



# Integrable generalizations of Schrödinger maps and Heisenberg spin models from Hamiltonian flows of curves and surfaces

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## ARTICLE INFO

### Article history:

Received 20 March 2010

Accepted 21 May 2010

Available online 1 June 2010

### Keywords:

Integrable vector model

Curve flow

Schrödinger map

Heisenberg model

bi-Hamiltonian

## ABSTRACT

A moving frame formulation of non-stretching geometric curve flows in Euclidean space is used to derive a  $1 + 1$  dimensional hierarchy of integrable  $SO(3)$ -invariant vector models containing the Heisenberg ferromagnetic spin model as well as a model given by a spin vector version of the mKdV equation. These models describe a geometric realization of the NLS hierarchy of soliton equations whose bi-Hamiltonian structure is shown to be encoded in the Frenet equations of the moving frame. This derivation yields an explicit bi-Hamiltonian structure, recursion operator, and constants of motion for each model in the hierarchy. A generalization of these results to geometric surface flows is presented, where the surfaces are non-stretching in one direction while stretching in all transverse directions. Through the Frenet equations of a moving frame, such surface flows are shown to encode a hierarchy of  $2 + 1$  dimensional integrable  $SO(3)$ -invariant vector models, along with their bi-Hamiltonian structure, recursion operator, and constants of motion, describing a geometric realization of  $2 + 1$  dimensional bi-Hamiltonian NLS and mKdV soliton equations. Based on the well-known equivalence between the Heisenberg model and the Schrödinger map equation in  $1 + 1$  dimensions, a geometrical formulation of these hierarchies of  $1 + 1$  and  $2 + 1$  vector models is given in terms of dynamical maps into the 2-sphere. In particular, this formulation yields a new integrable generalization of the Schrödinger map equation in  $2 + 1$  dimensions as well as a mKdV analog of this map equation corresponding to the mKdV spin model in  $1 + 1$  and  $2 + 1$  dimensions.

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## 1. Introduction and summary

Spin systems are an important class of dynamical vector models from both physical and mathematical points of view. In physics such models describe the nonlinear dynamics of magnetic materials, while in mathematics they give rise to associated geometric flows of curves where the unit tangent vector along a curve is identified with a dynamical spin vector.

A main example [1] is the Heisenberg model for the dynamics of an isotropic ferromagnet spin system in  $1 + 1$  dimensions. The geometric curve flow described by this  $SO(3)$ -invariant model corresponds to the equations of motion of a non-stretching vortex filament in Euclidean space. Remarkably, the vortex filament equations are an integrable Hamiltonian system that is equivalent to the  $1 + 1$  dimensional focusing nonlinear Schrödinger equation (NLS) through a change of dynamical variables known as a Hasimoto transformation [2].

The vortex filament equations are one example in an infinite hierarchy of non-stretching geometric flows of space curves whose equations of motion have a well-understood integrability: e.g. a Lax pair and an associated isospectral linear

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eigenvalue problem; an infinite set of symmetries and constants of motion; and exact solutions with solitonic properties. This integrability structure turns out to have a simple geometric origin. In particular, all of these equations of motion are generated through a recursion operator that can be derived geometrically [3,4] from the Serret–Frenet structure equations given by a  $SO(3)$  moving frame formulation for arbitrary non-stretching curve flows in Euclidean space, with the components of the frame connection matrix providing the dynamical variables that appear in the equations of motion. More recently, these  $SO(3)$  frame structure equations have been found to geometrically encode a pair of compatible Hamiltonian operators that yield a concrete bi-Hamiltonian structure for the equations of motion of each integrable curve flow in the hierarchy [5].

The explicit bi-Hamiltonian form of the resulting equations of motion depends on a choice of the  $SO(3)$  moving frame for the underlying space curve, which determines the form of the frame connection matrix and hence yields the dynamical variables in terms of the curve. In the case of the vortex filament equations, the dynamical variables consist of the curvature invariant,  $\kappa$ , and the torsion invariant,  $\tau$ , of the space curve, corresponding to the choice of a classical Frenet frame [6] given by the unit tangent vector, unit normal and bi-normal vectors, along the curve. Other geometrical choices of a moving frame can be made [7], since there is a  $SO(3)$  gauge freedom relating any two orthonormal frames along an arbitrary curve in Euclidean space. In particular, the Hasimoto transformation arises geometrically as a gauge transformation from a Frenet frame to a parallel frame [8], where the frame vectors in the normal space of the curve are chosen such that their derivative with respect to the arclength  $s$  along the curve lies in the tangent space of the curve. This choice of frame is unique up to rigid  $SO(2)$  rotations acting on the normal vectors by the same angle at all points along the curve, while leaving invariant the tangent vector. The corresponding pair of dynamical variables (defined by the connection matrix of a parallel frame) are naturally equivalent to a single complex-valued variable  $u = \kappa \exp(i \int \tau ds)$  that is determined by the curve only up to constant phase rotations  $u \rightarrow e^{i\phi} u$  (where  $\phi$  is independent of arclength  $s$ ). This dynamical variable  $u$  thus has the geometrical meaning [9] of a  $U(1) \simeq SO(2)$  covariant of the space curve. Importantly, the resulting Hamiltonian structure for the equations of motion looks simplest in terms of the covariant  $u$ , which directly incorporates the Hasimoto transformation, rather than using the classical invariants  $\kappa$  and  $\tau$ .

The purpose of the present paper will be to give some new applications of these ideas to the study of integrable vector models in  $1 + 1$  and  $2 + 1$  dimensions.

Firstly, from the hierarchy of non-stretching geometric space curve flows that contains the vortex filament equations, we derive the complete hierarchy of corresponding integrable  $SO(3)$ -invariant vector models in  $1 + 1$  dimensions, along with their bi-Hamiltonian integrability structure in explicit form. In addition to the Heisenberg model, this hierarchy will be seen to contain a model that describes a spin vector version of the mKdV equation. Our results provide a new derivation of the Hamiltonian structure, recursion operator, and constants of motion for these models.

Secondly, we extend the derivation to a geometrically analogous class of surface flows where the surface is non-stretching in one coordinate direction while stretching in all transverse directions. Such surfaces arise in a natural fashion from a spatial Hamiltonian flow of non-stretching space curves. This generalization will be shown to give rise to a class of  $2 + 1$  dimensional NLS and mKdV soliton equations with an explicit bi-Hamiltonian structure, yielding a hierarchy of integrable  $SO(3)$ -invariant vector models in  $2 + 1$  dimensions. In particular, this hierarchy includes  $2 + 1$  generalizations of the Heisenberg spin model and the mKdV spin model, which were found in earlier work by one of us [10–14]. Our derivation here, in contrast, yields the explicit bi-Hamiltonian structure, recursion operator, and constants of motion, which are new results for these models. We also write out the corresponding surface flows explicitly in terms of geometric variables given by [6] the geodesic and normal curvatures and the relative torsion of the non-stretching coordinate lines on the surface. The surface flow arising from the  $2 + 1$  integrable Heisenberg model will be seen to describe a sheet of non-stretching vortex filaments in Euclidean space.

Lastly, we also derive an interesting geometric formulation of these results by viewing the spin vector as a dynamical map into the 2-sphere in Euclidean space. This formulation is based on the well-known geometrical equivalence between the Heisenberg model and the Schrödinger map equation in  $1 + 1$  dimensions [15]. When applied to the  $1 + 1$  and  $2 + 1$  dimensional hierarchies of  $SO(3)$ -invariant vector models, our derivation yields a new integrable generalization of the Schrödinger map equation in  $2 + 1$  dimensions as well as a new mKdV analog of this map equation corresponding to the mKdV spin vector model in  $1 + 1$  and  $2 + 1$  dimensions.

The rest of the paper is organized as follows. In Section 2, we review from a unified point of view the mathematical relationships amongst  $1 + 1$  dimensional vector models, dynamical maps into the 2-sphere, non-stretching curve flows in Euclidean space, Frenet and parallel frames, and the Hasimoto transformation. In Section 3, we derive the NLS hierarchy of soliton equations in terms of the geometrical covariant  $u$  given by the Frenet equations of a moving parallel frame for non-stretching space curve flows. This approach directly yields the explicit bi-Hamiltonian structure of these soliton equations, including a formula for the Hamiltonians. As examples, the parallel-frame Frenet equations are used to show, firstly, how the NLS equation itself corresponds geometrically to the Heisenberg spin model and the Schrödinger map equation; and secondly, how the mKdV spin model and the mKdV map equation arise geometrically from the next soliton equation in the NLS hierarchy.

Section 4 contains several main results. We work out the equations of motion for the space curves corresponding to the NLS hierarchy and write down the induced flows on the curvature and torsion invariants  $\kappa$ ,  $\tau$ . Next we derive the resulting geometrical hierarchies of vector models and dynamical map equations, along with their bi-Hamiltonian structure, recursion operators, and constants of motion. This new derivation involves only the parallel-frame Frenet equations plus the bi-Hamiltonian structure of the NLS hierarchy. The explicit bi-Hamiltonian form of the Schrödinger map equation and Heisenberg model, including a geometric expression for the Hamiltonians, are presented as examples.

In Section 5, we consider surfaces generated by a spatial Hamiltonian flow of curves with a parallel framing in Euclidean space. The underlying Hamiltonian structure is shown to arise naturally from the Frenet equations of the induced frame along the surface. This formulation is then used in Section 6 to study surface flows expressed in terms of the covariant variable  $u$  geometrically associated with the non-stretching space curves that foliate the surface, where the surface is stretching in all directions transverse to these curves. We show that the bi-Hamiltonian structure for  $1 + 1$  flows on  $u$  has a natural extension to  $2 + 1$  flows based on the observation that the Hamiltonian operators involve only the coordinate in the non-stretching direction on the surface. This leads to a hierarchy of  $2 + 1$  flows on  $u$ , with the starting flow given geometrically by translations in the coordinate in the transverse direction, which yields a  $2 + 1$  generalization of the NLS hierarchy.

The final two sections of the paper contain our main new results. In Section 7, we use the surface Frenet equations to derive the complete hierarchies of integrable  $2 + 1$  vector models and dynamical maps arising from the  $2 + 1$  generalization of the NLS hierarchy. The derivation yields the explicit bi-Hamiltonian structure of these two hierarchies, in addition to their respective recursion operators and constants of motion. As examples, the integrable generalizations of the Heisenberg model and the mKdV spin model in  $2 + 1$  dimensions are written down in detail, as well as the corresponding new  $2 + 1$  dimensional integrable generalizations of the Schrödinger map equation and mKdV map equation. In Section 8, we work out the equations of motion for the surface flows that correspond to the previous hierarchies. These equations are obtained by means of a different framing defined in a purely geometrical fashion by the non-stretching coordinate direction on the surface and the orthogonal direction of the surface normal in Euclidean space. We also discuss aspects of both the intrinsic and extrinsic geometry of the resulting surface motions. In particular, we obtain a recursion operator, constants of motion, and explicit evolution equations formulated in terms of geometric variables given by the geodesic curvature, normal curvature, and relative torsion of the non-stretching coordinate lines on the surface.

Some concluding remarks on future extensions of this work are given in Section 9.

## 2. Vector models and space curve flows

We start from an arbitrary  $SO(3)$  vector model in  $1 + 1$  dimensions,

$$S_t = f(S, S_x, S_{xx}, \dots), \quad |S| = 1 \quad (2.1)$$

where  $S(t, x) = (S_1, S_2, S_3)$  is a dynamical unit vector in Euclidean space,  $f$  is a vector function  $\perp S$ , and  $x$  belongs to some one dimensional domain  $C$ . A running example will be the Heisenberg spin model

$$S_t = S \wedge S_{xx} = (S \wedge S_x)_x \quad (2.2)$$

with  $C$  being  $\mathbb{R}$  or  $S^1$ .

There are two different ways to associate a curve flow to Eq. (2.1). One formulation consists of intrinsically identifying  $S$  with a map  $\gamma$  into the unit sphere  $S^2 \subset \mathbb{R}^3$ . Then  $S_t$  and  $S_x$  correspond to  $\gamma_t$  and  $\gamma_x$ ;  $\partial_x + S(S_x \cdot)$  corresponds to the covariant derivative  $\nabla_x$  on the sphere with respect to the tangent direction  $\gamma_x$ ; and  $S \wedge$  corresponds to the Hodge dual  $* = J$  (i.e. a complex structure on the sphere). Under these identifications, each vector model (2.1) describes a curve flow

$$\gamma_t = F(\gamma_x, \nabla_x \gamma_x, \dots) \quad (2.3)$$

for  $\gamma(t, x)$  on  $S^2$ . Ex. the Heisenberg model (2.2) corresponds to

$$\gamma_t = J \nabla_x \gamma_x \quad (2.4)$$

which is the Schrödinger map equation on  $S^2$ .

Alternatively, in an extrinsic formulation,  $S$  can be identified with the unit tangent vector  $T$  along a non-stretching space curve given by a position vector  $\vec{r}$  in Euclidean space,

$$S = T = \vec{r}_x \quad (2.5)$$

where  $x$  is the arclength along the curve  $\vec{r}(x)$ . Then the equation of motion of  $\vec{r}$  is

$$\vec{r}_{tx} = f(\vec{r}_x, \vec{r}_{xx}, \dots), \quad |\vec{r}_x| = 1, \quad (2.6)$$

or equivalently

$$\vec{r}_t = \int^x f(\vec{r}_x, \vec{r}_{xx}, \dots) dx, \quad |\vec{r}_x| = 1, \quad (2.7)$$

under which the arclength of the curve is preserved, i.e.  $\int_C |\vec{r}_x| dx = \ell$  is a constant of the motion. Ex. the Heisenberg model (2.2) corresponds to

$$\vec{r}_t = \vec{r}_x \wedge \vec{r}_{xx} \quad (2.8)$$

with  $|\vec{r}_x| = 1$ . This is the equation of motion of a non-stretching vortex filament studied by Hasimoto [2].

To proceed we first introduce a Frenet frame  $\mathbf{E}$  along  $\vec{r}(x)$ . It is expressed in matrix column notation by

$$\mathbf{E} = \begin{pmatrix} T \\ N \\ B \end{pmatrix} \quad (2.9)$$

where

$$T = \vec{r}_x, \quad N = |T_x|^{-1} T_x = |\vec{r}_{xx}|^{-1} \vec{r}_{xx}, \quad B = T \wedge N = |\vec{r}_{xx}|^{-1} \vec{r}_x \wedge \vec{r}_{xx}. \quad (2.10)$$

Here  $N$  is the unit normal and  $B$  is the unit bi-normal of the space curve  $\vec{r}(x)$ . Note we have the relations

$$S = T, \quad S_x = \kappa N, \quad S \wedge S_x = \kappa B, \quad (2.11)$$

where

$$\kappa = T_x \cdot N = |S_x| \quad (2.12)$$

is the curvature of  $\vec{r}(x)$ , and

$$\tau = N_x \cdot B = |S_x|^{-2} S_{xx} \cdot (S \wedge S_x) \quad (2.13)$$

is the torsion of  $\vec{r}(x)$ . The Serret–Frenet equations of this frame (2.9) are given by

$$\mathbf{E}_x = \mathbf{K} \mathbf{E} \quad (2.14)$$

with

$$\mathbf{K} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \in \mathfrak{so}(3). \quad (2.15)$$

From the equation of motion (2.1) for  $S$  we obtain the frame evolution equation

$$\mathbf{E}_t = \mathbf{A} \mathbf{E}, \quad \mathbf{A} = \begin{pmatrix} 0 & a_2 & a_3 \\ -a_2 & 0 & a_1 \\ -a_3 & -a_1 & 0 \end{pmatrix} \in \mathfrak{so}(3) \quad (2.16)$$

where

$$\begin{aligned} a_1 &= f_x \cdot B / |S_x| = f_x \cdot (S \wedge S_x) / |S_x|^2, \\ a_2 &= f \cdot N = f \cdot S_x / |S_x|, \\ a_3 &= f \cdot B = f \cdot (S \wedge S_x) / |S_x|, \end{aligned}$$

are determined by taking the  $t$ -derivative of (2.11) and substituting (2.1), followed by applying respective projections orthogonal to  $T, N, B$ .

This evolution of the frame  $\mathbf{E}$  induces evolution equations for  $\kappa$  and  $\tau$  through the zero-curvature relation  $\mathbf{K}_t = \mathbf{A}_x + [\mathbf{A}, \mathbf{K}]$ . Ex. the Heisenberg model (2.2) gives the vortex filament equations in terms of the curvature and torsion [1]:

$$\begin{aligned} \kappa_t &= -\kappa \tau_x - 2\kappa_x \tau = -\frac{(\kappa^2 \tau)_x}{\kappa}, \\ \tau_t &= \frac{\kappa_{xxx}}{\kappa} - \frac{\kappa_{xx} \kappa_x}{\kappa^2} - 2\tau \tau_x + \kappa \kappa_x = \left( \frac{\kappa_{xx}}{\kappa} - \tau^2 + \frac{1}{2} \kappa^2 \right)_x. \end{aligned} \quad (2.17)$$

Next we perform a  $SO(2)$  gauge transformation on the normal vectors in the Frenet frame (2.9):

$$\tilde{E}_1 = E_1 = T, \quad \tilde{E}_2 = E_2 \cos \theta + E_3 \sin \theta, \quad \tilde{E}_3 = -E_2 \sin \theta + E_3 \cos \theta \quad (2.18)$$

with the rotation angle  $\theta$  defined by

$$\theta_x = -\tau \quad (2.19)$$

so thus

$$\tilde{E}_{1x} = \kappa \cos \theta \tilde{E}_2 - \kappa \sin \theta \tilde{E}_3 \perp T, \quad (2.20)$$

$$\tilde{E}_{2x} = -\kappa \cos \theta \tilde{E}_1 \parallel T, \quad \tilde{E}_{3x} = \kappa \sin \theta \tilde{E}_1 \parallel T. \quad (2.21)$$

This is called a *parallel framing* [8] of the space curve  $\vec{r}(x)$ . The frame vectors (2.18) are characterized by the geometrical property that along  $\vec{r}(x)$  their derivatives lie completely in the normal space (2.20) or in the tangent space (2.21). Such a frame is unique up to a rigid ( $x$ -independent) rotation

$$\theta \rightarrow \theta + \phi, \quad \phi = \text{const.} \quad (2.22)$$

acting on the pair of normal vectors.

