Lambda)$, $\lambda \mapsto [\{\lambda\}]$ is a map of skew topologies.

We simplify notations by writing $[\{\lambda\}] = [\lambda]$ and call such an element **classical** in $C(\Lambda)$.

A directed set A in Λ is **idempotently directed** if for every $a \in A$ there exists an $a' \in A \cap \mathrm{id}_\wedge(\Lambda)$ such that $a' \leq a$; in this case $[A] \in i_\wedge(C(\Lambda))$ but these elements of $C(\Lambda)$ may be thought of as obtained from a directed set having a cofinal subset of "commutative" opens (the idempotents belonging to the commutative shadow $S((\Lambda))$. We write $\mathrm{Id}_\wedge(C(\Lambda))$ for the subset of $\mathrm{id}_\wedge(C(\Lambda))$ consisting of the classes of idempotently directed subsets of Λ; the elements $[A]$ of $\mathrm{Id}_\wedge(\mathrm{Id}_\wedge(C(\Lambda)))$ are called **strongly idempotents**. We identify Λ and the image of $\Lambda \to C(\Lambda)$, then observe that $\mathrm{Id}_\wedge(C(\Lambda)) \cap \Lambda = \mathrm{id}_\wedge(\Lambda)$. We shall write $\prod(\Lambda)$ for the skew (noncommutative) topology obtained by taking $\wedge$-finite bracketed expression $P(\wedge, \vee, x_i)$ in terms of strong idempotents $x_i \in \mathrm{Id}_\wedge(C(\Lambda))$; similarly we write $T(\Lambda)$ for the skew (noncommutative) topology obtained by taking $\wedge$-finite bracketed expressions $p(\wedge, \vee, \lambda_i)$ in idempotents $\lambda_i \in \mathrm{id}_\wedge(\Lambda)$.

It is not hard to verify: $\prod(\Lambda) = \prod(T(\Lambda))$, so we just write $\prod$ to denote this. Moreover $C(T(\Lambda))$ satisfies the same axioms ((i)–(v)) as Λ and $T(\Lambda)$ but with respect to $\mathrm{Id}_\wedge(C(\Lambda))$. The **"strong" commutative shadow** of $\prod$ is obtained by restricting $\dot{\wedge}$ on $\mathrm{id}_\wedge(C(\Lambda))$ to $\mathrm{Id}_\wedge(C(\Lambda))$ and viewing $SL_s(\prod)$ as the lattice structure induced on $\mathrm{Id}_\wedge(C(\Lambda))$, where s in the notation refers to "strong".

1.3. Lemma

If Λ is a $\vee$-complete noncommutative topology such that $\vee$ is commutative then: $SL_s(\prod) = C(SL(\Lambda))$.

1.4. Definition. Generalized stone topology

Consider a skew topology Λ and $C(\Lambda)$. For $\lambda \in \Lambda$, let $O_\lambda \subset C(\Lambda)$ be given by $O_\lambda = \{[A], \lambda \in \overline{A}\}$. It is very easy to verify: $O_{\lambda\wedge\mu} \subset O_\lambda \cap O_\mu$, $O_{\lambda\vee\mu} \supset O_\lambda \cup O_\mu$, hence the O_λ define a basis for a topology on $C(\Lambda)$ termed: generalized Stone topology. This definition obtains a more classical meaning in terms of points and sets when related to suitable point-spectra constructed in $C(\Lambda)$.

We say that $[A]$ in $C(\Lambda)$ is a **minimal point of** Λ if $\overline{A}$ is a maximal filter, i.e. $\overline{A} \neq \Lambda$ but if $\overline{A} \subsetneq B$ where B is a filter then $B = \Lambda$; it follows that a minimal point is necessarily in $\mathrm{id}_\wedge(C(\Lambda))$ and it is indeed a minimal nonzero element of the poset $C(\Lambda)$. An **irreducible point** $[A]$ of Λ is characterized by either one of the following equivalent properties:

(i) $[A] \leq \vee\{[A_\alpha], \alpha \in \mathcal{A}\}$ yields $[A] \leq [A_\alpha]$ for some $\alpha \in \mathcal{A}$.
(ii) If $\vee\{\lambda_\alpha, \alpha \in \mathcal{A}\} \in \overline{A}$ then $\lambda_\alpha \in \overline{A}$ for some $\alpha \in \mathcal{A}$.

More general types of points may be considered, e.g. the elements of a so-called quantum-basis, cf [7], but we need not go into this here. Under some suitable condition often (but not always) present in examples, the irreducible points in $\mathrm{Id}_\wedge(C(\Lambda))$ are exactly those that are $\vee$-irreducible in $C(\Lambda)$ (e.g. if Λ satisfies the weak FDI property, cf. [1], Proposition 5.9.).

Define the (irreducible) **point-spectrum** by putting: $\mathrm{Sp}(\Lambda) = \{[p], [p]$ an (irreducible) point of $\Lambda\}$. Put $p(\lambda) = \{[p] \in \mathrm{Sp}(\Lambda), [p] \leq [\lambda]\}$ for $\lambda \in \Lambda$, then $p(\lambda \wedge \mu) \subset p(\lambda) \cap p(\mu)$, $p(\lambda \vee \mu) = p(\lambda) \cup p(\mu)$. Thus the $p(\lambda)$ define a basis for a topology on $\mathrm{Sp}(\lambda)$ called the point-topology. Write $SP(\Lambda)$ for $Sp(\Lambda) \cap \mathrm{Id}_\wedge(C(\Lambda))$ and refer to this as the **Point-spectrum** (capital P). For $\lambda \in \Lambda$ we consider $P(\lambda) = \{[P] \in \mathrm{SP}(\Lambda), [P] \leq [\lambda]\}$ and this indices the Point-topology on $\mathrm{SP}(\Lambda)$; this time we even have $P(\lambda \wedge \mu) = P(\lambda) \cap P(\mu)$ and this time we even have $P(\lambda \wedge \mu) = P(\lambda) \cap P(\mu)$ and this defines a topology on $\mathrm{SP}(\mathrm{SL}(C(\Lambda)))$. Similar constructions may be applied to the minimal point-spectrum $Q\mathrm{Sp}(\Lambda)$ on $\mathrm{Sp}(\Lambda)$ the generalized Stone topology is nothing but the point-topology. In the foregoing Λ may be replaced by $T(\Lambda)$ or $\prod(\Lambda)$ with topologies induced by the generalized Stone topology on the point spectra always again being called: generalized Stone topology. Finally the generalized Stone topology can also be defined on the commutative shadow $\mathrm{SL}(\Lambda)$, which is a modular lattice, and then we obtain $QSP(SL(\Lambda))$, where the topology induced on $Q\mathrm{SP}(\mathrm{SL}(\Lambda))$ is exactly the Stone topology of the Stone space of $\mathrm{SL}(\Lambda)$.

In the special case $\Lambda = L(H)$ (only a skew topology) the generalized Stone space defined on $Q\mathrm{SP}(L(H))$ is exactly the classical Stone space as used in Gelfand duality theory for $L(H)$. Warning: $L(H)$ does not satisfy the axiom v. and whereas $Q\mathrm{SP}(L(H))$ is rather big, $\mathrm{Sp}(L(H)) = \mathrm{SP}(L(H))$ is empty! Moreover over $L(H)$ there are no sheaves but there will be many sheaves over $C(L(H))$ making sheafification of a separated presheaf over $L(H)$ possible over $Q\mathrm{SP}(L(H))$.

The foregoing fixes a context for results in the sequel, however the methods in Sections 2–4 are rather generic and can be applied to other notions of noncommutative spaces.

2. Dynamical noncommutative topology

We propose to construct space as a dynamical noncommutative topological space and defining geometrical objects as existing over some parameter-intervals. Noncommutative continuity is introduced via the variation of an external parameter in a totally ordered set T (if one wants to consider this as a kind of time, fine … but then this time is an index, not a dimension). This is philosophically satisfying because momentary observations which are only abstractly possible (real measurement takes time) put us in the discrete-versus-continuous situation of noncommutative geometry, as well as quantum theory.

Let T be a totally ordered set and for every $t \in T$ we give a noncommutative space Λ_t. This can have several meanings, in the sequel we take this to mean that Λ_t is the generalized Stone space constructed on $C(X_t)$ for some skew topology X_t as in Section 1. This is just to fix ideas, in fact one could just as well restrict to topologies induced by the generalized Stone topology on point spectra of any type, see also Section 1 or [1], or take pattern topologies as introduced in [1,7], or go to other theories and take quantales etc …. For $t \le t'$ in T we have $\varphi_{tt'} : \Lambda_t \to \Lambda_{t'}$ which are poset maps respecting $\wedge$ and $\vee$; when $t = t'$ we take $\varphi_{tt} = 1_{\Lambda_t}$ to be the identity of Λ_t and when $t \le t' \le t''$ then we demand that $\varphi_{t't''} \circ \varphi_{tt'} = \varphi_{tt''}$, where notation for composition of maps is conventional i.e. writing the one acting last first. If $A_t \subset \Lambda_t$ is a directed set then $\varphi_{tt'}(A_t) \subset \Lambda_{t'}$, for $t \le t'$, is again a directed set.

