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Integrable motions of curves in projective geometries

Yanyan Li [a,b,*], Changzheng Qu [b,c], Shichang Shu [a]

[a] *Institute of Mathematics and Information Science, Xianyang Normal University, Xianyang 712000, PR China*
[b] *Center for Nonlinear Studies, Northwest University, Xi'an 710069, PR China*
[c] *Department of Mathematics, Northwest University, Xi'an 710069, PR China*

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A B S T R A C T

In this paper, integrable systems satisfied by the curvatures of curves under inextensible motions in projective geometries are identified. It is shown that motions of curves in two-, three- and four-dimensional projective geometries are respectively related to the Kaup–Kupershmidt hierarchy, a coupled Kaup–Kupershmidt–KdV system and their extension.

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1. Introduction

It has been known that integrable systems are closely related to the motions of curves or surfaces. There have been many works devoted to this subject. About the relationship between integrable systems and motions of curves in the plane, Goldstein and Petrich [1] in an intriguing paper showed that the celebrated mKdV equation naturally arises from inextensible motion of curves in Euclidean geometry, where "naturally" means that the normal velocity in the curve flow is the derivative of curvature of the curve with respect to the arc-length. It was of great interest that Nakayama, Segur and Wadati [2] set up a correspondence between mKdV hierarchy and inextensible motions of plane curves in Euclidean geometry. Later, it was also shown that the KdV hierarchy [3,4], Burgers hierarchy [5,6] and Sawada–Kotera hierarchy [7] naturally arise from inextensible motions of plane curves in centro-affine geometry, similarity geometry and affine geometry, respectively. They can be regarded as a centro-affine version, similarity version and affine version of the mKdV hierarchy.

As compared to the integrable motions of plane curves, it is of great interest to study motions of space curves in various geometric settings. The pioneering work concerning the relationship between integrable systems and invariant curve motions is due to Hasimoto [8]. He showed that the vortex filament flow is related to the non-linear Schrödinger equation via the so-called Hasimoto transformation. Indeed, the Schrödinger flow is an inextensible motion of space curves in Euclidean geometry. Lamb [9] used the Hasimoto transformation to connect other motions of curves to the mKdV and sine-Gordon equations. Lakshmanan [10] interpreted the dynamics of a nonlinear string of fixed length in $\mathbb{R}^3$ through consideration of the motion of an arbitrary rigid body along it, deriving the AKNS spectral problem without spectral parameter. Langer and Perline [11] showed that the dynamics of a non-stretching vortex filament in $\mathbb{R}^3$ gives the NLS hierarchy. Motions of curves on S^2 and S^3 were considered by Doliwa and Santini [12]. Langer et al. [13] obtained the system of mKdV equations from

* Corresponding author at: Institute of Mathematics and Information Science, Xianyang Normal University, Xianyang 712000, PR China.
E-mail address: muzilyy@163.com (Y. Li).

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inextensible motions of space curves in Euclidean geometries. Nakayama [14,15] showed that the defocusing nonlinear Schrödinger equation and a coupled system of KdV equations and their hyperbolic type arise from motions of curves in hyperboloids in the Minkowski space. In [6], integrable motions of space curves in similarity geometries were also studied, a coupled system of mKdV–Burgers hierarchy and their higher-dimensional extension were obtained. The well-known Boussinesq equation can be obtained from the inextensible motion of space curves in affine geometry and centro-affine geometries [7].

It is also of great interest to study motions of curves in various Manifolds and Homogeneous spaces. A number of integrable systems were obtained in [16–32]. More interestingly, the Hamiltonian structure can be identified clearly from the invariant geometric flows, which provides a nice geometric interpretation for Hamiltonian structure to integrable systems.

To obtain higher-dimensional integrable systems from invariant curves or surface motions in various geometries, Lakshmanan, Myrzakulov et al. [33–38] studied motions of curves in Euclidean geometries by adding an extra space variable; many higher-dimensional integrable systems can be obtained in this way. This provides a geometric interpretation for higher-dimensional integrable systems, although it is not natural.

It has been known that there are three integrable fifth-order evolution equations of the form

$$u_t = u_5 + auu_3 + bu_1u_2 + cu^2u_1$$

where a, b and c are constants, $u(t, x)$ is a function of t and x, which are the fifth-order KdV equation, the Sawada–Kotera equation and the Kaup–Kupershmidt equation. The fifth-order KdV equation and the Sawada–Kotera equation have been shown to associate with inextensible motions of curves respectively in centro-affine geometry and affine geometry. So a question arises: Does the Kaup–Kupershmidt equation arise from a motion of curves in certain geometries?

The purpose of this paper is to investigate the relationship between integrable systems and inextensible motions of curves in two-, three- and four-dimensional projective geometries. The paper is outlined as follows: In Section 2, we give a brief account of projective differential geometry of curves. In Section 3, we discuss the inextensible motion of plane curves in the two-dimensional projective geometry $\mathbb{P}^2$. It will be shown that the Kaup–Kupershmidt hierarchy naturally arises from such motion flow. In Section 4, we consider motion of curves in the three-dimensional projective geometry $\mathbb{P}^3$, a coupled Kaup–Kupershmidt and KdV system is obtained. Motion of curves in the four-dimensional projective geometry $\mathbb{P}^4$ is studied in Section 5. Finally, Section 6 gives concluding remarks on this work.

2. Projective differential geometry of curves

Projective differential geometry is an old subject; earlier work on this topic can be traced back to [39]. From Klein's Erlanger program, it is the study of the differential geometry which is invariant with respect to the projective group, i.e. homography.

Let us first give an account of projective differential geometry $\mathbb{P}^2$ of plane curves [39–42]. Let $\mathbf{A}$ be a point of γ parametrized by s with projective coordinates $x_0(s)$, $x_1(s)$ and $x_2(s)$. Denote it by $\mathbf{A} = [x_0(s), x_1(s), x_2(s)]^T$. The coordinates x_i of $\mathbf{A}$ satisfy the third order differential equation

$$\begin{vmatrix} z''' & z'' & z' & z \\ \mathbf{A}''' & \mathbf{A}'' & \mathbf{A}' & \mathbf{A} \end{vmatrix} = 0.$$

It can be written as

$$z''' + pz'' + qz' + rz = 0 \tag{2.1}$$

where

$$p = -\frac{|\mathbf{A}'''\ \mathbf{A}'\ \mathbf{A}|}{|\mathbf{A}''\ \mathbf{A}'\ \mathbf{A}|}, \qquad q = -\frac{|\mathbf{A}'''\ \mathbf{A}''\ \mathbf{A}|}{|\mathbf{A}''\ \mathbf{A}'\ \mathbf{A}|}, \qquad r = -\frac{|\mathbf{A}'''\ \mathbf{A}''\ \mathbf{A}'|}{|\mathbf{A}''\ \mathbf{A}'\ \mathbf{A}|}.$$

It is necessary to assume that $|\mathbf{A}''\ \mathbf{A}'\ \mathbf{A}| \neq 0$, which is equivalent to that the curve has no inflection points. Eq. (2.1) defines a class of curves projectively equal to each other and to the curve γ, but it does not necessarily give all curves projectively equal to γ. In fact, if we change $\mathbf{A}$ to $\bar{\mathbf{A}} = \lambda(s)\mathbf{A}$, we obtain a new equation (2.1) which from the same curve γ, also defines curves that are projectively equal to it. This also applies to the change of parameter $s \to \bar{s} = f(s)$. This allows us to simplify Eq. (2.1). We look for two functions $\lambda(s)$ and $f(s)$ in order to eliminate the terms in z' and z'' in the equation. We shall see that there are infinitely many possible ways, as $\bar{s}$ is only defined up to homography.

Now, we want to transform the equation

$$\mathbf{A}''' + p\mathbf{A}'' + q\mathbf{A}' + r\mathbf{A} = 0 \tag{2.2}$$

to the equation of the form

$$\bar{\mathbf{A}}''' + \bar{r}\bar{\mathbf{A}} = 0 \tag{2.3}$$

by the transformations

$$\bar{\mathbf{A}} = \lambda(s)\mathbf{A}, \qquad \bar{s} = f(s),$$

provided $f(s)$ and $\lambda(s)$ satisfy the system

$$3\left(\frac{\lambda'}{\lambda}-\frac{f''}{f'}\right)=p,\qquad \frac{3\lambda''}{\lambda}-\frac{6\lambda'}{\lambda}\frac{f''}{f'}-\frac{f'''}{f'}+\frac{3f''^2}{f'^2}=q, \tag{2.4}$$

where $\bar{r}$ satisfies

$$\frac{\lambda'''}{\lambda}-\frac{3\lambda''}{\lambda}\frac{f''}{f'}-\frac{\lambda'}{\lambda}\frac{f'''}{f'}+\frac{3\lambda'}{\lambda}\frac{f''^2}{f'^2}+\bar{r}f'^3=r. \tag{2.5}$$

From the above expression, we also have

$$\frac{1}{2}\frac{f'''}{f'}-\frac{3}{4}\frac{f''^2}{f'^2}=-\frac{1}{12}p^2-\frac{1}{4}p'+\frac{1}{4}q. \tag{2.6}$$

The left hand side of (2.6) is the Schwarzian of $\bar{s}$ with respect to s. Once $\bar{s}$ is known we deduce λ by the first one of (2.4). It is easily to see that $\bar{s}$ is defined only up to homography since the Schwarzian is invariant under homography.

If we take into account the expression of $\bar{s}$ and of $\lambda(s)$ in (2.5), this reduces to

$$\bar{r}f'^3=r-\frac{1}{3}pq+\frac{2}{27}p^3-\frac{1}{2}q'+\frac{1}{3}pp'+\frac{1}{6}p''\equiv H(s), \tag{2.7}$$

which can be written as

$$\bar{r}\mathrm{d}\bar{s}^3=H\mathrm{d}s^3.$$

Thus we have

$$\mathrm{d}\sigma=\bar{r}^{\frac{1}{3}}\mathrm{d}\bar{s}=H^{\frac{1}{3}}\mathrm{d}s.$$

Notice that H involves derivatives of the fourth order; it is easy to check that the arc-length is invariant under the projective transformation group.