In matrix notation the Serret–Frenet equations of a parallel frame are given by

$$\tilde{\mathbf{E}}_x = \mathbf{U} \tilde{\mathbf{E}} \quad (2.23)$$

with

$$\tilde{\mathbf{E}} = \begin{pmatrix} T \\ \cos \theta N + \sin \theta B \\ -\sin \theta N + \cos \theta B \end{pmatrix}, \quad \mathbf{U} = \begin{pmatrix} 0 & u_2 & u_3 \\ -u_2 & 0 & u_1 \\ -u_3 & -u_1 & 0 \end{pmatrix} \in \mathfrak{so}(3) \quad (2.24)$$

where

$$u_1 = 0, \quad u_2 = \kappa \cos \theta = \kappa \cos \left( \int \tau dx \right), \quad u_3 = -\kappa \sin \theta = \kappa \sin \left( \int \tau dx \right) \quad (2.25)$$

are the components of the principal normal  $T_x$  of  $\tilde{r}(x)$ . The evolution of this frame

$$\tilde{\mathbf{E}}_t = \mathbf{W}\tilde{\mathbf{E}} \quad (2.26)$$

is described by the matrix

$$\mathbf{W} = \begin{pmatrix} 0 & \varpi_2 & \varpi_3 \\ -\varpi_2 & 0 & \varpi_1 \\ -\varpi_3 & -\varpi_1 & 0 \end{pmatrix} \in \mathfrak{so}(3) \quad (2.27)$$

which is related to  $\mathbf{U}$  through the zero-curvature equation

$$\mathbf{U}_t - \mathbf{W}_x + [\mathbf{U}, \mathbf{W}] = 0. \quad (2.28)$$

Note  $\mathbf{W}$  can be determined directly from the model (2.1) via the relations (2.9) and (2.11).

It now becomes convenient to work in terms of a complex variable formalism

$$\varpi = \varpi_2 + i\varpi_3, \quad (2.29)$$

$$u = u_2 + iu_3 = \kappa e^{-i\theta} = \kappa \exp \left( i \int \tau dx \right), \quad (2.30)$$

encoding the well-known Hasimoto transformation [2]. Ex. in the Heisenberg model (2.2), the vortex filament equations on  $\kappa$  and  $\tau$  transform into the NLS equation on  $u$ :

$$-iu_t = u_{xx} + \frac{1}{2}|u|^2 u. \quad (2.31)$$

Thus, Hasimoto's transformation has the geometrical interpretation [4] of a  $SO(2)$  gauge transformation on the normal frame of the curve  $\tilde{r}(x)$ , relating a Frenet frame to a parallel frame.

**Remark.** Since the form (2.25) of a parallel frame is preserved by  $SO(2)$  rotations (2.22), the complex scalar variable (2.30) given by the Hasimoto transformation is uniquely determined by the curve  $\tilde{r}(x)$  up to rigid phase rotations  $u \rightarrow e^{-i\phi}u$ , depending on an arbitrary constant  $\phi$ . Therefore,  $u$  has the geometrical meaning of a *covariant* of the curve [9] relative to the group  $SO(2) \simeq U(1)$ , while  $|u| = \kappa$  and  $(\arg u)_x = \tau$  are invariants of the curve.

### 3. Bi-Hamiltonian flows and operators

For a general vector model (2.1) the zero-curvature equation (2.28) gives an evolution equation on  $u$ ,

$$u_t = \varpi_x - i\varpi_1 u, \quad (3.1)$$

plus an auxiliary equation relating  $\varpi_1$  to  $u$ ,

$$\varpi_{1x} = \text{Im}(\bar{\varpi}u). \quad (3.2)$$

From (3.2) we can eliminate  $\varpi_1 = D_x^{-1} \text{Im}(\bar{\varpi}u)$  in terms of  $u$  and  $\varpi$ , and then we see (3.1) yields

$$u_t = D_x \varpi - iuD_x^{-1} \text{Im}(\bar{\varpi}u) = \mathcal{H}(\varpi) \quad (3.3)$$

where  $\varpi$  is determined from (2.1) via the frame evolution equation (2.26).

#### Proposition 1.

$$\mathcal{H} = D_x - iuD_x^{-1} \text{Im}(u\bar{C}) \quad (3.4)$$

is a Hamiltonian operator with respect to the flow variable  $u(t, x)$ , whence the evolution equation (3.3) has a Hamiltonian structure

$$u_t = \mathcal{H}(\delta \mathfrak{H} / \delta \bar{u}) \quad (3.5)$$

iff

$$\varpi = \delta H / \delta \bar{u} \quad (3.6)$$

holds for some Hamiltonian

$$\mathfrak{H} = \int_C H(x, u, \bar{u}, u_x, \bar{u}_x, u_{xx}, \bar{u}_{xx}, \dots) dx. \quad (3.7)$$

Here  $\mathcal{C}$  is the complex conjugation operator, and  $C = \mathbb{R}$  or  $S^1$  is the domain of  $x$ . In the present setting, an operator  $\mathcal{D}$  is Hamiltonian if it defines an associated Poisson bracket

$$\{\mathfrak{H}, \mathfrak{G}\} = \int_C \operatorname{Re}(\mathcal{D}(\delta\mathfrak{H}/\delta\bar{u})\delta\mathfrak{G}/\delta u) dx \quad (3.8)$$

obeying skew-symmetry  $\{\mathfrak{H}, \mathfrak{G}\} = -\{\mathfrak{G}, \mathfrak{H}\}$  and the Jacobi identity  $\{\mathfrak{F}, \{\mathfrak{H}, \mathfrak{G}\}\} + \text{cyclic} = 0$ , for all real-valued functionals  $\mathfrak{F}, \mathfrak{G}, \mathfrak{H}$  on the  $x$ -jet space of the flow variable  $u$ .

**Proposition 2.** (i) The Hamiltonian operator  $\mathcal{H}$  is invariant with respect to  $U(1)$  phase rotations  $e^{i\lambda}\mathcal{H}e^{-i\lambda} = \mathcal{H}|_{u \rightarrow e^{i\lambda}u}$ . (ii) A second Hamiltonian operator is given by

$$\mathcal{I} = -i \quad (3.9)$$

which is similarly  $U(1)$ -invariant,  $e^{i\lambda}\mathcal{I}e^{-i\lambda} = \mathcal{I}$ . (iii) The operators  $\mathcal{H}$  and  $\mathcal{I}$  are a compatible Hamiltonian pair (i.e. every linear combination is again a Hamiltonian operator), and their compositions define  $U(1)$ -invariant hereditary recursion operators

$$\mathcal{R} = \mathcal{H}\mathcal{I}^{-1} = i(D_x + uD_x^{-1}\operatorname{Re}(u\mathcal{C})), \quad \mathcal{R}^* = \mathcal{I}^{-1}\mathcal{H} = iD_x - uD_x^{-1}\operatorname{Re}(iu\mathcal{C}). \quad (3.10)$$

(iv) Composition of  $\mathcal{R}$  and  $\mathcal{H}$  yields a third  $U(1)$ -invariant Hamiltonian operator

$$\begin{aligned} \mathcal{E} = \mathcal{R}\mathcal{H} &= iD_x^2 + D_x(uD_x^{-1}\operatorname{Im}(u\mathcal{C})) + iuD_x^{-1}\operatorname{Re}(uD_x\mathcal{C}) \\ &= iD_x^2 + i|u|^2 + u_xD_x^{-1}\operatorname{Im}(u\mathcal{C}) - iuD_x^{-1}\operatorname{Re}(u_x\mathcal{C}) \end{aligned} \quad (3.11)$$

satisfying  $e^{i\lambda}\mathcal{E}e^{-i\lambda} = \mathcal{E}|_{u \rightarrow e^{i\lambda}u}$ . In particular,  $\mathcal{E}, \mathcal{H}, \mathcal{I}$  form a compatible Hamiltonian triple.

These propositions are a special case of group-invariant bi-Hamiltonian operators derived from non-stretching curve flows in constant-curvature spaces and general symmetric spaces in recent work [16–18,9]. Moreover, in the present complex variable formalism, Proposition 2 provides a substantial simplification of some main results in [5] on Hamiltonian operators connected with non-stretching curve flows in Euclidean space.

Because phase rotation on  $u$  is a symmetry of both  $\mathcal{H}$  and  $\mathcal{I}$ , the recursion operator  $\mathcal{R}$  generates a hierarchy of commuting Hamiltonian vector fields given by

$$i\varpi^{(n)}\partial/\partial u = \mathcal{R}^n(iu)\partial/\partial u, \quad n = 0, 1, 2, \dots \quad (3.12)$$

where

$$\varpi^{(n)} = \delta H^{(n)}/\delta \bar{u} = \mathcal{R}^{*n}(u), \quad n = 0, 1, 2, \dots \quad (3.13)$$

are Hamiltonian derivatives, starting with

$$\varpi^{(0)} = u, \quad H^{(0)} = \bar{u}u = |u|^2 \quad (3.14)$$

which corresponds to phase-rotation  $iu\partial/\partial u$ . Next in the hierarchy comes

$$\varpi^{(1)} = iu_x, \quad H^{(1)} = \frac{i}{2}(\bar{u}u_x - u\bar{u}_x) = \operatorname{Im}(\bar{u}_xu), \quad (3.15)$$

followed by

$$\varpi^{(2)} = -\left(u_{xx} + \frac{1}{2}|u|^2u\right), \quad H^{(2)} = |u_x|^2 - \frac{1}{4}|u|^4, \quad (3.16)$$

corresponding to respective Hamiltonian vector fields  $-u_x\partial/\partial u$  which is  $x$ -translation and  $-i(u_{xx} + \frac{1}{2}|u|^2u)\partial/\partial u$  which is of NLS form.

Through Propositions 1 and 2, this hierarchy produces integrable evolution equations on  $u(t, x)$  with a tri-Hamiltonian structure. An explicit formulation of this result has not appeared previously in the literature.

**Theorem 1.** There is a hierarchy of integrable bi-Hamiltonian flows on  $u(t, x)$  given by

$$u_t = \mathcal{H}(\delta\mathfrak{H}^{(n)}/\delta\bar{u}) = \mathcal{I}(\delta\mathfrak{H}^{(n+1)}/\delta\bar{u}), \quad n = 0, 1, 2, \dots \quad (3.17)$$

(called the  $+n$  flow) in terms of Hamiltonians  $\mathfrak{H}^{(n)} = \int_C H^{(n)} dx$  where

$$H^{(n)} = \frac{2}{1+n}D_x^{-1}\operatorname{Im}(\bar{u}(i\mathcal{H})^{n+1}u) \quad n = 0, 1, 2, \dots \quad (3.18)$$

are local Hamiltonian densities. Moreover, all the flows for  $n \neq 0$  have a tri-Hamiltonian structure

$$u_t = \mathcal{E}(\delta \mathfrak{H}^{(n-1)} / \delta \bar{u}), \quad n = 1, 2, \dots \quad (3.19)$$

**Remarks.** Each flow  $n = 0, +1, +2, \dots$  in the hierarchy is  $U(1)$ -invariant under the phase rotation  $u \rightarrow e^{i\lambda} u$  and has scaling weight  $t \rightarrow \lambda^{1+n} t$  under the NLS scaling symmetry  $x \rightarrow \lambda x, u \rightarrow \lambda^{-1} u$ , where the scaling weight of  $H^{(n)}$  is  $-2 - n$ . Additionally, these flows on  $u(t, x)$  each admit constants of motion (under suitable boundary conditions)

$$D_t \int_C |u|^2 dx = 0, \quad D_t \int_C i \bar{u} u_x dx = 0, \quad D_t \int_C |u_x|^2 - \frac{1}{4} |u|^4 dx = 0, \quad \dots \quad (3.20)$$

and symmetries

$$-u_x \partial / \partial u, \quad -i \left( u_{xx} + \frac{1}{2} |u|^2 u \right) \partial / \partial u, \quad \left( u_{xxx} + \frac{3}{2} |u|^2 u \right) \partial / \partial u, \quad \dots \quad (3.21)$$

respectively comprising all of the Hamiltonians (3.18) in the hierarchy and all of the corresponding Hamiltonian vector fields (3.12).

At the bottom of the hierarchy, the 0 flow is given by a linear traveling wave equation  $u_t = u_x$ , and next the +1 flow produces the NLS equation (2.31). The +2 flow yields the complex mKdV equation

$$-u_t = u_{xxx} + \frac{3}{2} |u|^2 u \quad (3.22)$$

which corresponds to an mKdV analog of the vortex filament equations,

$$-\kappa_t = \left( \kappa_{xx} + \frac{1}{2} \kappa^3 \right)_x - \frac{3}{2} \frac{(\tau^2 \kappa^2)_x}{\kappa} = \kappa_{xxx} + \frac{3}{2} (\kappa^2 - 2\tau^2) \kappa_x - 3\kappa \tau \tau_x, \quad (3.23)$$

$$\begin{aligned} -\tau_t &= \left( \tau_{xx} + 3 \frac{(\tau \kappa_x)_x}{\kappa} + \frac{3}{2} \tau \kappa^2 - \tau^3 \right)_x \\ &= \tau_{xxx} + 3 \frac{\tau_{xx} \kappa_x}{\kappa} + \tau_x \left( 6 \frac{\kappa_{xx}}{\kappa} - 3 \frac{\kappa_x^2}{\kappa^2} - 3\tau^2 + \frac{3}{2} \kappa^2 \right) + \tau \left( 3 \frac{\kappa_{xxx}}{\kappa} - 3 \frac{\kappa_{xx} \kappa_x}{\kappa^2} + 3\kappa \kappa_x \right), \end{aligned} \quad (3.24)$$

as obtained through the Hasimoto transformation  $u = \kappa \exp(-i\theta)$ .

The evolution equations describing the  $0, +1, +2, \dots$  flows on  $u$  each arise from geometric space curve flows corresponding to  $SO(3)$ -invariant vector models (2.1). To make this correspondence explicit, it is convenient to introduce a complex frame notation

$$E^{\parallel} = \tilde{E}_1 = T, \quad E^{\perp} = \tilde{E}_2 + i\tilde{E}_3 = e^{-i\theta} (N + iB) \quad (3.25)$$

satisfying

$$E^{\parallel} \wedge E^{\perp} = -iE^{\perp} = \tilde{E}_3 - i\tilde{E}_2 = e^{-i\theta} (B - iN) \quad (3.26)$$

and

$$E^{\parallel} \cdot E^{\parallel} = 1, \quad E^{\perp} \cdot \bar{E}^{\perp} = 2, \quad E^{\parallel} \cdot E^{\perp} = 0 = E^{\perp} \cdot E^{\perp}. \quad (3.27)$$

The Frenet equations (2.14) become

$$E_x^{\parallel} = \text{Re}(\bar{u} E^{\perp}), \quad E_x^{\perp} = -u E^{\parallel}, \quad (3.28)$$

while from (2.27), (2.29), (3.2), the evolution of the frame is given by the equations

$$E_t^{\parallel} = \text{Re}(\bar{\omega} E^{\perp}), \quad E_t^{\perp} = iD_x^{-1} \text{Im}(\bar{\omega} \bar{u}) E^{\perp} - \omega E^{\parallel}. \quad (3.29)$$

Then any flow belonging to the general class

$$\omega = \omega(u, \bar{u}, u_x, \bar{u}_x, u_{xx}, \bar{u}_{xx}, \dots) \quad (3.30)$$

will determine a vector model (2.1) via

$$S = E^{\parallel}, \quad f = \text{Re}(\bar{\omega} E^{\perp}), \quad (3.31)$$

where  $f$  is expressed in terms of  $S, S_x, S_{xx}$ , etc., through the Frenet equations (2.23)–(2.24).

**Ex. 1.** The +1 flow  $\omega = iu_x$  yields

$$E_t^{\parallel} = -\text{Re}(i\bar{u}_x E^{\perp}). \quad (3.32)$$

By rewriting

$$\bar{u}_x E^\perp = (\bar{u} E^\perp)_x + \bar{u} u E^\parallel$$

we obtain

$$\operatorname{Re}(\bar{u} E^\perp) = -E^\parallel \wedge \operatorname{Re}(\bar{u} E^\perp) = -E^\parallel \wedge E_x^\parallel,$$

$$\operatorname{Re}(\bar{u} u E^\parallel) = \operatorname{Re}(i|u|^2) E^\parallel = 0,$$

and hence

$$E_t^\parallel = (E^\parallel \wedge E_x^\parallel)_x. \quad (3.33)$$

The identifications (3.31) then directly give the  $SO(3)$  Heisenberg model (2.2), which corresponds to the non-stretching space curve flow (2.17) or equivalently

$$\vec{r}_t = \kappa B, \quad |\vec{r}_x| = 1 \quad (3.34)$$

expressed as a geometric flow.

