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Some results on L-dendriform algebras

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ABSTRACT

We introduce the notion of an L-dendriform algebra due to several different motivations. L-dendriform algebras are regarded as the underlying algebraic structures of pseudo-Hessian structures on Lie groups and the algebraic structures behind the $\mathcal{O}$-operators of pre-Lie algebras and the related S-equation. As a direct consequence, they provide some explicit solutions of S-equations in certain pre-Lie algebras constructed from L-dendriform algebras. They also fit into a bigger framework as Lie algebraic analogues of dendriform algebras. Moreover, we introduce the notion of an $\mathcal{O}$-operator of an L-dendriform algebra which gives an algebraic equation regarded as an analogue of the classical Yang–Baxter equation in a Lie algebra.

1. Introduction

1.1. Motivations

In this paper, we introduce a new class of algebras, namely, L-dendriform algebras, due to the following different motivations.

(1) *Pseudo-Hessian structures.* An L-dendriform algebra is regarded as the underlying algebraic structure of a pseudo-Hessian structure on a Lie group. In geometry, a Hessian manifold M is a flat affine manifold provided with a Hessian metric g, that is, g is a Riemannian metric such that for each point $p \in M$ there exists a C^∞-function φ defined on a neighborhood of p such that $g_{ij} = \frac{\partial^2\varphi}{\partial x^i \partial x^j}$. The algebraic structure corresponding to an affine Lie group G with a G-invariant Hessian metric is a real pre-Lie algebra (see a survey article [1] for the study of pre-Lie algebras) with a symmetric and positive definite 2-cocycle [2]. The Hessian structures could be extended to pseudo-cases by replacing "positive definite" by "nondegenerate" over the real number field, which could be extended to the other fields at the level of algebraic structures. We will show that there exists a natural L-dendriform algebra structure on a pre-Lie algebra with a nondegenerate symmetric 2-cocycle.

(2) *$\mathcal{O}$-operators and S-equations in pre-Lie algebras.* In fact, pre-Lie algebras can be regarded as the algebraic structures behind the classical Yang–Baxter equation (CYBE) which plays an important role in integrable systems, quantum groups and so on ([3] and the references therein). This can be seen more clearly in terms of $\mathcal{O}$-operators of a Lie algebra introduced by Kupershmidt in [4] as generalizations of (the operator form of) the CYBE in a Lie algebra [5]. Explicitly, the $\mathcal{O}$-operators of Lie algebras provide a direct relationship between Lie algebras and pre-Lie algebras and in the invertible cases, they provide a necessary and sufficient condition for the existence of a compatible pre-Lie algebra structure on a Lie algebra. Moreover, a

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skew-symmetric solution of the CYBE and the Lie algebraic analogue of the Rota–Baxter operator [6,7] which gives exactly the operator form of the CYBE in [5] are understood as the $\mathcal{O}$-operators associated with the co-adjoint representation and adjoint representation respectively. Furthermore, there are some solutions of the CYBE in certain Lie algebras obtained from pre-Lie algebras [8].

On the other hand, an analogue of the CYBE in a pre-Lie algebra, namely, the S-equation, was introduced in [9], which is closely related to a kind of bialgebra structure on pre-Lie algebras. In order to understand the S-equation well, we introduce the notion of an $\mathcal{O}$-operator of a pre-Lie algebra in this paper and we will show that it plays a similar role to the $\mathcal{O}$-operator of a Lie algebra. Then it is natural to ask what the algebraic structures behind the $\mathcal{O}$-operators of pre-Lie algebras and the related S-equation are. The answer is L-dendriform algebras!

(3) *Dendriform algebras and Loday algebras*. In fact, L-dendriform algebras fit into a bigger framework. Recall that a *dendriform algebra* $(A, \prec, \succ)$ is a vector space A with two binary operations denoted by $\prec$ and $\succ$ satisfying (for any $x, y, z \in A$)

$$(x \prec y) \prec z = x \prec (y * z), \qquad (x \succ y) \prec z = x \succ (y \prec z), \qquad x \succ (y \succ z) = (x * y) \succ z, \tag{1.1}$$

where $x * y = x \prec y + x \succ y$. Note that $(A, *)$ is an associative algebra as a direct consequence.

The notion of a dendriform algebra was introduced by Loday [10] in 1995 with motivation from algebraic K-theory and has been studied quite extensively with connections to several areas in mathematics and physics, including operads, homology, Hopf algebras, Lie and Leibniz algebras, combinatorics, arithmetic and quantum field theory and so on (see [11] and the references therein). Moreover, there is the following relationship among Lie algebras, associative algebras, pre-Lie algebras and dendriform algebras in the sense of a commutative diagram of categories [12–14]:

$$\begin{array}{ccc} \text{Lie algebra} & \leftarrow & \text{Pre-Lie algebra} \\ \uparrow & & \uparrow \\ \text{Associative algebra} & \leftarrow & \text{Dendriform algebra} \end{array} \tag{1.2}$$

Later quite a few more similar algebra structures were introduced, such as quadri-algebras of Aguiar and Loday [15]. All of them are called Loday algebras [16]. These algebras have a common property of "splitting associativity", that is, expressing the multiplication of an associative algebra as the sum of a string of binary operations [17].

In order to extend the commutative diagram (1.2) at the level of associative algebras (the bottom level of the commutative diagram (1.2)) to the more Loday algebras, it is natural to find the corresponding algebraic structures at the level of Lie algebras which extends the top level of commutative diagram (1.2). We will show that the L-dendriform algebras are chosen in a certain sense such that the following diagram including the diagram (1.2) as a sub-diagram is commutative:

$$\begin{array}{cccccccc} \text{Lie algebra} & \leftarrow & \text{Pre-Lie algebra} & \leftarrow & \text{L-dendriform algebra} & & \\ & \nwarrow & & \nwarrow & & \nwarrow & \\ & & \text{Associative algebra} & \leftarrow & \text{Dendriform algebra} & \leftarrow & \text{Quadri-algebra} \end{array} \tag{1.3}$$

In this sense, L-dendriform algebras are regarded as the Lie algebraic analogues of dendriform algebras, which explains why we give the notion of an "L-dendriform algebra". Furthermore, it is reasonable to consider interpreting L-dendriform algebras in terms of the Manin black product of symmetric operads [16].

1.2. Layout of the paper

In Section 2, we recall some basic facts regarding pre-Lie algebras and introduce the notion of an $\mathcal{O}$-operator of a pre-Lie algebra interpreting the S-equation. In Section 3, we introduce the notion of an L-dendriform algebra and then study some fundamental properties of L-dendriform algebras in terms of the $\mathcal{O}$-operators of pre-Lie algebras. In particular, we interpret their relationships with pre-Lie algebras, S-equations, pseudo-Hessian structures and some Loday algebras. In Section 4, we introduce the notion of an $\mathcal{O}$-operator of an L-dendriform algebra which gives an algebraic equation regarded as an analogue of the CYBE in a Lie algebra. In Section 5, we discuss certain generalizations of the study in the previous sections.

1.3. Notation

Throughout this paper, all algebras are finite dimensional and over a field of characteristic zero. We also give some notation as follows. Let A be an algebra with a binary operation $*$.