Using the projective arc-length σ to parametrize the curve γ, and noting the expression (2.4) for p, Eq. (2.2) has no terms in $\mathrm{d}^2\mathbf{A}/\mathrm{d}\sigma^2$, so Eq. (2.2) can be written in the form

$$\frac{\mathrm{d}^3\mathbf{A}}{\mathrm{d}\sigma^3}+2\kappa\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}\sigma}+h\mathbf{A}=0. \tag{2.8}$$

The quantities h and κ are invariant with respect to a projective transformation, but they are not independent, since for any parameter s, we must have

$$\mathrm{d}\sigma=H^{\frac{1}{3}}\mathrm{d}s.$$

In particular, if we have $\mathrm{d}\sigma=\mathrm{d}s$, that is to say $H=1$ which, taking into account Eq. (2.6), enables us to write

$$h=\kappa'+1.$$

Hence if the curve is parametrized by arc-length, Eq. (2.8) becomes

$$\frac{\mathrm{d}^3\mathbf{A}}{\mathrm{d}\sigma^3}+2\kappa\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}\sigma}+(\kappa'+1)\mathbf{A}=0. \tag{2.9}$$

This leaves only one invariant κ in the equation, and it is the projective curvature introduced previously. The Schwarzian $\{\bar{s}\}_\sigma$ of $\bar{s}$ with respect to σ is given by

$$\{\bar{s}\}_\sigma=\frac{1}{2}\kappa$$

and we can deduce

$$\kappa=2\{\bar{s}\}_\sigma.$$

To determine κ we need the following result.

Proposition 2.1. *Given two functions $x(s)$ and $y(s)$ of the same variable s, we have*

$$\{y\}_x\mathrm{d}x^2=[\{y\}_s-\{x\}_s]\mathrm{d}s^2.$$

Applying this proposition to $x(s) = \sigma(s)$ and $y(s) = \bar{s}(s)$, we obtain

$$\{\bar{s}\}_\sigma \mathrm{d}\sigma^2 = [\{\bar{s}\}_s - \{\sigma\}_s]\mathrm{d}s^2.$$

Hence, taking into account the expression of κ, we have

$$\kappa = 2\frac{\{\bar{s}\}_s - \{\sigma\}_s}{(\frac{\mathrm{d}\sigma}{\mathrm{d}s})^2}.$$

Since $\{\bar{s}\}_s$ is known, we need to compute $\{\sigma\}_s$. In fact, we have

$$\frac{\mathrm{d}\sigma}{\mathrm{d}s} = H^{\frac{1}{3}}.$$

Deriving logarithmically

$$\frac{\sigma''}{\sigma'} = \frac{1}{3}\frac{H'}{H}$$

and a further derivative yields

$$\frac{\sigma'''}{\sigma'} - \frac{\sigma''^2}{\sigma'^2} = \frac{1}{3}\frac{H''}{H} - \frac{1}{3}\frac{H'^2}{H^2}.$$

Thus we have

$$\{\sigma\}_s = \frac{1}{2}\frac{\sigma'''}{\sigma'} - \frac{3}{4}\frac{\sigma''^2}{\sigma'^2} = \frac{1}{6}\frac{H''}{H} - \frac{7}{36}\frac{H'^2}{H^2}.$$

Plugging them into the expression for κ, we get

$$\kappa = H^{-\frac{2}{3}}\left[-\frac{1}{2}p' - \frac{1}{6}p^2 + \frac{1}{2}q - \frac{1}{3}\frac{H''}{H} + \frac{7}{18}\frac{H'^2}{H^2}\right]. \tag{2.10}$$

It is readily checked that the projective curvature is related to the affine curvature of the curve γ via

$$\kappa = \left(\frac{\phi}{2}\right)^{-\frac{2}{3}}\left[-\frac{\phi}{2} - \frac{1}{3}\frac{\phi'''}{\phi'} + \frac{7}{18}\left(\frac{\phi''}{\phi'}\right)^2\right],$$

where the $'$ denotes the derivative with respect to the affine arc-length.

The Frenet frame of the curve is the moving frame of $\mathbb{P}^2$ denoted by $\mathbf{A}, \mathbf{A}^{(1)}, \mathbf{A}^{(2)}$, defined by the following formulas

$$\mathbf{A}^{(1)} = \frac{\mathrm{d}\mathbf{A}}{\mathrm{d}\sigma},$$
$$\mathbf{A}^{(2)} = \frac{\mathrm{d}^2\mathbf{A}}{\mathrm{d}\sigma^2} + \kappa\mathbf{A}.$$

It gives a moving frame of $\mathbb{P}^2$, where the point $\mathbf{A}^{(1)}$ is the tangent to the curve in $\mathbf{A}$ and the line $\langle\mathbf{A}, \mathbf{A}^{(2)}\rangle$ is the projective normal. Notice that $\det(\mathbf{A}, \mathbf{A}^{(1)}, \mathbf{A}^{(2)}) = 1$. The following Frenet formulas are an immediate consequence of the definition of the Frenet frame and the third-order differential equation (2.8):

$$\begin{pmatrix}\mathbf{A}\\ \mathbf{A}^{(1)}\\ \mathbf{A}^{(2)}\end{pmatrix}_\sigma = \begin{pmatrix}0 & 1 & 0\\ \kappa & 0 & 1\\ -1 & \kappa & 0\end{pmatrix}\begin{pmatrix}\mathbf{A}\\ \mathbf{A}^{(1)}\\ \mathbf{A}^{(2)}\end{pmatrix}. \tag{2.11}$$

Now the Frenet frame is given by

$$\mathbf{A}^{(1)}(s) = \frac{\mathrm{d}\mathbf{A}}{\mathrm{d}\sigma} = H^{-\frac{1}{3}}\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}s},$$
$$\mathbf{A}^{(2)}(s) = \frac{\mathrm{d}^2\mathbf{A}}{\mathrm{d}\sigma^2} + \kappa\mathbf{A} = H^{-\frac{2}{3}}\frac{\mathrm{d}^2\mathbf{A}}{\mathrm{d}s^2} - \frac{1}{3}H^{-\frac{5}{3}}H'\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}s} + \kappa\mathbf{A}.$$

It is easy to verify the fundamental theorem for curves in $\mathbb{P}^2$.

Theorem 2.1. *Given a continuous function $\kappa(\sigma)$ on a closed interval $a \le \sigma \le b$. Then there exists a unique plane curve up to the homography, its arc-length is σ and curvature is $\kappa(\sigma)$.*

It follows from the fundamental Theorem 2.1 that the curves with constant projective curvature fall in three categories:

1. The general parabola: $y = x^m$, $m \notin \{2, \frac{1}{2}, -1\}$;
2. The logarithmic spiral: $\rho = e^{m\theta}$, $m \neq 0$;
3. The exponential: $y = e^x$.

Similarly, we can establish the theory of the curves in n-dimensional projective geometry $\mathbb{P}^n$ [42]. In the three-dimensional case, we obtain the Serrent–Frenet formulas for curves in $\mathbb{P}^3$ [42]

$$\begin{pmatrix} \mathbf{A} \\ \mathbf{A}^{(1)} \\ \mathbf{A}^{(2)} \\ \mathbf{A}^{(3)} \end{pmatrix}_\sigma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3\kappa & 0 & 1 & 0 \\ -\dfrac{16}{5} & 4\kappa & 0 & 1 \\ -\tau & -\dfrac{4}{5} & 3\kappa & 0 \end{pmatrix} \begin{pmatrix} \mathbf{A} \\ \mathbf{A}^{(1)} \\ \mathbf{A}^{(2)} \\ \mathbf{A}^{(3)} \end{pmatrix}, \tag{2.12}$$

where κ and τ are the projective curvatures of the curves in $\mathbb{P}^3$. In the four-dimensional case, the Serrent–Frenet formulas read as [42]

$$\begin{pmatrix} \mathbf{A} \\ \mathbf{A}^{(1)} \\ \mathbf{A}^{(2)} \\ \mathbf{A}^{(3)} \\ \mathbf{A}^{(4)} \end{pmatrix}_\sigma = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 4\kappa_1 & 0 & 1 & 0 & 0 \\ 264 & 6\kappa_1 & 0 & 1 & 0 \\ 104\kappa_2 & 108 & 6\kappa_1 & 0 & 1 \\ 24\kappa_3 & 8\kappa_2 & 48 & 4\kappa_1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{A} \\ \mathbf{A}^{(1)} \\ \mathbf{A}^{(2)} \\ \mathbf{A}^{(3)} \\ \mathbf{A}^{(4)} \end{pmatrix}, \tag{2.13}$$

where κ_1, κ_2 and κ_3 are the projective curvatures of the curves in $\mathbb{P}^4$. The curves are determined uniquely by the curvatures κ_1, κ_2 and κ_3 up to the projective transformation group.

3. Motion of curves in two-dimensional projective geometry $\mathbb{P}^2$

In this section, we consider inextensible motion of curves in two-dimensional projective geometry $\mathbb{P}^2$ by using the previous frame structure to obtain the Kaup–Kupershmidt hierarchy.

For convenience, instead of $\mathbf{A}$ by γ in (2.11), we have the Serret–Frenet formulas in $\mathbb{P}^2$

$$\begin{pmatrix} \gamma \\ \gamma^{(1)} \\ \gamma^{(2)} \end{pmatrix}_\sigma = \begin{pmatrix} 0 & 1 & 0 \\ \kappa & 0 & 1 \\ -1 & \kappa & 0 \end{pmatrix} \begin{pmatrix} \gamma \\ \gamma^{(1)} \\ \gamma^{(2)} \end{pmatrix}, \tag{3.1}$$

where $\sigma(p, t)$ is the projective arc-length and κ is the projective curvature of the curve $\gamma(t, \sigma)$, p is a free parameter, independent of time t. One may parametrize the curve γ by

$$[\gamma, \gamma^{(1)}, \gamma^{(2)}] = 1. \tag{3.2}$$

The projective curve flow in $\mathbb{P}^2$ is governed by

$$\gamma_t = F\gamma + G\gamma^{(1)} + H\gamma^{(2)}, \tag{3.3}$$

where F, G and H depend on the projective curvature κ and its derivatives.

Let L be the perimeter for a closed curve, then

$$L = \oint g\,\mathrm{d}p,$$

where $g = \mathrm{d}\sigma/\mathrm{d}p$. Normally, we require that the curve is not inextensible and the arc-length does not depend on time t. In view of the formula for the projective arc-length, this is the case if $\mathrm{d}/\mathrm{d}t[\gamma, \gamma^{(1)}, \gamma^{(2)}] = 0$. At the same time, using the projective Serret–Frenet formula (3.1), one can readily find

$$\frac{g_t}{g} = F + G_\sigma + \frac{1}{3}H_{\sigma\sigma} + \frac{1}{3}\kappa H.$$

Using it, one obtains the time variation of the perimeter

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \oint \left(F + G_\sigma + \frac{1}{3}H_{\sigma\sigma} + \frac{1}{3}\kappa H\right) \mathrm{d}\sigma.$$

So the inextensibility condition means

$$\begin{aligned} &\oint \left(F + \frac{1}{3}\kappa H\right) \mathrm{d}\sigma = 0, \\ &F = -G_\sigma - \frac{1}{3}H_{\sigma\sigma} - \frac{1}{3}\kappa H. \end{aligned} \tag{3.4}$$

The first equation in (3.4) means that the perimeter for a closed curve is invariant under the motion (3.3), and the second one means that the arc-length σ commutes with time t.