**Ex. 2.** The +2 flow  $\varpi = -(u_{xx} + \frac{1}{2}|u|^2 u)$  yields

$$-E_t^\parallel = \operatorname{Re} \left( \left( \bar{u}_{xx} + \frac{1}{2}|u|^2 \bar{u} \right) E^\perp \right) = \operatorname{Re}(\bar{u}_{xx} E^\perp) + \frac{1}{2}|u|^2 E_x^\parallel. \quad (3.35)$$

Here we can rewrite the first term as

$$\operatorname{Re}(\bar{u}_{xx} E^\perp) = \operatorname{Re}(\bar{u} E^\perp)_{xx} - \operatorname{Re}(\bar{u} E_x^\perp)_x - \operatorname{Re}(\bar{u}_x E_x^\perp) = E_{xxx}^\parallel + (|u|^2 E^\parallel)_x + \frac{1}{2}(\bar{u} u)_x E^\parallel,$$

with  $|u|^2 = |E_x^\parallel|^2$ , and thus

$$-E_t^\parallel = E_{xxx}^\parallel + \frac{3}{2}(|E_x^\parallel|^2 E^\parallel)_x. \quad (3.36)$$

Hence,  $E^\parallel = S$  gives

$$-S_t = S_{xxx} + \frac{3}{2}(|S_x|^2 S)_x \quad (3.37)$$

which can be viewed as an  $SO(3)$  mKdV model. The corresponding non-stretching space curve flow looks like

$$-\vec{r}_t = \vec{r}_{xxx} + \frac{3}{2}|\vec{r}_{xx}|^2 \vec{r}_x, \quad |\vec{r}_x| = 1. \quad (3.38)$$

This describes a geometric flow [19]

$$-\vec{r}_t = \frac{1}{2}\kappa^2 T + \kappa_x N + \kappa \tau B, \quad |\vec{r}_x| = 1 \quad (3.39)$$

which is equivalent to the evolution (3.23) and (3.24) on the curvature and torsion of  $\vec{r}(x)$ .

**Remark.** A different geometric derivation of the mKdV model (3.37) appears in work [7] on non-stretching flows of curves in three dimensional manifolds with constant curvature, i.e.  $S^3, H^3, \mathbb{R}^3$ , where the spin vector  $S$  is identified with the components of the unit tangent vector in a moving frame defined by parallel transport along the curve. The mKdV model also has been derived in [20] as a higher order symmetry of the Heisenberg model by non-geometric methods.

All of these  $SO(3)$  vector models describe dynamical maps  $\gamma$  on the unit sphere  $S^2 \subset \mathbb{R}^3$  by means of the identifications:

$$S_t \leftrightarrow \gamma_t, \quad S_x \leftrightarrow \gamma_x, \quad \partial_x + S(S_x \cdot) \leftrightarrow \nabla_x, \quad S \wedge \leftrightarrow J = * \quad (3.40)$$

and thus

$$\nabla_x \gamma_x \leftrightarrow S_{xx} + |S_x|^2 S, \quad \nabla_x^2 \gamma_x \leftrightarrow S_{xxx} + |S_x|^2 S_x + \frac{3}{2}(|S_x|^2)_x S, \quad (3.41)$$

$$J \gamma_x \leftrightarrow S \wedge S_x, \quad J \nabla_x \gamma_x \leftrightarrow S \wedge S_{xx}, \quad (3.42)$$

$$g(\gamma_x, \gamma_x) = |\gamma_x|_g^2 \leftrightarrow |S_x|^2 = S_x \cdot S_x \quad (3.43)$$

where  $g$  denotes the Riemannian metric on the sphere  $S^2$  (given by restricting the Euclidean inner product in  $\mathbb{R}^3$  to the tangent space of  $S^2 \subset \mathbb{R}^3$ ).



In particular, the  $SO(3)$  Heisenberg model yields the Schrödinger map equation (2.4) on  $S^2$ , while the  $SO(3)$  mKdV model (3.37) is identified with

$$-\gamma_t = \nabla_x^2 \gamma_x + \frac{1}{2} |\gamma_x|_g^2 \gamma_x \quad (3.44)$$

which is a mKdV map equation on  $S^2$  (i.e. a dynamical map version of the potential mKdV equation).

Thus, Theorem 1 provides a geometric realization of the hierarchies of integrable vector models and dynamical maps containing the Heisenberg model and the Schrödinger map as well as their mKdV counterparts.

#### 4. Geometric hierarchy of integrable vector models and dynamical maps

In general, any non-stretching space curve flow (2.7) can be written in terms of a Frenet frame (2.9) by an equation of motion of the form

$$\vec{r}_t = aT + bN + cB \quad (4.1)$$

such that

$$D_x a = \kappa b. \quad (4.2)$$

This relation between the tangential and normal components of the motion arises due to the non-stretching property

$$|\vec{r}_x| = 1 \quad (4.3)$$

by which the motion preserves the local arclength  $ds = |\vec{r}_x| dx$  of the space curve if (and only if)  $\vec{r}_x \cdot \vec{r}_{tx} = 0$ . As a consequence, through the Serret–Frenet equations (2.14)–(2.15), the tangent vector  $T = \vec{r}_x$  along the space curve obeys the equation of motion

$$T_t = \hat{f}_1 N + \hat{f}_2 B = f \perp T \quad (4.4)$$

given by a linear combination of the normal and bi-normal vectors with coefficients

$$\hat{f}_1 = D_x b - \tau c + \kappa a, \quad \hat{f}_2 = D_x c + \tau b. \quad (4.5)$$

Now we consider a Hasimoto transformation (2.18)–(2.19) from the Frenet frame (2.9) to a parallel frame (3.25). The equation of motion (4.1) on  $\vec{r}$  takes the form

$$\vec{r}_t = h_{\parallel} E^{\parallel} + \text{Re}(\bar{h}_{\perp} E^{\perp}) = h_{\parallel} T + \text{Re}(h_{\perp} e^{i\theta}) N + \text{Im}(h_{\perp} e^{i\theta}) B \quad (4.6)$$

in terms of the tangential and normal components given by

$$h_{\perp} = (b + ic)e^{-i\theta}, \quad h_{\parallel} = a, \quad (4.7)$$

with these components satisfying the relation (4.2) given by

$$D_x h_{\parallel} = \text{Re}(\bar{u} h_{\perp}) \quad (4.8)$$

where  $u = \kappa e^{-i\theta}$ . Correspondingly, from the Frenet equations (3.28) of the parallel frame, the equation of motion (4.4) for the tangent vector  $T = \vec{r}_x$  has the form

$$T_t = \text{Re}(\bar{\varpi} E^{\perp}) = \text{Re}(\varpi e^{i\theta}) N + \text{Im}(\varpi e^{i\theta}) B \quad (4.9)$$

in terms of

$$\varpi = D_x h_{\perp} + h_{\parallel} u \quad (4.10)$$

which encodes the normal and bi-normal components

$$\hat{f}_1 + i\hat{f}_2 = \varpi e^{i\theta}. \quad (4.11)$$

The evolution of  $T$  is thus specified by the variable  $\varpi$ , while the underlying evolution of  $\vec{r}$  is specified in terms of the variable  $h_{\perp}$ , with  $h_{\parallel}$  given by the non-stretching condition (4.8). From Eq. (4.10) these variables are related by

$$\varpi = D_x h_{\perp} + u D_x^{-1} \text{Re}(\bar{u} h_{\perp}) = \mathcal{J}(h_{\perp}). \quad (4.12)$$

The operator here

$$\mathcal{J} = D_x + u D_x^{-1} \text{Re}(u \mathcal{C}) \quad (4.13)$$

is related to the Hamiltonian operator  $\mathcal{I} = -i$  by the properties

$$-\mathcal{J} = \mathcal{R}^* \mathcal{I}^{-1} = \mathcal{I}^{-1} \mathcal{R} \quad \text{and} \quad -\mathcal{J}^{-1} = \mathcal{R}^{-1} \mathcal{I} = \mathcal{I} \mathcal{R}^{*-1}, \quad (4.14)$$

where  $\mathcal{R}$  and  $\mathcal{R}^*$  are the recursion operators (3.10). Consequently,  $\mathcal{J}^{-1}$  is a formal Hamiltonian operator compatible with  $\mathcal{I}$ .

**Proposition 3.** The evolution (4.1) of a non-stretching space curve  $\vec{r}(x)$  can be expressed in terms of a geometrical variable that determines the corresponding evolution (4.4) of the tangent vector  $T = \vec{r}_x$  through the relation  $T_t = (\vec{r}_t)_x$ . In particular,

$$h_\perp = \mathcal{J}^{-1}(\varpi) = \mathcal{R}^{-1}(i\varpi) = i\mathcal{R}^{*-1}(\varpi) \quad (4.15)$$

yields the normal components of the evolution vector  $\vec{r}_t$  in a parallel frame, where  $\varpi$  represents the frame components of  $T_t$ . The curvature  $\kappa$  and torsion  $\tau$  of  $\vec{r}(x)$  correspondingly have the evolution

$$\kappa_t = D_x \hat{f}_1 - \tau \hat{f}_2 = \text{Re}(e^{i\theta} D_x \varpi) \quad (4.16)$$

$$\tau_t = D_x(\kappa^{-1} D_x \hat{f}_2 + \tau \kappa^{-1} \hat{f}_1) + \kappa \hat{f}_2 = D_x(\kappa^{-1} \text{Im}(e^{i\theta} D_x \varpi)) + \kappa \text{Im}(e^{i\theta} \varpi) \quad (4.17)$$

which can be expressed in terms of the Frenet frame coefficients  $a, b, c$  of  $\vec{r}_t$  through the relations (4.5).

Conditions will now be stated within the general class of flows (3.30) on  $u$  such that the various evolutions (4.16)–(4.17), (4.9), (4.6), (4.4), (4.1), and (3.5)–(3.6) each define a geometric flow.

**Theorem 2.** For a non-stretching flow of a space curve  $\vec{r}(x)$  in  $\mathbb{R}^3$ , the following conditions are equivalent:

(i) Its tangent vector  $T = \vec{r}_x = S$  obeys a  $SO(3)$ -invariant vector model iff  $\hat{f}_1$  and  $\hat{f}_2$  are functions of scalar invariants formed out of  $S$  and its  $x$  derivatives (modulo differential consequences of  $S \cdot S = 1$ ), i.e.

$$\hat{f}_1 + i\hat{f}_2 = \hat{f}(S_x \cdot S_x, S_x \cdot S_{xx}, S_{xx} \cdot S_{xx}, \dots, S_{xx} \cdot (S \wedge S_x), S_{xxx} \cdot (S \wedge S_x), S_{xxx} \cdot (S \wedge S_{xx}), \dots). \quad (4.18)$$

(ii) Its principal normal component  $u = T_x \cdot E^\perp$  in a parallel frame (3.25) satisfies a  $U(1)$ -invariant evolution equation iff  $\varpi$  is an equivariant function of  $u, \bar{u}$ , and  $x$  derivatives of  $u$  and  $\bar{u}$ , under the action of a rigid ( $x$ -independent)  $U(1)$  rotation group  $u \rightarrow e^{-i\phi} u$  (with  $\phi = \text{const.}$ ), i.e.

$$\varpi = u\hat{f}(|u|, |u|_x, |u|_{xx}, \dots, (\arg u)_x, (\arg u)_{xx}, \dots). \quad (4.19)$$

(iii) Its curvature  $\kappa$  and torsion  $\tau$  satisfy geometric evolution equations in terms of invariants and differential invariants of  $\vec{r}(x)$  iff  $\varpi e^{i\theta}$  is a function of  $\kappa, \tau$ , and their  $x$  derivatives, i.e.,

$$\varpi = e^{-i\theta} \kappa \hat{f}(\kappa, \kappa_x, \kappa_{xx}, \dots, \tau, \tau_x, \tau_{xx}, \dots). \quad (4.20)$$

(iv) Its equation of motion is invariant under the Euclidean isometry group  $SO(3) \rtimes \mathbb{R}^3$  iff  $a, b, c$  are scalar functions of the curvature  $\kappa$ , torsion  $\tau$ , and their  $x$ -derivatives, subject to the non-stretching condition (4.2).

The proof of this proposition amounts to enumerating the Euclidean (differential) invariants of a space curve  $\vec{r}(x)$  with an arclength parameterization  $x$ , as shown in Appendix A.

We are now able to derive the entire hierarchy of  $SO(3)$ -invariant vector models and geometric space curve motions that correspond to all of the  $U(1)$ -invariant flows on  $u$  in Theorem 1.

From Eq. (4.9) combined with the hierarchy (3.6), the evolution of the spin vector  $S = T = \vec{r}_x$  can be written as

$$S_t = \text{Re}(\bar{E}^\perp \mathcal{R}^{*n}(u)), \quad n = 0, 1, 2, \dots \quad (4.21)$$

as generated via the recursion operator  $\mathcal{R}^* = iD_x - uD_x^{-1} \text{Re}(iu\mathcal{C})$ . The main step is now to establish the operator identity

$$\text{Re}(\bar{E}^\perp \mathcal{R}^*) = \mathcal{S} \text{Re}(E^\perp \mathcal{C}) \quad (4.22)$$

where  $\mathcal{S}$  is a spin vector operator corresponding to  $\mathcal{R}^*$ , and  $\text{Re}(E^\perp \mathcal{C})$  is the operator that produces a vector  $f$  in the perp space of  $S$  in  $\mathbb{R}^3$  when applied to the components of  $f$  with respect to  $E^\perp$  (i.e.  $f = \text{Re}(E^\perp \mathcal{C} \hat{f})$  if  $f \cdot S = 0$ , where  $\hat{f} = f \cdot E^\perp$ ). Through the Frenet equations (3.28) and the orthonormality relations (3.27) on  $E^\perp$  and  $E^\parallel$ , we straightforwardly find

$$\text{Re}(\bar{E}^\perp u) = E_x^\parallel = S_x \quad (4.23)$$

and

$$\begin{aligned} \text{Re}(\bar{E}^\perp iD_x \hat{f}) &= E^\parallel \wedge \text{Re}(\bar{E}^\perp D_x \hat{f}) = S \wedge D_x \text{Re}(E^\perp \mathcal{C} \hat{f}) = S \wedge D_x f \\ D_x^{-1} \text{Re}(iu\mathcal{C} \hat{f}) &= D_x^{-1} \text{Re}((E^\parallel \wedge E_x^\parallel) \cdot E^\perp \mathcal{C} \hat{f}) = D_x^{-1} \text{Re}((S \wedge S_x) \cdot f) \end{aligned}$$

for any vector  $f(x)$ , orthogonal to  $S$  in  $\mathbb{R}^3$ , with components  $\hat{f}(x) = E^\perp \cdot f(x)$ . Hence, this yields the vector operator

$$\mathcal{S} = S \wedge D_x - S_x D_x^{-1} (S \wedge S_x). \quad (4.24)$$

**Theorem 3.** The bi-Hamiltonian flows (3.17) on  $u(t, x)$  correspond to a hierarchy of integrable  $SO(3)$ -invariant vector models

$$S_t = (S \wedge D_x - S_x D_x^{-1} (S \wedge S_x))^n S_x = f^{(n)}, \quad n = 0, 1, 2, \dots \quad (4.25)$$

generated by the recursion operator (4.24). These models have the equivalent geometrical formulation

$$\gamma_t = (J\nabla_x - \gamma_x D_x^{-1} g(J\gamma_x, \cdot))^n \gamma_x = F^{(n)}, \quad n = 0, 1, 2, \dots \quad (4.26)$$

expressed as dynamical maps  $\gamma(t, x)$  into the 2-sphere  $S^2 \subset \mathbb{R}^3$ . Moreover, the Hamiltonians (3.18) for all the flows on  $u(t, x)$  correspond to a set of constants of motion  $\mathfrak{H}^{(0)} = \int_C H^{(0)} dx$ ,  $\mathfrak{H}^{(1)} = \int_C H^{(1)} dx$ ,  $\mathfrak{H}^{(2)} = \int_C H^{(2)} dx$ , etc., for each vector model (4.25) and each dynamical map equation (4.26). This entire set has the explicit form given by the densities (modulo total  $x$ -derivatives)

$$(1 + n)H^{(n)} = D_x^{-1}(S_x \cdot D_x F^{(n)}) = D_x^{-1}g(\gamma_x, \nabla_x F^{(n)}), \quad n = 0, 1, 2, \dots \quad (4.27)$$

which are scalar polynomials formed out of  $SO(3)$ -invariant wedge products and dot products of  $S, S_x, S_{xx}, \dots$  in terms of the equations of motion for  $S(t, x)$ , or equivalently, scalar inner products of  $\gamma_x, J\gamma_x, \nabla_x \gamma_x, J\nabla_x \gamma_x, \dots$  formed in terms of the equations of motion for  $\gamma(t, x)$ .

The vector models (4.25) have been derived previously in [21] by non-geometric methods (based on a Lax pair representation). As one new result, Theorem 3 derives the equivalent dynamical map equations (4.26), along with their recursion operator and provides an explicit expression for the constants of motion, for all of these integrable generalizations of the Heisenberg spin model (2.2) and the mKdV spin model (3.37). In particular, from the hierarchy (4.25), higher derivative versions of the Heisenberg spin model  $n = 1$  as given by  $n = 3, 5, \dots$  are seen to describe higher derivative Schrödinger maps; similarly, higher derivative versions of the mKdV spin model  $n = 2$  as given by  $n = 4, 6, \dots$  are found to describe higher derivative mKdV maps.

**Ex.**  $n = 3$  and  $n = 4$  respectively yield a 4th order Heisenberg vector model

$$S_t = S \wedge S_{xxxx} + \frac{5}{2}(|S_x|^2 S \wedge S_x)_x \quad (4.28)$$

and a 5th order mKdV vector model

$$-S_t = S_{xxxxx} - \frac{5}{2} \left( |S_{xx}|^2 S + |S_x|^2 S_x - \frac{7}{4} |S_x|^4 S \right)_x + \frac{5}{2} (|S_x|^2 S)_{xxx} \quad (4.29)$$

which are described geometrically by a 4th order Schrödinger map equation

$$-\gamma_t = J\nabla_x^3 \gamma_x + \frac{1}{2} \nabla_x (|\gamma_x|_g^2 J\gamma_x) - \frac{1}{2} g(J\gamma_x, \nabla_x \gamma_x) \gamma_x \quad (4.30)$$

and a 5th order mKdV map equation

$$\gamma_t = \nabla_x^4 \gamma_x + \frac{3}{4} (|\gamma_x|_g^2)_x \nabla_x \gamma_x + \left( \frac{5}{4} (|\gamma_x|_g^2)_{xx} - \frac{3}{2} |\nabla_x \gamma_x|_g^2 + \frac{3}{8} |\gamma_x|_g^4 \right) \gamma_x. \quad (4.31)$$

**Remark.** The equations of motion of these dynamical maps  $\gamma(t, x)$  do not locally preserve the arclength  $ds = |\gamma_x|_g dx$  in the  $x$  direction along  $\gamma$  on  $S^2$ , namely  $|\gamma_x|_g$  has a non-trivial time evolution for any of the dynamical map equations (4.26), and additionally, the total arclength  $\int_C |\gamma_x|_g dx$  is time dependent.