(1) Let $L_*(x)$ and $R_*(x)$ denote the left and right multiplication operators respectively, that is, $L_*(x)y = R_*(y)x = x * y$ for any $x, y \in A$. We also simply denote them by $L(x)$ and $R(x)$ respectively without confusion. Moreover let $L_*, R_* : A \to gl(A)$ be two linear maps with $x \to L_*(x)$ and $x \to R_*(x)$ respectively. In particular, when A is a Lie algebra, we let $\mathrm{ad}(x)$ denote the adjoint operator, that is, $\mathrm{ad}(x)y = [x, y]$ for any $x, y \in A$.

(2) Let $r = \sum_i a_i \otimes b_i \in A \otimes A$. Set

$$r_{12} = \sum_i a_i \otimes b_i \otimes 1, \qquad r_{13} = \sum_i a_i \otimes 1 \otimes b_i, \qquad r_{23} = \sum_i 1 \otimes a_i \otimes b_i, \tag{1.4}$$

where 1 is the unit if $(A, *)$ is unital or a symbol playing a similar role to the unit for the non-unital cases. The operation between two rs is given in an obvious way. For example,

$$r_{12} * r_{13} = \sum_{i,j} a_i * a_j \otimes b_i \otimes b_j, \qquad r_{13} * r_{23} = \sum_{i,j} a_i \otimes a_j \otimes b_i * b_j, \qquad r_{23} * r_{12} = \sum_{i,j} a_j \otimes a_i * b_j \otimes b_i, \tag{1.5}$$

and so on. Note that Eq. (1.5) is independent of the existence of the unit.
(3) Let V be a vector space. Let $\sigma : V \otimes V \to V \otimes V$ be the exchanging operator defined as

$$\sigma(x \otimes y) = y \otimes x, \quad \forall x, y \in V. \tag{1.6}$$

On the other hand, for any $r \in A \otimes A$, define a linear map $F_r : A^* \to A$ by

$$\langle F_r(u^*), v^* \rangle = \langle r, u^* \otimes v^* \rangle, \quad \forall u^*, v^* \in A^*, \tag{1.7}$$

where $\langle , \rangle$ is the ordinary pair between the vector space V and the dual space V^*. This defines an invertible linear map $F : A \otimes A \to \mathrm{Hom}(A^*, A)$ and thus allows us to identify r with F_r which we still denote by r for simplicity of notation. Moreover, any invertible linear map $T : V^* \to V$ can induce a nondegenerate bilinear form $\mathcal{B}(,)$ on V through

$$\mathcal{B}(u, v) = \langle T^{-1}u, v \rangle, \quad \forall u, v \in V. \tag{1.8}$$

(4) Let V be a vector space. For any linear map $\rho : A \to gl(V)$, define a linear map $\rho^* : A \to gl(V^*)$ by

$$\langle \rho^*(x)v^*, u \rangle = -\langle v^*, \rho(x)u \rangle, \quad \forall x \in A, u \in V, v^* \in V^*. \tag{1.9}$$

2. Pre-Lie algebras

2.1. Some fundamental properties of pre-Lie algebras

Definition 2.1. Let A be a vector space with a binary operation denoted by $\circ : A \otimes A \to A$. $(A, \circ)$ is called a *pre-Lie algebra* if for any $x, y, z \in A$, the associator

$$(x, y, z) = (x \circ y) \circ z - x \circ (y \circ z) \tag{2.1}$$

is symmetric in x, y, that is,

$$(x, y, z) = (y, x, z), \text{ or equivalently } (x \circ y) \circ z - x \circ (y \circ z) = (y \circ x) \circ z - y \circ (x \circ z), \quad \forall x, y, z \in A. \tag{2.2}$$

Proposition 2.2 (*Cf.* [18,19]). *Let* $(A, \circ)$ *be a pre-Lie algebra.*

(1) *The commutator*

$$[x, y] = x \circ y - y \circ x, \quad \forall x, y \in A \tag{2.3}$$

defines a Lie algebra $\mathfrak{g}(A)$, which is called the sub-adjacent Lie algebra of A and A is also called a compatible pre-Lie algebra structure on the Lie algebra $\mathfrak{g}(A)$.

(2) *$L_\circ$ gives a representation of the Lie algebra $\mathfrak{g}(A)$, that is,*

$$L_\circ([x, y]) = L_\circ(x)L_\circ(y) - L_\circ(y)L_\circ(x), \quad \forall x, y \in A. \tag{2.4}$$

Proposition 2.3. *Let $\mathfrak{g}$ be a vector space with a binary operation $\circ$. Then $(\mathfrak{g}, \circ)$ is a pre-Lie algebra if and only if $(\mathfrak{g}, [,])$ defined by Eq. (2.3) is a Lie algebra and $(L_\circ, \mathfrak{g})$ is a representation.*

Definition 2.4 ([9]). Let $(A, \circ)$ be a pre-Lie algebra and V be a vector space. Let $l, r : A \to gl(V)$ be two linear maps. (l, r, V) is called a *module of* $(A, \circ)$ if

$$l(x)l(y) - l(x \circ y) = l(y)l(x) - l(y \circ x), \tag{2.5}$$

$$l(x)r(y) - r(y)l(x) = r(x \circ y) - r(y)r(x), \quad \forall x, y \in A. \tag{2.6}$$

In fact, (l, r, V) is a module of a pre-Lie algebra $(A, \circ)$ if and only if the direct sum $A \oplus V$ of the underlying vector spaces of A and V is turned into a pre-Lie algebra (the *semidirect sum*) by defining multiplication in $A \oplus V$ by

$$(x + u) * (y + v) = x \circ y + (l(x)v + r(y)u), \quad \forall x, y \in A, u, v \in V. \tag{2.7}$$

We denote it by $A \ltimes_{l,r} V$.

Proposition 2.5 ([9]). *Let (l, r, V) be a module of a pre-Lie algebra $(A, \circ)$. Then $(l^* - r^*, -r^*, V^*)$ is a module of $(A, \circ)$.*

Definition 2.6. Let $(A, \circ)$ be a pre-Lie algebra and $\mathcal{B}$ be a bilinear form on A. $\mathcal{B}$ is called a 2-*cocycle of* A if $\mathcal{B}$ satisfies

$$\mathcal{B}(x \circ y, z) - \mathcal{B}(x, y \circ z) = \mathcal{B}(y \circ x, z) - \mathcal{B}(y, x \circ z), \quad \forall x, y, z \in A. \tag{2.8}$$

Remark 2.7. A real pre-Lie algebra A endowed with a symmetric and definite positive 2-cocycle corresponds to an affine Lie group G with a G-invariant Hessian metric [2].

Definition 2.8. Let $(A, \circ)$ be a pre-Lie algebra and $r \in A \otimes A$. The following equation is called an *S-equation in* $(A, \circ)$:

$$-r_{12} \circ r_{13} + r_{12} \circ r_{23} + [r_{13}, r_{23}] = 0. \tag{2.9}$$

The S-equation in a pre-Lie algebra is an analogue of the CYBE in a Lie algebra, which is related to the study of pre-Lie bialgebras [9].