In view of Eqs. (3.1) and (3.3), one obtains the time evolution of the frame in two-dimensional projective geometry

$$\begin{pmatrix}\gamma\\ \gamma^{(1)}\\ \gamma^{(2)}\end{pmatrix}_t=\begin{pmatrix}F & G & H\\ F_1 & G_1 & H_1\\ F_2 & G_2 & H_2\end{pmatrix}\begin{pmatrix}\gamma\\ \gamma^{(1)}\\ \gamma^{(2)}\end{pmatrix},\tag{3.5}$$

where F, G, H, F_i, G_i and H_i, $i = 1, 2$ are functions of curvature κ to be determined.

Eqs. (3.1) and (3.5) are compatible, that is to say

$$\begin{pmatrix}\gamma\\ \gamma^{(1)}\\ \gamma^{(2)}\end{pmatrix}_{t\sigma}=\begin{pmatrix}\gamma\\ \gamma^{(1)}\\ \gamma^{(2)}\end{pmatrix}_{\sigma t},$$

one has

$$\begin{aligned}
F_1 &= F_\sigma + \kappa G - H,\\
G_1 &= F + G_\sigma + \kappa H,\\
H_1 &= G + H_\sigma,\\
F_2 &= -\kappa_t - \kappa F + F_{1\sigma} + \kappa G_1 - H_1,\\
G_2 &= F_1 - \kappa G + G_{1\sigma} + \kappa H_1,\\
H_2 &= G_1 - \kappa H + H_{1\sigma},
\end{aligned}$$

and

$$\begin{aligned}
&-F + \kappa F_1 - F_{2\sigma} - \kappa G_2 + H_2 = 0,\\
&\kappa_t = F_2 + G - \kappa G_1 + G_{2\sigma} + \kappa H_2,\\
&G_2 + H - \kappa H_1 + H_{2\sigma} = 0.
\end{aligned}\tag{3.6}$$

Clearing up and simplifying the above equations, one arrives at the following equations

$$\begin{aligned}
F &= -G_\sigma - \frac{1}{3}H_{\sigma\sigma} - \frac{1}{3}\kappa H,\\
G &= \frac{1}{18}H_{\sigma\sigma\sigma\sigma} - \frac{5}{9}\kappa H_{\sigma\sigma} - \left(\frac{5}{18}\kappa_\sigma + \frac{1}{2}\right)H_\sigma - \frac{1}{9}(\kappa_{\sigma\sigma} - 4\kappa^2)H - \frac{1}{9}\partial_\sigma^{-1}[(\kappa_{\sigma\sigma} - 4\kappa^2)H_\sigma],\\
F_1 &= F_\sigma + \kappa G - H,\\
G_1 &= F + G_\sigma + \kappa H,\\
H_1 &= G + H_\sigma,\\
F_2 &= -\frac{1}{2}F_{\sigma\sigma} - \frac{1}{2}G_{\sigma\sigma\sigma} - G - \frac{3}{2}\kappa H_{\sigma\sigma} - \frac{1}{2}(3\kappa_\sigma + 1)H_\sigma - \left(\frac{1}{2}\kappa_{\sigma\sigma} - \kappa^2\right)H,\\
G_2 &= 2F_\sigma + G_{\sigma\sigma} + \kappa G + 2\kappa H_\sigma + (\kappa_\sigma - 1)H,\\
H_2 &= F + 2G_\sigma + H_{\sigma\sigma},\\
\kappa_t &= \frac{3}{2}F_{\sigma\sigma} + \frac{1}{2}G_{\sigma\sigma\sigma} + 2\kappa G_\sigma + \kappa_\sigma G + \frac{3}{2}\kappa H_{\sigma\sigma} + \frac{3}{2}(\kappa_\sigma - 1)H_\sigma + \frac{1}{2}\kappa_{\sigma\sigma}H.
\end{aligned}\tag{3.7}$$

Substituting F and G into the last equation of system (3.7), we obtain the evolution equation satisfied by the curvature κ

$$\kappa_t = -18\Omega H_\sigma,\tag{3.8}$$

where

$$\begin{aligned}
\Omega = {}& \partial_\sigma^6 - 12\kappa\partial_\sigma^4 - 36\kappa_\sigma\partial_\sigma^3 - 49\kappa_{\sigma\sigma}\partial_\sigma^2 + 36\kappa^2\partial_\sigma^2 - 35\kappa_{\sigma\sigma\sigma}\partial_\sigma + 120\kappa\kappa_\sigma\partial_\sigma - 13\kappa_{\sigma\sigma\sigma\sigma}\\
&+ 82\kappa\kappa_{\sigma\sigma} + 69\kappa_\sigma^2 - 32\kappa^3 + 27 - (2\kappa_{\sigma\sigma\sigma\sigma\sigma} - 20\kappa\kappa_{\sigma\sigma\sigma} - 50\kappa_\sigma\kappa_{\sigma\sigma} + 40\kappa^2\kappa_\sigma)\partial_\sigma^{-1} + 2\kappa_\sigma\partial_\sigma^{-1}(\kappa_{\sigma\sigma} - 4\kappa^2).
\end{aligned}$$

Indeed, taking $\kappa = -\frac{1}{2}\tilde{\kappa}$, we arrive at

$$\begin{aligned}
\Omega = {}& \partial_\sigma^6 + 6\tilde{\kappa}\partial_\sigma^4 + 18\tilde{\kappa}_\sigma\partial_\sigma^3 + \frac{49}{2}\tilde{\kappa}_{\sigma\sigma}\partial_\sigma^2 + 9\tilde{\kappa}^2\partial_\sigma^2 + \frac{35}{2}\tilde{\kappa}_{\sigma\sigma\sigma}\partial_\sigma + 30\tilde{\kappa}\tilde{\kappa}_\sigma\partial_\sigma + \frac{13}{2}\tilde{\kappa}_{\sigma\sigma\sigma\sigma}\\
&+ \frac{41}{2}\tilde{\kappa}\tilde{\kappa}_{\sigma\sigma} + \frac{69}{4}\tilde{\kappa}_\sigma^2 + 4\tilde{\kappa}^3 + 27 + \left(\tilde{\kappa}_{\sigma\sigma\sigma\sigma\sigma} + 5\tilde{\kappa}\tilde{\kappa}_{\sigma\sigma\sigma} + \frac{25}{2}\tilde{\kappa}_\sigma\tilde{\kappa}_{\sigma\sigma} + 5\tilde{\kappa}^2\tilde{\kappa}_\sigma\right)\partial_\sigma^{-1} + \frac{1}{2}\tilde{\kappa}_\sigma\partial_\sigma^{-1}(\tilde{\kappa}_{\sigma\sigma} + 2\tilde{\kappa}^2).
\end{aligned}$$

It is the recursion operator of the Kaup–Kupershmidt equation [43]

$$\kappa_t = \kappa_{\sigma\sigma\sigma\sigma\sigma} - 10\kappa\kappa_{\sigma\sigma\sigma} - 25\kappa_\sigma\kappa_{\sigma\sigma} + 20\kappa^2\kappa_\sigma. \tag{3.9}$$

Its integrability has been explored extensively.

Letting $H_\sigma = -(1/18)\Omega^{n-1}\kappa_\sigma$, we get the Kaup–Kupershmidt hierarchy

$$\kappa_t = \Omega^n\kappa_\sigma, \quad n \geq 1.$$

Taking $H = -18\kappa$, we arrive at the seventh-order Kaup–Kupershmidt equation

$$\kappa_t = \kappa_{\sigma\sigma\sigma\sigma\sigma\sigma\sigma} - 14\kappa\kappa_{\sigma\sigma\sigma\sigma\sigma} - 49\kappa_\sigma\kappa_{\sigma\sigma\sigma\sigma} - 84\kappa_{\sigma\sigma}\kappa_{\sigma\sigma\sigma} + 56\kappa^2\kappa_{\sigma\sigma\sigma} + 252\kappa\kappa_\sigma\kappa_{\sigma\sigma} + 70\kappa_\sigma^3 - \frac{224}{3}\kappa^3\kappa_\sigma + 27\kappa_\sigma.$$

4. Motion of curves in three-dimensional projective geometry $\mathbb{P}^3$

Now we study the motion of space curves in three-dimensional projective geometry $\mathbb{P}^3$.

In three-dimensional projective geometry, the Serret–Frenet formulas for curves read as [42]

$$\begin{pmatrix} \gamma \\ \gamma^{(1)} \\ \gamma^{(2)} \\ \gamma^{(3)} \end{pmatrix}_\sigma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3\kappa & 0 & 1 & 0 \\ -\frac{16}{5} & 4\kappa & 0 & 1 \\ -\tau & -\frac{4}{5} & 3\kappa & 0 \end{pmatrix} \begin{pmatrix} \gamma \\ \gamma^{(1)} \\ \gamma^{(2)} \\ \gamma^{(3)} \end{pmatrix}. \tag{4.1}$$

The three-dimensional projective curve flow is governed by

$$\gamma_t = E\gamma + F\gamma^{(1)} + G\gamma^{(2)} + H\gamma^{(3)}, \tag{4.2}$$

where E, F, G and H are velocities of the curve motion, depending on the projective curvatures κ, τ and their derivatives with respect to the projective arc-length σ.

In terms of (4.1) and (4.2), the time evolution of the frame in three-dimensional projective geometry is given by

$$\begin{pmatrix} \gamma \\ \gamma^{(1)} \\ \gamma^{(2)} \\ \gamma^{(3)} \end{pmatrix}_t = \begin{pmatrix} E & F & G & H \\ E_1 & F_1 & G_1 & H_1 \\ E_2 & F_2 & G_2 & H_2 \\ E_3 & F_3 & G_3 & H_3 \end{pmatrix} \begin{pmatrix} \gamma \\ \gamma^{(1)} \\ \gamma^{(2)} \\ \gamma^{(3)} \end{pmatrix}, \tag{4.3}$$

where $E, F, G, H, E_i, F_i, G_i$ and H_i, $i = 1, 2, 3$ are functions of the curvatures, to be determined.