The spin vector recursion operator (4.24) has the factorization

$$\mathcal{R} = (D_x + SS_x \cdot + S_x D_x^{-1} S_x \cdot) (S \wedge) \quad (4.32)$$

where, as shown by the results in [21], the operators  $D_x + SS_x \cdot + S_x D_x^{-1} S_x \cdot$  and  $(S \wedge)^{-1} = -S \wedge$  constitute a compatible Hamiltonian pair with respect to the spin vector variable  $S(t, x)$ . By comparison, using the present geometric framework, we now directly derive the explicit bi-Hamiltonian structure of the hierarchy of vector models (4.25) through the bi-Hamiltonian flow equation (3.17) on  $u(t, x)$  in Theorem 1.

The starting point is the variational identity

$$\delta H^{(n)} \equiv \delta S \cdot (\delta H^{(n)}) / \delta S \equiv 2\text{Re}(\bar{\omega}^{(n)} \delta u) \quad (4.33)$$

holding modulo total  $x$ -derivatives and modulo (differential consequences of)  $S \cdot \delta S = 0$ , with the Hamiltonian densities given in the equivalent forms (3.18) and (4.27), and with  $\bar{\omega}^{(n)}$  given by the hierarchy (3.13) generated through the recursion operator  $\mathcal{R}^*$ . To begin we derive an explicit expression for  $\delta u$  in terms of  $\delta S = \delta E^\parallel$  by taking the Frechet derivative of the Frenet equation (3.28) for the normal vectors in a parallel frame with respect to  $u$ :

$$D_x \delta E^\perp = -\delta u E^\parallel - u \delta E^\parallel. \quad (4.34)$$

The  $\bar{E}^\perp$  component of (4.34) gives

$$D_x (\bar{E}^\perp \cdot \delta E^\perp) = -u \bar{E}^\perp \cdot \delta E^\parallel - \bar{u} E^\parallel \cdot \delta E^\perp = -2i\text{Re}(i\bar{u} E^\perp \cdot \delta E^\parallel)$$

via  $E^\parallel \cdot \delta E^\perp = -E^\perp \cdot \delta E^\parallel$  due to the orthogonality of  $E^\perp, E^\parallel$ . Similarly, the  $E^\parallel$  component of (4.34) yields

$$\delta u = D_x(E^\perp \cdot \delta E^\parallel) + \frac{1}{2}u\bar{E}^\perp \cdot \delta E^\perp.$$

Combining these two expressions, we obtain

$$\delta u = D_x(E^\perp \cdot \delta E^\parallel) - iuD_x^{-1}\text{Re}(i\bar{u}E^\perp \cdot \delta E^\parallel) \quad (4.35)$$

whence

$$\text{Re}(\bar{\omega}^{(n)}\delta u) = \text{Re}(\bar{\omega}^{(n)}D_x(E^\perp \cdot \delta E^\parallel)) + \text{Im}(\bar{\omega}^{(n)}u)D_x^{-1}\text{Re}(i\bar{u}E^\perp \cdot \delta E^\parallel). \quad (4.36)$$

Integration by parts on both terms in (4.36) then yields

$$\begin{aligned} \text{Re}(\bar{\omega}^{(n)}\delta u) &= -\text{Re}(E^\perp \cdot \delta E^\parallel D_x\bar{\omega}^{(n)}) - \text{Re}(i\bar{u}E^\perp \cdot \delta E^\parallel D_x^{-1}\text{Im}(\bar{\omega}^{(n)}u)) \\ &\equiv -\delta E^\parallel \cdot \text{Re}(\bar{E}^\perp \mathcal{H}(\bar{\omega}^{(n)})) \end{aligned} \quad (4.37)$$

modulo total  $x$ -derivatives. Next, after use of the relations  $i\mathcal{R}^* = -\mathcal{H}$  and  $i\bar{E}^\perp = E^\parallel \wedge \bar{E}^\perp$  which follow from (3.10) and (3.26), we get

$$\text{Re}(\bar{\omega}^{(n)}\delta u) \equiv \delta E^\parallel \cdot (E^\parallel \wedge \text{Re}(\bar{E}^\perp \omega^{(n+1)})). \quad (4.38)$$

Hence, equating (4.33) and (4.38) yields

$$\delta S \cdot (\delta H^{(n)}/\delta S - 2S \wedge \text{Re}(\bar{E}^\perp \omega^{(n+1)})) \equiv 0$$

from which we obtain the variational relation

$$-S \wedge (\delta H^{(n)}/\delta S) = 2\text{Re}(\bar{E}^\perp \omega^{(n+1)}) = 2\text{Re}(\bar{E}^\perp \mathcal{R}^{*n+1}(u)) = 2f^{(n+1)} \quad (4.39)$$

where the final equality comes from the equation of motion (4.21) combined with the hierarchy (4.25). This result (4.39), together with the factorization of the recursion operator (4.32), leads to the following Hamiltonian structure.

**Theorem 4.** In terms of the Hamiltonian densities (4.27), the hierarchy of vector models (4.25) for  $S(t, x)$  has two Hamiltonian structures

$$S_t = -S \wedge (\delta H^{(n-1)}/\delta S) = f^{(n)}, \quad n = 1, 2, \dots \quad (4.40)$$

and, for  $n \neq 1$ ,

$$S_t = D_x(\delta H^{(n-2)}/\delta S) + SD_x^{-1}(S_x \cdot \delta H^{(n-2)}/\delta S) = f^{(n)}, \quad n = 2, 3, \dots \quad (4.41)$$

where

$$S \wedge \quad \text{and} \quad D_x + SS_x \cdot + S_x D_x^{-1} S_x \cdot \quad (4.42)$$

are a compatible pair of Hamiltonian operators. Correspondingly, the hierarchy of dynamical maps (4.26) on  $\gamma(t, x)$  has the Hamiltonian structures

$$\gamma_t = -J(\delta H^{(n-1)}/\delta \gamma) = F^{(n)}, \quad n = 1, 2, \dots \quad (4.43)$$

$$= \nabla_x(\delta H^{(n-2)}/\delta \gamma) + \gamma_x D_x^{-1} g(\gamma_x, \delta H^{(n-2)}/\delta \gamma), \quad n = 2, 3, \dots \quad (4.44)$$

in terms of the compatible Hamiltonian operators

$$J \quad \text{and} \quad \nabla_x + \gamma_x D_x^{-1} g(\gamma_x, \cdot) \quad (4.45)$$

as given by the geometrical identifications (3.40). Moreover, these bi-Hamiltonian pairs (4.42) and (4.45) are geometrically equivalent to the pair of compatible symplectic (inverse Hamiltonian) operators  $-\mathcal{I}^{-1}$  and  $\mathcal{J} = -\mathcal{R}^* \mathcal{I}^{-1}$  with respect to the flow variable  $u(t, x)$  in the hierarchy (3.17)–(3.18) (cf. Theorems 1 and 3).

These results provide an explicit geometrical formulation of the abstract symplectic structure given in [22] for the Schrödinger map equation (2.4), i.e. ( $n = 1$ )

$$\gamma_t = J\nabla_x \gamma_x = -J(\delta H^{(0)}/\delta \gamma), \quad H^{(0)} = \frac{1}{2}g(\gamma_x, \gamma_x), \quad (4.46)$$

and its higher order generalizations ( $n = 3, 5, \dots$ ); the Hamiltonian structure of the mKdV map, i.e. ( $n = 2$ )

$$\gamma_t = -\nabla_x^2 \gamma_x - \frac{1}{2}|\gamma_x|_g^2 \gamma_x = \nabla_x(\delta H^{(0)}/\delta \gamma) + \gamma_x D_x^{-1} g(\gamma_x, \delta H^{(0)}/\delta \gamma) \quad (4.47)$$

$$= -J(\delta H^{(1)}/\delta \gamma), \quad H^{(1)} = -\frac{1}{2}g(\nabla_x \gamma_x, J\gamma_x), \quad (4.48)$$

and its higher order generalizations ( $n = 4, 6, \dots$ ) has not appeared previously in the literature. Note the  $n = 1$  and  $n = 2$  vector models respectively yield the well-known Hamiltonian structure of the Heisenberg spin model (2.2),

$$S_t = S \wedge S_{xx} = -S \wedge (\delta H^{(0)} / \delta S), \quad H^{(0)} = \frac{1}{2} |S_x|^2, \quad (4.49)$$

and the mKdV spin model (3.37),

$$S_t = -S_{xxx} - \frac{3}{2} (|S_x|^2 S)_x = D_x (\delta H^{(0)} / \delta S + S D_x^{-1} (S_x \cdot \delta H^{(0)} / \delta S)) \quad (4.50)$$

$$= -S \wedge (\delta H^{(1)} / \delta S), \quad H^{(1)} = \frac{1}{2} S_x \cdot (S \wedge S_{xx}). \quad (4.51)$$

**Remark.** There is a second Hamiltonian structure for both the Schrödinger map equation and the Heisenberg spin model. This structure, in contrast to the first Hamiltonian structure (4.46) and (4.49), turns out to involve a *non-polynomial* Hamiltonian density defined as follows. Let  $\xi(\gamma)$  be a vector field on  $S^2$  such that its divergence is constant at all points  $\gamma \in S^2$ . (This property geometrically characterizes  $\xi(\gamma)$  as a homothetic vector with respect to the metric-normalized volume form  $\epsilon_g$  on  $S^2$ , i.e.  $\mathcal{L}_{\xi} \epsilon_g = c \epsilon_g$  for some constant  $c \neq 0$ .) Then the Hamiltonian density given by

$$H^{(-1)} = g(\xi(\gamma), J \nabla \gamma_x), \quad \text{div}_g \xi(\gamma) = 1 \quad (4.52)$$

can be shown to satisfy (see Appendix B)

$$\delta H^{(-1)} \equiv g(\delta \gamma, J \nabla \gamma_x) \quad (4.53)$$

modulo total  $x$ -derivatives, so thus

$$\gamma_t = \nabla_x (\delta H^{(-1)} / \delta \gamma) + \gamma_x D_x^{-1} g(\gamma_x, \delta H^{(-1)} / \delta \gamma) = \nabla_x (J \gamma_x) \quad (4.54)$$

yields a second Hamiltonian structure for the Schrödinger map (2.4). The corresponding second Hamiltonian structure for the Heisenberg spin model (2.2) is given by

$$S_t = D_x (\delta H^{(-1)} / \delta S + S D_x^{-1} (S_x \cdot \delta H^{(-1)} / \delta S)) = (S \wedge S_x)_x, \quad H^{(-1)} = \xi(S) \cdot (S \wedge S_x) \quad (4.55)$$

in terms of a vector function  $\xi(S)$  such that

$$S \cdot \xi(S) = 0, \quad \partial_S^\perp \cdot \xi(S) = 1, \quad (4.56)$$

where the operator  $\partial_S^\perp = \partial_S - S(S \cdot \partial_S)$  is the orthogonal projection of a gradient with respect to the components of  $S$ . From the properties

$$S \cdot \partial_S^\perp = 0, \quad \partial_S^\perp (S \cdot S) = 0, \quad (4.57)$$

the Hamiltonian density can be shown to satisfy

$$\delta H^{(-1)} \equiv \delta S \cdot (S \wedge S_x) \quad (4.58)$$

modulo total  $x$ -derivatives (see Appendix C). Thus, both the Heisenberg spin model and the Schrödinger map equation are Hamiltonian equations of motion with respect to the corresponding bi-Hamiltonian pairs (4.42) and (4.45).

To conclude, a counterpart of Theorem 3 will now be stated for the underlying space curve motions on  $\vec{r}$ .

**Theorem 5.** The  $SO(3)$ -invariant vector models (4.25) on  $S(t, x)$  correspond to a hierarchy of integrable flows of non-stretching space curves

$$\vec{r}_t = a^{(n-1)} T + b^{(n-1)} N + c^{(n-1)} B, \quad |\vec{r}_x| = 1, \quad n = 1, 2, \dots \quad (4.59)$$

with geometric coefficients

$$c^{(n)} = \text{Re}(\mathcal{Q}^n \kappa), \quad b^{(n)} = -\text{Im}(\mathcal{Q}^n \kappa), \quad (4.60)$$

$$a^{(n)} = -D_x^{-1} (\kappa \text{Im}(\mathcal{Q}^n \kappa)) \quad (4.61)$$

given in terms of the recursion operator

$$\mathcal{Q} = iD_x - \tau - \kappa D_x^{-1} (\kappa \text{Im}) = e^{i\theta} \mathcal{R}^* e^{-i\theta}, \quad (4.62)$$

where  $\kappa, \tau$  are the curvature and torsion of  $\vec{r}$ . The tangential coefficients (4.61) yield a set of non-trivial constants of motion  $\int_C a^{(1)} dx = -\int_C \frac{1}{2} \kappa^2 dx$ ,  $\int_C a^{(2)} dx = \int_C \tau \kappa^2 dx$ , etc., for each curve flow (4.59) in the hierarchy (under suitable boundary conditions), where  $C = \mathbb{R}$  or  $S^1$  is the coordinate domain of  $x$ .

The equations of motion (4.59) arise from writing the flow equation (4.6) on  $\vec{r}$  in terms of the hierarchy (3.6) by means of the relation (4.15). This yields

$$a^{(n)} = D_x^{-1} \operatorname{Re}(i\bar{u}\varpi^{(n-1)}) = -D_x^{-1} \operatorname{Im}(\bar{u}\varpi^{(n-1)}), \quad (4.63)$$

$$b^{(n)} = \operatorname{Re}(ie^{i\theta}\varpi^{(n-1)}) = -\operatorname{Im}(e^{i\theta}\varpi^{(n-1)}), \quad (4.64)$$

$$c^{(n)} = \operatorname{Im}(ie^{i\theta}\varpi^{(n-1)}) = \operatorname{Re}(e^{i\theta}\varpi^{(n-1)}). \quad (4.65)$$

The geometric form (4.60)–(4.61) for these coefficients is then obtained through the identity

$$e^{i\theta}\varpi^{(n-1)} = e^{i\theta}\mathcal{R}^{*n-1}u = (e^{i\theta}\mathcal{R}^*e^{-i\theta})^{n-1}e^{i\theta}u \quad (4.66)$$

combined with the Hasimoto transformation

$$\kappa = e^{i\theta}u, \quad \tau = -\theta_x. \quad (4.67)$$

## 5. Surfaces and spatial Hamiltonian curve flows

The results in Theorem 1 for  $1+1$  flows can be generalized in a natural way to  $2+1$  flows by considering surfaces  $\vec{r}(x, y)$  that are foliated by space curves with a parallel framing in  $\mathbb{R}^3$ . Here  $y$  will denote a coordinate assumed to be transverse to these curves, and  $x$  will denote the arclength coordinate along the curves, so thus

$$|\vec{r}_x| = 1. \quad (5.1)$$

In this setting, a parallel frame consists of a triple of unit vectors whose derivatives along each  $y = \text{const.}$  coordinate line on the surface  $\vec{r}(x, y)$  lie completely in either the tangent space or the normal plane of this line in  $\mathbb{R}^3$ . The explicit form of such a frame is given by the vectors

$$\tilde{E}_1 = T = \vec{r}_x, \quad (5.2)$$

$$\tilde{E}_2 = \operatorname{Re}(e^{-i\theta}(N + iB)) = |\vec{r}_{xx}|^{-1}(\cos\theta\vec{r}_{xx} + \sin\theta\vec{r}_x \wedge \vec{r}_{xx}), \quad (5.3)$$

$$\tilde{E}_3 = \operatorname{Im}(e^{-i\theta}(N + iB)) = |\vec{r}_{xx}|^{-1}(-\sin\theta\vec{r}_{xx} + \cos\theta\vec{r}_x \wedge \vec{r}_{xx}), \quad (5.4)$$

with

$$\theta_x = -|\vec{r}_{xx}|^{-2}(\vec{r}_x \wedge \vec{r}_{xx}) \cdot \vec{r}_{xxx} \quad (5.5)$$

where  $T, N, B$  respectively denote the unit tangent vector, unit normal and bi-normal vectors of the  $y = \text{const.}$  coordinate lines.