2.2. $\mathcal{O}$-operators of pre-Lie algebras and the S-equation

Definition 2.9. Let $(A, \circ)$ be a pre-Lie algebra and (l, r, V) be a module. A linear map $T : V \to A$ is called an *$\mathcal{O}$-operator associated with* (l, r, V) if T satisfies

$$T(u) \circ T(v) = T(l(T(u))v + r(T(v))u), \quad \forall u, v \in V. \tag{2.10}$$

Example 2.10. Let $(A, \circ)$ be a pre-Lie algebra. A linear map $R : A \to A$ is called a *Rota–Baxter operator* (*of weight* 0) on A if R is an $\mathcal{O}$-operator associated with the module (L, R, A), that is, R satisfies [20]

$$R(x) \circ R(y) = R(R(x) \circ y + x \circ R(y)), \quad \forall x, y \in A. \tag{2.11}$$

Remark 2.11. In the case of associative algebras, the linear map T satisfying Eq. (2.10) was introduced independently in [21] through the notion of a *generalized Rota–Baxter operator*.

Theorem 2.12. *Let $(A, \circ)$ be a pre-Lie algebra and $r \in A \otimes A$ be symmetric. Then r is a solution of the S-equation in A if and only if r is an $\mathcal{O}$-operator of $(A, \circ)$ associated with $(L_\circ^* - R_\circ^*, -R_\circ^*, A^*)$.*

Proof. Let $\{e_1, \dots, e_n\}$ be a basis of A and $\{e_1^*, \dots, e_n^*\}$ be the dual basis. Suppose that $e_i \circ e_j = \sum_{k=1}^n c_{ij}^k e_k$ and $r = \sum_{i,j=1}^n a_{ij} e_i \otimes e_j$, $a_{ij} = a_{ji}$. Hence $r(e_i^*) = \sum_{k=1}^n a_{ik} e_k$. Then the coefficient of e_k in

$$r(e_i^*) \circ r(e_j^*) - r((L_\circ^* - R_\circ^*)(r(e_i^*))e_j^* - R_\circ^*(r(e_j^*))e_i^*) = 0,$$

is

$$\sum_{t,l=1}^n (a_{it} a_{jl} c_{tl}^k + a_{it} a_{lk} c_{tl}^j - a_{it} a_{lk} c_{lt}^j - a_{jt} a_{lk} c_{lt}^i) = 0,$$

which is precisely the coefficient of $e_i \otimes e_j \otimes e_k$ in the equation

$$r_{13} \circ r_{23} + [r_{12}, r_{23}] - r_{13} \circ r_{12} = 0,$$

which is a form equivalent to the S-equation [9]. Therefore the conclusion holds. □

Theorem 2.13. *Let $(A, \circ)$ be a pre-Lie algebra and (l, r, V) be a module. Let $T : V \to A$ be a linear map which is identified as an element in the vector space $(A \oplus V^*) \otimes (A \oplus V^*)$. Then $r = T + \sigma(T)$ is a symmetric solution of the S-equation in the pre-Lie algebra $A \ltimes_{l^* - r^*, -r^*} V^*$ if and only if T is an $\mathcal{O}$-operator of $(A, \circ)$ associated with (l, r, V).*

Proof. Let $\{v_1, \dots, v_m\}$ be a basis of V and $\{v_1^*, \dots, v_m^*\}$ be the dual basis. Then

$$T = \sum_{i=1}^m T(v_i) \otimes v_i^* \in T(V) \otimes V^* \subset (A \oplus V^*) \otimes (A \oplus V^*).$$

By Eq. (1.9), we show that

$$l^*(T(v_i))v_j = -\sum_{k=1}^m v_j(l(T(v_i))v_k)v_k^*, \qquad r^*(T(v_i))v_j = -\sum_{k=1}^m v_j(r(T(v_i))v_k)v_k^*.$$

Therefore we get

$$\begin{aligned} -r_{12} \circ r_{13} &= \sum_{i,j=1}^m [-(T(v_i) \circ T(v_j) \otimes v_i^* \otimes v_j^* + (l^* - r^*)(T(v_i))v_j^* \otimes v_i^* \otimes T(v_j) - r^*(T(v_j))v_i^* \otimes T(v_i) \otimes v_j^*)] \\ &= \sum_{i,j=1}^m [-T(v_i) \circ T(v_j) \otimes v_i^* \otimes v_j^* + v_j^* \otimes v_i^* \otimes T((l - r)(T(v_i))v_j) - v_i^* \otimes T(r(T(v_j))v_i) \otimes v_j^*]. \end{aligned}$$

Similarly, we have

$$r_{12}\circ r_{23}=\sum_{i,j=1}^{m}[T(r(T(v_j))v_i)\otimes v_i^*\otimes v_j^*-v_i^*\otimes v_j^*\otimes T((l-r)(T(v_i))v_j)+v_i^*\otimes T(v_i)\circ T(v_j)\otimes v_j^*],$$

$$[r_{13},r_{23}]=\sum_{i,j=1}^{m}(T(l(T(v_j))v_i)\otimes v_j^*\otimes v_i^*-v_i^*\otimes T(l(T(v_i))v_j)\otimes v_j^*+v_i^*\otimes v_j^*\otimes [T(v_i),T(v_j)]).$$

So r is a symmetric solution of the S-equation in the pre-Lie algebra $A\ltimes_{l^*-r^*,-r^*}V^*$ if and only if T is an $\mathcal{O}$-operator of $(A,\circ)$ associated with (l,r,V). $\square$

Combining Theorems 2.12 and 2.13, we have the following conclusion:

Corollary 2.14. *Let $(A,\circ)$ be a pre-Lie algebra and (l,r,V) be a module. Set $\hat{A}=A\ltimes_{l^*-r^*,-r^*}V^*$. Let $T:V\to A$ be a linear map. Then the following conditions are equivalent:*

(1) *T is an $\mathcal{O}$-operator of $(A,\circ)$ associated with (l,r,V).*
(2) *$T+\sigma(T)$ is a symmetric solution of the S-equation in the pre-Lie algebra $\hat{A}$.*
(3) *$T+\sigma(T)$ is an $\mathcal{O}$-operator of $\hat{A}$ associated with $(L_{\hat{A}}^*-R_{\hat{A}}^*,-R_{\hat{A}}^*,\hat{A}^*)$.*

3. L-dendriform algebras

3.1. The definition and some basic properties

Definition 3.1. Let A be a vector space with two binary operations denoted by $\rhd$ and $\lhd: A\otimes A\to A$. $(A,\rhd,\lhd)$ is called an *L-dendriform algebra* if for any $x,y,z\in A$,

$$x\rhd(y\rhd z)=(x\rhd y)\rhd z+(x\lhd y)\rhd z+y\rhd(x\rhd z)-(y\lhd x)\rhd z-(y\rhd x)\rhd z, \tag{3.1}$$

$$x\rhd(y\lhd z)=(x\rhd y)\lhd z+y\lhd(x\rhd z)+y\lhd(x\lhd z)-(y\lhd x)\lhd z. \tag{3.2}$$

Proposition 3.2. *Let $(A,\rhd,\lhd)$ be an L-dendriform algebra.*

(1) *The binary operation $\bullet: A\otimes A\to A$ given by*

$$x\bullet y=x\rhd y+x\lhd y,\quad \forall x,y\in A, \tag{3.3}$$

defines a pre-Lie algebra. $(A,\bullet)$ is called the associated horizontal pre-Lie algebra of $(A,\rhd,\lhd)$ and $(A,\rhd,\lhd)$ is called a compatible L-dendriform algebra structure on the pre-Lie algebra $(A,\bullet)$.