The commutativity condition $\partial_\sigma\partial_t = \partial_t\partial_\sigma$ leads to the following equations

$$\begin{aligned}
E_1 &= E_\sigma + 3\kappa F - \frac{16}{5}G - \tau H, \\
F_1 &= E + F_\sigma + 4\kappa G - \frac{4}{5}H, \\
G_1 &= F + G_\sigma + 3\kappa H, \\
H_1 &= G + H_\sigma, \\
E_2 &= -3\kappa_t - 3\kappa E + E_{1\sigma} + 3\kappa F_1 - \frac{16}{5}G_1 - \tau H_1, \\
F_2 &= E_1 - 3\kappa F + F_{1\sigma} + 4\kappa G_1 - \frac{4}{5}H_1, \\
G_2 &= F_1 - 3\kappa G + G_{1\sigma} + 3\kappa H_1, \\
H_2 &= G_1 - 3\kappa H + H_{1\sigma}, \\
E_3 &= \frac{16}{5}E - 4\kappa E_1 + E_{2\sigma} + 3\kappa F_2 - \frac{16}{5}G_2 - \tau H_2, \\
F_3 &= E_2 - 4\kappa_t + \frac{16}{5}F - 4\kappa F_1 + F_{2\sigma} + 4\kappa G_2 - \frac{4}{5}H_2, \\
G_3 &= F_2 + \frac{16}{5}G - 4\kappa G_1 + G_{2\sigma} + 3\kappa H_2, \\
H_3 &= G_2 + \frac{16}{5}H - 4\kappa H_1 + H_{2\sigma},
\end{aligned}$$

and

$$E_3+\tau F+\frac{4}{5}F_1-3\kappa F_2+F_{3\sigma}+4\kappa G_3-\frac{4}{5}H_3=0,$$
$$G_3+\tau H+\frac{4}{5}H_1-3\kappa H_2+H_{3\sigma}=0,$$
$$\kappa_t=\frac{1}{3}(F_3+\tau G+\frac{4}{5}G_1-3\kappa G_2+G_{3\sigma}+3\kappa H_3),$$
$$\tau_t=-\tau E-\frac{4}{5}E_1+3\kappa E_2-E_{3\sigma}-3\kappa F_3+\frac{16}{5}G_3+\tau H_3.$$

Simplifying the above equations, one has the following equations

$$E=-\frac{3}{2}F_\sigma-G_{\sigma\sigma}-2\kappa G-\frac{1}{4}H_{\sigma\sigma\sigma}-2\kappa H_\sigma-\frac{3}{2}\kappa_\sigma H-\frac{1}{5}H,$$
$$\begin{aligned}F\;=\;&\frac{1}{12}G_{\sigma\sigma\sigma\sigma}-\frac{5}{3}\kappa G_{\sigma\sigma}-\left(\frac{5}{6}\kappa_\sigma+1\right)G_\sigma-\left(\frac{1}{3}\kappa_{\sigma\sigma}-\frac{8}{3}\kappa^2+\frac{1}{6}\tau\right)G-\partial_\sigma^{-1}\left[\left(\frac{1}{3}\kappa_{\sigma\sigma}-\frac{8}{3}\kappa^2+\frac{1}{6}\tau\right)G_\sigma\right]\\&+\frac{1}{24}H_{\sigma\sigma\sigma\sigma\sigma}-\frac{5}{6}\kappa H_{\sigma\sigma\sigma}+\left(-\frac{5}{12}\kappa_\sigma+\frac{1}{6}\right)H_{\sigma\sigma}-\left(\frac{1}{6}\kappa_{\sigma\sigma}-\frac{4}{3}\kappa^2+\frac{1}{12}\tau\right)H_\sigma-3\kappa H\\&-\partial_\sigma^{-1}\left[\left(\frac{1}{6}\kappa_{\sigma\sigma}-\frac{4}{3}\kappa^2+\frac{1}{12}\tau\right)H_{\sigma\sigma}+\frac{5}{3}\kappa H_\sigma\right],\end{aligned}$$
$$E_1=E_\sigma+3\kappa F-\frac{16}{5}G-\tau H,$$
$$F_1=E+F_\sigma+4\kappa G-\frac{4}{5}H,$$
$$G_1=F+G_\sigma+3\kappa H,$$
$$H_1=G+H_\sigma,$$
$$E_2=\frac{7}{10}E_{1\sigma}-\frac{21}{10}\kappa E-\frac{3}{10}F_{2\sigma}+\frac{33}{10}\kappa F_1-\frac{24}{25}F-\frac{62}{25}G_1-\frac{3}{10}\kappa G_2-\frac{3}{10}G_{3\sigma}-\frac{3}{10}\tau G-\frac{7}{10}\tau H_1+\frac{6}{25}H_2-\frac{9}{10}\kappa H_3,$$
$$F_2=E_1-3\kappa F+F_{1\sigma}+4\kappa G_1-\frac{4}{5}H_1,$$
$$G_2=F_1-3\kappa G+G_{1\sigma}+3\kappa H_1,$$
$$H_2=G_1-3\kappa H+H_{1\sigma},$$
$$E_3=\frac{16}{5}E-4\kappa E_1+E_{2\sigma}+3\kappa F_2-\frac{16}{5}G_2-\tau H_2,$$
$$F_3=\frac{3}{10}E_{1\sigma}-\frac{9}{10}\kappa E+\frac{3}{10}F_{2\sigma}-\frac{3}{10}\kappa F_1+\frac{24}{25}F-\frac{38}{25}G_1+\frac{33}{10}\kappa G_2-\frac{7}{10}\tau G-\frac{7}{10}G_{3\sigma}-\frac{3}{10}\tau H_1-\frac{6}{25}H_2-\frac{21}{10}\kappa H_3,$$
$$G_3=F_2+\frac{16}{5}G-4\kappa G_1+G_{2\sigma}+3\kappa H_2,$$
$$H_3=G_2+\frac{16}{5}H-4\kappa H_1+H_{2\sigma},$$
$$\begin{aligned}\kappa_t\;=\;&-\frac{1}{2}F_{\sigma\sigma\sigma}+2\kappa F_\sigma+\kappa_\sigma F-\frac{1}{2}G_{\sigma\sigma\sigma\sigma}+2\kappa G_{\sigma\sigma}+\left(\kappa_\sigma-\frac{6}{5}\right)G_\sigma-\frac{3}{20}H_{\sigma\sigma\sigma\sigma\sigma}-\left(\frac{3}{2}\kappa_\sigma+\frac{3}{5}\right)H_{\sigma\sigma}\\&-\left(\frac{9}{5}\kappa_{\sigma\sigma}-\frac{12}{5}\kappa^2+\frac{2}{5}\tau\right)H_\sigma-\left(\frac{3}{5}\kappa_{\sigma\sigma\sigma}-\frac{18}{5}\kappa\kappa_\sigma+\frac{3}{10}\tau_\sigma\right)H,\end{aligned}$$
$$\begin{aligned}\tau_t\;=\;&6F_{\sigma\sigma}+4\tau F_\sigma+\tau_\sigma F-\frac{1}{2}G_{\sigma\sigma\sigma\sigma\sigma\sigma}+10\kappa G_{\sigma\sigma\sigma\sigma}+\left(25\kappa_\sigma+\frac{2}{5}\right)G_{\sigma\sigma\sigma}+(24\kappa_{\sigma\sigma}-32\kappa^2+6\tau)G_{\sigma\sigma}\\&+\left(11\kappa_{\sigma\sigma\sigma}-96\kappa\kappa_\sigma+\frac{208}{5}\kappa+4\tau_\sigma\right)G_\sigma+(2\kappa_{\sigma\sigma\sigma\sigma}-32\kappa\kappa_{\sigma\sigma}-32\kappa_\sigma^2+20\kappa_\sigma+\tau_{\sigma\sigma})G-\frac{1}{5}H_{\sigma\sigma\sigma\sigma\sigma\sigma\sigma}\\&+\frac{11}{5}\kappa H_{\sigma\sigma\sigma\sigma\sigma}+\left(\frac{11}{2}\kappa_\sigma-\frac{19}{5}\right)H_{\sigma\sigma\sigma\sigma}+\left(\frac{18}{5}\kappa_{\sigma\sigma}+\frac{116}{5}\kappa^2+\frac{14}{5}\tau\right)H_{\sigma\sigma\sigma}\\&+\left(-\frac{1}{10}\kappa_{\sigma\sigma\sigma}+\frac{348}{5}\kappa\kappa_\sigma+\frac{244}{5}\kappa+\frac{27}{10}\tau_\sigma\right)H_{\sigma\sigma}+\left(-\kappa_{\sigma\sigma\sigma\sigma}+\frac{272}{5}\kappa\kappa_{\sigma\sigma}+56\kappa_\sigma+43\kappa_\sigma^2-\frac{4}{5}\kappa\tau-\frac{576}{5}\kappa^3\right.\\&\left.-\;12+\tau_{\sigma\sigma}\right)H_\sigma+\left(-\frac{3}{10}\kappa_{\sigma\sigma\sigma\sigma\sigma}+\frac{78}{5}\kappa\kappa_{\sigma\sigma\sigma}+\frac{177}{5}\kappa_\sigma\kappa_{\sigma\sigma}+18\kappa_{\sigma\sigma}+2\kappa_\sigma\tau\right.\\&\left.-\;\frac{864}{5}\kappa^2\kappa_\sigma-\frac{8}{5}\kappa\tau_\sigma+\frac{1}{10}\tau_{\sigma\sigma\sigma}\right)H.\end{aligned}$$

Substituting the expression for F into the equations for the projective curvatures κ and τ, one obtains the following equations

$$\begin{pmatrix}\kappa\\ \tau\end{pmatrix}_t=\begin{pmatrix}A_1 & B_1\\ A_2 & B_2\end{pmatrix}\begin{pmatrix}G_\sigma\\ H_\sigma\end{pmatrix},\tag{4.4}$$