In matrix notation these frame vectors satisfy the Frenet equations

$$\tilde{\mathbf{E}}_x = \mathbf{U}\tilde{\mathbf{E}}, \quad \tilde{\mathbf{E}}_y = \mathbf{V}\tilde{\mathbf{E}} \quad (5.6)$$

given by

$$\tilde{\mathbf{E}} = \begin{pmatrix} \tilde{E}_1 \\ \tilde{E}_2 \\ \tilde{E}_3 \end{pmatrix}, \quad \mathbf{U} = \begin{pmatrix} 0 & u_2 & u_3 \\ -u_2 & 0 & 0 \\ -u_3 & 0 & 0 \end{pmatrix} \in \mathfrak{so}(3), \quad \mathbf{V} = \begin{pmatrix} 0 & v_2 & v_3 \\ -v_2 & 0 & v_1 \\ -v_3 & -v_1 & 0 \end{pmatrix} \in \mathfrak{so}(3), \quad (5.7)$$

where

$$u_2 + iu_3 = u = (\tilde{E}_2 + i\tilde{E}_3) \cdot \tilde{E}_{1x}, \quad v_2 + iv_3 = v = (\tilde{E}_2 + i\tilde{E}_3) \cdot \tilde{E}_{1y} \quad (5.8)$$

are complex scalar variables, and

$$v_1 = \frac{1}{2}(\tilde{E}_3 + i\tilde{E}_2) \cdot (\tilde{E}_{2y} + i\tilde{E}_{3y}) = \bar{v}_1 \quad (5.9)$$

is a real scalar variable. Note, similarly to the case of space curves, here we have

$$(\tilde{E}_3 + i\tilde{E}_2) \cdot (\tilde{E}_{2x} + i\tilde{E}_{3x}) = 0 \quad (5.10)$$

which characterizes  $\tilde{\mathbf{E}}(x, y)$  as a parallel frame adapted to the foliation of the surface  $\vec{r}(x, y)$  by  $x$ -coordinate lines, i.e.  $\tilde{E}_{1x} \perp T, \tilde{E}_{2x} \parallel T, \tilde{E}_{3x} \parallel T$ . From (5.5) we see this choice of framing is geometrically unique up to rigid rotations that act on the normal vectors (5.3)–(5.4):

$$\theta \rightarrow \theta + \phi, \quad \tilde{E}_2 + i\tilde{E}_3 \rightarrow \exp(-i\phi)(\tilde{E}_2 + i\tilde{E}_3), \quad \text{with } \phi = \text{const.} \quad (5.11)$$

Under such rotations,  $v_1$  is invariant, while  $u$  and  $v$  undergo a rigid phase rotation,

$$u \rightarrow e^{-i\phi}u, \quad v \rightarrow e^{-i\phi}v. \quad (5.12)$$

To proceed we need to write down the tangent vectors of  $\vec{r}(x, y)$  in terms of the parallel framing,

$$\vec{r}_x = \tilde{E}_1, \quad \vec{r}_y = q_1 \tilde{E}_1 + q_2 \tilde{E}_2 + q_3 \tilde{E}_3. \quad (5.13)$$

The respective projections of  $\vec{r}_y$  orthogonal and parallel to  $\vec{r}_x$  are given by the scalar variables

$$q_2 + iq_3 = q = (\tilde{E}_2 + i\tilde{E}_3) \cdot \vec{r}_y, \quad q_1 = \tilde{E}_1 \cdot \vec{r}_y, \quad (5.14)$$

where, under rigid rotations (5.11) on the normal vectors in the parallel frame,  $q_1$  is invariant while  $q$  transforms by a rigid phase rotation,

$$q \rightarrow e^{-i\phi} q. \quad (5.15)$$

**Remark.** Due to the transformation properties (5.12) and (5.15), the variables  $u, v, q$  represent  $U(1)$ -covariants of  $\vec{r}(x, y)$  as geometrically defined with respect to the  $x$ -coordinate lines, where  $U(1)$  is the equivalence group of rigid rotations (5.11) that preserves the form of the framing (5.2)–(5.5) for the surface  $\vec{r}(x, y)$ . These variables will be seen later (cf. Propositions 7 and 8) to encode both the intrinsic and extrinsic surface geometry of  $\vec{r}(x, y)$  in  $\mathbb{R}^3$ .

We will now show that all surfaces  $\vec{r}(x, y)$  with the  $x$  coordinate satisfying the non-stretching property (5.1) have a natural geometrical interpretation as a spatial Hamiltonian curve flow with respect the  $y$  coordinate. This interpretation arises directly from the structure equations satisfied by the parallel frame (5.2)–(5.4) and the tangent vectors (5.13) adapted to these coordinates. Firstly, the frame connection matrices given in (5.7) obey the zero-curvature condition  $\mathbf{U}_y - \mathbf{V}_x + [\mathbf{U}, \mathbf{V}] = 0$ , which yields the structure equations

$$u_y - v_x + iv_1 u = 0, \quad v_{1x} - \text{Im}(\bar{v}u) = 0. \quad (5.16)$$

Secondly, the frame expansions of the tangent vectors given by (5.13) obey the zero-torsion condition  $(\vec{r}_x)_y = (\vec{r}_y)_x$ , leading to the structure equations

$$v - q_x - q_1 u = 0, \quad q_{1x} - \text{Re}(\bar{q}u) = 0. \quad (5.17)$$

**Proposition 4.** Through the frame structure Eqs. (5.16) and (5.17), the  $U(1)$ -covariants  $u, v, q$  of  $\vec{r}(x, y)$  are related by the Hamiltonian equations

$$u_y = \mathcal{H}(v), \quad v = \mathcal{J}(q), \quad (5.18)$$

with  $\mathcal{J} = -\mathcal{I}^{-1}\mathcal{R} = -\mathcal{R}^*\mathcal{I}^{-1}$ , where  $\mathcal{H} = D_x - iuD_x^{-1}\text{Im}(u\mathcal{C})$  and  $\mathcal{I} = -i$  are a pair of compatible Hamiltonian operators with respect to  $u$ , and  $\mathcal{R} = \mathcal{H}\mathcal{I}^{-1} = i(D_x + uD_x^{-1}\text{Re}(u\mathcal{C}))$  and  $\mathcal{R}^* = \mathcal{I}^{-1}\mathcal{H} = iD_x - uD_x^{-1}\text{Re}(iu\mathcal{C})$  are the associated recursion operators.

This leads to a main preliminary geometric result.

**Lemma 1.** Let  $u$  be an arbitrary complex-valued function of  $x$  and  $y$ , and define

$$v = \mathcal{H}^{-1}(u_y) = i\mathcal{R}^{-1}(u_y), \quad q = \mathcal{J}^{-1}(v) = -\mathcal{R}^{-2}(u_y), \quad (5.19)$$

$$v_1 = D_x^{-1}\text{Im}(\bar{v}u), \quad q_1 = \text{Re}(\bar{q}u) \quad (5.20)$$

in terms of the function  $u$  and the formal inverse operators  $\mathcal{H}^{-1}$  and  $\mathcal{J}^{-1}$ , where  $\mathcal{R}^{-1}$  is the formal inverse of the recursion operator  $\mathcal{R} = \mathcal{H}\mathcal{I} = i\mathcal{J}$ . Then the matrix equations (5.6) and the vector equations (5.13) constitute a consistent linear system of 1st order PDEs that determine a surface  $\vec{r}(x, y)$  with a parallel framing of the  $y = \text{const.}$  coordinate lines for which  $x$  is the arclength.

For a given function  $u(x, y)$ , the resulting surface  $\vec{r}(x, y)$  and frame  $\tilde{\mathbf{E}}(x, y)$  are unique up to Euclidean isometries.

## 6. Surface flows and 2 + 1 soliton equations

The derivation of bi-Hamiltonian soliton equations from non-stretching space curve flows in Theorem 1 and their geometrical correspondence to integrable vector models and dynamical map equations in Theorems 3 and 4 will now be generalized to geometrically analogous surface flows in  $\mathbb{R}^3$ . This will mean we consider surfaces that are non-stretching along one coordinate direction yet stretching in all transverse directions. (No assumptions will be place on the topology of the surface.)

Such surface flows  $\vec{r}(t, x, y)$  can be written naturally in terms of a parallel frame (5.2)–(5.4) adapted to the non-stretching coordinate lines, for which  $x$  will be the arclength coordinate and  $y$  will be a transverse coordinate, by an equation of motion

$$\vec{r}_t = h_1 \tilde{E}_1 + h_2 \tilde{E}_2 + h_3 \tilde{E}_3, \quad |\vec{r}_x| = 1, \quad (6.1)$$

with

$$h_1 = D_x^{-1}\text{Re}(\bar{u}h), \quad h_2 + ih_3 = h. \quad (6.2)$$

Here  $h_1$  and  $h$  respectively determine the components of the flow that are tangential and orthogonal to the non-stretching  $x$ -coordinate lines on the surface. The relation (6.2) imposes the non-stretching property by which the flow (6.1) preserves the local arclength  $ds = |\vec{r}_x|dx$  along these coordinate lines, with  $u = u_2 + iu_3$  given by the components of the parallel frame connection matrix with respect to  $x$  from (5.7).

The corresponding evolution of the parallel frame is given by the same matrix equations (2.26)–(2.27) that govern a parallel framing of non-stretching flows of space curves. Equivalently, in the complex variable notation (2.29) for the components of the evolution matrix (2.27), the evolution equations on the frame vectors consist of

$$\tilde{E}_{1t} = \text{Re}(\varpi(\tilde{E}_2 - i\tilde{E}_3)), \quad \tilde{E}_{2t} + i\tilde{E}_{3t} = -\varpi\tilde{E}_1 + \varpi_1(\tilde{E}_3 - i\tilde{E}_2), \quad (6.3)$$

where

$$\varpi = D_x h + \varpi_1 u, \quad \varpi_{1x} = \text{Re}(\bar{u}h) \quad (6.4)$$

are given in terms of  $h$  and  $u$ . These equations have the following Hamiltonian interpretation.

**Proposition 5.** The respective evolution Eqs. (6.1) and (6.3) for the surface and the parallel frame are related by

$$h = \mathcal{J}^{-1}(\varpi) = \mathcal{R}^{-1}(i\varpi) = i\mathcal{R}^{*-1}(\varpi) \quad (6.5)$$

where  $\mathcal{J}^{-1}$  is a formal Hamiltonian operator compatible with the Hamiltonian pair  $\mathcal{H}$  and  $\mathcal{I}$ , and where  $\mathcal{R}^{-1}$ ,  $\mathcal{R}^{*-1}$  are inverse recursion operators (cf. Proposition 4).

Hence, surface flows of the type (6.1)–(6.2) can be expressed in terms of the complex scalar variable  $\varpi$ . This variable also determines the resulting evolution of the frame connection matrices (5.7) through the pair of zero-curvature equations

$$\mathbf{U}_t - \mathbf{W}_x + [\mathbf{U}, \mathbf{W}] = 0 \quad (6.6)$$

$$\mathbf{V}_t - \mathbf{W}_y + [\mathbf{V}, \mathbf{W}] = 0 \quad (6.7)$$

which express the compatibility between the Frenet equations (5.6) and the evolution equations of the parallel frame in matrix form (2.26)–(2.27). From (6.6) we see  $u$  satisfies the same evolution equation (3.3) in terms of  $\varpi$  as holds for non-stretching space curve flows in Proposition 1. We then find that the evolution equation obtained from (6.7) holds identically as a consequence of Lemma 1. This result leads to the following Hamiltonian structure derived from surface flows (6.1)–(6.2).

**Lemma 2.** The Hamiltonian structure (3.5)–(3.7) for  $1 + 1$  flows generalizes to  $2 + 1$  flows on  $u(t, x, y)$  given by Hamiltonians of the form

$$\mathcal{H} = \iint_C H(x, y, u, \bar{u}, u_x, \bar{u}_x, u_y, \bar{u}_y, u_{xx}, \bar{u}_{xx}, u_{xy}, \bar{u}_{xy}, u_{yy}, \bar{u}_{yy}, \dots) dx dy \quad (6.8)$$

(where  $C$  denotes the coordinate domain of  $(x, y)$ ). In particular,

$$\varpi = \delta H / \delta \bar{u} \quad (6.9)$$

yields a  $2 + 1$  Hamiltonian evolution equation

$$u_t = \mathcal{H}(\varpi), \quad \mathcal{H} = D_x - iuD_x^{-1}\text{Im}(u\mathcal{C}), \quad (6.10)$$

corresponding to a surface flow given by (6.1), (6.2), (6.5).

Since the Hamiltonian operator  $\mathcal{H}$  does not contain the  $y$  coordinate, it obviously has  $y$ -translation symmetry. Hence starting from  $-u_y \partial / \partial u$ , there will be a hierarchy of commuting Hamiltonian vector fields

$$i\varpi^{(n)} \partial / \partial u = -\mathcal{R}^n(u_y) \partial / \partial u, \quad n = 0, 1, 2, \dots \quad (6.11)$$

where

$$\varpi^{(n)} = \delta H^{(n)} / \delta \bar{u} = \mathcal{R}^{*n}(iu_y), \quad n = 0, 1, 2, \dots \quad (6.12)$$

are derivatives of Hamiltonian densities  $H^{(n)}$ . An explicit expression for these densities can be derived by applying the scaling symmetry methods in [23,9].

**Theorem 6.** The recursion operator  $\mathcal{R} = \mathcal{H}\mathcal{I}^{-1} = i(D_x + uD_x^{-1}\text{Re}(u\mathcal{C}))$  produces a hierarchy of integrable bi-Hamiltonian  $2 + 1$  flows

$$u_t = \mathcal{H}(\delta \mathcal{H}^{(n-1)} / \delta \bar{u}) = \mathcal{I}(\delta \mathcal{H}^{(n)} / \delta \bar{u}), \quad n = 1, 2, \dots \quad (6.13)$$

(called the  $+n$  flow) in terms of the compatible Hamiltonian operators  $\mathcal{H}$  and  $\mathcal{I}$ , with the Hamiltonians  $\mathcal{H}^{(n)} = \iint_C H^{(n)} dx dy$  given by

$$H^{(n)} = \frac{-2}{1+n} D_x^{-1} \text{Re}(\bar{u} \mathcal{R}^{n+1}(u_y)), \quad n = 0, 1, 2, \dots \quad (6.14)$$

(modulo total  $x, y$ -derivatives).



At the bottom of this hierarchy,

$$H^{(0)} = \text{Re}(i\bar{u}u_y), \quad \delta H^{(0)}/\delta \bar{u} = iu_y = \varpi^{(0)} \quad (6.15)$$

yields the +1 flow

$$-iu_t = u_{xy} + v_0u, \quad v_{0x} = |u||u|_y. \quad (6.16)$$

This is a 2 + 1 nonlocal bi-Hamiltonian NLS equation which was first derived from Lax pair methods by Zhakarov [24] and Strachan [25].

Next in the hierarchy is

$$H^{(1)} = \text{Re}(\bar{u}_xu_y) - \frac{1}{2}v_0|u|^2, \quad \delta H^{(1)}/\delta \bar{u} = -(u_{xy} + v_0u) = \varpi^{(1)}. \quad (6.17)$$

This yields the +2 flow

$$-u_t = u_{xy} + |u|^2u_y + v_0u_x + iv_1u, \quad v_{1x} = \text{Im}(\bar{u}_yu_x) \quad (6.18)$$

which is a 2 + 1 nonlocal bi-Hamiltonian mKdV equation. It can be written in the equivalent form

$$-u_t = u_{xy} + (v_0u)_x + iv_2u, \quad v_{2x} = \text{Im}(\bar{u}u_{xy}) \quad (6.19)$$

studied in work of Calogero [26] and Strachan [27].

These 2 + 1 flow equations have the following integrability properties.

**Proposition 6.** *The hierarchy (6.13)–(6.14) displays  $U(1)$ -invariance under phase rotations  $u \rightarrow e^{i\lambda}u$  and homogeneity under scalings  $x \rightarrow \lambda x$ ,  $y \rightarrow \lambda y$ ,  $u \rightarrow \lambda^{-1}u$ , with  $t \rightarrow \lambda^{2+n}t$  for the + $n$  flow, where the scaling weight of  $H^{(n)}$  is  $-3 - n$ . Each of the evolution equations (6.13) in this hierarchy admits the constants of motion*

$$D_t \iint_C i\bar{u}u_y \, dx dy = 0, \quad D_t \iint_C \bar{u}_xu_y - \frac{1}{2}v_0|u|^2 \, dx dy = 0, \quad \dots \quad (6.20)$$

and

$$D_t \iint_C |u|^2 \, dx dy = 0, \quad D_t \iint_C i\bar{u}u_x \, dx dy = 0, \quad D_t \iint_C |u_x|^2 - \frac{1}{4}|u|^4 \, dx dy = 0, \quad \dots \quad (6.21)$$

comprising, respectively, all of the 2 + 1 Hamiltonians (6.14) plus the 1 + 1 Hamiltonians (3.18) extended to two spatial dimensions (under suitable boundary conditions depending on the coordinate domain  $C$  of  $(x, y)$ ). Additionally, these evolution equations (6.13) each admit the corresponding Hamiltonian symmetries

$$-u_y\partial/\partial u, \quad -i(u_{xy} + v_0u)\partial/\partial u, \quad \dots \quad (6.22)$$

plus

$$iu\partial/\partial u, \quad -u_x\partial/\partial u, \quad -i\left(u_{xx} + \frac{1}{2}|u|^2u\right)\partial/\partial u, \quad \dots \quad (6.23)$$

We note the constants of motion (6.21) and symmetries (6.23) are inherited from the 1 + 1 integrability properties in Theorem 1 as a consequence of the fact that the Hamiltonian phase-rotation vector field  $iu\partial/\partial u$  (which generates the hierarchy of 1 + 1 flows (3.17)) commutes with the Hamiltonian  $y$ -translation vector field  $-u_y\partial/\partial u$  (which generates the hierarchy of 2 + 1 flows (6.13)).

## 7. 2 + 1 vector models and dynamical maps

Each evolution equation (6.13) in the hierarchy presented in Theorem 6 determines a surface flow  $\vec{r}(t, x, y)$  and a corresponding 2 + 1 vector model for  $S(t, x, y) = \vec{r}_x = \vec{E}_1$  through the frame evolution equations (6.3) in a similar manner to the derivation of flows of space curves and 1 + 1 vector models.

The +1 flow yields the geometric  $SO(3)$  vector model known as the M–I equation [10,11,28]

$$S_t = S \wedge S_{xy} + v_1S_x = (S \wedge S_y + v_1S)_x, \quad v_{1x} = -S \cdot (S_x \wedge S_y), \quad (7.1)$$

which is a 2 + 1 integrable generalization of the  $SO(3)$  Heisenberg model. It corresponds to the surface flow

$$\vec{r}_t = \vec{r}_x \wedge \vec{r}_{xy} + v_1\vec{r}_x, \quad v_{1x} = -\vec{r}_x \cdot (\vec{r}_{xx} \wedge \vec{r}_{xy}), \quad |\vec{r}_x| = 1. \quad (7.2)$$

This flow equation describes the motion of a sheet of non-stretching filaments in Euclidean space, in analogy with the form of the vortex filament equations (2.8). Some properties of the model (7.1) have been studied recently in [29,30].