(2) *The binary operation $\circ: A\otimes A\to A$ given by*

$$x\circ y=x\rhd y-y\lhd x,\quad \forall x,y\in A, \tag{3.4}$$

defines a pre-Lie algebra. $(A,\circ)$ is called the associated vertical pre-Lie algebra of $(A,\rhd,\lhd)$ and $(A,\rhd,\lhd)$ is called a compatible L-dendriform algebra structure on the pre-Lie algebra $(A,\circ)$.

(3) *Both $(A,\bullet)$ and $(A,\circ)$ have the same sub-adjacent Lie algebra $\mathfrak{g}(A)$ defined by*

$$[x,y]=x\rhd y+x\lhd y-y\rhd x-y\lhd x,\quad \forall x,y\in A. \tag{3.5}$$

Proof. It is straightforward. $\square$

Remark 3.3. Let $(A,\rhd,\lhd)$ be an L-dendriform algebra. Then Eqs. (3.1) and (3.2) can be rewritten (for any $x,y,z\in A$) as

$$x\rhd(y\rhd z)-(x\bullet y)\rhd z=y\rhd(x\rhd z)-(y\bullet x)\rhd z, \tag{3.6}$$

$$x\rhd(y\lhd z)-(x\rhd y)\lhd z=y\lhd(x\bullet z)-(y\lhd x)\lhd z. \tag{3.7}$$

The both sides of the above two equations can be regarded as kinds of "generalized associators". In this sense, Eqs. (3.6) and (3.7) express a certain "generalized left-symmetry" of the "generalized associators".

Proposition 3.4. *Let A be a vector space with two binary operations denoted by $\rhd,\lhd: A\otimes A\to A$.*

(1) *$(A,\rhd,\lhd)$ is an L-dendriform algebra if and only if $(A,\bullet)$ defined by Eq. (3.3) is a pre-Lie algebra and $(L_\rhd,R_\lhd,A)$ is a module.*
(2) *$(A,\rhd,\lhd)$ is an L-dendriform algebra if and only if $(A,\circ)$ defined by Eq. (3.4) is a pre-Lie algebra and $(L_\rhd,-L_\lhd,A)$ is a module.*

Proof. The conclusions can be obtained by straightforward computation or a proof similar to that of Theorem 3.8. $\square$

Corollary 3.5. *Let $(A,\rhd,\lhd)$ be an L-dendriform algebra. Then $(L_\rhd^*-R_\lhd^*,-R_\lhd^*,A^*)$ is a module of the associated horizontal pre-Lie algebra $(A,\bullet)$ and $(L_\rhd^*+L_\lhd^*,\ L_\lhd^*,A^*)$ is a module of the associated vertical pre-Lie algebra $(A,\circ)$.*

Proof. It follows from Propositions 2.5 and 3.4. $\square$

Proposition 3.6. *Let $(A, \rhd, \lhd)$ be an L-dendriform algebra. Define two binary operations $\rhd^t, \lhd^t : A \otimes A \to A$ by*

$$x \rhd^t y = x \rhd y, \qquad x \lhd^t y = -y \lhd x, \quad \forall x, y \in A. \tag{3.8}$$

Then $(A, \rhd^t, \lhd^t)$ is an L-dendriform algebra. The associated horizontal pre-Lie algebra of $(A, \rhd^t, \lhd^t)$ is the associated vertical pre-Lie algebra $(A, \circ)$ of $(A, \rhd, \lhd)$ and the associated vertical pre-Lie algebra of $(A, \rhd^t, \lhd^t)$ is the associated horizontal pre-Lie algebra $(A, \bullet)$ of $(A, \rhd, \lhd)$, that is,

$$\bullet^t = \circ, \qquad \circ^t = \bullet. \tag{3.9}$$

Proof. It is straightforward. □

Definition 3.7. Let $(A, \rhd, \lhd)$ be an L-dendriform algebra. The L-dendriform algebra $(A, \rhd^t, \lhd^t)$ given by Eq. (3.8) is called the *transpose* of $(A, \rhd, \lhd)$.

For brevity, in the following (sub)sections, we only give the study involving the associated vertical pre-Lie algebras. The corresponding study on the associated horizontal pre-Lie algebras can be obtained using the transposes of the L-dendriform algebras through Proposition 3.6.

3.2. L-dendriform algebras and $\mathcal{O}$-operators of pre-Lie algebras

Theorem 3.8. *Let $(A, \circ)$ be a pre-Lie algebra and (l, r, V) be a module. If T is an $\mathcal{O}$-operator associated with (l, r, V), then there exists an L-dendriform algebra structure on V defined by*

$$u \rhd v = l(T(u))v, \qquad u \lhd v = -r(T(u))v, \quad \forall u, v \in V. \tag{3.10}$$

Therefore there is a pre-Lie algebra structure on V defined by Eq. (3.4) as the associated vertical pre-Lie algebra of $(V, \rhd, \lhd)$ and T is a homomorphism of pre-Lie algebras. Furthermore, $T(V) = \{T(v) \mid v \in V\} \subset A$ is a pre-Lie subalgebra of $(A, \circ)$ and there is an induced L-dendriform algebra structure on $T(V)$ given by

$$T(u) \rhd T(v) = T(u \rhd v), \qquad T(u) \lhd T(v) = T(u \lhd v), \quad \forall u, v \in V. \tag{3.11}$$

Moreover, the corresponding associated vertical pre-Lie algebra structure on $T(V)$ is a pre-Lie subalgebra of $(A, \circ)$ and T is a homomorphism of L-dendriform algebras.

Proof. For any $u, v, w \in V$, we have

$$\begin{aligned}
&u \rhd (v \rhd w) = l(T(u))l(T(v))w, \qquad (u \rhd v) \rhd w = l(T(l(T(u))v))w, \qquad (u \lhd v) \rhd w = -l(T(r(T(u))v))w,\\
&v \rhd (u \rhd w) = l(T(v))l(T(u))w, \qquad (v \lhd u) \rhd w = -l(T(r(T(v))u))w, \qquad (v \rhd u) \rhd w = l(T(l(T(v))u))w.\\
&(u \rhd v) \lhd w = -r(T(l(T(u))v))w, \qquad u \rhd (v \lhd w) = -l(T(u))r(T(v))w, \qquad v \lhd (u \rhd w) = -r(T(v))l(T(u))w,\\
&v \lhd (u \lhd w) = r(T(v))r(T(u))w, \qquad (v \lhd u) \lhd w = r(T(r(T(v))u))w.
\end{aligned}$$

Hence

$$\begin{aligned}
&(u \rhd v) \rhd w + (u \lhd v) \rhd w + v \rhd (u \rhd w) - (v \lhd u) \rhd w - (v \rhd u) \rhd w - u \rhd (v \rhd w)\\
&\quad = l(T(u))l(T(v))w - l(T(v))l(T(u))w - l(T(u) \circ T(v))T(v)w + l(T(v) \circ T(u))w = 0,\\
&(u \rhd v) \lhd w + v \lhd (u \rhd w) + v \lhd (u \lhd w) - (v \lhd u) \lhd w - u \rhd (v \lhd w)\\
&\quad = -r(T(l(T(u))v))w + r(T(u) \circ T(v))w - r(T(r(T(v))u))w = 0.
\end{aligned}$$

Therefore $(V, \rhd, \lhd)$ is an L-dendriform algebra. The other conclusions follow easily. □

Corollary 3.9. *Let $(A, \circ)$ be a pre-Lie algebra and R be a Rota–Baxter operator of weight zero. Then the binary operations given by*

$$x \rhd y = R(x) \circ y, \qquad x \lhd y = -y \circ R(x), \quad \forall x, y \in A, \tag{3.12}$$

define an L-dendriform algebra structure on A.