It is easily verified that if we start from a system $\{X_t, \Psi_{tt'}, T\}$ defined as above, we obtain a similar system $\{C(X_t), \Psi^e_{tt'}, T\}$ where $\Psi^e_{tt'} : C(X_t) \to C(X_{t'})$ is given by putting: $\Psi^e_{tt'}([A]) = [\Psi_{tt'}(A)]$, for $[A] \in C(X_t)$ and $t \in t'$ in T. In case it is interesting to view Λ_t as coming from some X_t via $C(X_t)$ we may restrict attention to systems given by $\varphi_{tt'}$, $t \le t'$ in T, stemming from $\psi_{tt'}$ on X_t by extension as indicated above. Note that not every system $\{C(X_t), \varphi_{tt'}, T\}$ has to derive from a system $\{X_t, \psi_{tt'}, T\}$ in general.

2.1. Lemma

Any system of poset maps $\varphi_{tt'}$, $t \le t'$ in T, defines a system of poset maps $\varphi^e_{tt'}$, $t \le t'$ in T. If the maps $\varphi_{tt'}$ respect the operations $\wedge$ and $\vee$ in the Λ_t then so does $\varphi^e_{tt'}$ for $C(\Lambda_t)$, $t \in T$. In this situation $\varphi_{tt'}$ maps $\wedge$-idempotent elements of Λ_t to $\wedge$-idempotent elements of $\Lambda_{t'}$ (also $\vee$-idempotent to $\vee$-idempotent) moreover if $[A_t]$ is strongly idempotent in $C(\Lambda_t)$ then $[\varphi_{tt'}(A_t)]$ is a strongly idempotent element of $C(\Lambda_{t'})$, for every $t \le t'$ in T,

Proof. First statements follow obviously from: for directed sets A and B,

$$\varphi^e_{tt'}([A] \wedge [B]) = \varphi^e_{tt'}([A \dot{\wedge} B]) = [\varphi_{tt'}(A \dot{\wedge} B)] = [\varphi_{tt'}(A) \dot{\wedge} \varphi_{tt'}(B)] = [\varphi_{tt'}(A)] \wedge [\varphi_{tt'}(B)]$$

for $t \le t'$ in T. Similar with respect to $\vee$, using $\dot{\vee}$. In case $\lambda \in \Lambda_t$ is idempotent in Λ_t then $\varphi_{tt'}(\lambda) \wedge \varphi_{tt'}(\lambda) = \varphi_{tt'}(\lambda \wedge \lambda) = \varphi_{tt'}(\lambda)$ for $t \le t'$ in T. Finally if A is idempotently directed look at $\varphi_{tt'}(a)$ for $a \in A_t$; by assumption there exists some $\mu \in \mathrm{id}_\wedge(\Lambda_t)$ such that $\mu \le a$, thus $\varphi_{tt'}(\mu) \le \varphi_{tt'}(a)$ and $\varphi_{tt'}(\mu) \in \mathrm{id}_\wedge(\varphi_{tt'}(A_t))$, for $t \le t'$ in T. Consequently $\varphi_{tt'}(A_t)$ is idempotently directed too. □

The skew topology $\prod_t$, introduced after Lemma 1.2. is called the pattern topology of X_t, i.e. it is obtained by taking all $\wedge$-finite bracketed expressions with respect to $\vee$ and $\wedge$ in the letters of $\mathrm{Id}_\wedge(C(\Lambda_t))$.

2.2. Corollary

The system $\{\Lambda_t, \varphi_{tt'}T\}$ induces a system $\{\prod_t, \varphi_{tt'} \mid \prod_t, T\}$, satisfying the same properties, on the pattern topologies.

In general the $\varphi_{tt'}$, $t \le t'$, do not map points of Λ_t to points of $\Lambda_{t'}$, $t \le t'$, neither does $\varphi_{tt'}$ respect the operation $\underset{\cdot}{\wedge}$ of the commutative shadow $SL(\Lambda_t)$, i.e. the $\varphi_{tt'}$ do not necessarily induce a system on the commutative shadows.

2.3. Axioms for dynamical noncommutative topology (DNT)

A system $\{\Lambda_t, \varphi_{tt'}, T\}$ is called a DNT if the following five conditions are satisfied:

DNT.1 Writing 0, resp. 1, for the minimal, resp. maximal element of Λ_t (we shall assume these exist throughout) then $\varphi_{tt'}(0) = 0$, $\varphi_{tt'}(1) = 1$ for all $t \le t'$ in T.

DNT.2 For all $t \in T$, $\varphi_{tt} = 1_{\Lambda_t}$ and for $t \leq t' \leq t''$ in T, $\varphi_{t't''} \circ \varphi_{tt'} = \varphi_{tt''}$. Moreover, all $\varphi_{tt'}$ preserve $\wedge$ and $\vee$. Hence DNT.1. and 2. just restate that $\{\Lambda_t, \varphi_{tt'}, T\}$ is as before.

DNT.3 If for some $t \in T$, $0 < x < y$ in Λ_t, then there is a $t < t_1$ in T such that for $z_1 \in \Lambda_{t_1}$ satisfying $\varphi_{tt_1}(x) < z_1 < \varphi_{tt_1}(y)$ there is a $z \in \Lambda_t$, $x < z < y$, such that $\varphi_{tt_1}(z) = z_1$.

DNT.4 For every $t \in T$ and $0 < x < z < y$ in Λ_t there exist $t_1, t_2 \in T$ such that $t_1 < t < t_2$ and for every $t' \in]t_1, t_2[$ we have either $t \leq t'$ and $\varphi_{tt'}(x) < \varphi_{tt'}(z) < \varphi_{tt'}(y)$, or $t' \leq t$ and then if $x' < y'$ in $\Lambda_{t'}$ exist such that $\varphi_{t't}(x') = x$, $\varphi_{t't}(y') = y$ then there also exist z' in $\Lambda_{t'}$ such that $x' < z' < y'$ and $\varphi_{t't}(z') = z$. Taking the special case $y = 1$ and $y' = 1$ then we see that a nontrivial strict relation in Λ_t is valid in an open T-interval containing t.

DNT.5 T-**local unambiguity**. In the situation of DNT.3, resp. DNT.4, the $t_1 \in T$, resp. t_1 and t_2, may be chosen such that $z \in \Lambda_t$ is the unique element such that $\varphi_{tt_1}(z) = z_1$, resp. x', y', z' in Λ_t, are the unique elements such that $\varphi_{t't}(z') = z$, $\varphi_{t't}(y') = y$, $\varphi_{t't}(x') = x$, or when $t \leq t'$ the x, y, z, are unique elements mapping to $\varphi_{tt'}(z)$, $\varphi_{tt'}(y)$, $y_{tt'}(x)$ resp.

Since we are able to take finite intersection of open T-intervals in the totally ordered set T, we may extend the foregoing to finite chains of $0 < x_1 < x_2 < \cdots < x_n$, $n \geq 3$.

Observe that the axioms allow that non-interacting elements, i.e. x such that $0 < x < 1$ is the only order relation it is involved in, may appear and disappear momentarily. Here disappearing means going to 0 under all $\varphi_{tt'}$, $t \leq t'$, if $x \in \Lambda_t$.

Very often properties studied are only preserved in some T-interval, in particular this happens often when trying to derive a property of a related system from another one that may be globally defined for T. This leads to an interesting phenomenon, already encoding some aspect of the moment-spaces to be defined later.

The DNT axioms are set-theoretical properties phrased in terms of noncommutative Λ_t and disregarding where these come from and how the transition maps $\varphi_{tt'}$ are obtained. When trying to construct such systems for Λ_t deriving from noncommutative algebras it is immediately clear that it is not that easy to realize the main axiom making everything expressible in temporal points. In the noncommutative algebraic geometry checking that something is a point comes down to checking whether a certain localization is corresponding to a prime torsion theory: to guarantee that a certain localization becomes a prime localization under some transition map may not be a very restricting condition but it is hard to verify. This should be easier in situations where some extra information is available from some associated commutative objects e.g. for almost commutative algebras (including Weyl algebras, rings of differential operators or varieties, ...) the quantum-site obtained by passing to microlocalizations (by completing localizations suitably) yields a commutative topology, see [1] for detail. On the other hand dealing with systems of Noetherian algebras presents no problem because for such rings every torsion theory is covered by prime torsion theories. It is important to keep in mind that in general points are in $C(\Lambda_t)$ not in Λ_t and DNT will afterwards be viewed as systems of generalized Stone spaces; this explains also the introduction of traditional system in 3.1. hereafter.

2.4. Definition of observed truth

A statement in terms of finitely many ingredients of a DNT and depending on parametrization by $t \in T$ is said to be an **observed truth** at $t_0 \in T$ if there is an open T-interval $]t_1, t_2[$ containing t_0, such that the statement holds for parameter values in this interval. The notion "statement" may be understood in any desired generality, most often we may understand it as an expression in some predicate logic. Hereafter the statements are always simple mathematical formulas.

It seems that mathematical statements about a DNT turn into "observed truth" when one tries to check them in the commutative shadow, meaning on the negative side that many global (over T) properties of a DNT cannot be established globally over T in the commutative world.