The +2 flow produces a 2 + 1 integrable generalization of the geometric SO(3) mKdV model,

$$-S_t = S_{xy} + ((S_x \cdot S_y + v_0)S - v_1 S \wedge S_x)_x, \quad v_{0x} = |S_x| |S_x|_y, \quad (7.3)$$

(the so-called M–XXIX equation [28]) which describes the surface flow

$$-\vec{r}_t = \vec{r}_{xy} + (\vec{r}_{xx} \cdot \vec{r}_{xy} + v_0)\vec{r}_x - v_1 \vec{r}_x \wedge \vec{r}_{xx}, \quad v_{0x} = |\vec{r}_{xx}| |\vec{r}_{xx}|_y, \quad |\vec{r}_x| = 1. \quad (7.4)$$

Each of these surface flows  $\vec{r}(t, x, y)$  in  $\mathbb{R}^3$  geometrically corresponds to a dynamical map  $\gamma(t, x, y)$  on the unit sphere  $S^2 \subset \mathbb{R}^3$  through extending the identifications (3.41) and (3.42) as follows:

$$S_y \leftrightarrow \gamma_y, \quad \partial_y + S(S_y \cdot) \leftrightarrow \nabla_y, \quad (7.5)$$

$$S_{xy} + (S_x \cdot S_y)S \leftrightarrow \nabla_x \gamma_y = \nabla_y \gamma_x, \quad (7.6)$$

$$S \wedge S_{xy} \leftrightarrow J \nabla_x \gamma_y = J \nabla_y \gamma_x. \quad (7.7)$$

The +1 and +2 flows thereby yield, respectively,

$$\gamma_t = J \nabla_x \gamma_y + v_1 \gamma_x, \quad v_{1x} = g(\gamma_x, J \gamma_y), \quad (7.8)$$

and

$$-\gamma_t = \nabla_x^2 \gamma_y + |\gamma_x|_g^2 \gamma_y - v_1 J \nabla_x \gamma_x - v_2 \gamma_x, \quad v_{2x} = g(\gamma_y, \nabla_x \gamma_x), \quad (7.9)$$

which are new nonlocal 2 + 1 integrable generalizations of the Schrödinger map equation (2.4) on  $S^2$  and the mKdV map equation (3.44) on  $S^2$ .

The complete hierarchy of vector models and dynamical map equations in 2 + 1 dimensions can be written down in the same manner as in 1 + 1 dimensions (cf. Theorems 3 and 4) by means of the spin vector recursion operator (4.24) and its Hamiltonian factorization (4.32). In particular, the obvious y-translation invariance of this operator provides the geometric origin for the 2 + 1 generalization of the Heisenberg spin model and the mKdV spin model.

**Theorem 7.** (i) The bi-Hamiltonian flows (6.13) correspond to a hierarchy of integrable SO(3)-invariant vector models in 2 + 1 dimensions

$$S_t = (S \wedge D_x - S_x D_x^{-1} (S \wedge S_x) \cdot)^n S_y = f^{(n)}, \quad n = 1, 2, \dots \quad (7.10)$$

which are geometrically equivalent to 2 + 1 dimensional dynamical maps  $\gamma$  into the 2-sphere  $S^2 \subset \mathbb{R}^3$

$$\gamma_t = (J \nabla_x - \gamma_x D_x^{-1} g(J \gamma_x, \cdot))^n \gamma_y = F^{(n)}, \quad n = 1, 2, \dots \quad (7.11)$$

(ii) Each vector model and dynamical map in the hierarchy (7.10)–(7.11) possesses a set of polynomial constants of motion that correspond to all of the Hamiltonians (6.14) for the +1, +2, ... flows (6.13), i.e.  $\mathfrak{H}^{(0)} = \iint_C H^{(0)} dx dy$ ,  $\mathfrak{H}^{(1)} = \iint_C H^{(1)} dx dy$ , etc., as obtained from the Hamiltonian densities (modulo total x, y-derivatives)

$$(1 + n)H^{(n)} = -D_x^{-1} (S_x \cdot D_x f^{(n+1)}) = -D_x^{-1} g(\gamma_x, \nabla_x F^{(n+1)}), \quad n = 0, 1, 2, \dots \quad (7.12)$$

given in terms of the equations of motion (7.10) for  $S(t, x)$  and (7.11) for  $\gamma(t, x)$ . In addition, the vector models (7.10) and dynamical maps (7.11) each possess two non-polynomial constants of motion  $\mathfrak{H}^{(-1)} = \iint_C H^{(-1)} dx dy$  and  $\mathfrak{H}^{(-2)} = \iint_C H^{(-2)} dx dy$  explicitly given by

$$H^{(-2)} = \frac{1}{2} \xi(S) \cdot (S \wedge S_y) = \frac{1}{2} g(\xi(\gamma), J \gamma_y), \quad (7.13)$$

$$H^{(-1)} = \frac{1}{2} S_x \cdot S_y + \frac{1}{2} v_1 \xi(S) \cdot (S \wedge S_x) = \frac{1}{2} g(\gamma_x, \gamma_y) + \frac{1}{2} v_1 g(\xi(\gamma), J \gamma_x), \quad (7.14)$$

where  $\xi(\gamma)$  is a vector field with covariantly constant divergence  $\text{div}_g \xi(\gamma) = 1$  at all points  $\gamma \in S^2$ , and where  $\xi(S)$  is an analogous vector function satisfying  $S \cdot \xi(S) = 0$ ,  $\partial_S^\perp \cdot \xi(S) = 1$ , in terms of the component-wise gradient operator  $\partial_S^\perp = \partial_S - S(S \cdot \partial_S)$  with properties (4.57). These Hamiltonian densities (7.13)–(7.14) correspond to two compatible nonlocal Hamiltonian structures for the +0 flow

$$u_t = \mathcal{E}(\delta \mathfrak{H}^{(-2)} / \bar{u}) = \mathcal{H}(\delta \mathfrak{H}^{(-1)} / \delta \bar{u}) = -u_y \quad (7.15)$$

with (cf. Lemma 1)

$$-\delta H^{(-1)} / \bar{u} = \mathcal{R}^{*-1}(iu_y) = v, \quad -\delta H^{(-2)} / \bar{u} = \mathcal{R}^{*-2}(iu_y) = -iq. \quad (7.16)$$

(iii) In terms of the Hamiltonian densities (7.12)–(7.14), all the  $2 + 1$  vector models (7.10) and dynamical map equations (7.11) have the bi-Hamiltonian structure

$$\begin{aligned} S_t &= -S \wedge (\delta H^{(n-2)} / \delta S) = D_x (\delta H^{(n-3)} / \delta S + S D_x^{-1} (S_x \cdot \delta H^{(n-3)} / \delta S)) \\ &= f^{(n)}, \quad n = 1, 2, \dots \end{aligned} \quad (7.17)$$

and

$$\begin{aligned} \gamma_t &= -J (\delta H^{(n-2)} / \delta \gamma) = \nabla_x (\delta H^{(n-3)} / \delta \gamma) + \gamma_x D_x^{-1} g(\gamma_x, \delta H^{(n-3)} / \delta \gamma) \\ &= F^{(n)}, \quad n = 1, 2, \dots \end{aligned} \quad (7.18)$$

given by the respective pairs (4.42) and (4.45) of compatible Hamiltonian operators.

**Remark.** Explicit bi-Hamiltonian formulations for the  $2 + 1$  generalization of the Heisenberg spin model (7.1) and the geometrically corresponding new  $2 + 1$  integrable Schrödinger map (7.8) are given by

$$S_t = -S \wedge (-S_{xy} + v_1 S \wedge S_x) = D_x (S \wedge S_y + S D_x^{-1} (S_x \cdot (S \wedge S_y))) \quad (7.19)$$

and

$$\gamma_t = -J (-\nabla_y \gamma_x + v_1 J \gamma_x) = \nabla_x (J \gamma_y) + \gamma_x D_x^{-1} g(\gamma_x, J \gamma_y) \quad (7.20)$$

where

$$\delta H^{(-2)} = \delta S \cdot (S \wedge S_y) = g(\delta \gamma, J \gamma_y) \quad (7.21)$$

$$\delta H^{(-1)} = \delta S \cdot (-S_{xy} + v_1 S \wedge S_x) = g(\delta \gamma, -\nabla_y \gamma_x + v_1 J \gamma_x) \quad (7.22)$$

yield the respective derivatives of the non-polynomial Hamiltonian densities (7.13) and (7.14). These two densities in addition to all the polynomial densities (7.12) give a set of constants of motion for the Hamiltonian equations (7.19) and (7.20). In particular, the first four constants of motion are explicitly given by the integrals

$$D_t \iint_C \xi(S) \cdot (S \wedge S_y) dx dy = D_t \iint_C g(\xi(\gamma), J \gamma_y) dx dy = 0, \quad (7.23)$$

$$D_t \iint_C S_x \cdot S_y + v_1 \xi(S) \cdot (S \wedge S_x) dx dy = D_t \iint_C g(\gamma_x, \gamma_y) + v_1 g(\xi(\gamma), J \gamma_x) dx dy = 0, \quad (7.24)$$

$$D_t \iint_C -S \cdot (S_x \wedge S_{xy}) + v_1 |S_x|^2 dx dy = D_t \iint_C g(\nabla_y \gamma_x, J \gamma_x) + v_1 |\gamma_x|_g^2 dx dy = 0, \quad (7.25)$$

$$\begin{aligned} D_t \iint_C S_{xx} \cdot S_{xy} - |S_x|^2 \left( S_x \cdot S_y + \frac{1}{2} v_0 \right) - v_1 S \cdot (S_x \wedge S_{xx}) dx dy \\ = D_t \iint_C g(\nabla_y \gamma_x, \nabla_x \gamma_x) - \frac{1}{2} v_0 |\gamma_x|_g^2 - v_1 g(J \gamma_x, \nabla_x \gamma_x) dx dy = 0 \end{aligned} \quad (7.26)$$

under suitable boundary conditions (where  $C$  denotes the coordinate domain of  $(x, y)$ ). The vector field  $\xi(\gamma)$  on  $S^2$ , or equivalently the vector function  $\xi(S)$ , in the non-polynomial constant of motion (7.23) has the geometrical meaning of a homothetic vector with respect to the metric-normalized volume form  $\epsilon_g$  on  $S^2$ , i.e.  $\mathcal{L}_{\xi} \epsilon_g = \epsilon_g$ .

**Theorem 7** is established as follows. In parts (i) and (ii), the derivation of the equations of motion (7.10) and Hamiltonians (7.12) for  $S(t, x)$  involves combining the evolution equation (6.3) for the frame vector  $\tilde{E}_1 = S$  with the hierarchy (6.11) for the variable  $\varpi$  by means of the identities

$$u(\tilde{E}_2 - i\tilde{E}_3) = S_x - iS \wedge S_x, \quad (7.27)$$

$$u_y(\tilde{E}_2 - i\tilde{E}_3) = -(S \wedge \mathcal{S}(S_y) + i\mathcal{S}(S_y)), \quad (7.28)$$

in addition to

$$\text{Re}(\tilde{u}\hat{f}) = S_x \cdot f, \quad \text{Im}(\tilde{u}\hat{f}) = (S \wedge S_x) \cdot f, \quad (7.29)$$

$$\text{Re}(\tilde{u}_y\hat{f}) = \mathcal{S}(S_y) \cdot (S \wedge f), \quad \text{Im}(\tilde{u}_y\hat{f}) = \mathcal{S}(S_y) \cdot f \quad (7.30)$$

holding for vectors  $f$  in  $\mathbb{R}^3$ , with components  $\hat{f} = (\tilde{E}_2 + i\tilde{E}_3) \cdot f$ , such that  $f \cdot S = 0$ ; here  $\mathcal{S}$  is the recursion operator (4.24). Similarly, the derivation of the Hamiltonian structures (7.17) in part (iii) for  $n \neq 1$  relies on applying the previous identities to the bi-Hamiltonian structure (6.13) for the flow equations on  $u(t, x)$ . The  $n = 1$  case reduces to computing the Hamiltonian derivatives (7.21)–(7.22), which is carried out in Appendices B and C. Finally, all of the corresponding results for  $\gamma(t, x)$  are an immediate consequence of the geometric identifications (7.5)–(7.7) in addition to (3.41)–(3.42).

## 8. Geometric formulation

There is a natural geometric formulation for the surface flows (6.1) corresponding to the  $2 + 1$  vector models (7.10) and  $2 + 1$  dynamical maps (7.11) in Theorem 7. We begin by writing down the intrinsic and extrinsic surface geometry of  $\vec{r}(x, y)$  in terms of the variables  $u, v, q, v_1, q_1$  appearing in the structure equations of the parallel framing for the non-stretching  $x$ -coordinate lines.

**Proposition 7.** Let  $\vec{r}(x, y)$  in  $\mathbb{R}^3$  be a surface with a parallel framing (5.1)–(5.5) adapted to the  $x$  coordinate lines, satisfying the structure equations (5.16) and (5.17). Then, on the surface  $\vec{r}(x, y)$ , the infinitesimal arclength is given by the line element

$$ds^2 = (dx + q_1 dy)^2 + |q|^2 dy^2, \quad (8.1)$$

and the infinitesimal surface area is given by the area element

$$dA = |q| dx \wedge dy. \quad (8.2)$$

All other aspects of the intrinsic surface geometry can be derived from the line element (8.1). In particular, the 1st fundamental form (i.e. the surface metric tensor) is simply  $dx dx + 2q_1 dx dy + |q|^2 dy dy$ , from which the Gauss curvature can be directly computed in terms of the  $x, y$  coordinates [6].

The extrinsic surface geometry can be determined through the surface normal vector

$$\vec{n} = \vec{r}_x \wedge \vec{r}_y = \text{Im}(\tilde{q}(\tilde{E}_2 + i\tilde{E}_3)), \quad |\vec{n}| = |q|, \quad (8.3)$$

as given by the expression (5.13) for the surface tangent vectors  $\vec{r}_x$  and  $\vec{r}_y$  in terms of the parallel frame along the  $x$ -coordinate lines. This normal vector (8.3) depends on a choice of the transverse coordinate  $y$  due to its normalization factor  $|q|$ . To proceed, we use the following natural geometric framing [6] that is defined entirely by the non-stretching direction on the surface and the orthogonal direction of the surface unit normal in  $\mathbb{R}^3$ :

$$e^\parallel = \vec{r}_x = \tilde{E}_1, \quad (8.4)$$

$$e^\perp = |\vec{n}|^{-1} \vec{n} = \text{Im}(e^{i\psi}(\tilde{E}_2 + i\tilde{E}_3)), \quad (8.5)$$

$$*e^\parallel = |\vec{n}|^{-1} \vec{n} \wedge \vec{r}_x = \text{Re}(e^{i\psi}(\tilde{E}_2 + i\tilde{E}_3)), \quad (8.6)$$

where

$$\psi = \arg(q). \quad (8.7)$$

(Here  $*$  denotes the Hodge dual acting in the tangent plane at each point on the surface.) Note this frame (8.4)–(8.6) differs from a parallel frame by a rotation through the angle (8.7) applied to the frame vectors in the normal plane relative to the  $x$ -coordinate lines, with  $e^\parallel$  and  $*e^\parallel$  being an orthogonal pair of unit tangent vectors on the surface  $\vec{r}(x, y)$ , and  $e^\perp$  being a unit normal vector for the surface.

The Frenet equations of the frame  $e^\parallel, *e^\parallel, e^\perp$  directly encode the extrinsic geometry of the surface  $\vec{r}(x, y)$ . In matrix notation, the Frenet equations with respect to the  $x, y$  coordinates are given by

$$\begin{pmatrix} e^\parallel \\ *e^\parallel \\ e^\perp \end{pmatrix}_x = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix} \begin{pmatrix} e^\parallel \\ *e^\parallel \\ e^\perp \end{pmatrix}, \quad \begin{pmatrix} e^\parallel \\ *e^\parallel \\ e^\perp \end{pmatrix}_y = \begin{pmatrix} 0 & \mu & \rho \\ -\mu & 0 & \sigma \\ -\rho & -\sigma & 0 \end{pmatrix} \begin{pmatrix} e^\parallel \\ *e^\parallel \\ e^\perp \end{pmatrix}, \quad (8.8)$$

where

$$\alpha = *e^\parallel \cdot e_x^\parallel = \text{Re}(ue^{i\psi}), \quad \beta = e^\perp \cdot e_x^\parallel = \text{Im}(ue^{i\psi}), \quad (8.9)$$

$$\gamma = e^\perp \cdot *e_x^\parallel = |q|^{-1} \text{Im}(q_x e^{i\psi}) = -\psi_x \quad (8.10)$$

and

$$\mu = *e^\parallel \cdot e_y^\parallel = \text{Re}(ve^{i\psi}) = q_1 \alpha + |q|_x, \quad \rho = e^\perp \cdot e_y^\parallel = \text{Im}(ve^{i\psi}) = q_1 \beta - |q| \psi_x, \quad (8.11)$$

$$\sigma = e^\perp \cdot *e_y^\parallel = |q|^{-1} \text{Im}((q_y + iv_1 q) e^{i\psi}) = v_1 - \psi_y \quad (8.12)$$

are obtained through the relations (5.8), (5.9), (5.14) together with the structure equations (5.16), (5.17). With respect to the  $x, y$  coordinates on the surface, the scalars  $\alpha$  and  $\mu$  are known as the *geodesic curvatures*;  $\beta$  and  $\rho$  as the *normal curvatures*;  $\gamma$  and  $\sigma$  as the *relative torsions* [6].