Proof. It follows immediately from Theorem 3.8 by taking $V = A$, $l = L$ and $r = R$. □

Recall that a linear map $T : V \to \mathfrak{g}$ is called an *$\mathcal{O}$-operator of a Lie algebra $\mathfrak{g}$ associated with a representation (ρ, V)* if T satisfies

$$[T(u), T(v)] = T(\rho(T(u))v - \rho(T(v))u), \quad \forall u, v \in V. \tag{3.13}$$

In particular, if R is an $\mathcal{O}$-operator of $\mathfrak{g}$ associated with the representation $(\mathrm{ad}, \mathfrak{g})$, it is known [22] that there exists a pre-Lie algebra structure on $\mathfrak{g}$ given by

$$x \circ y = [R(x), y], \quad \forall x, y \in \mathfrak{g}. \tag{3.14}$$

Corollary 3.10. *Let $\mathfrak{g}$ be a Lie algebra and $\{R_1, R_2\}$ be a pair of commuting $\mathcal{O}$-operators of $\mathfrak{g}$ associated with $(\mathrm{ad}, \mathfrak{g})$. Then there exists an L-dendriform algebra structure on $\mathfrak{g}$ defined by*

$$x \rhd y = [R_1(R_2(x)), y], \qquad x \lhd y = [R_2(x), R_1(y)], \quad \forall x, y \in \mathfrak{g}. \tag{3.15}$$

Proof. There exists a pre-Lie algebra structure on $\mathfrak{g}$ defined by Eq. (3.14) with the $\mathcal{O}$-operator R_1 of the Lie algebra $\mathfrak{g}$ associated with $(\mathrm{ad}, \mathfrak{g})$. It is straightforward to show that R_2 is a Rota–Baxter operator of weight zero on this pre-Lie algebra if R_2 as an $\mathcal{O}$-operator of the Lie algebra $\mathfrak{g}$ associated with $(\mathrm{ad}, \mathfrak{g})$ is commutative with R_1. Then the result follows from Corollary 3.9. □

Theorem 3.11. *Let $(A, \circ)$ be a pre-Lie algebra. Then there exists a compatible L-dendriform algebra structure on $(A, \circ)$ such that $(A, \circ)$ is the associated vertical pre-Lie algebra if and only if there exists an invertible $\mathcal{O}$-operator of $(A, \circ)$.*

Proof. Suppose that there exists an invertible $\mathcal{O}$-operator of $(A, \circ)$ associated with a module (l, r, V). By Theorem 3.8, there exists an L-dendriform algebra structure on V given by Eq. (3.10). Therefore we define an L-dendriform algebra structure on A by Eq. (3.11) such that T is an isomorphism of L-dendriform algebras, that is,

$$x \rhd y = T(l(x)T^{-1}(y)), \qquad x \lhd y = -T(r(x)T^{-1}(y)), \quad \forall x, y \in A.$$

Moreover it is a compatible L-dendriform algebra structure on $(A, \circ)$ since

$$x \rhd y - y \lhd x = T(l(x)T^{-1}(y) + r(y)T^{-1}(x)) = T(T^{-1}(x) \circ T^{-1}(y)) = x \circ y, \quad \forall x, y \in A.$$

Conversely, let $(A, \rhd, \lhd)$ be an L-dendriform algebra and $(A, \circ)$ be the associated vertical pre-Lie algebra. Then $(L_\rhd, -L_\lhd, A)$ is a module of $(A, \circ)$ and the identity map $id : A \to A$ is an $\mathcal{O}$-operator of $(A, \circ)$ associated with $(L_\rhd, -L_\lhd, A)$. □

The following conclusion reveals the relationship between L-dendriform algebras and pseudo-Hessian structures (that is, the pre-Lie algebras with a nondegenerate symmetric 2-cocycle):

Corollary 3.12. *Let $(A, \circ)$ be a pre-Lie algebra with a nondegenerate symmetric 2-cocycle $\mathcal{B}$. Then there exists a compatible L-dendriform algebra structure on $(A, \circ)$ given by*

$$\mathcal{B}(x \rhd y, z) = -\mathcal{B}(y, [x, z]), \qquad \mathcal{B}(x \lhd y, z) = -\mathcal{B}(y, z \circ x), \quad \forall x, y, z \in A. \tag{3.16}$$

such that $(A, \circ)$ is the associated vertical pre-Lie algebra.

Proof. It is straightforward to show that the invertible linear map $T : A^* \to A$ defined by Eq. (1.8) is an invertible $\mathcal{O}$-operator of $(A, \circ)$ associated with the module $(L_\circ^* - R_\circ^*, -R_\circ^*, A^*)$. By Theorem 3.11, there is a compatible L-dendriform algebra structure on A defined by (for any $x, y, z \in A$)

$$\begin{aligned} \mathcal{B}(x \rhd y, z) &= \mathcal{B}(T((L_\circ^* - R_\circ^*)(x)T^{-1}(y)), z) = \langle (L_\circ^* - R_\circ^*)(x)T^{-1}(y), z\rangle = -\langle T^{-1}(y), [x, z]\rangle \\ &= -\mathcal{B}(y, [x, z]); \end{aligned}$$

$$\mathcal{B}(x \lhd y, z) = \mathcal{B}(T(R_\circ^*(x)T^{-1}(y)), z) = \langle R_\circ^*(x)T^{-1}(y), z\rangle = -\langle T^{-1}(y), z \circ x\rangle = -\mathcal{B}(y, z \circ x).$$

such that $(A, \circ)$ is the associated vertical pre-Lie algebra. Hence the conclusion holds. □

The following conclusion provides a construction of solutions of the S-equation in certain pre-Lie algebras from L-dendriform algebras:

Corollary 3.13. *Let $(A, \rhd, \lhd)$ be an L-dendriform algebra and $(A, \circ)$, $(A, \bullet)$ be the associated vertical and horizontal pre-Lie algebras respectively. Let $\{e_1, \ldots, e_n\}$ be a basis of A and $\{e_1^*, \ldots, e_n^*\}$ be the dual basis. Then $r = \sum_{i=1}^n (e_i \otimes e_i^* + e_i^* \otimes e_i)$ is a symmetric solution of the S-equation in the pre-Lie algebras $A \ltimes_{L_\rhd^* + L_\lhd^*, L_\lhd^*} A^*$ and $A \ltimes_{L_\rhd^* - R_\lhd^*, -R_\lhd^*} A^*$.*

Proof. Since id is an $\mathcal{O}$-operator of both the pre-Lie algebra $(A, \circ)$ associated with the module $(L_\rhd, -L_\lhd, A)$ and the pre-Lie algebra $(A, \bullet)$ associated with the module $(L_\rhd, R_\lhd, A)$, the conclusion follows from Theorem 2.13. □

3.3. Relationships with dendriform algebras and quadri-algebras

Proposition 3.14. *Any dendriform algebra $(A, \succ, \prec)$ is an L-dendriform algebra on letting $x \rhd y = x \succ y$, $x \lhd y = x \prec y$.*

Proof. In fact, for a dendriform algebra, both sides of Eqs. (3.6) and (3.7), which are the equivalent identities of an L-dendriform algebra, are zero. □

Remark 3.15. In the above sense, associative algebras are the special pre-Lie algebras whose associators are zero, whereas dendriform algebras are the special L-dendriform algebras whose "generalized associators" (see Remark 3.3) are zero.