The noncommutative topologies Λ_t considered in the sequel will be such that $\vee$ is commutative and $\vee$ of arbitrary families exist; in fact one may restrict to so-called "virtual topologies" as introduced in [7]; here we do not need to assume axiom (v) with respect to global covers, we may want to restrict to this case when needed. We refer to the special case mentioned as a DVT.

2.5. Proposition

Let $\{\Lambda_t, \varphi_{tt'}, T\}$ be a DVT and let $SL(\Lambda_t)$ be the commutative shadow of Λ_t with maps $\varphi_{tt'} : SL(\Lambda_t) \to SL(\Lambda_{t'})$, $t \leq t'$ in T, just being the restrictions of the $\varphi_{tt'}$ (using same notation). Then the statement that $\{SL(\Lambda_t), \varphi_{tt'}, T\}$ is a DVT too is an observed truth at every $t_0 \in T$.

Proof. All $\varphi_{tt'}$ map $\wedge$-idempotents to $\wedge$-idempotents, cf. Lemma 2.1., so DNT.1. is obvious. For DNT.2 we have to check that $\varphi_{tt'}$ preserves $\underset{\cdot}{\wedge}$ on $\mathrm{id}_\wedge(\Lambda_t)$, since $\underset{\cdot}{\vee} = \vee$ now we have nothing to check for $\underset{\cdot}{\vee}$. Look at $t_0 \in T$, $\varphi_{t_0t} : \Lambda_{t_0} \to \Lambda_t$ and σ, τ in $\mathrm{id}_\wedge(\Lambda_{t_0})$. If $\sigma < \tau$ then $\varphi_{t_0t}(\sigma) \leq \varphi_{t_0t}(\tau)$ and $\varphi_{t_0t}(\sigma) \underset{\cdot}{\wedge} \varphi_{t_0t}(\tau) = \varphi_{t_0t}(\sigma) = \varphi_{t_0t}(\sigma \underset{\cdot}{\wedge} \tau)$, interchange the role of σ and τ in case $\tau < \sigma$. So we may assume σ and τ to be incomparable. Restricting t to a suitable T-interval (DNT 5) we may assume that $\varphi_{t_0t}(\sigma) \neq \varphi_{t_0t}(\tau)$. Assume $\varphi_{t_0t}(\sigma \underset{\cdot}{\wedge} \tau) < \varphi_{t_0t}(\sigma) \underset{\cdot}{\wedge} \varphi_{t_0t}(\tau)$.

If $\varphi_{t_0t}(\sigma \underset{\cdot}{\wedge} \varphi_{t_0t}(\tau) = \varphi_{t_0t}(\sigma))$ (a similar argument will hold if σ and τ are interchanged) then $\varphi_{t_0t}(\sigma) \leq \varphi_{t_0t}(\tau)$, hence $\varphi_{t_0t}(\sigma) < \varphi_{t_0t}(\tau)$. Using DNT.5 again, taking t close enough to t_0, we obtain $\sigma \underset{\cdot}{\wedge} \tau < \sigma_1 < \tau$ such that $\varphi_{t_0t}(\sigma \underset{\cdot}{\wedge} \tau) < \varphi_{t_0t}(\sigma_1) = \varphi_{t_0t}(\sigma) < \varphi_{t_0t}(\tau)$. Passing to $[t_0, t]$ small enough in order to have unambiguity for $\varphi_{t_0t}(\sigma)$, we arrive at $\sigma_1 = \sigma$, contradicting the incomparability of σ and τ. Therefore we arrive at strict relations: $\varphi_{t_0t}(\sigma \underset{\cdot}{\wedge} \tau) < \varphi_{t_0t}(\sigma) \underset{\cdot}{\wedge}$

$\varphi_{t_0t}(\tau) < \varphi_{t_0t}(\sigma), \varphi_{t_0t}(\tau)$. We may moreover assume (DNT.3) that t is close enough to t_0 so that there is a $z \in \Lambda_{t_0}$ such that $\sigma \dot{\wedge} \tau < \sigma, \tau$ and $\varphi_{t_0t}(z) = \varphi_{t_0t}(\sigma) \dot{\wedge} \varphi_{t_0t}(\tau)$. If z is not $\wedge$-idempotent, then $\sigma \dot{\wedge} \tau < z \wedge z < \sigma$ would lead to $\varphi_{t_0t}(z \wedge z) = \varphi_{t_0t}(\sigma) \dot{\wedge} \sigma_{t_0t}(\tau)$ because φ_{t_0t} respects $\wedge$ and the latter is idempotent in Λ_t; then $\varphi_{t_0t}(z) = \varphi_{t_0t}(z \wedge z)$ but the unambiguity guaranteed by the choice of t close enough to t_0 (DNT.5) then yields $z = z \wedge z$ or $z \in \mathrm{id}_\wedge(\Lambda_{t_0})$. Thus $z = \sigma \dot{\wedge} \tau$ by definition, a contradiction. Consequently, for t in some small enough T-interval containing t_0 we have obtained:$\varphi_{t_0t}(\sigma \dot{\wedge} \tau) = \varphi_{t_0t}(\sigma) \dot{\wedge} \varphi_{t_0t}(\tau)$, thus DNT.2 is an observed truth. To check DNT.3, take $\sigma < \tau$ in $\mathrm{id}_\wedge(\Lambda_{t_1})$, $t < t_1$ such that $z_1 \in \mathrm{id}_\wedge(\Lambda_{t_1})$ exists such that we have $\varphi_{tt_1}(\sigma) < z_1 < \varphi_{tt_1}(\tau)$. Now by DNT.3. for $\{\Lambda_t, \varphi_{tt'}, T\}$ there is a $z \in \Lambda_t, z < \tau$, such that $\varphi_{tt_1}(z) = z_1$, and DNT.5 for $(\Lambda_t, \varphi_{tt'}, T\}$, used as in foregoing part of the proof, yields $\varphi_{tt_1}(z) = z_1$ with z also $\wedge$-idempotent in Λ_t, for t_1 close enough to t. The proof of DNT.4 follows in the same way and DNT.5 is equally obvious because unambiguity in a suitable T-interval allows to pull back idempotency. Therefore all DNT-axioms hold for $\{SL(\Lambda_t), \varphi_{tt'}, \tau)$ in a suitable T-interval, hence we have the observed truth statement that $\{SL(\Lambda_t), \varphi_{tt'}, T\}$ is DNT. $\square$

Now fix a notion of point i.e. either minimal point or irreducible point as in Section 1. We say that $\lambda_t \in \Lambda_t$ is a **temporal point** if $t \in]t_0, t_1[$ such that for some $t' \in]t_0, t_1[$ there is a point $p_{t'} \in \Lambda_{t'}$ such that: either $t \le t'$ and $\varphi_{tt'}(\lambda_t) = p_{t'}$, or $t' \le t$ and $\varphi_{t't}(p_{t'}) = \lambda_t$; in the first case we say λ_t is a **future point**, in the second case a **past point**. The system $\{\Lambda_t, \varphi_{tt'}, T\}$ is said to be **temporally pointed** if for every $t \in T$ and $\lambda_t, \in \Lambda_t$ there exists a family of temporal points $\{p_{\alpha,t}; \alpha \in \mathcal{T}\}$ in Λ_t such that λ_t is covered by it, i.e. $\lambda_t = \vee\{p_{\alpha,t}, \alpha \in \mathcal{A}\}$. Write $T\mathcal{P}(\Lambda_t)$ for the set of temporal points of Λ_t, if we write $\mathrm{Spec}(\Lambda_t) = \{p_{t'}$ point in $\Lambda_{t'}, p_{t'}$ defines a temporal point of $\Lambda_t\}$ then $T\mathcal{P}(\Lambda_t)$ may also be written as $T\mathrm{Spec}(\Lambda_t)$ (note $T\mathrm{Spec}(\Lambda_t)$ is in Λ_t but $\mathrm{Spec}(\Lambda_t)$ not).

We need to build in more "continuity" aspects in the DVT-axioms without using functions or extra assumptions on T e.g. that it should be a group. A temporary pointed system $\{\Lambda_t, \varphi_{tt'}, T\}$ is a **space continuum** if the following conditions hold:

SC.1 There is a minimal closed interval T_t containing t in T such that $T\mathrm{Spec}(\Lambda_t)$ has support in I_t. The set of points in $\Lambda_{t'}$ with $t' \in I_t$, representing temporary points in Λ_t is then called the **minimal spectrum** for $T\mathrm{Spec}(\Lambda_t)$, denoted by $\mathrm{Spec}(\Lambda_t, I_t)$.

SC.2 For any open T-interval I such that $I_t \subset I$ there exists an open T-interval I_t^* with $t \in I_t^*$, such that for $t' \in I_t^*$ we have $I_{t'} \subset I$.

SC.3 For intervals $[t_1, t_2]$ and $[t_3, t_4]$ we write $[t_1, t_2] < [t_3, t_4]$ if $t_1 \le t_3$ and $t_2 \le t_4$ (similarly for open intervals). If $t \le t'$ in T then $I_t < I_{t'}$. This provides an "orientation" of the variation of the minimal spectra!