**Proposition 8.** For a surface  $\vec{r}(x, y)$  with a parallel framing (5.1)–(5.5) adapted to the  $x$  coordinate lines, satisfying the structure equations (5.16) and (5.17), the 2nd fundamental form is given by

$$II = (\vec{r}_x dx + \vec{r}_y dy) \cdot (e_x^\perp dx + e_y^\perp dy) = -(\beta dx dx + 2\rho dx dy + (\sigma - q_1 \gamma) |q| dy dy). \quad (8.13)$$

The components of  $\Pi$  with respect to the surface tangent frame  $e^\parallel, *e^\parallel$  yield the extrinsic curvature scalars

$$k_{11} = e^\parallel \cdot D_\parallel e^\perp = -\beta, \quad (8.14)$$

$$k_{12} = e^\parallel \cdot D_{*\parallel} e^\perp = k_{21} = *e^\parallel \cdot D_\parallel e^\perp = -\gamma, \quad (8.15)$$

$$k_{22} = *e^\parallel \cdot D_{*\parallel} e^\perp = -|q|^{-1}(\sigma - q_1 \gamma), \quad (8.16)$$

where  $D_\parallel = e^\parallel \lrcorner D = D_x$  and  $D_{*\parallel} = *e^\parallel \lrcorner D = |q|^{-1}(D_y - q_1 D_x)$  denote the projections of the total exterior derivative  $D$  on the surface in the directions tangential and orthogonal to the  $x$ -coordinate lines.

All aspects of the extrinsic surface geometry of  $\vec{r}(x, y)$  can be determined from the extrinsic curvature matrix  $\begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}$ . In particular, the mean curvature of the surface [6]

$$H = (k_{11} + k_{22})/2 \quad (8.17)$$

is given by the normalized trace of the extrinsic curvature matrix.

Now, in terms of the frame vectors (8.4)–(8.6), any surface flow (6.1)–(6.2) in which the  $x$ -coordinate lines are non-stretching can be written in the form

$$\vec{r}_t = a e^\parallel + b *e^\parallel + c e^\perp, \quad |\vec{r}_x| = 1 \quad (8.18)$$

with

$$a = D_x^{-1} \operatorname{Re}(\bar{u}h), \quad b = \operatorname{Re}(e^{i\psi} h), \quad c = \operatorname{Im}(e^{i\psi} h). \quad (8.19)$$

Through Proposition 5, we then obtain a hierarchy of flows on  $\vec{r}$  corresponding to the bi-Hamiltonian  $2 + 1$  flows on  $u$  in Theorem 6, as given by

$$b^{(n)} + i c^{(n)} = -e^{i\psi} \mathcal{R}^{n-1}(u_y), \quad a^{(n)} = -D_x^{-1} \operatorname{Re}(\bar{u} \mathcal{R}^{n-1}(u_y)) \quad (8.20)$$

in terms of the recursion operator  $\mathcal{R} = i(D_x + u D_x^{-1} \operatorname{Re}(u \mathcal{C}))$ . Moreover, these coefficients (8.20) have a geometrical formulation derived from the operator identity

$$e^{i\psi} \mathcal{R} e^{-i\psi} = i D_x + \psi_x + i u e^{i\psi} D_x^{-1} \operatorname{Re}(u e^{i\psi} \mathcal{C}) = i D_x - \gamma - (\beta - i\alpha) D_x^{-1} \operatorname{Re}((\alpha + i\beta) \mathcal{C}) \quad (8.21)$$

combined with

$$e^{i\psi} u_y = (e^{i\psi} u)_y - i \psi_y e^{i\psi} u = \alpha_y + \beta \psi_y + i(\beta_y - \alpha \psi_y). \quad (8.22)$$

This leads to the following geometric counterpart of Theorem 7.

**Theorem 8.** The integrable  $2 + 1$  vector models (7.10) and integrable  $2 + 1$  dynamical maps (7.11) correspond to a hierarchy of surface flows in  $\mathbb{R}^3$ ,

$$\vec{r}_t = a^{(n-1)} e^\parallel + b^{(n-1)} *e^\parallel + c^{(n-1)} e^\perp, \quad |\vec{r}_x| = 1, \quad n = 1, 2, \dots \quad (8.23)$$

with geometric coefficients

$$b^{(n)} + i c^{(n)} = -\mathcal{P}^{n-1}(\alpha_y + \beta \psi_y + i(\beta_y - \alpha \psi_y)), \quad a^{(n)} = D_x^{-1}(\alpha b^{(n)} + \beta c^{(n)}), \quad (8.24)$$

given in terms of the recursion operator

$$\mathcal{P} = i D_x - \gamma - (\beta - i\alpha) D_x^{-1} \operatorname{Re}((\alpha + i\beta) \mathcal{C}). \quad (8.25)$$

Here  $\alpha$  and  $\beta$  are the geodesic curvature and normal curvature of the non-stretching  $x$ -coordinate lines on the surface  $\vec{r}(t, x, y)$ ,  $\gamma$  is the relative torsion of these lines, and  $\psi = -\int \gamma dx$ .

**Remark.** The bottom flow in this hierarchy can be written in the alternative form

$$b^{(0)} = -\rho, \quad c^{(0)} = \mu, \quad a^{(0)} = D_x^{-1}(\beta \mu - \alpha \rho) = v_1 \quad (8.26)$$

by means of the relation

$$\mathcal{P}^{-1}(\alpha_y + \beta \psi_y + i(\beta_y - \alpha \psi_y)) = \rho - i\mu \quad (8.27)$$

expressed in terms of the geodesic curvature  $\mu$  and normal curvature  $\rho$  of the  $y$ -coordinate lines on the surface  $\vec{r}(t, x, y)$ . This relation (8.27) is obtained from  $-ie^{i\psi} v = \mathcal{P}^{-1}(e^{i\psi} u_y)$  which is a straightforward consequence of Lemma 1.

Geometric properties of the surface flows in the hierarchy (8.23) can be straightforwardly derived from the results in Propositions 7 and 8 combined with the explicit evolution equations for the variables  $q$  and  $v$  (as determined by Lemma 1).

In terms of the surface flow equations (8.18)–(8.19) and the operator identity (8.21), these evolution equations are given by

$$iv_t = w_1 v + e^{-i\psi} (D_y + i\sigma) \mathcal{P}(b + ic), \quad (8.28)$$

$$q_t = a q_x - i q w_1 + e^{-i\psi} (D_y - q_1 D_x + i(\sigma - q_1 \gamma)) (b + ic), \quad (8.29)$$

with

$$w_{1x} = \operatorname{Re}((\alpha - i\beta) \mathcal{P}(b + ic)), \quad q_{1x} = |q| \alpha = -\operatorname{Re}((\alpha - i\beta) \mathcal{P}(\rho - i\mu)), \quad (8.30)$$

from which we obtain the evolution of the geodesic curvature, normal curvature, and relative torsion of the  $x$ -coordinate lines:

$$\alpha_t = D_x(a\alpha) + \operatorname{Re}(\mathcal{D}_1^2(b + ic)) + \beta \operatorname{Im}(\mathcal{D}_2(b + ic)), \quad (8.31)$$

$$\beta_t = D_x(a\beta) + \operatorname{Im}(\mathcal{D}_1^2(b + ic)) - \alpha \operatorname{Im}(\mathcal{D}_2(b + ic)), \quad (8.32)$$

$$\gamma_t = D_x(a\gamma + \operatorname{Im}(\mathcal{D}_2(b + ic))) - \operatorname{Re}((\beta + i\alpha) \mathcal{D}_1(b + ic)), \quad (8.33)$$

where

$$\mathcal{D}_1 = D_{\parallel} - ik_{12}, \quad \mathcal{D}_2 = D_{*\parallel} - ik_{22} \quad (8.34)$$

are a pair of geometric derivative operators associated with the  $x$ -coordinate lines on the surface (cf. Proposition 8). Similarly, we find the area element of the surface has the evolution

$$(dA)_t = a(dA)_x + \operatorname{Re}(\mathcal{D}_2(b + ic)) dA = \varepsilon dA + \mathcal{L}_X dA \quad (8.35)$$

which consists of an infinitesimal change due to the tangential part of the surface flow  $X = ae^{\parallel} + b * e^{\parallel}$  plus a multiplicative expansion/contraction factor  $\varepsilon = 2Hc$  related to the mean curvature (8.17) of the surface through the normal part of the surface flow  $\mathcal{L}_{e^{\perp}} dA = 2H dA$ . These developments now lead to the following geometric results.

**Theorem 9.** Under each flow  $n = 1, 2, \dots$  in the hierarchy (8.23), the surface  $\vec{r}(t, x, y)$  is non-stretching the  $x$ -coordinate direction while stretching in all transverse directions, such that surface area locally expands/contracts by the dynamical factor

$$\varepsilon^{(n)} = 2H \operatorname{Im}(\mathcal{P}^{n-1}(\rho - i\mu))$$

where  $H$  is the mean curvature (8.17) of the surface. The geodesic curvature  $\alpha$ , normal curvature  $\beta$ , and relative torsion  $\gamma = -\psi_x$  of the  $x$ -coordinate lines in each surface flow (8.23) satisfy the integrable system of evolution equations

$$\alpha_t + i\beta_t = D_x(a^{(n-1)}(\alpha + i\beta)) - \mathcal{D}_1^2 \mathcal{P}^{n-1}(\rho - i\mu) - (\beta - i\alpha) \operatorname{Im}(\mathcal{D}_2 \mathcal{P}^{n-1}(\rho - i\mu)) \quad (8.36)$$

$$\psi_t = a^{(n-1)} \psi_x + a^{(n)} + \operatorname{Im}(\mathcal{D}_2 \mathcal{P}^{n-1}(\rho - i\mu)) \quad (8.37)$$

in terms of the recursion operator (8.25) and the pair of geometric operators (8.34), with the geodesic curvature  $\mu$  and normal curvature  $\rho$  of the  $y$ -coordinate lines given by (8.27), and where

$$a^{(n)} = -D_x^{-1} \operatorname{Re}((\alpha - i\beta) \mathcal{P}^n(\rho - i\mu)) \quad (8.38)$$

yields (modulo total  $x, y$ -derivatives) a set of non-trivial constants of motion  $\iint_C a^{(2)} dx dy = \frac{1}{2} \iint_C \alpha \beta_y - \beta \alpha_y - \psi_y (\alpha^2 + \beta^2) dx dy$ ,  $\iint_C a^{(3)} dx dy = \frac{1}{2} \iint_C \alpha_x \alpha_y + \beta_x \beta_y + \psi_x \psi_y (\alpha^2 + \beta^2) + \psi_x (\alpha_y \beta - \beta_y \alpha) + \psi_y (\alpha_x \beta - \beta_x \alpha) dx dy$ , etc., for the system (8.36)–(8.37). (Here  $C$  denotes the coordinate domain of  $(x, y)$ .)

We conclude by pointing out that the surface flows (8.23) in Theorem 8 and the corresponding integrable systems (8.36)–(8.37) in Theorem 9 provide a geometric realization for the  $2 + 1$  vector models (7.10) and  $2 + 1$  dynamical maps (7.11) in Theorem 7.

**Ex. 1.** The  $+1$  surface flow is given by

$$\vec{r}_t = v_1 \vec{r}_x + \rho \vec{r}_x \wedge \hat{n} + \mu \hat{n}, \quad |\vec{r}_x| = 1, \quad (8.39)$$

with  $v_{1x} = \beta \mu - \alpha \rho$ , where  $\hat{n} = \sqrt{|\vec{r}_y|^2 - (\vec{r}_x \cdot \vec{r}_y)^2}^{-1} \vec{r}_x \wedge \vec{r}_y$  denotes the surface unit normal. This flow is a geometric realization of the  $2 + 1$  Heisenberg model (7.1), corresponding to the new integrable  $2 + 1$  generalization of the Schrödinger map (7.8).

**Ex. 2.** Similarly, a geometric realization of the  $2 + 1$  mKdV vector model (7.3) and the corresponding mKdV map (7.9) is provided by the  $+2$  surface flow

$$-\vec{r}_t = v_{0x} \vec{r}_x + (\alpha_y + \psi_y \beta) \vec{r}_x \wedge \hat{n} + (\beta_y - \psi_y \alpha) \hat{n}, \quad |\vec{r}_x| = 1, \quad (8.40)$$

with  $v_{0x} = \alpha \alpha_y + \beta \beta_y$ .

The corresponding geometric integrable systems on the geodesic curvature  $\alpha$ , normal curvature  $\beta$ , and relative torsion  $\gamma = -\psi_x$  of the non-stretching  $x$ -coordinate lines in these surface flows (8.39) and (8.40) are respectively given by

$$\alpha_t + i\beta_t = v_1(\alpha_x + i\beta_x) - \mathcal{D}_1^2(\rho - i\mu) + (\alpha + i\beta)(\mu\beta - \rho\alpha + \text{ilm}(\mathcal{D}_2(\rho - i\mu))) \quad (8.41)$$

$$\psi_t = v_1\psi_x - v_0 + \text{Im}(\mathcal{D}_2(\rho - i\mu)) \quad (8.42)$$

and

$$\begin{aligned} \alpha_t + i\beta_t = & -v_0(\alpha_x + i\beta_x) - \mathcal{D}_1^2(\alpha_y + \beta\psi_y + i(\beta_y - \alpha\psi_y)) \\ & - (\alpha + i\beta) \left( \frac{1}{2}(\alpha^2 + \beta^2)_y - \text{ilm}(\mathcal{D}_2(\alpha_y + \beta\psi_y + i(\beta_y - \alpha\psi_y))) \right) \end{aligned} \quad (8.43)$$

$$\psi_t = -v_0\psi_x + v_2 + \text{Im}(\mathcal{D}_2(\alpha_y + \beta\psi_y + i(\beta_y - \alpha\psi_y))) \quad (8.44)$$

with  $v_{2x} = \alpha\beta_{xy} - \beta\alpha_{xy} - \psi_y(\alpha\alpha_x + \beta\beta_x) - \psi_x(\alpha\alpha_y + \beta\beta_y) - \psi_{xy}(\alpha^2 + \beta^2)$ .

## 9. Concluding remarks

There are some directions in which to extend the geometrical correspondence among integrable vector models, Hamiltonian curve and surface flows, and bi-Hamiltonian soliton equations presented in this paper.

First, it would be of interest to generalize this correspondence to integrable models for spin vectors  $S$  in Euclidean spaces  $\mathbb{R}^N$  for  $N \geq 3$ . In particular, the Heisenberg model  $S_t = S \wedge S_{xx}$ ,  $S \cdot S = 1$ , is well known to have a natural generalization where the vector wedge product and dot product in  $\mathbb{R}^3$  are replaced by a Lie bracket  $[\cdot, \cdot]$  and (negative definite) Killing form  $\langle \cdot, \cdot \rangle$  of any non-abelian semisimple Lie algebra on  $\mathbb{R}^N$ , i.e.  $S_t = [S, S_{xx}]$ ,  $-\langle S, S \rangle = 1$ . All of our results in this paper have a direct extension to such a Lie algebra setting by applying the methods of Ref. [9] to non-stretching curve flows in semisimple Lie algebras viewed as flat Klein geometries. This will lead to a large class of integrable  $2 + 1$  generalizations of the Heisenberg model for spin vectors  $S$  in semisimple Lie algebras.

Second, an interesting open problem is to find a similar geometric derivation for non-isotropic spin vector models in  $\mathbb{R}^3$  as described by the Landau–Lifshitz equation  $S_t = S \wedge (S_{xx} + JS)$ ,  $S \cdot S = 1$ , where  $J = \text{diag}(j_1, j_2, j_3)$  is a constant matrix which measures the deviation from isotropy. This equation has two compatible Hamiltonian structures [21], one of which uses the same Hamiltonian operator  $S \wedge$  that arises in the isotropic case (i.e. in the Heisenberg model). The second Hamiltonian operator, however, involves the anisotropy matrix  $J$ , which cannot be derived from the frame structure equations for non-stretching curve flows in Euclidean space. This suggests a non-Euclidean geometric setting will be needed instead.

## Acknowledgement

S.C.A. is supported by an NSERC research grant.

## Appendix A. Proof of Theorem 2

Let  $\vec{r}(x)$  be a space curve with  $x$  as the arclength, i.e.  $|\vec{r}'(x)| = 1$ , and let  $T, N, B$  be its Frenet frame (2.10). To prove Theorem 2, we will enumerate the Euclidean invariants of  $\vec{r}(x)$ .

Firstly, since the curvature  $\kappa = T_x \cdot N = |T_x|$  and torsion  $\tau = N_x \cdot B = |T_x|^{-2} T_{xx} \cdot (T \wedge T_x)$  are invariantly defined in terms of the unit tangent vector  $T = \vec{r}'_x$  along  $\vec{r}(x)$ , so are all of their  $x$  derivatives. This establishes part (iii) of the theorem.

Secondly, these (differential) invariants generate all possible scalar expressions formed out of  $T, T_x, T_{xx}, \dots$  by dot products and wedge products, as shown from a recursive application of the Frenet equations (2.14)–(2.15). Specifically,

$$T_x = \kappa N, \quad (A.1)$$

$$T_{xx} = -\kappa^2 T + \kappa_x N + \kappa \tau B, \quad (A.2)$$

$$T_{xxx} = -3\kappa\kappa_x T + (\kappa_{xx} - \kappa^3 - \kappa\tau^2)N + (2\kappa_x\tau + \tau_x\kappa)B, \quad (A.3)$$

etc.,

yields

$$T_x \cdot T_x = -T_{xx} \cdot T = \kappa^2, \quad (A.4)$$

$$T_{xx} \cdot T_x = \frac{1}{2}(T_x \cdot T_x)_x = -\frac{1}{3}T_{xxx} \cdot T = \kappa\kappa_x, \quad (A.5)$$

$$T_{xxx} \cdot T_x = \frac{1}{2}(T_x \cdot T_x)_{xx} - T_{xx} \cdot T_{xx} = -\frac{1}{4}T_{xxxx} \cdot T = \kappa\kappa_{xx} - \kappa^4 - \kappa^2\tau^2, \quad (A.6)$$

etc.,

and

$$T_{xx} \cdot (T \wedge T_x) = \kappa^2 \tau, \quad (\text{A.7})$$

$$T_{xxx} \cdot (T \wedge T_x) = (T_{xx} \cdot (T \wedge T_x))_x = \kappa^2 \tau_x + 2\tau \kappa \kappa_x, \quad (\text{A.8})$$

$$T_{xxx} \cdot (T \wedge T_{xx}) = -\kappa \tau (\kappa_{xx} - \kappa^3 - \kappa \tau^2) + \kappa_x (2\kappa_x \tau + \tau_x \kappa), \quad (\text{A.9})$$

etc.,

which thus establishes parts (i) and (iv) of the theorem.