where

$$\begin{aligned}
A_1 &= \Omega_1+\frac{1}{3}\tau_{\sigma\sigma}+\frac{5}{12}\tau_\sigma\partial_\sigma+\frac{1}{6}\tau\partial_\sigma^2-\frac{2}{3}\kappa\tau-\frac{1}{6}\kappa_\sigma\partial_\sigma^{-1}\tau+\frac{1}{12}\Omega_2(\tau_\sigma\partial_\sigma^{-1}),\\
B_1 &= \left(\frac{1}{2}\Omega_1+\frac{1}{12}\tau\Omega_2\right)\partial_\sigma-\frac{7}{30}\partial_\sigma^4+\frac{8}{3}\kappa\partial_\sigma^2+\frac{29}{6}\kappa_\sigma\partial_\sigma+\frac{53}{15}\kappa_{\sigma\sigma}-\frac{104}{15}\kappa^2+\left(\frac{9}{10}\kappa_{\sigma\sigma\sigma}-\frac{27}{5}\kappa\kappa_\sigma\right)\partial_\sigma^{-1}\\
&\quad -\frac{5}{3}\kappa_\sigma\partial_\sigma^{-1}\kappa+\frac{1}{24}\tau_{\sigma\sigma\sigma}+\frac{1}{6}\tau_{\sigma\sigma}\partial_\sigma+\tau_\sigma\left(\frac{5}{24}\partial_\sigma^2-\frac{1}{6}\kappa-\frac{3}{10}\partial_\sigma^{-1}\right)+\left(\frac{1}{12}\kappa_\sigma-\frac{2}{5}\right)\tau-\frac{1}{12}\kappa_\sigma\partial_\sigma^{-1}\tau\partial_\sigma,\\
A_2 &= \frac{1}{3}\tau\partial_\sigma^4+\frac{1}{12}\tau_\sigma\partial_\sigma^3-\left(\frac{20}{3}\kappa\tau+\frac{28}{5}\right)\partial_\sigma^2-\left(10\kappa_\sigma\tau+\frac{5}{3}\kappa\tau_\sigma\right)\partial_\sigma-\left(6\kappa_{\sigma\sigma}\tau+\frac{5}{6}\kappa_\sigma\tau_\sigma-\frac{64}{3}\kappa^2\tau-\frac{208}{5}\kappa\right.\\
&\quad \left.+\frac{4}{3}\tau^2\right)-\left(\frac{4}{3}\kappa_{\sigma\sigma\sigma}\tau+\frac{1}{3}\kappa_{\sigma\sigma}\tau_\sigma-\frac{64}{3}\kappa\kappa_\sigma\tau-20\kappa_\sigma-\frac{8}{3}\kappa^2\tau_\sigma+\frac{5}{6}\tau\tau_\sigma\right)\partial_\sigma^{-1}-\frac{1}{6}\tau_\sigma\partial_\sigma^{-1}(2\kappa_{\sigma\sigma}-16\kappa^2+\tau),\\
B_2 &= \frac{1}{20}\partial_\sigma^6+\frac{1}{6}\tau\partial_\sigma^5-\left(\frac{14}{5}\kappa-\frac{1}{24}\tau_\sigma\right)\partial_\sigma^4-\left(7\kappa_\sigma+\frac{10}{3}\kappa\tau+\frac{14}{5}\right)\partial_\sigma^3-\left(\frac{42}{5}\kappa_{\sigma\sigma}+5\kappa_\sigma\tau-\frac{196}{5}\kappa^2+\frac{5}{6}\kappa\tau_\sigma\right.\\
&\quad \left.-\frac{37}{15}\tau\right)\partial_\sigma^2-\left(\frac{28}{5}\kappa_{\sigma\sigma\sigma}+3\kappa_{\sigma\sigma}\tau-\frac{588}{5}\kappa\kappa_\sigma+\frac{5}{12}\kappa_\sigma\tau_\sigma-\frac{32}{3}\kappa^2\tau-\frac{104}{5}\kappa-\frac{41}{30}\tau_\sigma+\frac{2}{3}\tau^2\right)\partial_\sigma-2\kappa_{\sigma\sigma\sigma\sigma}\\
&\quad -\frac{2}{3}\kappa_{\sigma\sigma\sigma}\tau+\frac{352}{5}\kappa\kappa_{\sigma\sigma}-\frac{1}{6}\kappa_{\sigma\sigma}\tau_\sigma+\frac{32}{3}\kappa\kappa_\sigma\tau+59\kappa_\sigma^2+10\kappa_\sigma-\frac{576}{5}\kappa^3+\frac{4}{3}\kappa^2\tau_\sigma-\frac{292}{15}\kappa\tau-12+\frac{1}{2}\tau_{\sigma\sigma}\\
&\quad -\frac{5}{12}\tau\tau_\sigma-\left(\frac{3}{10}\kappa_{\sigma\sigma\sigma\sigma\sigma}-\frac{78}{5}\kappa\kappa_{\sigma\sigma\sigma}-\frac{177}{5}\kappa_\sigma\kappa_{\sigma\sigma}+\frac{864}{5}\kappa^2\kappa_\sigma+10\kappa_\sigma\tau+\frac{23}{5}\kappa\tau_\sigma-\frac{1}{10}\tau_{\sigma\sigma\sigma}\right)\partial_\sigma^{-1}\\
&\quad -\frac{1}{12}\tau_\sigma\partial_\sigma^{-1}((2\kappa_{\sigma\sigma}-16\kappa^2+\tau)\partial_\sigma+20\kappa),
\end{aligned}$$

where

$$\begin{aligned}
\Omega_1 &= -\frac{1}{24}\partial_\sigma^6+\kappa\partial_\sigma^4+3\kappa_\sigma\partial_\sigma^3+\frac{49}{12}\kappa_{\sigma\sigma}\partial_\sigma^2-6\kappa^2\partial_\sigma^2+\frac{35}{12}\kappa_{\sigma\sigma\sigma}\partial_\sigma-20\kappa\kappa_\sigma\partial_\sigma+\frac{13}{12}\kappa_{\sigma\sigma\sigma\sigma}-\frac{41}{3}\kappa\kappa_{\sigma\sigma}-\frac{23}{2}\kappa_\sigma^2\\
&\quad +\frac{32}{3}\kappa^3-\frac{6}{5}+\left(\frac{1}{6}\kappa_{\sigma\sigma\sigma\sigma\sigma}-\frac{10}{3}\kappa\kappa_{\sigma\sigma\sigma}-\frac{25}{3}\kappa_\sigma\kappa_{\sigma\sigma}+\frac{40}{3}\kappa^2\kappa_\sigma\right)\partial_\sigma^{-1}-\frac{1}{3}\kappa_\sigma\partial_\sigma^{-1}(\kappa_{\sigma\sigma}-8\kappa^2)
\end{aligned}$$

is the recursion operator of the Kaup–Kupershmidt equation, and

$$\Omega_2=\partial_\sigma^2-4\kappa-2\kappa_\sigma\partial_\sigma^{-1}$$

is the recursion operator of the KdV equation.

Letting $G=6$, $H=0$, one gets the following system

$$\begin{aligned}
\kappa_t &= \kappa_{\sigma\sigma\sigma\sigma\sigma}-20\kappa\kappa_{\sigma\sigma\sigma}-50\kappa_\sigma\kappa_{\sigma\sigma}+80\kappa^2\kappa_\sigma+\frac{1}{2}\tau_{\sigma\sigma\sigma}-2\kappa\tau_\sigma-\kappa_\sigma\tau,\\
\tau_t &= -8\kappa_{\sigma\sigma\sigma}\tau-2\kappa_{\sigma\sigma}\tau_\sigma+120\kappa_\sigma+128\kappa\kappa_\sigma\tau-5\tau\tau_\sigma+16\kappa^2\tau_\sigma,
\end{aligned}\tag{4.5}$$

which can be written as

$$\begin{aligned}
\kappa_t &= \kappa_{\sigma\sigma\sigma\sigma\sigma}-20\kappa\kappa_{\sigma\sigma\sigma}-50\kappa_\sigma\kappa_{\sigma\sigma}+80\kappa^2\kappa_\sigma+\frac{1}{2}(\partial_\sigma^2-4\kappa-2\kappa_\sigma\partial_\sigma^{-1})\tau_\sigma,\\
\tau_t &= -(2\kappa_{\sigma\sigma}+5\tau-16\kappa^2)\tau_\sigma-8\tau(\partial_\sigma^2-4\kappa-2\kappa_\sigma\partial_\sigma^{-1})\kappa_\sigma+(80\kappa\tau+120)\kappa_\sigma.
\end{aligned}$$

Note that

$$\kappa_t=\kappa_{\sigma\sigma\sigma\sigma\sigma}-20\kappa\kappa_{\sigma\sigma\sigma}-50\kappa_\sigma\kappa_{\sigma\sigma}+80\kappa^2\kappa_\sigma$$

is the fifth-order Kaup–Kupershmidt equation and $\partial_\sigma^2-4\kappa-2\kappa_\sigma\partial_\sigma^{-1}$ is the recursion operator of the KdV equation

$$\kappa_t=\kappa_{\sigma\sigma\sigma}-6\kappa\kappa_\sigma.$$

So the system (4.5) can be regarded as a coupled Kaup–Kupershmidt–KdV system.

5. Motion of curves in four-dimensional projective geometry $\mathbb{P}^4$

In four-dimensional projective geometry, the Serret–Frenet formulas for curves read as [42]

$$\begin{pmatrix} \gamma \\ \gamma^{(1)} \\ \gamma^{(2)} \\ \gamma^{(3)} \\ \gamma^{(4)} \end{pmatrix}_\sigma = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 4\kappa_1 & 0 & 1 & 0 & 0 \\ 264 & 6\kappa_1 & 0 & 1 & 0 \\ 104\kappa_2 & 108 & 6\kappa_1 & 0 & 1 \\ 24\kappa_3 & 8\kappa_2 & 48 & 4\kappa_1 & 0 \end{pmatrix} \begin{pmatrix} \gamma \\ \gamma^{(1)} \\ \gamma^{(2)} \\ \gamma^{(3)} \\ \gamma^{(4)} \end{pmatrix}. \tag{5.1}$$

The four-dimensional projective curve flow is governed by

$$\gamma_t = E\gamma + F\gamma^{(1)} + G\gamma^{(2)} + H\gamma^{(3)} + L\gamma^{(4)}, \tag{5.2}$$

where E, F,G, H and L are velocities, depending on the projective curvatures κ_i, $i = 1, 2, 3$.

In terms of (5.1) and (5.2), the time evolution in four-dimensional projective geometry is given by

$$\begin{pmatrix} \gamma \\ \gamma^{(1)} \\ \gamma^{(2)} \\ \gamma^{(3)} \\ \gamma^{(4)} \end{pmatrix}_t = \begin{pmatrix} E & F & G & H & L \\ E_1 & F_1 & G_1 & H_1 & L_1 \\ E_2 & F_2 & G_2 & H_2 & L_2 \\ E_3 & F_3 & G_3 & H_3 & L_3 \\ E_4 & F_4 & G_4 & H_4 & L_4 \end{pmatrix} \begin{pmatrix} \gamma \\ \gamma^{(1)} \\ \gamma^{(2)} \\ \gamma^{(3)} \\ \gamma^{(4)} \end{pmatrix} \tag{5.3}$$

where $E, F, G, H, L, E_i, F_i, G_i, H_i$ and L_i, $i = 1, 2, 3, 4$ are functions of κ_i, $i = 1, 2, 3$, to be determined.