SC.4 **Local Preservation of Directed Sets**. For given $t \le t'$ in I_t and any directed set A_t in Λ_t, the subset $\{\gamma_t \in A_t$, there exists $\xi_t < \gamma_t$ in A_t such that $\varphi_{tt'}(\xi_t) < \varphi_{tt'}(\gamma_t)\}$ is cofinal in A_t (defines the same limit $[A_t]$). For $t'' \le t$ in I_t there is a directed set $A_{t''}$ in $\Lambda_{t''}$ mapped by $\varphi_{t''t}$ to a cofinal subset of A_t.

A subset J of T is **relative open around** $t \in T$ if it is intersection of I_t and an open T-interval. For $x = (\ldots, x_t, \ldots) \in \prod_{t \in T} \Lambda_t$ we put $\sup(x) = \{t \in T, x_t \neq 0\}$. We say that such an x is **topologically accessible** if all $x_{t'}t \in \sup(x)$, are classical i.e. $x_t = [\chi_t]$ (for some $\chi_t \in X_t$ and $\Lambda_t = C(X_t)$. In case we do not consider Λ_t as coming from some X_t the condition becomes void. An x as before is said to be t-**accessible** if $\sup(x) = J$ is relative open around t and for all $t' \le t''$ in J we have $\varphi_{t't''}(x_{t'}) \le x_{t''}$. When Λ_t has enough points i.e. if $I_t = \{t\}$, then the points in an open for the point topology would be characterized by $\{p, p \le [\chi_t]\} = U(\chi_t)$ for some $\chi_t \in X_t$. When Λ_t does not have enough points then we have to modify the definition of point spectrum and point topology correspondingly. If $x = (\ldots, x_t, \ldots)$ is t accessible and $p_{t'} \in \mathrm{Spec}(\Lambda_t, I_t)$ then we say $p_{t'} \in x$ if $t' \in J = \sup(x)$ and there exists an open T-interval $J_1 \subset J$ with $t' \in J_1$ such that for $t'' \in J_1$ we have: if $t' \le t''$ then $p_{t''} = \varphi_{t't''}(p_{t'}) \le x_{t''}$, or if $t'' \le t'$ there is a $p_{t''} \in \Lambda_{t''}$ such that $\varphi_{t''t'}(p_{t''}) = p_{t'}, p_{t''} \le x_{t''}$, i.e. $\{p_{t''}, t'' \in J_1\}$ is the restriction of a temporal point representing $p_{t'}$ defined over a bigger T-interval $]t_0, t_1[$ containing both t' and t (note: J_1 need not contain t).

2.6. Theorem

The empty set together with the sets $U_t(x) = \{p_{t'}, p_{t'} \in x$ for some $t' \in I_t\} \subset \mathrm{Spec}(\Lambda_t, I_t)$, x being t-accessible in $\prod_{t \in T} \Lambda_t$, form a topology on $\mathrm{Spec}(\Lambda_t, I_t)$, called **spectral topology** at $t \in T$.

Proof. Consider $x \neq y$ both t-accessible with respective T-intervals J, resp. J' contained in I_t. If $p_{t'} \in U_t(x) \cap U_t(y)$ then $t' \in J \cap J'$ and for every $t_1 \in J$, $p_{t_1} \le x_{t_1}$, for every $t_2 \in J'$, $p_{t_2} \le y_{t_2}$. Of course the interval $J \cap J'$ is relative open around t. If $t' \le t''$ with $t'' \in J \cap J'$ then $o_{t''} = \varphi_{t't''}(p_{t'})$ is idempotent in $\Lambda_{t''}$ because $p_{t'}$ is in $\Lambda_{t'}$ as it is a point. Hence we obtain:

$$p_{t''} = p_{t''} \wedge p_{t''} \le x_{t''} \wedge y_{t''}.$$

Obviously for all $t'' \le t'''$ in $J \wedge J'$ we do have: $\varphi_{t''t'''}(x_{t''} \wedge y_{t''}) \le x_{t'''} \wedge y_{t'''}$. On the other hand, for $t'' \le t'$ we obtain: $\varphi_{t''t'}(p_{t''}) = p_{t'}$ and therefore $p_{t'} \le \varphi_{t''t'}(x_{t''}) \le x_{t'}$, as well as $p_{t'} \le \varphi_{t''t'}(y_{t''}) \le y_{t'}$. Hence, again by idempotency of $p_{t'}$ in $\Lambda_{t'}$ we arrive at $p_{t'} \le x_{t'} \wedge y_{t'}$. By restricting $J \cap J'$ to the interval obtained by allowing only those $t'' \le t'$ which belong to an (open) unambiguity interval for $p_{t'}$ we arrive at a relative open around t, say $J'' \subset J \cap J'$, containing t'.

Now for $p_{t''}$ with $t'' \in J''$ it follows that $p_{t''}$ is idempotent because both $p_{t''}$ and $p_{t''} \wedge p_{t''}$ map to $p_{t'}$ via $\varphi_{t''t'}$ for $t'' \le t'$ (other t'' in J'' are no problem). Thus for t'' in J'' we do arrive at $p_{t''} \le x_{t''} \wedge y_{t''}$. Define w by putting $w_{t''} = x_{t''} \wedge y_{t''}$ for $t'' \in J''$. Clearly, w is t-accessible and $p_{t'} \in U_t(w)$. Conversely if $p_{t'} \in U_t(w)$ then $p_{t'} \in U_t(x) \cap U_t(y)$ is clear because J'' used in the definition of w is open in $J \cap J'$. Now we look at a union of $U_{i,t} = U_t(x_i)$ for $i \in J$ and each x_i being A-accessible with corresponding relative open interval J_i in I_t. Define w over the "interval" $J = \cup_i\{J_i, i \in \mathcal{J}\}$ by putting $w_t = \vee\{x_{i,t}, i \in \mathcal{J}\}$ for $t \in J$. It is clear that J is relative open around t and for all $t_1 \leq t_2$ in J we have $\varphi_{t_1t_2}(w_{t_1}) \leq w_{t_2}$ because φ_{t,t_2} respects arbitrary $\vee$. Now $p_{t'} \in w$ means that $p_{t''} \leq \vee\{x_{i,t''}, i \in \mathcal{J}\}$ for t'' in some relative open containing t', say $J_1 \subset J$. We use relative open sets in T because I_t was closed and there are two situations to consider concerning $t' \in I_t$. First if t' is the lowest element of I_t then for all $t'' \in J_1$ we have that $p_{t''} = \varphi_{t't''}(p_{t'}) \leq \varphi_{t't''}(\vee\{x_{i,t'}, i \in \mathcal{J}\})$ and for all $t' \leq t_1 \leq t''$ we also obtain: $p_{t_1} \leq \varphi_{t't_1}(\vee\{x_{i,t'}, i \in \mathcal{J}\})$ and $p_{t''} \leq \varphi_{t_1t''}(\vee\{x_{t,t_1}, i \in \mathcal{J}\})$. Otherwise, if t' is not the lowest element of I_t then we may restrict J_1 to be an open interval $]t_0, t_0'[$ containing t' with $t_0 \in J$. The same reasoning as in the first case yields for all $t'' \in]t_0, t_0'[$ so that: $p_{t''} \leq \varphi_{t_0t''}(\vee\{x_{i,t_0}, \tau \in \mathcal{J}\})$ and for any $t' \leq t_1 \leq t'' p_{t''} \leq \varphi_{t_1t''}(\vee\{x_{i,t_1}, i \in \mathcal{J}\}) = \vee\{\varphi_{t_1t''}(x_{i,t_1}), i \in \mathcal{J}\}$. Since $t' \in J_1$ we obtain $p_{t'} \leq \vee\{\varphi_{t_1t'}(x_{i,t_1}), i \in J\}$ for all $t_1 \in [t_0, t']$.

Since $p_{t'}$ is a point in $\Lambda_{t'}$ there is an $i_0 \in J$ such that $p_{t'} \leq \varphi_{t_0t'}(x_{i_0,t_0})$ and therefore we have that $p_{t'} \leq \varphi_{t_1t'}(x_{i_0,t_1})$ with $t_1 \in [t_0, t']$, the gain being that i_0 does not depend on t_1 here! Now for $t'' \geq t'$ in $J_1 \cap J_{i_0}$ (note that this is not empty because x_{i_0} is nonzero at t_0 because $p_{t'} \leq \varphi_{t_0t''}(x_{i_0,t_0})$ would then make $p_{t'}$ zero and we do not look at the zero (the empty set) as a point of Λ_t) we obtain:

$$p_{t''} = \varphi_{t't''}(p_{t'}) \leq \varphi_{t't''}(x_{i_0,t'}) \leq x_{i_0,t''}. \tag{$*$}$$

In the other situation $t'' \leq t'$ in $J_1 \cap J_{i_0}$ we have $\varphi_{t''t'}(p_{t''}) = p_{t'}, \varphi_{t''t'}(x_{i_0,t''}) \leq x_{i_0,t'}$. By restricting $J_1 \cap J_{i_0}$ further so that the $t'' \leq t'$ are only varying in an (open) unambiguity interval for $p_{t'}$, say $J_2 \subset J_1 \cap J_{i_0}$, we arrive at one of two cases: either $p_{t''} = x_{i_0,t''}$ or else $p_{t''} \neq x_{i_0,t''}$ and also $p_{t'} < x_{i_0,t'}$. In the first case $p_{t'} \in x_{i_0}$ follows because $p_{t_1} = \varphi_{t''t_1}(x_{i_0,t''}) \leq x_{i_0,t_1}$ for t_1 in $]t'', 1[\cap J_2$, the latter interval containing t' is relative open again. In the second case we may look at $p_{t'} < x_{i_0,t'} < 1$, hence there exists a $z_{t''}$ such that $p_{t''} < z_{t''} < 1$ and $\varphi_{t''t'}(z_{t''}) = x_{i_0,t'}$. Again we have to distinguish two cases, first $\varphi_{t''t'}(x_{i_0t''}) = x_{i_0,t'}$ or $\varphi_{t''t'}(x_{i_0,t''}) < x_{i_0,t'}$.