By Propositions 3.2 and 3.14, the following result is obvious:

Corollary 3.16. *Let $(A, \succ, \prec)$ be a dendriform algebra.*

(1) [10] *The binary operation given by Eq. (3.3) defines a pre-Lie algebra (in fact, it is an associative algebra).*
(2) [13,14] *The binary operation given by Eq. (3.4) defines a pre-Lie algebra.*

Definition 3.17 (*[15]*). Let A be a vector space with four binary operations denoted by $\searrow$, $\nearrow$, $\nwarrow$ and $\swarrow$: $A \otimes A \to A$. $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ is called a *quadri-algebra* if for any $x, y, z \in A$,

$$(x \nwarrow y) \nwarrow z = x \nwarrow (y * z), \qquad (x \nearrow y) \nwarrow z = x \nearrow (y \prec z), \qquad (x \wedge y) \nearrow z = x \nearrow (y \succ z), \tag{3.17}$$

$$(x \swarrow y) \nwarrow z = x \swarrow (y \wedge z), \qquad (x \searrow y) \nwarrow z = x \searrow (y \nwarrow z), \qquad (x \vee y) \nearrow z = x \searrow (y \nearrow z), \tag{3.18}$$

$$(x \prec y) \swarrow z = x \swarrow (y \vee z), \qquad (x \succ y) \swarrow z = x \searrow (y \swarrow z), \qquad (x * y) \searrow z = x \searrow (y \searrow z), \tag{3.19}$$

where

$$x \succ y = x \nearrow y + x \searrow y, \qquad x \prec y = x \nwarrow y + x \swarrow y, \tag{3.20}$$

$$x \vee y = x \searrow y + x \swarrow y, \qquad x \wedge y = x \nearrow y + x \nwarrow y, \tag{3.21}$$

$$x * y = x \searrow y + x \nearrow y + x \nwarrow y + x \swarrow y = x \succ y + x \prec y = x \vee y + x \wedge y. \tag{3.22}$$

Proposition 3.18 (*[15]*). *Let $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ be a quadri-algebra.*

(1) *The binary operations given by Eq. (3.20) define a dendriform algebra $(A, \succ, \prec)$.*
(2) *The binary operations given by Eq. (3.21) define a dendriform algebra $(A, \vee, \wedge)$.*
(3) *The binary operation given by Eq. (3.22) defines an associative algebra $(A, *)$.*

There is an additional relationship between quadri-algebras and L-dendriform algebras given as follows:

Proposition 3.19. *Let $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ be a quadri-algebra. The binary operations given by*

$$x \triangleright y = x \searrow y - y \nwarrow x, \qquad x \triangleleft y = x \nearrow y - y \swarrow x, \quad \forall x, y \in A, \tag{3.23}$$

define an L-dendriform algebra $(A, \triangleright, \triangleleft)$.

Proof. It is straightforward. □

Corollary 3.20. *Let $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ be a quadri-algebra.*

(1) *The binary operation given by*

$$x \circ y = x \searrow y + x \swarrow y - y \nwarrow x - y \nearrow x = x \triangleright y - y \triangleleft x = x \vee y - y \wedge x, \quad \forall x, y \in A, \tag{3.24}$$

defines a pre-Lie algebra $(A, \circ)$.
(2) *The binary operation given by Eq. (3.22) defines an associative algebra $(A, *)$.*
(3) *The binary operation given by*

$$x \bullet y = x \searrow y + x \nearrow y - y \nwarrow x - y \swarrow x = x \triangleright y + x \triangleleft y = x \succ y - y \prec x, \quad \forall x, y \in A, \tag{3.25}$$

defines a pre-Lie algebra $(A, \bullet)$.
(4) *The binary operation given by*

$$[x, y] = x \searrow y + x \swarrow y + x \nearrow y + x \nwarrow y - (y \searrow x + y \swarrow x + y \nearrow x + y \nwarrow x), \quad \forall x, y \in A, \tag{3.26}$$

defines a Lie algebra $(\mathfrak{g}(A), [,])$.

Proof. (1) and (3) follow from Propositions 3.18 and 3.19 and Corollary 3.16. (2) is exactly the conclusion (3) in Proposition 3.18 which follows from the conclusions (1) and (2) in Proposition 3.18 and Corollary 3.16. (4) follows from (1), (2), (3) and Proposition 2.2. □

Summarizing the above study in this subsection, we have the following commutative diagram:

$$\begin{tikzcd}
\text{Lie algebra} & \text{Pre-Lie algebra} \arrow[l, "-"'] & \text{L-dendriform algebra} \arrow[l, "{-,+}"'] & \\
& \text{Associative algebra} \arrow[ul, "-"'] \arrow[u, Rightarrow, "\in"'] & \text{Dendriform algebra} \arrow[l, "+"'] \arrow[ul, "-"'] \arrow[u, Rightarrow, "\in"'] & \text{Quadri-algebra} \arrow[l, "+"'] \arrow[ul, "-"']
\end{tikzcd} \tag{3.27}$$

where "$\Uparrow\in$" means inclusion. "+" means the binary operation $x \circ_1 y + x \circ_2 y$ and "−" means the binary operation $x \circ_1 y - y \circ_2 x$.

4. $\mathcal{O}$-operators of L-dendriform algebras and the *LD*-equation

Definition 4.1. Let $(A, \triangleright, \triangleleft)$ be an L-dendriform algebra and V be a vector space. Let $l_\triangleright, r_\triangleright, l_\triangleleft, r_\triangleleft : A \to gl(V)$ be four linear maps. $(l_\triangleright, r_\triangleright, l_\triangleleft, r_\triangleleft, V)$ is called a *module of* $(A, \triangleright, \triangleleft)$ if

$$[l_\triangleright(x), l_\triangleright(y)] = l_\triangleright[x, y]; \tag{4.1}$$
$$[l_\triangleright(x), l_\triangleleft(y)] = l_\triangleleft(x \circ y) + l_\triangleleft(y)l_\triangleleft(x); \tag{4.2}$$
$$r_\triangleright(x \triangleright y) = r_\triangleright(y)r_\triangleright(x) + r_\triangleright(y)r_\triangleleft(x) + [l_\triangleright(x), r_\triangleright(y)] - r_\triangleright(y)l_\triangleleft(x); \tag{4.3}$$
$$r_\triangleright(x \triangleleft y) = r_\triangleleft(y)r_\triangleright(x) + l_\triangleleft(x)r_\triangleright(y) + [l_\triangleleft(x), r_\triangleleft(y)]; \tag{4.4}$$
$$[l_\triangleright(x), r_\triangleleft(y)] = r_\triangleleft(x \bullet y) - r_\triangleleft(y)r_\triangleleft(x), \quad \forall x, y \in A, \tag{4.5}$$

where $x \circ y = x \triangleright y - y \triangleleft x$, and $x \bullet y = x \triangleright y + x \triangleleft y$.