The commutativity condition $\partial_\sigma \partial_t = \partial_t \partial_\sigma$ leads to the following equations

$$\begin{aligned}
&E_1 = E_\sigma + 4\kappa_1 F + 264G + 104\kappa_2 H + 24\kappa_3 L, \\
&F_1 = E + F_\sigma + 6\kappa_1 G + 108H + 8\kappa_2 L, \\
&G_1 = F + G_\sigma + 6\kappa_1 H + 48L, \\
&H_1 = G + H_\sigma + 4\kappa_1 L, \\
&L_1 = H + L_\sigma, \\
&E_2 = -4\kappa_{1t} - 4\kappa_1 E + E_{1\sigma} + 4\kappa_1 F_1 + 264G_1 + 104\kappa_2 H_1 + 24\kappa_3 L_1, \\
&F_2 = E_1 - 4\kappa_1 F + F_{1\sigma} + 6\kappa_1 G_1 + 108H_1 + 8\kappa_2 L_1, \\
&G_2 = F_1 - 4\kappa_1 G + G_{1\sigma} + 6\kappa_1 H_1 + 48L_1, \\
&H_2 = G_1 - 4\kappa_1 H + H_{1\sigma} + 4\kappa_1 L_1, \\
&L_2 = H_1 - 4\kappa_1 L + L_{1\sigma}, \\
&E_3 = -264E - 6\kappa_1 E_1 + E_{2\sigma} + 4\kappa_1 F_2 + 264G_2 + 104\kappa_2 H_2 + 24\kappa_3 L_2, \\
&F_3 = -6\kappa_{1t} + E_2 - 264F - 6\kappa_1 F_1 + F_{2\sigma} + 6\kappa_1 G_2 + 108H_2 + 8\kappa_2 L_2, \\
&G_3 = F_2 - 264G - 6\kappa_1 G_1 + G_{2\sigma} + 6\kappa_1 H_2 + 48L_2, \\
&H_3 = G_2 - 264H - 6\kappa_1 H_1 + H_{2\sigma} + 4\kappa_1 L_2, \\
&L_3 = H_2 - 264L - 6\kappa_1 L_1 + L_{2\sigma}, \\
&E_4 = -104\kappa_{2t} - 104\kappa_2 E - 108E_1 - 6\kappa_1 E_2 + E_{3\sigma} + 4\kappa_1 F_3 + 264G_3 + 104\kappa_2 H_3 + 24\kappa_3 L_3, \\
&F_4 = E_3 - 104\kappa_2 F - 108F_1 - 6\kappa_1 F_2 + F_{3\sigma} + 6\kappa_1 G_3 + 108H_3 + 8\kappa_2 L_3, \\
&G_4 = -6\kappa_{1t} + F_3 - 104\kappa_2 G - 108G_1 - 6\kappa_1 G_2 + G_{3\sigma} + 6\kappa_1 H_3 + 48L_3, \\
&H_4 = G_3 - 104\kappa_2 H - 108H_1 - 6\kappa_1 H_2 + H_{3\sigma} + 4\kappa_1 L_3, \\
&L_4 = H_3 - 104\kappa_2 L - 108L_1 - 6\kappa_1 L_2 + L_{3\sigma},
\end{aligned}$$

and

$$
\begin{aligned}
&F_4 - 24\kappa_3 G - 8\kappa_2 G_1 - 48G_2 - 4\kappa_1 G_3 + G_{4\sigma} + 6\kappa_1 H_4 + 48L_4 = 0,\\
&H_4 - 24\kappa_3 L - 8\kappa_2 L_1 - 48L_2 - 4\kappa_1 L_3 + L_{4\sigma} = 0,\\
&\kappa_{1t} = -\frac{1}{5}\kappa_1 E + \frac{1}{20}E_{1\sigma} - \frac{66}{5}F - \frac{1}{10}\kappa_1 F_1 + \frac{1}{20}F_{2\sigma} - \frac{26}{5}\kappa_2 G + \frac{39}{5}G_1 + \frac{1}{20}G_{3\sigma}\\
&\qquad - \frac{6}{5}\kappa_3 H + \frac{24}{5}\kappa_2 H_1 + 3H_2 + \frac{1}{10}\kappa_1 H_3 + \frac{1}{20}H_{4\sigma} + \frac{6}{5}\kappa_3 L_1 + \frac{2}{5}\kappa_2 L_2 + \frac{12}{5}L_3 + \frac{1}{5}\kappa_1 L_4,\\
&\kappa_{2t} = -\frac{13}{14}\kappa_2 E - \frac{27}{28}E_1 - \frac{3}{56}\kappa_1 E_2 + \frac{1}{112}E_{3\sigma} - \frac{3}{14}\kappa_3 F - \frac{1}{14}\kappa_2 F_1 - \frac{3}{7}F_2\\
&\qquad + \frac{1}{112}F_{4\sigma} + \frac{33}{14}G_3 + \frac{3}{56}\kappa_1 G_4 + \frac{13}{14}\kappa_2 H_3 + \frac{27}{28}H_4 + \frac{3}{14}\kappa_3 L_3 + \frac{1}{14}\kappa_2 L_4,\\
&\kappa_{3t} = -\kappa_3 E - \frac{1}{3}\kappa_2 E_1 - 2E_2 - \frac{1}{6}\kappa_1 E_3 + \frac{1}{24}E_{4\sigma} + \frac{1}{6}\kappa_1 F_4 + 11G_4 + \frac{13}{3}\kappa_2 H_4 + \kappa_3 L_4.
\end{aligned}
\tag{5.4}
$$

Simplifying the above equations, one has the following equations

$$
\begin{aligned}
&\frac{16}{5}F_{\sigma\sigma\sigma\sigma} - \frac{32}{5}\kappa_1 F_{\sigma\sigma} - \left(\frac{64}{5}\kappa_{1\sigma} - 1260\right)F_\sigma - \frac{32}{5}\kappa_{1\sigma\sigma}F + \frac{67}{10}G_{\sigma\sigma\sigma\sigma\sigma} - \frac{318}{5}\kappa_1 G_{\sigma\sigma\sigma} - \left(\frac{429}{5}\kappa_{1\sigma} - \frac{7338}{5}\right)G_{\sigma\sigma}\\
&- \left(\frac{219}{5}\kappa_{1\sigma\sigma} - 224\kappa_1^2 - 448\kappa_2\right)G_\sigma - \left(\frac{38}{5}\kappa_{1\sigma\sigma\sigma} - 224\kappa_1\kappa_{1\sigma} - 224\kappa_{2\sigma}\right)G + \frac{51}{10}H_{\sigma\sigma\sigma\sigma\sigma\sigma} - \frac{286}{5}\kappa_1 H_{\sigma\sigma\sigma\sigma}\\
&- \left(57\kappa_{1\sigma} + \frac{1146}{5}\right)H_{\sigma\sigma\sigma} \quad + \left(\frac{21}{5}\kappa_{1\sigma\sigma} + 224\kappa_1^2 + \frac{1408}{5}\kappa_2\right)H_{\sigma\sigma}\\
&+ \left(\frac{138}{5}\kappa_{1\sigma\sigma\sigma} + 224\kappa_1\kappa_{1\sigma} + 13080\kappa_1 + \frac{656}{5}\kappa_{2\sigma} + 120\kappa_3\right)H_\sigma + \left(\frac{48}{5}\kappa_{1\sigma\sigma\sigma\sigma} + 8856\kappa_{1\sigma}\right.\\
&\left.- \frac{112}{5}\kappa_{2\sigma\sigma} + 72\kappa_{3\sigma}\right)H + \frac{33}{25}L_{\sigma\sigma\sigma\sigma\sigma\sigma\sigma} - \frac{38}{5}\kappa_1 L_{\sigma\sigma\sigma\sigma\sigma} + \left(\frac{144}{5}\kappa_{1\sigma} - \frac{2004}{5}\right)L_{\sigma\sigma\sigma\sigma}\\
&+ \left(\frac{2346}{25}\kappa_{1\sigma\sigma} - \frac{1208}{25}\kappa_1^2 - \frac{3376}{25}\kappa_2\right)L_{\sigma\sigma\sigma} + \left(\frac{2568}{25}\kappa_{1\sigma\sigma\sigma} - \frac{10748}{25}\kappa_1\kappa_{1\sigma} - \frac{10628}{25}\kappa_{2\sigma} + 7800\kappa_1\right.\\
&\left.+ \frac{108}{5}\kappa_3\right)L_{\sigma\sigma} + \left(\frac{1278}{25}\kappa_{1\sigma\sigma\sigma\sigma} - \frac{7848}{25}\kappa_{1\sigma}^2 - \frac{204}{25}\kappa_{3\sigma} - \frac{10548}{25}\kappa_1\kappa_{1\sigma\sigma} + 8208\kappa_{1\sigma} + 60480 + 2688\kappa_1\kappa_2\right.\\
&\left.- \frac{10428}{25}\kappa_{2\sigma\sigma} + 384\kappa_1^3\right)L_\sigma + \left(\frac{246}{25}\kappa_{1\sigma\sigma\sigma\sigma\sigma} - \frac{3216}{25}\kappa_1\kappa_{1\sigma\sigma\sigma} - \frac{6048}{25}\kappa_{1\sigma}\kappa_{1\sigma\sigma} + 2520\kappa_{1\sigma\sigma} + 768\kappa_1^2\kappa_{1\sigma}\right.\\
&\left.- \frac{3376}{25}\kappa_{2\sigma\sigma\sigma} + 1792\kappa_{1\sigma}\kappa_2 + 1792\kappa_1\kappa_{2\sigma} - \frac{192}{5}\kappa_{3\sigma\sigma}\right)L = 0,
\end{aligned}
\tag{5.5}
$$

and

$$
\begin{pmatrix}\kappa_1\\ \kappa_2\\ \kappa_3\end{pmatrix}_t = \begin{pmatrix}A_1 & B_1 & C_1 & D_1\\ A_2 & B_2 & C_2 & D_2\\ A_3 & B_3 & C_3 & D_3\end{pmatrix}\begin{pmatrix}F_\sigma\\ G_\sigma\\ H_\sigma\\ L_\sigma\end{pmatrix},
\tag{5.6}
$$

where

$$
\begin{aligned}
A_1 &= -\frac{1}{2}\Omega,\\
B_1 &= -\frac{3}{4}\Omega\partial_\sigma + 63,\\
C_1 &= -\frac{9}{20}\partial_\sigma^4 - \left(\frac{9}{2}\kappa_{1\sigma} - 63\right)\partial_\sigma - \frac{1}{5}(27\kappa_{1\sigma\sigma} - 36\kappa_1^2 - 112\kappa_2) - \frac{1}{5}(9\kappa_{1\sigma\sigma\sigma} - 54\kappa_1\kappa_{1\sigma} - 8\kappa_{2\sigma})\partial_\sigma^{-1},\\
D_1 &= -\frac{1}{10}\partial_\sigma^5 - \kappa_1\partial_\sigma^3 - (5\kappa_{1\sigma} - 18)\partial_\sigma^2 - \frac{1}{5}(36\kappa_{1\sigma\sigma} - 28\kappa_1^2 - 56\kappa_2)\partial_\sigma - \left(\frac{22}{5}\kappa_{1\sigma\sigma\sigma} - \frac{62}{5}\kappa_1\kappa_{1\sigma}\right.\\
&\quad\left. - 36\kappa_1 - \frac{102}{5}\kappa_{2\sigma} - 6\kappa_3\right) - \left(\kappa_{1\sigma\sigma\sigma\sigma} - 4\kappa_1\kappa_{1\sigma\sigma} - 2\kappa_{1\sigma}^2 - \frac{252}{5}\kappa_{1\sigma} - \frac{48}{5}\kappa_{2\sigma\sigma} - \frac{24}{5}\kappa_{3\sigma}\right)\partial_\sigma^{-1},\\
A_2 &= \frac{2}{35}\partial_\sigma^4 - \frac{2}{7}\kappa_1\partial_\sigma^2 - \left(\frac{12}{35}\kappa_{1\sigma} - \frac{45}{4}\right)\partial_\sigma - \left(\frac{12}{35}\kappa_{1\sigma\sigma} - \frac{12}{35}\kappa_1^2 - 4\kappa_2\right) - \left(\frac{4}{35}\kappa_{1\sigma\sigma\sigma} - \frac{12}{35}\kappa_1\kappa_{1\sigma} - \kappa_{2\sigma}\right)\partial_\sigma^{-1},
\end{aligned}
$$