In the first case $z_{t''}$ and $x_{i_0,t''}$ map to the same element via $\varphi_{t''t'}$, hence up to restricting the interval further such that t'' stays within an unambiguity interval for $x_{i_0,t'}$, we may conclude $z_{t''} = x_{i_0,t''}$ in this case and then $p_{t''} < x_{i_0,t''}$. In the second case we may look at: $p_{t'} \leq \varphi_{t''t'}(x_{i_0,t''}) < x_{i_0,t'} < 1$ (where the first inequality stems from ($*$) above). Again restricting the interval further (but open) we find a $z'_{t''}$ in $\Lambda_{t''}$ such that $x_{i,t''} < z'_{t''} < 1$ such that $\varphi_{t''t'}(z'_{t''}) = x_{i_0,t'}$. Since we are dealing with the case $p_{t''} \neq x_{i_0,t''}$ and we are in an unambiguity interval for $p_{t'}$ it follows that $p_{t'} < \varphi_{t''t'}(x_{i_0,t''})$. Look at: $p_{t'} < \varphi_{t''t'}(x_{i_0,t''}) < x_{i_0,t'}$ with $\varphi_{t''t'}(p_{t''}) = p_{t'}$ and $\varphi_{t''t'}(z'_{t''}) = x_{i_0,t'}$; by restricting the interval (open) further if necessary we obtain the existence of $z''_{t''}$ such that, $p_{t''} < z''_{\prime\prime} < z'_{t''}$ such that $\varphi_{t''t'}(z''_{t''}) = \varphi_{t''t'}(x_{i_0,t''})$. Finally restricting again the $t'' \leq t'$ to vary in an unambiguity interval for $\varphi_{t''t'}(x_{i_0,t''})$ it follows that $z''_{t''} = xz_{i_0,t''}$ and hence $z''_{t''} \geq p_{t''}$ yields $x_{i_0,t''} \geq p_{t''}$ for t'' in a suitable relative open around t containing t'. This also in the case we arrive at $p_{t'} \in x_{i_0}$ or $p_{t'} \in U_t(x_{i_0})$. It follows that $\mathcal{U}_t(w) = U\{U_{i,t}, i \in \mathcal{J}\}$ establishing that arbitrary unions of opens are open. By taking $\mathcal{U}_t(1)$ we obtain the whole spectrum at t as an open too. □

In case a DNT would be obtained from a system of algebra morphisms in some suitable system of algebras the constructed moment spaces need not correspond to algebra spectra so one should **not** expect that some commutative algebras or algebras with a commutative geometry (for example twisted homogeneous coordinate rings) are approximating the noncommutative algebras defining Λ_t, $t \in T$.

2.7. A possible relation to M-theory?

In noncommutative topology and derived point topologies the gen-topology appears naturally (and it is a classical i.e. commutative topology (cf. [1])). Moreover continuity in the gen-topology also appears naturally in noncommutative geometry of associative algebras but we did not ask the $\varphi_{tt'}$ in the DNT-axioms to be continuous in the gen topology. However one may prove that in general "continuity of the $\varphi_{tt'}$ in the gen-topologies of Λ_t resp. $\Lambda_{t'}$ is an observed truth! Consequently for t' close enough to t the $\varphi_{tt'}$ is continuous with respect to the gen-topologies (cf. [7]).

In the mathematical theory all Λ_t may be different and there is no reason to aim at $SL(\Lambda_t)$ nor $\mathrm{Spec}(\Lambda_t, I_t)$ to be invariant under t-variation. From the point of view of Physics one may reason that only one case is important i.e. the case we see as "reality". This being the utmost special case it is then not far-fetched to assume that the dynamic noncommutative space has as commutative shadow the abstract mathematical frame we reason in about reality. for example identified with 3- or 4-dimensional space or space–time. Moreover there should be an observational mathematical frame where calculations about measurements are executed, thus we let $\mathrm{Spec}(\Lambda_t, I_t)$ be identified to an 11-dimensional space (for M-theory) or any other one fitting physical interpretations in some theory one chooses to believe in. An observed point in $\mathrm{Spec}(\Lambda_t, I_t)$ is then given by a string of elements say $p_{t'} \in \Lambda_{t'}$, $t' \in J \subset I_t$ with $p_t \in \Lambda_t$ a temporal point. If $p_{t'}$ is a point then for all $t' \leq t''$, $\varphi_{t't''}(p_{t'})$ is idempotent so appears in the commutative shadow $SL(\Lambda_{t''})$. Hence an observed point in $\mathrm{Spec}(\Lambda_t, I_t)$ appears as a string in the base space $SL(\Lambda_{t''})(t' \leq t'')$, identified with n-dimensional space but the string may "start after" t when the point was "observed". On the other hand, the assumption that the system $(\Lambda_t, \varphi_{tt'}, T\}$ is temporally pointed leads to a decomposition of of every $p_{t'}$ into temporal points of $\Lambda_{t'}$ realizing it as an open of $\mathrm{Spec}[\Lambda_{t'}, I_{t'}]$. Thus in the spectral space (identified with a certain m-dimensional space say), the observed point appears as a "string" connecting opens i.e. a possibly more dimensional string that can be thought of as a tube. The difference between the dimensions m and n has to be accounted for by the "rank" of T (e.g. if T were a group like $\mathbb{R}^d_+$, d would be the rank) which allowed to create the extra points in Spec(Λ_t, I_t) when compared to $SL(\Lambda_t)$. Note that even when the $\varphi_{tt'}$ do not necessarily define maps between Spec(Λ_t, I_t) and Spec$(\Lambda_{t'}, I_{t'})$ or between $SL(\Lambda_t)$ and $SL(\Lambda_{t'})$, the given strings at the Λ_t-level do define sequences of elements or opens in the Spec(Λ_t, I_t) resp. $SL(\Lambda_t)$ that may be viewed as strings, resp. tubes. Two more intriguing observations:

(i) Identifying Spec(Λ_t, I_t) to one fixed commutative world and $SL(\Lambda_t)$ to another allows strings and tubes to be open or closed.

(ii) Only temporal points corresponding to future points can be non-idempotent, therefore all noncommutativity is due to future points and uncertainty may be seen as an effect of the possibility that the interval needed to realize the temporal point p_t by a point $p_{t''} \in \Lambda_{t''}$ for $t \leq t''$ is **larger** than the unambiguity interval for $p_{t''}$. Passing from a commutative frame ($SL(\Lambda)$) to noncommutative (dynamical) geometry and phrasing theories and calculations in Spec(Λ_t, I_t) at the price of having to work in higher dimension seems to fit quantum theories. Of course this is at the level of mathematical formalism, for suitable interpretations within physics the physical entity connected to the notion of point in Spec(Λ_t, I_t) should be the smallest possible, i.e. a kind of building block of everything, so that observing it as a point in the moment spaces is acceptable; that these points are mathematically described as strings or higher-dimensional strings via the noncommutative geometry is a "Deus ex Machina" pointing at an unsuspected possibility of embedding M-theory in our approach. No further speculation about this here, perhaps specialists in string theory may be interested in investigating further this formal incidence.

3. Moment presheaves and sheaves

Continuing the point of view put forward in the short introduction to Section 2, points or more precisely functions defined in a set theoretic spirit, should be replaced by a generalization of "germs of functions" obtained from limit constructions in classical topology terms to noncommutative structures. Thus the notion of point is replaced by an avatar of the notion "stalk" of a given sheaf, more correctly when different (pre)sheaves over a noncommutative space are being considered, say with values in some nice category of objects $\underline{\mathcal{C}}$, then a "point" is a suitable limit functor on $\underline{\mathcal{C}}$-objects generalizing the classical construction of localization (functor) at a point. Assuming that a suitable topological space and satisfactory sheaf of "functions" on it have been identified, satisfactory in the sense that it allows to study the desired geometric phenomena one is aiming at, then the notion of point via stalks should be suitable too. For example, prime ideals would be identified via stalks of the structure sheaf of a commutative Noetherian ring without having to check a primeness condition of the corresponding localization functor. More on the definition of noncommutative geometry via (localization) functors can be found in [1] where it is introduced as a functor geometry over a noncommutative topology, also [8] and [10].

In this section we fix a category $\underline{\mathcal{C}}$ allowing limits and colimits; we might restrict to Abelian or even Grothendieck categories but that is not essential. In fact, the reader who wants to fix ideas on a concrete situation may choose to work in the category of abelian groups.