In fact, $(l_\triangleright, r_\triangleright, l_\triangleleft, r_\triangleleft, V)$ is a module of an L-dendriform algebra $(A, \triangleright, \triangleleft)$ if and only if there exists an L-dendriform algebra structure on the direct sum $A \oplus V$ of the underlying vector spaces of A and V given by (for any $x, y \in A$, $u, v \in V$)

$$(x + u) \triangleright (y + v) = x \triangleright y + l_\triangleright(x)v + r_\triangleright(y)u, \quad (x + u) \triangleleft (y + v) = x \triangleleft y + l_\triangleleft(x)v + r_\triangleleft(y)u. \tag{4.6}$$

We denote it by $A \ltimes_{l_\triangleright, r_\triangleright, l_\triangleleft, r_\triangleleft} V$.

Proposition 4.2. *Let $(l_\triangleright, r_\triangleright, l_\triangleleft, r_\triangleleft, V)$ be a module of an L-dendriform algebra $(A, \triangleright, \triangleleft)$. Then $(l^*_\triangleright + l^*_\triangleleft - r^*_\triangleright - r^*_\triangleleft,\ r^*_\triangleright, r^*_\triangleright - l^*_\triangleleft,\ -(r^*_\triangleright + r^*_\triangleleft), V^*)$ is a module of $(A, \triangleright, \triangleleft)$.*

Proof. It is straightforward. □

Definition 4.3. Let $(A, \triangleright, \triangleleft)$ be an L-dendriform algebra and $(l_\triangleright, r_\triangleright, l_\triangleleft, r_\triangleleft, V)$ be a module. A linear map $T : V \to A$ is called an *$\mathcal{O}$-operator of $(A, \triangleright, \triangleleft)$ associated with $(l_\triangleright, r_\triangleright, l_\triangleleft, r_\triangleleft, V)$* if T satisfies (for any $u, v \in V$)

$$T(u) \triangleright T(v) = T[l_\triangleright(T(u))v + r_\triangleright(T(v))u], \qquad T(u) \triangleleft T(v) = T[l_\triangleleft(T(u))v + r_\triangleleft(T(v))u]. \tag{4.7}$$

By a proof similar to that of Theorem 2.12, we obtain the following two conclusions:

Proposition 4.4. *Let $(A, \triangleright, \triangleleft)$ be an L-dendriform algebra and $r \in A \otimes A$ be skew-symmetric. Let $(A, \circ)$ and $(A, \bullet)$ be the associated vertical and horizontal pre-Lie algebras respectively. Then the following conditions are equivalent:*

(1) *r is an $\mathcal{O}$-operator of $(A, \circ)$ associated with $(L^*_\triangleright + L^*_\triangleleft, L^*_\triangleleft, A^*)$.*
(2) *r is an $\mathcal{O}$-operator of $(A, \bullet)$ associated with $(L^*_\triangleright - R^*_\triangleleft, -R^*_\triangleleft, A^*)$.*
(3) *r satisfies*

$$r_{13} \circ r_{23} + r_{12} \bullet r_{23} - r_{12} \triangleleft r_{13} = 0. \tag{4.8}$$

Proposition 4.5. *Let $(A, \triangleright, \triangleleft)$ be an L-dendriform algebra and $r \in A \otimes A$ be skew-symmetric. Then r is an $\mathcal{O}$-operator of $(A, \triangleright, \triangleleft)$ associated with $(L^*_\triangleright + L^*_\triangleleft - R^*_\triangleright - R^*_\triangleleft, R^*_\triangleright, R^*_\triangleright - L^*_\triangleleft, -(R^*_\triangleright + R^*_\triangleleft), A^*)$ if and only if r satisfies the following equations:*

$$r_{13} \triangleright r_{23} = -[r_{12}, r_{23}] + r_{13} \triangleright r_{12}, \tag{4.9}$$
$$r_{23} \triangleleft r_{13} = r_{13} \circ r_{12} + r_{23} \bullet r_{12}. \tag{4.10}$$

Lemma 4.6. *Let $(A, \triangleright, \triangleleft)$ be an L-dendriform algebra and $r \in A \otimes A$ be skew-symmetric. Then the following conditions are equivalent:*

(1) *r satisfies Eq. (4.8);*
(2) *r satisfies Eq. (4.10);*
(3) *r satisfies one of the following equations:*

$$r_{23} \circ r_{13} - r_{12} \bullet r_{13} + r_{12} \triangleleft r_{23} = 0; \tag{4.11}$$
$$r_{23} \circ r_{12} + r_{13} \bullet r_{12} + r_{13} \triangleleft r_{23} = 0; \tag{4.12}$$
$$r_{12} \circ r_{23} + r_{13} \bullet r_{23} + r_{13} \triangleleft r_{12} = 0; \tag{4.13}$$
$$r_{12} \circ r_{13} - r_{23} \bullet r_{13} + r_{23} \triangleleft r_{12} = 0. \tag{4.14}$$

Proof. Let σ be any element in the permutation group Σ_3 acting on $\{1, 2, 3\}$. Then σ induces a linear map from $A \otimes A \otimes A$ to $A \otimes A \otimes A$ by (we still denote it by σ)

$$\sigma(x_1 \otimes x_2 \otimes x_3) = x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes x_{\sigma(3)}, \quad \forall x_1, x_2, x_3 \in A.$$

Hence it is straightforward to check that Eqs. (4.10)–(4.14) are Eq. (4.8) under the action of the non-unit elements $\sigma \in \Sigma_3$ respectively. Note that the skew-symmetry of r is used. □

Corollary 4.7. *Let $(A, \triangleright, \triangleleft)$ be an L-dendriform algebra and $r \in A \otimes A$ be skew-symmetric. Then Eq. (4.9) holds if Eq. (4.10) holds.*

Proof. Note that Eq. (4.9) is exactly the difference between Eqs. (4.12) and (4.13). Then the conclusion follows immediately. □

Definition 4.8. Let $(A, \triangleright, \triangleleft)$ be an L-dendriform algebra and $r \in A \otimes A$. Eq. (4.8) is called the *LD-equation* in $(A, \triangleright, \triangleleft)$.