$$\begin{aligned}
B_2 = {} & \frac{99}{1120}\partial_\sigma^5 - \frac{191}{280}\kappa_1\partial_\sigma^3 - \left(\frac{619}{560}\kappa_{1\sigma} + \frac{747}{280}\right)\partial_\sigma^2 - \left(\frac{57}{70}\kappa_{1\sigma\sigma} - \frac{58}{35}\kappa_1^2 - 10\kappa_2\right)\partial_\sigma \\
& - \left(\frac{129}{560}\kappa_{1\sigma\sigma\sigma} - \frac{15}{14}\kappa_3 - \frac{135}{14}\kappa_{2\sigma} - \frac{186}{35}\kappa_1\kappa_{1\sigma} - \frac{3462}{35}\kappa_1\right) \\
& - \left(\frac{3}{280}\kappa_{1\sigma\sigma\sigma\sigma} - \frac{58}{35}\kappa_1\kappa_{1\sigma\sigma} - 2\kappa_{1\sigma}^2 - \frac{264}{7}\kappa_{1\sigma} - \frac{20}{7}\kappa_{2\sigma\sigma} - \frac{3}{7}\kappa_{3\sigma}\right)\partial_\sigma^{-1}, \\
C_2 = {} & \frac{9}{160}\partial_\sigma^6 - \frac{79}{280}\kappa_1\partial_\sigma^4 + \left(\frac{13}{560}\kappa_{1\sigma} - \frac{3981}{280}\right)\partial_\sigma^3 + \left(\frac{81}{70}\kappa_{1\sigma\sigma} - \frac{52}{35}\kappa_1^2 + \frac{24}{7}\kappa_2\right)\partial_\sigma^2 \\
& + \left(\frac{927}{560}\kappa_{1\sigma\sigma\sigma} - \frac{30}{7}\kappa_1\kappa_{1\sigma} + \frac{1941}{10}\kappa_1 + \frac{33}{70}\kappa_{2\sigma} + \frac{15}{7}\kappa_3\right)\partial_\sigma \\
& + \left(\frac{261}{280}\kappa_{1\sigma\sigma\sigma\sigma} - \frac{166}{35}\kappa_1\kappa_{1\sigma\sigma} - \frac{31}{14}\kappa_{1\sigma}^2 + \frac{1635}{7}\kappa_{1\sigma} + \frac{288}{35}\kappa_1^3 + \frac{1208}{35}\kappa_1\kappa_2 - \frac{127}{35}\kappa_{2\sigma\sigma} + \frac{15}{7}\kappa_{3\sigma} + 1890\right) \\
& + \left(\frac{27}{140}\kappa_{1\sigma\sigma\sigma\sigma\sigma} - \frac{57}{35}\kappa_1\kappa_{1\sigma\sigma\sigma} - \frac{177}{70}\kappa_{1\sigma}\kappa_{1\sigma\sigma} + \frac{1107}{14}\kappa_{1\sigma\sigma} + \frac{432}{35}\kappa_1^2\kappa_{1\sigma}\right. \\
& \left. + 20\kappa_{1\sigma}\kappa_2 + \frac{634}{35}\kappa_1\kappa_{2\sigma} - \frac{66}{35}\kappa_{2\sigma\sigma\sigma} + \frac{9}{14}\kappa_{3\sigma\sigma}\right)\partial_\sigma^{-1}, \\
D_2 = {} & \frac{9}{700}\partial_\sigma^7 + \frac{101}{1400}\kappa_1\partial_\sigma^5 + \left(\frac{211}{280}\kappa_{1\sigma} - \frac{687}{140}\right)\partial_\sigma^4 + \left(\frac{423}{200}\kappa_{1\sigma\sigma} - \frac{737}{350}\kappa_1^2 - \frac{212}{175}\kappa_2\right)\partial_\sigma^3 \\
& + \left(\frac{567}{200}\kappa_{1\sigma\sigma\sigma} - \frac{6687}{700}\kappa_1\kappa_{1\sigma} + \frac{363}{70}\kappa_1 - \frac{3667}{700}\kappa_{2\sigma} + \frac{69}{140}\kappa_3\right)\partial_\sigma^2 + \left(\frac{2871}{1400}\kappa_{1\sigma\sigma\sigma\sigma} - \frac{1903}{175}\kappa_1\kappa_{1\sigma\sigma}\right. \\
& \left. - \frac{1387}{175}\kappa_{1\sigma}^2 - \frac{627}{70}\kappa_{1\sigma} + \frac{1248}{175}\kappa_1^3 + \frac{7356}{175}\kappa_1\kappa_2 - \frac{2729}{350}\kappa_{2\sigma\sigma} - \frac{33}{70}\kappa_{3\sigma} + 1485\right)\partial_\sigma + \frac{1089}{1400}\kappa_{1\sigma\sigma\sigma\sigma\sigma} \\
& - \frac{117}{20}\kappa_1\kappa_{1\sigma\sigma\sigma} - \frac{3903}{350}\kappa_{1\sigma}\kappa_{1\sigma\sigma} + \frac{3792}{175}\kappa_1^2\kappa_{1\sigma} - \frac{972}{35}\kappa_{1\sigma\sigma} + \frac{12}{35}\kappa_1\kappa_3 + \frac{1116}{25}\kappa_1\kappa_{2\sigma} \\
& + \frac{3312}{7}\kappa_1^2 + 444\kappa_2 - \frac{27}{20}\kappa_{3\sigma\sigma} - \frac{3557}{700}\kappa_{2\sigma\sigma\sigma} + 62\kappa_{1\sigma}\kappa_2 + \left(\frac{171}{1400}\kappa_{1\sigma\sigma\sigma\sigma\sigma\sigma} - \frac{837}{70}\kappa_{1\sigma\sigma\sigma} - \frac{306}{175}\kappa_{1\sigma\sigma}^2\right. \\
& + 300\kappa_{2\sigma} - \frac{3}{5}\kappa_{3\sigma\sigma\sigma} - \frac{31}{25}\kappa_{2\sigma\sigma\sigma\sigma} - \frac{213}{175}\kappa_1\kappa_{1\sigma\sigma\sigma\sigma} + \frac{2256}{175}\kappa_1\kappa_{1\sigma}^2 + 24\kappa_{1\sigma\sigma}\kappa_2 + \frac{24}{35}\kappa_1\kappa_{3\sigma} \\
& \left. + 34\kappa_{1\sigma}\kappa_{2\sigma} + \frac{19296}{35}\kappa_1\kappa_{1\sigma} - \frac{69}{25}\kappa_{1\sigma}\kappa_{1\sigma\sigma\sigma} + \frac{2536}{175}\kappa_1\kappa_{2\sigma\sigma} + \frac{1056}{175}\kappa_1^2\kappa_{1\sigma\sigma}\right)\partial_\sigma^{-1}, \\
A_3 = {} & -\frac{4}{35}\partial_\sigma^5 + \frac{6}{35}\kappa_1\partial_\sigma^3 + \left(\frac{6}{7}\kappa_{1\sigma} - \frac{923}{20}\right)\partial_\sigma^2 + \left(\frac{48}{35}\kappa_{1\sigma\sigma} + \frac{4}{35}\kappa_1^2 - 8\kappa_2\right)\partial_\sigma \\
& + \frac{32}{35}\kappa_{1\sigma\sigma\sigma} + \frac{12}{35}\kappa_1\kappa_{1\sigma} + \frac{334}{5}\kappa_1 + 5\kappa_3 + \left(\frac{8}{35}\kappa_{1\sigma\sigma\sigma\sigma} + \frac{4}{35}\kappa_1\kappa_{1\sigma\sigma} + \frac{4}{35}\kappa_{1\sigma}^2 + \frac{64}{5}\kappa_{1\sigma} + \kappa_{3\sigma}\right)\partial_\sigma^{-1}, \\
B_3 = {} & -\frac{233}{1120}\partial_\sigma^6 + \frac{103}{56}\kappa_1\partial_\sigma^4 + \left(\frac{429}{80}\kappa_{1\sigma} - \frac{7543}{280}\right)\partial_\sigma^3 + \left(\frac{773}{112}\kappa_{1\sigma\sigma} - \frac{389}{35}\kappa_1^2 - 34\kappa_2\right)\partial_\sigma^2 \\
& + \left(\frac{479}{112}\kappa_{1\sigma\sigma\sigma} - \frac{2657}{70}\kappa_1\kappa_{1\sigma} - \frac{2689}{35}\kappa_1 - \frac{585}{14}\kappa_{2\sigma} + \frac{75}{14}\kappa_3\right)\partial_\sigma + \frac{145}{112}\kappa_{1\sigma\sigma\sigma\sigma} \\
& - \frac{2019}{70}\kappa_1\kappa_{1\sigma\sigma} - \frac{848}{35}\kappa_{1\sigma}^2 - \frac{8612}{35}\kappa_{1\sigma} + 16\kappa_1^3 + 112\kappa_1\kappa_2 - \frac{65}{2}\kappa_{2\sigma\sigma} - \frac{3}{2}\kappa_{3\sigma} + \frac{20604}{5} \\
& + \left(\frac{41}{280}\kappa_{1\sigma\sigma\sigma\sigma\sigma} - \frac{249}{35}\kappa_1\kappa_{1\sigma\sigma\sigma} - \frac{3088}{35}\kappa_{1\sigma\sigma} - \frac{634}{35}\kappa_{1\sigma}\kappa_{1\sigma\sigma} + 16\kappa_1^2\kappa_{1\sigma}\right. \\
& \left. + \frac{112}{3}\kappa_1\kappa_{2\sigma} + \frac{112}{3}\kappa_{1\sigma}\kappa_2 - \frac{169}{21}\kappa_{2\sigma\sigma\sigma} - \frac{6}{7}\kappa_{3\sigma\sigma}\right)\partial_\sigma^{-1}, \\
C_3 = {} & -\frac{23}{160}\partial_\sigma^7 + \frac{347}{280}\kappa_1\partial_\sigma^5 + \left(\frac{1403}{560}\kappa_{1\sigma} + \frac{8599}{280}\right)\partial_\sigma^4 + \left(\frac{393}{560}\kappa_{1\sigma\sigma} - \frac{197}{35}\kappa_1^2 - \frac{146}{7}\kappa_2\right)\partial_\sigma^3 \\
& - \left(\frac{55}{16}\kappa_{1\sigma\sigma\sigma} + \frac{741}{70}\kappa_1\kappa_{1\sigma} + \frac{8989}{10}\kappa_1 + \frac{153}{10}\kappa_{2\sigma} - \frac{5}{7}\kappa_3\right)\partial_\sigma^2 \\
& - \left(\frac{349}{80}\kappa_{1\sigma\sigma\sigma\sigma} + \frac{199}{70}\kappa_1\kappa_{1\sigma\sigma} - \frac{3}{5}\kappa_{1\sigma}^2 + \frac{121439}{70}\kappa_{1\sigma} + \frac{16}{35}\kappa_1^3 - \frac{2232}{35}\kappa_1\kappa_2 + \frac{293}{70}\kappa_{2\sigma\sigma} + \frac{60}{7}\kappa_{3\sigma} + \frac{2328}{5}\right)\partial_\sigma
\end{aligned}$$