For every $t \in T$, Γ_t is a presheaf over Λ_t and for $t \leq t'$ in T there is a $\phi_{tt'} : T_t \to T_{t'}$, defined by morphisms in $\underline{\mathcal{C}}$ as follows:

(i) For $\lambda_t \in \Lambda_t$ there is a $\phi_{tt'}(\lambda_t) : \Gamma_t(\lambda_t) \to \Gamma_{t'}(\varphi_{tt'}(\lambda_t))$

(ii) for $\mu_t \leq \lambda_t$ in Λ_t we have commutative diagrams in $\underline{\mathcal{C}}$:

$$\begin{tikzcd} \Gamma_t(\lambda_t) \arrow[r, "\phi_{tt'}(\lambda_t)"] \arrow[d, "\rho^t_{\lambda_t,\mu_t}"] & \Gamma_{t'}(\varphi_{tt'}(\lambda_t)) \arrow[d, "\rho^{t'}_{\lambda_{t'},\mu_{t'}}"] \\ \Gamma_t(\mu_t) \arrow[r, "\phi_{tt'}(\mu_t)"'] & \Gamma_{t'}(\varphi_{tt'}(\mu_t)) \end{tikzcd}$$

where we have written $\lambda_{t'}, \mu_{t'}$ for $\varphi_{tt'}(\lambda_t)$, resp. $\varphi_{tt'}(\mu_t)$ and $\rho^t_{\lambda_{t'}\mu_t}$ for the restriction morphism of Γ_t.

(iii) $\phi_{tt}(\lambda_t) = I_{\Gamma_t(\lambda_t)}$ for all $t \in T$, and for $t \leq t'$ and let $t' \leq t''$ we have $\phi_{t't''}(\phi_{tt'}(T_t(\lambda_t))) = \phi_{tt''}(T_t(\lambda_t))$ for all $\lambda_t \in \Lambda_t$.

The system $\{\Gamma_t, \phi_{tt'}, T\}$ is called a (global) dynamical presheaf over the DNT $\{\Lambda_t, \varphi_{tt'}, T\}$.

Since sheaves on a noncommutative topology do not form a topos it is a problem to define a suitable sheafification i.e.: can a presheaf Γ on Λ be sheafified to a sheaf $\underline{\underline{a}}\Gamma$ on the same Λ via a suitable notion of "stalk", then allowing interpretations in terms of "points"? In fact, the axioms of DNT allow to give a solution to the sheafification problem at the price that the sheaf $\underline{\underline{a}}\Gamma_t$ has to be constructed over Spec(Λ_t, I_t)!

There are categorical methods of sheafification and a general theory of pretopologies and sieves does exist. In [7] we point out some differences between different sheafifications, however when striving for a decent replacement for the use of functions as well as sufficient level of geometric intuition in terms of "points" those general techniques do not yield the desired structural results. The usefulness of our approach is highlighted by Theorem 3.4. expressing what I would call the commutative approximation property of the moment spaces in terms of sheaf theory.

From hereon we let $\underline{\Lambda} = \{\Lambda_t, \varphi_{tt'}, T\}$ be a temporally pointed system which is a space continuum. We refer to $Y_t = \text{Spec}(\Lambda_t, I_t)$ with its spectral topology as the **moment space** at $t \in T$.

For $p_{t'} \in Y_t$ we may calculate (in $\underline{\mathcal{C}}$): $\Gamma_{t',p_{t'}} = \lim_{\longrightarrow} \Gamma_{t'}(x_t)$ where $\lim_{\longrightarrow}$ is over $x_{t'} \in \Lambda_{t'}$ such that $p_{t'} \leq x_{t'}$, and where $x = (\ldots, x_t, \ldots)$ is t-accessible, in fact we have $p_{t'} \in x$. In the foregoing we did not demand $\underline{\Lambda}$ to derive from a system $\underline{X}$ and passing from X_t to Λ_t as a generalized Stone space or pattern topology via $C(X_t)$. We preferred not to dwell upon the formal comparison of dynamical theories for the X_t and the Λ_t. In order to keep trace of a possible original X_t if it was considered in the construction of Λ_t one may if desired use the following.

3.1. Definition

We say that $u_t \in \Lambda_t$ is **classical** if $u_t = [\chi_t]$ for $\chi_t \in X_t$. If u_t is classical then there is an open interval containing t in T, say L, such that for every $t' \in L$ we have that $u_{t'}$ is classical, where for $t \leq t'$ we have $u_{t'} = \varphi_{tt'}(u_t)$ and for $t' \leq t$, $u_{t'}$ is a suitably chosen representative for u_t, $\varphi_{t't}(u_{t'}) = u_t$. Restricting further to an unambiguity interval of u_t, the representations $u_{t'}$ for t' in that interval are unique. Since points of X_t are by definition elements in $C(X_t)$, the filter in Λ_t defined by that point has a cofinal directed subset of classical elements. When in $\{\Lambda_t, \varphi_{tt'}, T\}$ we restrict attention to classical x, i.e. every x_t classical Λ_t, then we say that we look at a **traditional system**.

3.2. Lemma

For a traditional space continuum with dynamical presheaf $\{\Gamma_t, \phi_{tt'}, T\}$, the stalk for $p_{t'} \in Y_t$ of $\Gamma_{t'}$ is exactly $\Gamma_{t',p_{t'}}$ as defined above.

Proof. In calculating $\lim_{\longrightarrow}\{\Gamma_{t'}(u_{t'}), p_{t'} \leq u_{t'}\}$ we may restrict to classical $u_{t'}$ in $\Lambda_{t'}$. It now suffices to establish the existence of a t-accessible y such that $p_{t'} \in y$ and $y_{t'} \leq u_{t'}$. From $p_{t'} \in U_t(x)$ we obtain $(\ldots, x_{t'}, \ldots)$ with a relative open T-interval J, $t' \in J$, such that $p_{t''} \leq x_{t''}$ for every $t'' \in J$. Since $u_{t'}$ and $x_{t'}$ are classical, so is $u_{t'} \wedge x_{t'}$ and moreover $p_{t'} \leq u_{t'} \wedge x_{t'}$ because $p_{t'}$ is a point in $\Lambda_{t'}$ (hence idempotent!). Let J_1 be an open T-interval containing t' such that $u_{t'} \wedge x_{t'}$ has a representative $u_{t''}$ in $\Lambda_{t''}$ such that $\varphi_{t''t'} = u_{t'} \wedge x_{t'}$. Since $x_{t'} \neq u_{t'} \wedge x_{t'}$ may be assumed (otherwise put $y = x$) we arrive at $p_{t''} \leq u_{t''} < x_{t''}$. Using the intersection of J_1 and the interval around t' allowing to select classical $u_{t''}$, call this interval J_2, we put $y_{t''} = u_{t''}$ for $t'' \leq t'$ in J_2 and $y_{t_1} = x_{t_1}$ for $t' < t_1$ in J. Then y is t-accessible with respect to the relative open T-interval around t just defined: we have $y_{t'} \leq u_{t'}$ and $p_{t'} \in y$. Consequently: $\lim_{\longrightarrow_{p_{t'} \leq u_{t'}}} \Gamma_{t'}(u_{t'}) = \lim_{\longrightarrow_{p_{t'} \in x}} \Gamma_{t'}(x_{t'})$. □

In the sequel we assume objects in $\underline{\mathcal{C}}$ are sets but let us even restrict to abelian groups. Again let $\{\Gamma_t, \phi_{tt'}, T\}$ be a dynamical presheaf over a traditional space continuum. On Y_t we define a presheaf, with respect to the spectral topology, by taking for $\mathcal{P}(U_t(x))$ the abelian group in $\coprod_{t' \in I_t} \Gamma_{t'}(x_{t'})$ formed by strings over $\sup(x) = \{t' \in I_t, x_{t'} \neq 0\}$. i.e. $\{\gamma_{t'}, t' \in \sup(x), \phi_{t''t'}(\gamma_{t''}) = \gamma_{t'}$ for $t'' \leq t'$ in $\sup(x)\}$. Let us write $x < y$ if $x_{t'} \leq y_{t'}$ for all t' in I_t, in particular $x < y$ means $\sup(x) \subset \sup(y)$. In sheaf theory sections are usually defined over non-empty opens, here it would mean to exclude the $o \in \Lambda_t$ at every t, and it makes sense to do that here as well. However one may define at every $t' \in T$, $\Gamma_{t'}(0) = \lim_{\longrightarrow}\{\Gamma_{t'}(x_{t'}), x_{t'}$ classical in $\Lambda_{t'}\}$ and all statements made in the sequel will remain consistent, hence we may disregard the special role of o without harm. in the sequel.

If $x < y$ then we have restriction morphisms $\rho^{t'}_{y_{t'},x_{t'}}, : \Gamma_{t'}(y_{t'}) \to \Gamma_{t'}(x_{t'})$. Commutativity of the diagrams in the beginning of the section yield corresponding morphisms on the strings over the respective supports: $\rho_{y,x} : \mathcal{P}(U_t(y)) \to \mathcal{P}(U_t(x))$. For a point $p_{t'}$ we let $\eta(p_{t'})$ be the set of $U_t(x)$ such that we have $p_{t'} \in U_t(x)$ i.e. $p_{t'} \in x$; in particular $t' \in J_x$ where J_x is the relative open around t in the definition of x and consequently: $t' \in \cap\{\sup(x), \eta(p_{t'})$ contains $U_t(x)\}$. For the dynamical sheaf theory we may want to impose coherence conditions on the system assuming some relations between $\Gamma_{t''}$ and $\Gamma_{t'}$ if t' and t'' are close enough in T. We shall restrict here to only one extra assumption, in some sense dual to the unambiguity interval assumption for the underlying DNT.