Combining Propositions 4.4, 4.5, Lemma 4.6 and Corollary 4.7, we obtain the following conclusion:

Corollary 4.9. *Let $(A, \triangleright, \triangleleft)$ be an L-dendriform algebra and $r \in A\otimes A$ be skew-symmetric. Let $(A, \circ)$ and $(A, \bullet)$ be the associated vertical and horizontal pre-Lie algebras respectively. Then the following conditions are equivalent.*

(1) *r is a solution of the LD-equation in $(A, \triangleright, \triangleleft)$.*
(2) *r is an $\mathcal{O}$-operator of $(A, \triangleright, \triangleleft)$ associated with $(L_\triangleright^* + L_\triangleleft^* - R_\triangleright^* - R_\triangleleft^*, R_\triangleright^*, R_\triangleright^* - L_\triangleleft^*, -(R_\triangleright^* + R_\triangleleft^*), A^*)$.*
(3) *r is an $\mathcal{O}$-operator of $(A, \circ)$ associated with $(L_\triangleright^* + L_\triangleleft^*, L_\triangleleft^*, A^*)$.*
(4) *r is an $\mathcal{O}$-operator of $(A, \bullet)$ associated with $(L_\triangleright^* - R_\triangleleft^*, -R_\triangleleft^*, A^*)$.*

Remark 4.10. Due to the above result, it is reasonable to regard the *LD*-equation in an L-dendriform algebra as an analogue of the CYBE in a Lie algebra (also see [8,9] and Theorem 2.12).

Lemma 4.11. *Let $(A, \triangleright, \triangleleft)$ be an L-dendriform algebra and $T : A^* \to A$ be an invertible linear map. Then T is an $\mathcal{O}$-operator of $(A, \triangleright, \triangleleft)$ associated with $(L_\triangleright^* + L_\triangleleft^* - R_\triangleright^* - R_\triangleleft^*, R_\triangleright^*, R_\triangleright^* - L_\triangleleft^*, -(R_\triangleright^* + R_\triangleleft^*), A^*)$ if and only if the bilinear form $\mathcal{B}$ induced by T through Eq. (1.8) satisfies*

$$\mathcal{B}(x \triangleright y, z) = -\mathcal{B}(y, [x, z]) - \mathcal{B}(x, z \triangleright y), \tag{4.15}$$

$$\mathcal{B}(x \triangleleft y, z) = -\mathcal{B}(y, z \circ x) + \mathcal{B}(x, z \bullet y), \quad \forall x, y, z \in A \tag{4.16}$$

where $x \circ y = x \triangleright y - y \triangleleft x$, $x \bullet y = x \triangleright y + x \triangleleft y$, $[x, y] = x \circ y - y \circ x = x \bullet y - y \bullet x$. If, in addition, $\mathcal{B}$ is skew-symmetric, then $\mathcal{B}$ satisfies Eq. (4.15) if $\mathcal{B}$ satisfies Eq. (4.16).

Proof. It is straightforward. □

Corollary 4.12. *Let $(A, \triangleright, \triangleleft)$ be an L-dendriform algebra and $r \in A \otimes A$. Suppose that r is skew-symmetric and invertible. Then r is a solution of the LD-equation in $(A, \triangleright, \triangleleft)$ if and only if the nondegenerate bilinear form $\mathcal{B}$ induced by r through Eq. (1.8) satisfies Eq. (4.16).*

Proof. It follows from Lemma 4.11 and Corollary 4.9. □

Definition 4.13. Let $(A, \triangleright, \triangleleft)$ be an L-dendriform algebra. A skew-symmetric bilinear form $\mathcal{B}$ satisfying Eq. (4.16) is called a 2-*cocycle* of $(A, \triangleright, \triangleleft)$.

By a proof similar to that of Theorem 2.13. we have the following conclusion:

Theorem 4.14. *Let $(A, \triangleright, \triangleleft)$ be an L-dendriform algebra and $(l_\triangleright, r_\triangleright, l_\triangleleft, r_\triangleleft, V)$ be a module. Let $T : V \to A$ be a linear map which can be identified as an element in the vector space $(A \oplus V^*) \otimes (A \oplus V^*)$. Then $r = T - \sigma(T)$ is a skew-symmetric solution of the LD-equation in the L-dendriform algebra $A \ltimes_{l_\triangleright^* + l_\triangleleft^* - r_\triangleright^* - r_\triangleleft^*,\ r_\triangleright^*, r_\triangleright^* - l_\triangleleft^*,\ -(r_\triangleright^* + r_\triangleleft^*)} V^*$ if and only if T is an $\mathcal{O}$-operator of $(A, \triangleright, \triangleleft)$ associated with $(l_\triangleright, r_\triangleright, l_\triangleleft, r_\triangleleft, V)$.*

5. Generalization

The study in the previous sections motivates us to consider the following structures: let $(X, [,])$ be a Lie algebra and $(*_i)_{1\le i\le N} : X \otimes X \to X$ be a family of binary operations on X. Then the Lie bracket $[,]$ splits into the commutator of N binary operations $*_1, \ldots, *_N$ if

$$[x, y] = \sum_{i=1}^{N} (x *_i y - y *_i x), \quad \forall x, y \in X. \tag{5.1}$$

Note that when $N = 1$, the algebra with the binary operation $*_1 = *$ is a Lie-admissible algebra.

Like for the Loday algebras, only Eq. (5.1) is too general for getting more interesting structures. So some additional conditions for the binary operations $*_i$ are necessary. We pay our main attention to the case where $N = 2^n, n = 0, 1, 2, \ldots$. Like the study in the cases of associative algebras given in [23] and the study in the previous sections, a "rule" for constructing the binary operations $*_i$ is defined as follows: the 2^{n+1} binary operations give a natural module structure of an algebra with the 2^n binary operations on the underlying vector space of the algebra itself, which is the beauty of such algebra structures.

That is, by induction, for the algebra $(A, *_i)_{1\le i\le 2^n}$, besides the natural module of A on the underlying vector space of A itself given by the left and right multiplication operators, one can introduce the 2^{n+1} binary operations $\{*_{i_1}, *_{i_2}\}_{1\le i\le 2^n}$ such that

$$x *_i y = x *_{i_1} y - y *_{i_2} x, \quad \forall x, y \in A,\ \ 1 \le i \le 2^n, \tag{5.2}$$

and their left or right multiplication operators give a module of $(A, *_i)_{1\le i\le 2^n}$ by acting on the underlying vector space of A itself.

In particular, when $N = 1$ and $N = 2$, the corresponding algebra $(A, *_i)_{1\le i\le N}$, according to the above rule, is exactly a pre-Lie algebra and an L-dendriform algebra respectively. On the other hand, note that for $n \ge 1$ ($N \ge 2$), in order to make Eq. (5.1) satisfied, there is an alternative (sum) form of Eq. (5.2):

$$x *_i y = x *_{i_1} y + x *'_{i_2} y, \quad \forall x, y \in A,\ \ 1 \le i \le 2^n, \tag{5.3}$$

obtained by letting $x *'_{i_2} y = -y *_{i_2} x$ for any $x, y \in A$. In particular, in such a situation, it can be regarded as a binary operation $*$ of a pre-Lie algebra that splits into the $N = 2^n$ ($n = 1, 2, \ldots$) binary operations $*_1, \ldots, *_N$. In this sense, an L-dendriform algebra is also regarded as a "pre-Lie algebraic analogue" of a dendriform algebra.

We would like to point out that in the case of associative algebras, there is an outline of a study by induction on the algebra systems with more binary operations given in [23], which is still valid for the case of Lie algebras.

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