$$
\begin{aligned}
&-\left(\frac{17}{8}\kappa_{1\sigma\sigma\sigma\sigma\sigma}-\frac{159}{35}\kappa_1\kappa_{1\sigma\sigma\sigma}-\frac{67}{5}\kappa_1\kappa_{1\sigma\sigma}+\frac{89641}{70}\kappa_{1\sigma\sigma}+\frac{2032}{35}\kappa_1^2\kappa_{1\sigma}+\frac{5344}{105}\kappa_{1\sigma}\kappa_2+\frac{664}{105}\kappa_1\kappa_{2\sigma}\right.\\
&\left.-10\kappa_1\kappa_3-\frac{12012}{5}\kappa_1^2-\frac{598}{105}\kappa_{2\sigma\sigma\sigma}-\frac{13368}{5}\kappa_2+\frac{99}{14}\kappa_{3\sigma\sigma}\right)-\left(\frac{27}{70}\kappa_{1\sigma\sigma\sigma\sigma\sigma\sigma}+\frac{248}{7}\kappa_{1\sigma}\kappa_{2\sigma}\right.\\
&-\frac{249}{35}\kappa_{1\sigma}\kappa_{1\sigma\sigma\sigma}+\frac{864}{35}\kappa_1^2\kappa_{1\sigma\sigma}+\frac{80}{7}\kappa_1\kappa_{2\sigma\sigma}-10\kappa_{1\sigma}\kappa_3+\frac{1728}{35}\kappa_1\kappa_{1\sigma}^2-\frac{10188}{5}\kappa_1\kappa_{1\sigma}+40\kappa_2\kappa_{1\sigma\sigma}-2\kappa_1\kappa_{3\sigma}\\
&\left.-\frac{72}{35}\kappa_1\kappa_{1\sigma\sigma\sigma\sigma}+\frac{23271}{70}\kappa_{1\sigma\sigma\sigma}-\frac{6032}{5}\kappa_{2\sigma}-\frac{177}{35}\kappa_{1\sigma\sigma}^2+\frac{25}{14}\kappa_{3\sigma\sigma\sigma}-\frac{97}{35}\kappa_{2\sigma\sigma\sigma\sigma}\right)\partial_\sigma^{-1},\\
D_3={}&-\frac{143}{4200}\partial_\sigma^8-\frac{83}{1400}\kappa_1\partial_\sigma^6-\left(\frac{296}{175}\kappa_{1\sigma}-\frac{11357}{700}\right)\partial_\sigma^5-\left(\frac{2253}{350}\kappa_{1\sigma\sigma}-\frac{1929}{350}\kappa_1^2+\frac{66}{175}\kappa_2\right)\partial_\sigma^4\\
&-\left(\frac{233}{20}\kappa_{1\sigma\sigma\sigma}-\frac{343}{20}\kappa_{2\sigma}+\frac{4201}{14}\kappa_1-\frac{5933}{140}\kappa_1\kappa_{1\sigma}-\frac{37}{140}\kappa_3\right)\partial_\sigma^3\\
&+\left(\frac{8107}{140}\kappa_{1\sigma}^2+\frac{2981}{84}\kappa_{2\sigma\sigma}-\frac{412}{35}\kappa_{1\sigma\sigma\sigma\sigma}-\frac{18608}{35}\kappa_{1\sigma}-\frac{37724}{525}\kappa_1\kappa_2+\frac{46637}{700}\kappa_1\kappa_{1\sigma\sigma}-\frac{26637}{5}-\frac{41}{140}\kappa_{3\sigma}\right.\\
&\left.-\frac{28292}{525}\kappa_1^3\right)\partial_\sigma^2+\left(\frac{14843}{420}\kappa_{2\sigma\sigma\sigma}+\frac{8212}{25}\kappa_2+\frac{51372}{175}\kappa_1^2-\frac{152713}{350}\kappa_{1\sigma\sigma}-\frac{102922}{525}\kappa_1\kappa_{2\sigma}+\frac{494}{35}\kappa_1\kappa_3\right.\\
&\left.+\frac{23113}{175}\kappa_{1\sigma}\kappa_{1\sigma\sigma}+\frac{39101}{700}\kappa_1\kappa_{1\sigma\sigma\sigma}-\frac{134674}{525}\kappa_{1\sigma}\kappa_2-\frac{51598}{175}\kappa_1^2\kappa_{1\sigma}+\frac{81}{28}\kappa_{3\sigma\sigma}-\frac{967}{140}\kappa_{1\sigma\sigma\sigma\sigma\sigma}\right)\partial_\sigma\\
&-\frac{35356}{105}\kappa_1\kappa_{1\sigma}^2+\frac{1792}{15}\kappa_1^2\kappa_2+\frac{243}{5}\kappa_{1\sigma\sigma}^2+\frac{487}{28}\kappa_{2\sigma\sigma\sigma\sigma}-\frac{53691}{350}\kappa_{1\sigma\sigma\sigma}+\frac{73}{20}\kappa_{3\sigma\sigma\sigma}+\frac{844}{35}\kappa_{1\sigma}\kappa_3\\
&-\frac{46896}{25}\kappa_1\kappa_{1\sigma}-\frac{7988}{25}\kappa_{1\sigma}\kappa_{2\sigma}+\frac{334}{35}\kappa_1\kappa_{3\sigma}-\frac{87302}{525}\kappa_1\kappa_{2\sigma\sigma}-\frac{1238}{5}\kappa_{1\sigma\sigma}\kappa_2+\frac{147587}{2100}\kappa_{1\sigma}\kappa_{1\sigma\sigma\sigma}\\
&-\frac{106382}{525}\kappa_1^2\kappa_{1\sigma\sigma}+\frac{50563}{2100}\kappa_1\kappa_{1\sigma\sigma\sigma\sigma}+43152\kappa_1+\frac{576}{5}\kappa_1^4+\frac{6272}{15}\kappa_2^2-\frac{1056}{5}\kappa_3-\frac{38376}{25}\kappa_{2\sigma}\\
&-\frac{133}{60}\kappa_{1\sigma\sigma\sigma\sigma\sigma\sigma}+\left(\frac{1152}{5}\kappa_1^3\kappa_{1\sigma}-\frac{26104}{525}\kappa_1^2\kappa_{1\sigma\sigma\sigma}+\frac{256}{75}\kappa_{2\sigma\sigma\sigma\sigma\sigma}-\frac{1271}{4200}\kappa_{1\sigma\sigma\sigma\sigma\sigma\sigma\sigma}-\frac{19968}{25}\kappa_{2\sigma\sigma}\right.\\
&-\frac{6393}{350}\kappa_{1\sigma\sigma\sigma\sigma}+\frac{38}{7}\kappa_{1\sigma}\kappa_{3\sigma}+\frac{896}{15}\kappa_1^2\kappa_{2\sigma}-\frac{66904}{525}\kappa_{1\sigma}\kappa_{2\sigma\sigma}-\frac{23504}{525}\kappa_1\kappa_{2\sigma\sigma\sigma}-\frac{704}{5}\kappa_{1\sigma\sigma}\kappa_{2\sigma}\\
&-\frac{1112}{15}\kappa_{1\sigma\sigma\sigma}\kappa_2+\frac{24}{7}\kappa_1\kappa_{3\sigma\sigma}-\frac{179436}{175}\kappa_1\kappa_{1\sigma\sigma}+\frac{2533}{175}\kappa_{1\sigma}\kappa_{1\sigma\sigma\sigma\sigma}+\frac{2216}{525}\kappa_1\kappa_{1\sigma\sigma\sigma\sigma\sigma}+\frac{2588}{105}\kappa_{1\sigma\sigma}\kappa_{1\sigma\sigma\sigma}\\
&+\frac{6272}{15}\kappa_2\kappa_{2\sigma}+\frac{1792}{15}\kappa_1\kappa_{1\sigma}\kappa_2-\frac{198336}{175}\kappa_{1\sigma}^2+\frac{109824}{5}\kappa_{1\sigma}+\frac{6}{5}\kappa_{3\sigma\sigma\sigma\sigma}\\
&\left.-\frac{34144}{525}\kappa_{1\sigma}^3-\frac{228}{5}\kappa_{3\sigma}-\frac{120496}{525}\kappa_1\kappa_{1\sigma}\kappa_{1\sigma\sigma}\right)\partial_\sigma^{-1},
\end{aligned}
$$

where $\Omega=\partial_\sigma^2-4\kappa_1-2\kappa_{1\sigma}\partial_\sigma^{-1}$ is the recursion operator of the KdV equation.

For a given (E, F, G, H, L), the motion of the curves in four-dimensional projective geometry is determined by Eqs. (5.5) and (5.6).

6. Concluding remarks

In this paper, we have carried out a study of inextensible motions of curves in two-, three- and four-dimensional projective geometries. In fact, the Kaup–Kupershmidt hierarchy, a coupled Kaup–Kupershmidt–KdV system and their extension can be obtained from such motions. So we can set up a correspondence between the Kaup–Kupershmidt hierarchy and the inextensible motions of curves in projective geometry. Similarly, we can discuss motions of inextensible space curves in $\mathbb{P}^n$ $(n \geq 5)$ and motions of surfaces in $\mathbb{P}^n$ $(n \geq 2)$; we believe that many $1+1$- and $2+1$-dimensional integrable systems are associated with such motions.

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