3.3. Definition

The dynamical presheaf $\{\Gamma_t, \phi_{tt'}, T\}$ on a traditional space continuum is locally temporally flabby at $t \in T$ if for t-accessible x such that $p_{t'} \in x$ and $s_{t'} \in \Gamma_{t'}(x_{t'})$ there exists a t-accessible $y < x$ with $p_{t'} \in y$ and a string $\bar{s} \in \mathcal{P}(U_t(y))$ such that $\bar{s}_{t'} = \rho^{t'}_{x_{t'},y_{t'}}(s_{t'})$.

3.4. Theorem

To a dynamical presheaf on a traditional space continuum there corresponds for every $t \in T$ a presheaf $\mathcal{P}_t$ on the moment space $\mathrm{Spec}(\Lambda_t, I_t)$ with its spectral topology given by the $U_t(x)$ for t accessible x. In case all $\Gamma_{t'}$, $t' \in I_t$, are separated presheaves then $\mathcal{P}_t$ is separated too. The sheafification $\underline{\underline{a}}\mathcal{P}_t$ of $\mathcal{P}_t$ on the moment space $\mathrm{Spec}(\Lambda_t, I_t) = Y_t$ is called the **moment sheaf** of **spectral sheaf** at $t \in T$. In case the dynamical presheaf is locally temporally flabby (LTF) then for any point $p_{t'} \in Y_t$ the stalk $\mathcal{P}_{t,p_{t'}}$ may be identified with $\Gamma_{t',p_{t'}}$.

Proof. At every $t \in T$, $\mathcal{P}_t$ is the spectral presheaf constructed on $\mathrm{Spec}(\Lambda_t, I_t)$ with its spectral topology. Now suppose all Γ_t are separated presheaves and look at a finite cover $U_t(x) = U_t(x_1) \cup \cdots \cup U_t(x_n)$ and a $\gamma \in \Gamma_t(U_t(x))$ such that for $i = 1, \ldots, n$, $\rho_{x.x_i}(\gamma) = 0$. We have seen before that the union $U_t(x_1) \cup \cdots \cup U_t(x_n)$ corresponds to the t accessible element $x_1 \wedge \cdots \wedge x_n$ obtained as the string over $\sup(x_1) \wedge \cdots \wedge \sup(x_n)$ given by the $x_{1,t'} \cup \cdots \cup x_{n.t'}$ in $\Lambda_{t'}$. For all $t' \in \sup(x)$ we obtain, in view of the compatibility diagrams for restrictions and $\phi_{t't''}$, $t' \leq t^n$: $\rho^{t'}_{x_{t'},x_{i,t'}}(\gamma_{t'}) = 0$, for $i = 1, \ldots, n$. The assumed separatedness of $\Gamma_{t'}$, for all t' then leads to $\gamma_{t'} = 0$ for all $t' \in \sup(x)$ and therefore $\gamma = 0$ as a string over $\sup(x)$. Consequently $\mathcal{P}_t$ is separated, for all $t \in T$. In order to calculate the stalk at $p_{t'} \in \mathrm{Spec}(\Lambda_t, I_t)$ for $\mathcal{P}_t$ we have to calculate: $\lim_{\longrightarrow_{p_{t'} \in x}} \Gamma_t(U_t(x)) = E_{t'}$.

Starting with $p_{t'} \in x$ for some t-accessible x we have a representative $\gamma_x \in \Gamma_t(U_t(x))$ being a string over $\sup(x)$ and the latter containing a relative open $J(x)$ around t containing t'. So an element $e_{t'}$ in $E_{t'}$ may be viewed as given by a direct family $\{\gamma_x, p_{t'} \in x, \rho_{x,y}(\gamma_x) = \gamma_y \text{ for } y < x\}$. At t', which is in $\sup(x)$ for all x appearing in the forgoing family (as $U_t(x)$ varies over $\eta(p_{t'})$), we obtain $\{(\gamma_x)_{t'}, p_{t'} \leq x_{t'}, \rho^{t'}_{x_{t'},y_{t'}}((\gamma_x)_{t'}) = (\gamma_y)_{t'}\}$ which defines an element of $\Gamma_{t',p_{t'}}$, say $\bar{e}_{t'}$. We have a well-defined map $\pi_{t'} : E_{t'} \to \Gamma_{t',p_{t'}}$, $e_{t'} \mapsto \bar{e}_{t'}$. Without further assumptions we therefore arrive at a sheaf $\underline{\underline{a}}\mathcal{P}_t$ with stalk $E_{t'}$ at $p_{t'}$ and a presheaf map $\mathcal{P}_t \to \underline{\underline{a}}\mathcal{P}_t$ which is "injective" in case all $\Gamma_{t''}$ are separated. Now we have to make use of the locally temporally flabbiness (LTF). Look at a germ $s_{t'} \in (\Gamma_{t'})_{p_{t'}}$. In view of Lemma 3.2. there exists a t-accessible x such that $s_{t'} \in \Gamma_{t'}((x_{t'}))$ with $p_{t'} \in x$, in particular $p_{t'} \leq x_{t'}$.

The LTF-condition allows to select a t-accessible $y < x$ with $p_{t'} \in y$ together with a string, $\vec{s(y)} \in \mathcal{P}(U_t(y))$ such that $\vec{s_{t'}}(y) = \rho^{t'}_{x_{t'},y_{t'}}(s_{t'})$. The element $e_{t'}$ in $E_{t'}$ defined by the directed family obtained by taking restrictions of $\vec{s_{t'}}(y)$ has $\bar{e}_{t'}$ exactly $s_{t'}$ (note that t' supports all the restrictions of $\vec{s_{t'}}(y)$ because y varies in $\eta(p_{t'})$). Thus $\pi_{t'} : E_{t'} \to \Gamma_{t',p_{t'}}$ is epimorphic. If $e_{t'}$ and $e'_{t'}$ have the same image under $\pi_{t'}$ then there is a t-accessible y such that $e_{t'} - e'_{t'}$ is represented by the zero-string over $\sup(y)$; in fact this follows by taking $s_{t'} = 0$ in the foregoing. Leading to a t-accessible y as above that may be restricted to a t-accessible y' defined by taking for $\sup(y')$ the relative open J containing t' in the support of y where $\vec{s_{t'}}(y) = 0$. Therefore, $\pi_{t'}$ is also injective. □

Can one avoid a condition like *LTF* in the foregoing theorem? It seems that the idea of "germ" appearing in the notion of stalks spatially needs an extension in the temporal direction, so probably some condition close to *LTF* is really necessary here.

4. Some remarks on spectral families and observables

Let Γ be any totally ordered abelian group. On a noncommutative topology Λ we define a Γ-**filtration** by a family $\{\lambda_\alpha, \alpha \in \Gamma\}$ such that for $\alpha \leq \beta$ in Γ, $\lambda_\alpha \leq \lambda_\beta$ in Λ and $\vee\{\lambda_\alpha, \alpha \in \Gamma\} = 1$. A Γ-filtration is said to be separated if from $\gamma = \inf\{\gamma_\alpha, \alpha \in \mathcal{A}\}$ in Γ it follows that $\lambda_\gamma = \wedge\{\lambda_\alpha, \alpha \in \mathcal{A}\}$ and $0 = \wedge\{\lambda_\gamma, \gamma \in \Gamma\}$. **A** Γ-**spectral family in** Λ is just a separated Γ-filtration, it may be seen as a map $F : \Gamma \to \Lambda$, $\gamma \mapsto \lambda_\gamma$, where F is a poset map satisfying the separatedness condition. Note that by definition the order in $\wedge\{\lambda_\gamma, \gamma \in \Gamma\}$ does not matter but λ_{γ_α} need not be idempotent in Λ. Taking $\Gamma = \mathbb{R}_+$ and $\Lambda = L(H)$ the lattice of a Hilbert space H, we recover the usual notion of a spectral family. We say that a Γ-spectral family on Λ is idempotent if $\lambda_\gamma \in \mathrm{id}_\wedge(\Lambda)$ for every $\gamma \in \Gamma$.

Observation If Γ is indiscrete, i.e. for all $\gamma \in \Gamma$, $\gamma = \inf\{\tau, \gamma < \tau\}$ (example $\Gamma = \mathbb{R}^n_+$), then every Γ-spectral family is idempotent.

4.1. *Proposition*

With notation as above:

(i) Let us consider a Γ-spectral family on Λ, then for $\gamma, \tau \in \Gamma : \lambda_\gamma \wedge \lambda_\tau = \lambda_\tau \wedge \lambda_\gamma = \lambda_\delta$, where $\delta = \min\{\tau, \gamma\}$.

(ii) If the Γ-spectral family is idempotent then for $\gamma, \tau \in \Gamma$, $\lambda_\gamma \wedge \lambda_\tau = \lambda_\tau \wedge \lambda_\gamma$ and the Γ-spectral family on Γ is in fact a Γ-spectral family of the commutative shadow $SL(\Lambda)$.

Proof. Easy enough. □

A filtration F on a noncommutative Λ is said to be **right bounded** if $\lambda_\gamma = 1$ for some $\gamma \in \Gamma$, F is **left bounded** if $\lambda_\delta = 0$ for some $\delta \in \Gamma$.

For a right bounded Γ-filtration $F : \Gamma \to \Lambda$ we may define for every $\mu \in \Lambda$ the induced filtration $F|_\mu : \Gamma \to \Lambda(\mu)$ where we use $\mu = 1_{\wedge(\mu)}$, $\Lambda(\mu) = \{\lambda \in \Lambda, \lambda \leq \mu\}$. Note that $F|_\mu$ need **not** be separated whenever F is, indeed if $\delta = \inf\{\delta_\alpha, \alpha \in A\}$ in Γ then $\lambda_\delta = \wedge\{\lambda_{\delta_\alpha}, \alpha \in \mathcal{A}\}$ in Γ then $\lambda_\delta = \wedge\{\lambda_{\delta_\alpha}, \alpha \in \mathcal{A}\}$ but $\mu \wedge \lambda_\delta$, and $\wedge\{\mu \wedge \lambda_\alpha, \alpha \in \mathcal{A}\}$ need not be equal in general.

4.2. Proposition

If F defines a right bounded Γ-spectral family on Λ then $F|_\mu$ is a spectral family of $\Lambda(\mu)$ in each of the following cases:

a. $\mu \in \mathrm{id}_\wedge(\Lambda)$ and μ commutes with all $\lambda_\alpha, \alpha \in \Gamma$.
b. $\mu \wedge \lambda_\alpha$ is idempotent for each $\alpha \in \Gamma$.

Proof. Easy and straightforward. $\square$

An element μ with property a. as above is called an F-**centralizer** of Λ.

4.3. Corollary

In case Λ is a lattice then for every $\mu \in \Lambda$ a right bounded Γ-spectral family of Λ induces a right bounded spectral family on $\Lambda(\mu)$.

Let F be a Γ-spectral family on a noncommutative Λ. To $\lambda \in \Lambda$ associate $\sigma(\lambda) \in \Gamma \cup \{\infty\}$ where $\sigma(\lambda) = \inf\{\gamma, \lambda \leq \lambda_\gamma\}$ and we agree to write $\inf \phi = \infty$. The map $\sigma : \Lambda \to \Gamma \cup \{\infty\}$ is a generalization of the principal symbol map in the theory of filtered rings and their associated graded rings. We refer to σ as the **observable function** of F.

$$\text{Clearly}: \quad \begin{array}{l} \sigma(\lambda \wedge \mu) \leq \min\{\sigma(\lambda), \sigma(\mu)\} \\ \sigma(\lambda \vee \mu) \leq \max\{\sigma(\lambda), \sigma(\mu)\}. \end{array}$$

The **domain of** σ is $\cup\{[0, \lambda_\gamma], \gamma \in \Gamma\}$ and observe that $\vee\{\lambda_\gamma, \gamma \in \Gamma\} = 1$ does not imply that $D(\sigma) = \Lambda$ (can even be checked for $\Gamma = \mathbb{R}_+, \Lambda = L(H)$).

If $F : T \to L(H)$ is a Γ-spectral family and $V \subset H$ a linear subspace, then we may define $\gamma_\vee \in \Gamma, \gamma_V = \inf\{\gamma \in \Gamma, \subset L(H)_\gamma\}$, again putting $\inf \phi = \infty$. The map $\rho : L(H) \to \Gamma \cup \{\infty\}, U \mapsto \rho(U) = \gamma_U$ is well-defined. One easily verifies for $\cup$ and $\vee$ in H:

$$\rho(U + V) \leq \max\{\rho(U), \rho(V)\}$$
$$\rho(U \cap V) \leq \min\{\rho(U), \rho(V)\}.$$

The function ρ defines a $\underline{\rho}$ defined on H by putting $\rho(x) = \rho(\mathbb{C}x)$. We denote $\underline{\rho}$ again by ρ and call it the **pseudo-place of the** Γ**-spectral family**. Then any Γ-spectral family defines a function on the projective Hilbert space $\mathbb{P}(H)$ described on the lines in H by $\overline{\rho} : \mathbb{P}(H) \to \Gamma, \underline{\mathbb{C}v} \mapsto \rho(\mathbb{C}v)$, where we wrote $\underline{\mathbb{C}v}$ for $\mathbb{C}v$ as an object in $\mathbb{P}(H)$.

The pseudo-place aspect of ρ translates to $\overline{\rho}$ in the following sense: $\mathbb{C}w \subset \mathbb{C}v + \mathbb{C}u$, then we have $\underline{\rho}(\underline{\mathbb{C}w}) \leq \max\{\overline{\rho}(\underline{\mathbb{C}v}), \overline{\rho}(\underline{\mathbb{C}u})\}$.

A linear subspace $U \subset H$ such that $\mathbb{P}(U) \subset \overline{\rho}^{-1}(]-\infty, \gamma])$ allows for $u \neq 0$ in $U : \rho(\mathbb{C}u) \leq \gamma$ i.e. $u \in L(H)_\gamma$. Hence, the largest U in H such that $\mathbb{P}(U)$ is in $\overline{\rho}^{-1}(]-\infty, \gamma])$ is exactly $L(H)_\gamma$; this means that the filtration F may be reconstructed from the knowledge of $\overline{\rho}$. One easily recovers the classical result that maximal abelian Von Neumann regular subalgebras of $\mathcal{L}(H)$ correspond bijectively to maximal distributive lattices in $L(H)$. Since any Γ-spectral family is a directed set in Λ it defines an element of $C(\Lambda)$ which we call a Γ-**point**. The set of Γ-points of Λ is denoted $[\Gamma] \subset C(\Lambda)$. We may for example think of $[\mathbb{R}] \subset C(L(H))$ as being identified via the Riemann–Stieltjes integral to the set of self-adjoint operators on H.

Let $\sigma : \Lambda \to \Gamma \cup \{\infty\}$ be the observable functions of a Γ-spectral family on Λ defined by $F : \Gamma \to \Lambda$. Put $\mathcal{F} : C(\Lambda) \to \Gamma \cup \{\infty\}, [A] \mapsto \inf\{\gamma \in \Gamma, \lambda_\gamma \in \overline{A}\}, \overline{A}$ the filter of A. Then $\widehat{\sigma}$ is the observable corresponding to the Γ-filtration on $C(\Lambda)$ defined by $[A]_\gamma$, where for $\gamma \in \Gamma, [A]_\gamma$ is the class of the smallest filter containing λ_γ i.e. the filter $\{\mu \in \Lambda, \lambda_\gamma \leq \mu\}$. This is clearly a Γ-spectral family because in fact $[A]_\gamma < [\lambda_\gamma]$. We define $[\Gamma] \cap \mathrm{Sp}(\Lambda) = \Gamma - \mathrm{Sp}(\Lambda), [\Gamma] \cap QSp(\Lambda)[\Gamma] \cap QSp(\Lambda) = \Gamma - QSp(\Lambda)$ and similarly with p replaced by P when $\mathrm{Id}_\wedge(C[\Lambda])$ is considered instead of $\mathrm{id}_\wedge(C(\Lambda_-))$ (see Section 1).

In view of Proposition 4.1(i) a Γ-spectral family is contained in a sublattice (that is with commutative $\wedge$) of the noncommutative Λ, in fact $\{\lambda_\gamma, \gamma \in \Gamma\}$ is such a sublattice. If $Ab(\Lambda)$ is the set of maximal commutative sublattices of Λ then every Γ-spectral family in Λ is a Γ-spectral family in some $B \in Ab(\Lambda)$ (B refers to Boolean sector in case $\Gamma = \mathbb{R}_+, \Lambda = L(H)$). The above remarks may be seen as a generalization of the result concerning maximal commutative Von Neumann regular subalgebras in $\mathcal{L}(H)$ quoted above.

Γ-spectral families may be defined on the moment spaces $\mathrm{Spec}(\Lambda_t, T_t)$ in exactly the way described above as filtrations $\{U_t(x_\gamma), \gamma \in \Gamma\}$, where each x_γ is t-accessible, defining a separated Γ-filtration. For $t'' \in I_t$ we may look at $V_t(x_\gamma) = \{p_{t''}, p_{t''} \in U_t(x_\gamma)\}$, again $p_{t''} = \varphi_{t't''}(p_{t'})$ or $\varphi_{t't''}(p_{t'}) = p_{t''}$ depending whether $t' \leq t''$ or $t'' \leq t'$. The family $\{V_t(x_\gamma), \gamma \in \Gamma\}$ need not (!) be a Γ-spectral family at $t'' \in T$. A stronger notion of **dynamical spectral family** may be obtained by demanding the existence of stringwise spectral families in a relative open T-interval J around t. Then indeed at $t'' \in J \subset I(t)$ such a stringwise Γ-spectral family induces a γ-spectral family in $\Lambda_{t''}$ but not immediately on $\mathrm{Spec}(\Lambda_{t''}, I_{t^n})$ unless a more stringent relation is put on I_{t^n} and its comparison with respect to I_t. We just point out the interesting problems arising with respect to observables when passing to moment spaces but this is work in progress.

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