Finally, on the other hand, since a parallel frame along  $\vec{r}(x)$  is unique up to a rigid ( $x$ -independent) rotation on the normal vectors, the corresponding components of the principal normal  $T_x$  of  $\vec{r}(x)$  given by  $u = T_x \cdot E^\perp = \kappa e^{i\theta}$  are invariantly defined only up to rotations  $\theta \rightarrow \theta + \phi$ , with  $\phi = \text{const.}$ , on  $E^\perp = (N + iB)e^{i\theta}$  (and likewise for the components of  $T_{xx}$ ,  $T_{xxx}$ , etc.). Such  $U(1)$  rotations comprise all transformations preserving the parallel property (2.20)–(2.21) of this framing. Consequently, the actual invariants of  $\vec{r}(x)$  will correspond to  $U(1)$ -invariants formed out of  $u, \bar{u}, u_x, \bar{u}_x, \dots$  via the relations  $\kappa = |u|$ ,  $\tau = (\arg u)_x$ . This establishes part (ii) of the theorem.

## Appendix B. Hamiltonian structure of the 1 + 1 and 2 + 1 Schrödinger maps

We will first verify the second Hamiltonian for the Schrödinger map equation (4.54) and its 2 + 1 generalization (7.20). The following preliminaries concerning the tangent space structure of  $S^2$  will be needed. Here  $u, v, w$  will be any triple of tangent vectors.

(1) The metric tensor  $g$ , complex structure tensor  $J$ , and metric-normalized volume form  $\epsilon_g$  on  $S^2$  satisfy the identities

$$g(u, Jv) = \epsilon_g(u, v), \quad (\text{B.1})$$

$$\epsilon_g(u, v)w + \epsilon_g(v, w)u + \epsilon_g(w, u)v = 0. \quad (\text{B.2})$$

(2) In local coordinates on  $S^2$ , the metric-compatible covariant derivative  $\nabla$  (i.e. Riemannian connection) and the associated covariant divergence operator  $\text{div}_g$  are given by

$$\nabla_v u = \partial_v u + \Gamma_v u, \quad (\text{B.3})$$

$$\text{div}_g u = \text{div} u + \text{tr}(\Gamma_u), \quad (\text{B.4})$$

where  $\Gamma$  denotes the Christoffel symbol [6] determined from  $g$  by the properties

$$\Gamma_v u = \Gamma_u v, \quad (\partial_w g)(u, v) = g(v, \Gamma_w u) + g(u, \Gamma_w v). \quad (\text{B.5})$$

(3) For an arbitrary variation  $\delta\gamma$  of the map  $\gamma$  into  $S^2$ , geometrically represented by a tangent vector field, the variation of  $g|_\gamma$  and  $\epsilon_g|_\gamma$  (as induced by their evaluation at  $\gamma$ ) is given by

$$\delta g|_\gamma(u, v) = \partial_{\delta\gamma} g(u, v) = g(u, \Gamma_{\delta\gamma} v) + g(v, \Gamma_{\delta\gamma} u), \quad (\text{B.6})$$

$$\delta \epsilon_g|_\gamma(u, v) = \left( \sqrt{\det g}^{-1} \partial_{\delta\gamma} \sqrt{\det g} \right) \epsilon_g(u, v) = \text{tr}(\Gamma_{\delta\gamma}) \epsilon_g(u, v). \quad (\text{B.7})$$

Now, consider the 1 + 1 Hamiltonian density (4.52) defined in terms of a vector field  $\xi(\gamma)$  with covariantly constant divergence,  $\text{div}_g \xi(\gamma) = 1$ . Through the identity (B.1), this density can be written more conveniently as

$$H^{(-1)} = \epsilon_g(\xi(\gamma), \gamma_x). \quad (\text{B.8})$$

Its variation is given by

$$\delta H^{(-1)} = \text{tr}(\Gamma_{\delta\gamma}) \epsilon_g(\xi(\gamma), \gamma_x) + \epsilon_g(\partial_{\delta\gamma} \xi(\gamma), \gamma_x) + \epsilon_g(\xi(\gamma), D_x \delta\gamma). \quad (\text{B.9})$$

Integration by parts on the third term in (B.9) yields

$$\epsilon_g(\xi(\gamma), D_x \delta\gamma) = D_x(\epsilon_g(\xi(\gamma), \delta\gamma)) - \epsilon_g(\partial_{\gamma_x} \xi(\gamma), \delta\gamma) - \text{tr}(\Gamma_{\gamma_x}) \epsilon_g(\xi(\gamma), \delta\gamma). \quad (\text{B.10})$$

By combining the middle terms in (B.9) and (B.10) via the identity (B.2), we get

$$\epsilon_g(\partial_{\delta\gamma} \xi(\gamma), \gamma_x) - \epsilon_g(\partial_{\gamma_x} \xi(\gamma), \delta\gamma) = \epsilon_g(\delta\gamma, \gamma_x) \text{div}_g \xi(\gamma). \quad (\text{B.11})$$

Likewise, combining the first term in (B.9) with the third term in (B.10), we obtain

$$\epsilon_g(\xi(\gamma), \gamma_x) \text{tr}(\Gamma_{\delta\gamma}) - \epsilon_g(\xi(\gamma), \delta\gamma) \text{tr}(\Gamma_{\gamma_x}) = \epsilon_g(\delta\gamma, \gamma_x) \text{tr}(\Gamma_{\xi(\gamma)}). \quad (\text{B.12})$$

Hence, modulo total  $x$ -derivatives, (B.11) and (B.12) combine to give

$$\delta H^{(-1)} \equiv \epsilon_g(\delta\gamma, \gamma_x) \text{div}_g \xi(\gamma) = g(\delta\gamma, J\gamma_x)$$

which yields the Hamiltonian derivative (4.53).



Next, consider the 2 + 1 Hamiltonian densities (7.13) and (7.14). The previous derivation applies verbatim to the density (7.13), yielding its derivative (7.21). For the second density (7.14), we look at its two terms  $H^{(-1)} = H_1 + H_2$  separately:

$$H_1 = \frac{1}{2}g(\gamma_x, \gamma_y), \quad H_2 = \frac{1}{2}v_1\epsilon_g(\xi(\gamma), \gamma_x), \quad (\text{B.13})$$

with

$$v_{1x} = \epsilon_g(\gamma_x, \gamma_y). \quad (\text{B.14})$$

First, the variation of  $H_1$  is given by

$$\begin{aligned} \delta H_1 &= \frac{1}{2}g(\gamma_x, D_y\delta\gamma) + \frac{1}{2}g(D_x\delta\gamma, \gamma_y) + \frac{1}{2}g(\gamma_x, \Gamma_{\delta\gamma}\gamma_y) + \frac{1}{2}g(\gamma_y, \Gamma_{\delta\gamma}\gamma_x) \\ &= \frac{1}{2}g(\gamma_x, \nabla_y\delta\gamma) + \frac{1}{2}g(\gamma_y, \nabla_x\delta\gamma) \end{aligned} \quad (\text{B.15})$$

through (B.3) and (B.5). Integration by parts on these terms yields

$$g(\gamma_x, \nabla_y\delta\gamma) = D_y(g(\gamma_x, \delta\gamma)) - g(\delta\gamma, \nabla_y\gamma_x), \quad g(\gamma_y, \nabla_x\delta\gamma) = D_x(g(\gamma_y, \delta\gamma)) - g(\delta\gamma, \nabla_x\gamma_y) \quad (\text{B.16})$$

where, on scalar expressions, a covariant derivative reduces to an ordinary total derivative. Hence, modulo total  $x, y$ -derivatives, substitution of (B.16) into (B.15) yields

$$\delta H_1 \equiv -\frac{1}{2}g(\delta\gamma, \nabla_x\gamma_y + \nabla_y\gamma_x) = g(\delta\gamma, -\nabla_y\gamma_x) \quad (\text{B.17})$$

after we use the commutativity identity  $\nabla_x\gamma_y = \nabla_y\gamma_x$  which is a consequence of the first property in (B.5).

Second, the variation of  $H_2$  is given by

$$\delta H_2 = \frac{1}{2}\epsilon_g(\xi(\gamma), \gamma_x)\delta v_1 + \frac{1}{2}v_1D_x(\epsilon_g(\xi(\gamma), \delta\gamma)) + \frac{1}{2}v_1\epsilon_g(\delta\gamma, \gamma_x)\text{div}_g\xi(\gamma) \quad (\text{B.18})$$

where the last two terms come from (B.10)–(B.12). To evaluate the first term in (B.18), we start with

$$\delta v_{1x} = \epsilon_g(D_x\delta\gamma, \gamma_y) + \epsilon_g(\gamma_x, D_y\delta\gamma) + \text{tr}(\Gamma_{\delta\gamma})\epsilon_g(\gamma_x, \gamma_y) \quad (\text{B.19})$$

and use integration by parts to expand the first two terms, giving

$$\epsilon_g(D_x\delta\gamma, \gamma_y) = D_x(\epsilon_g(\delta\gamma, \gamma_y)) - \epsilon_g(\delta\gamma, D_x\gamma_y) - \text{tr}(\Gamma_{\gamma_x})\epsilon_g(\delta\gamma, \gamma_y), \quad (\text{B.20})$$

$$\epsilon_g(\gamma_x, D_y\delta\gamma) = D_y(\epsilon_g(\gamma_x, \delta\gamma)) - \epsilon_g(D_y\gamma_x, \delta\gamma) - \text{tr}(\Gamma_{\gamma_y})\epsilon_g(\gamma_x, \delta\gamma). \quad (\text{B.21})$$

Using the identity (B.2), we note the Christoffel terms in (B.19), (B.20), (B.21) combine to give 0, while the middle terms in (B.20) and (B.21) cancel due to

$$D_x\gamma_y - D_y\gamma_x = \nabla_x\gamma_y - \nabla_y\gamma_x = 0. \quad (\text{B.22})$$

Hence, (B.19) simplifies to a sum of total  $x, y$ -derivatives

$$\delta v_{1x} = D_x(\epsilon_g(\delta\gamma, \gamma_y)) + D_y(\epsilon_g(\gamma_x, \delta\gamma)).$$

As a result, the first term in the variation (B.18) becomes

$$\frac{1}{2}\epsilon_g(\xi(\gamma), \gamma_x)\delta v_1 = \frac{1}{2}\epsilon_g(\delta\gamma, \gamma_y)\epsilon_g(\xi(\gamma), \gamma_x) + \frac{1}{2}D_x^{-1}(D_y\epsilon_g(\gamma_x, \delta\gamma))\epsilon_g(\xi(\gamma), \gamma_x). \quad (\text{B.23})$$

Integration by parts on the second term in (B.23) yields

$$\frac{1}{2}D_x^{-1}D_y(\epsilon_g(\gamma_x, \delta\gamma))\epsilon_g(\xi(\gamma), \gamma_x) \equiv \frac{1}{2}\epsilon_g(\gamma_x, \delta\gamma)D_x^{-1}D_y(\epsilon_g(\xi(\gamma), \gamma_x)). \quad (\text{B.24})$$

Now we use the relation

$$\begin{aligned} D_y\epsilon_g(\xi(\gamma), \gamma_x) - D_x\epsilon_g(\xi(\gamma), \gamma_y) &= \epsilon_g(\partial_{\gamma_y}\xi(\gamma), \gamma_x) - \epsilon_g(\partial_{\gamma_x}\xi(\gamma), \gamma_y) + \epsilon_g(\xi(\gamma), D_y\gamma_x - D_x\gamma_y) \\ &\quad + \epsilon_g(\xi(\gamma), \gamma_x)\text{tr}(\Gamma_{\gamma_y}) - \epsilon_g(\xi(\gamma), \gamma_y)\text{tr}(\Gamma_{\gamma_x}) \\ &= \epsilon_g(\gamma_y, \gamma_x)(\text{div}\xi(\gamma) + \text{tr}(\Gamma_{\xi(\gamma)})) = -v_{1x}\text{div}_g\xi(\gamma) \end{aligned}$$

obtained via the identities (B.2) and (B.22). Thus, (B.24) reduces to

$$\frac{1}{2}D_x^{-1}D_y(\epsilon_g(\gamma_x, \delta\gamma))\epsilon_g(\xi(\gamma), \gamma_x) \equiv \frac{1}{2}\epsilon_g(\gamma_x, \delta\gamma)\epsilon_g(\xi(\gamma), \gamma_y) + \frac{1}{2}v_1\epsilon_g(\delta\gamma, \gamma_x) \quad (\text{B.25})$$

which combines with the first term in (B.23) by use of (B.2) to give

$$\frac{1}{2}\epsilon_g(\xi(\gamma), \gamma_x)\delta v_1 \equiv \frac{1}{2}\epsilon_g(\xi(\gamma), \delta\gamma)\epsilon_g(\gamma_x, \gamma_y) + \frac{1}{2}v_1\epsilon_g(\delta\gamma, \gamma_x). \quad (\text{B.26})$$

Substituting (B.26) into the variation (B.18), and using (B.14), we get

$$\delta H_2 \equiv \frac{1}{2}v_{1x}\epsilon_g(\xi(\gamma), \delta\gamma) + \frac{1}{2}v_1D_x(\epsilon_g(\xi(\gamma), \delta\gamma)) + v_1\epsilon_g(\delta\gamma, \gamma_x) \equiv g(\delta\gamma, v_1J\gamma_x). \quad (\text{B.27})$$

Finally, we combine the separate variations (B.17) and (B.27) to obtain

$$\delta H^{(-1)} = \delta H_1 + \delta H_2 \equiv g(\delta\gamma, -\nabla_y\gamma_x + v_1J\gamma_x) \quad (\text{B.28})$$

which yields the Hamiltonian derivative (7.22).

### Appendix C. Hamiltonian structure of the 1 + 1 and 2 + 1 Heisenberg models

We will next verify the second Hamiltonian for the 1 + 1 Heisenberg model (4.55), given by the density

$$H^{(-1)} = \xi(S) \cdot (S \wedge S_x). \quad (\text{C.1})$$

Here  $\xi(S)$  is a vector function, defined in terms of the spin vector  $S$ , such that

$$S \cdot \xi(S) = 0 \quad (\text{C.2})$$

and

$$\partial_S^\perp \cdot \xi(S) = 1, \quad (\text{C.3})$$

where  $\partial_S^\perp = \partial_S - S(S \cdot \partial_S)$  is a component-wise gradient operator satisfying the properties (4.57). We note that, due to these properties,  $\partial_S^\perp$  has a well-defined action on any function of  $S$  with  $S \cdot S = 1$ . To proceed, the following algebraic preliminaries will be needed.

(1) A variation of  $S$  consists of an arbitrary vector  $\delta S \perp S$ , i.e.  $S \cdot \delta S = 0$ .

(2) The variation of  $\xi(S)$  induced by  $\delta S$  is given by

$$\delta\xi(S) = (\delta S \cdot \partial_S^\perp)\xi(S). \quad (\text{C.4})$$

Similarly, the total  $x$ -derivative of  $\xi(S)$  is given by

$$D_x\xi(S) = (S_x \cdot \partial_S^\perp)\xi(S). \quad (\text{C.5})$$

(3) Since the subspace of vectors orthogonal to  $S$  in  $\mathbb{R}^3$  is two dimensional,  $\delta S \wedge S_x$  lies in the one dimensional perp space, so thus

$$\delta S \wedge S_x = -(\delta S \cdot (S \wedge S_x))S. \quad (\text{C.6})$$

Now, the variation of the density (C.1) is given by

$$\delta H^{(-1)} = \delta\xi(S) \cdot (S \wedge S_x) + \xi(S) \cdot (\delta S \wedge S_x) + \xi(S) \cdot (S \wedge D_x\delta S). \quad (\text{C.7})$$

Integration by parts on the third term in (C.7) yields

$$\xi(S) \cdot (S \wedge D_x\delta S) = D_x(S \cdot (\delta S \wedge \xi(S))) - \xi(S) \cdot (S_x \wedge \delta S) - \delta S \cdot (D_x\xi(S) \wedge S) \quad (\text{C.8})$$

with the middle terms in (C.7) and (C.8) each vanishing due to (C.6). Hence, modulo total  $x$ -derivatives, (C.7) reduces to

$$\delta H^{(-1)} \equiv \delta\xi(S) \cdot (S \wedge S_x) - \delta S \cdot (D_x\xi(S) \wedge S) = S \cdot (S_x \wedge (\delta S \cdot \partial_S^\perp)\xi(S) - \delta S \wedge (S_x \cdot \partial_S^\perp)\xi(S)) \quad (\text{C.9})$$

via (C.4) and (C.5). To simplify the terms in (C.9), we first rewrite

$$S_x(\delta S \cdot \partial_S^\perp) - \delta S(S_x \cdot \partial_S^\perp) = (\delta S \wedge S_x) \wedge \partial_S^\perp = -\delta S \cdot (S \wedge S_x)(S \wedge \partial_S^\perp) \quad (\text{C.10})$$

by means of standard vector cross-product identities in addition to identity (C.6). Thus, (C.9) becomes

$$\delta H^{(-1)} \equiv (\delta S \cdot (S \wedge S_x))S \cdot (-S \wedge \partial_S^\perp) \wedge \xi(S) \quad (\text{C.11})$$

and we again apply vector cross-product identities to rewrite the term

$$(S \wedge \partial_S^\perp) \wedge \xi(S) = \partial_S^\perp(S \cdot \xi(S)) - \xi(S)(\partial_S^\perp \cdot S) - S(\partial_S^\perp \cdot \xi(S)). \quad (\text{C.12})$$

Then, since  $S \cdot \partial_S^\perp = 0 = S \cdot \xi(S)$ , we have

$$S \cdot (-(S \wedge \partial_S^\perp) \wedge \xi(S)) = \partial_S^\perp \cdot \xi(S) = 1 \quad (\text{C.13})$$

whence (C.11) simplifies to

$$\delta H^{(-1)} \equiv \delta S \cdot (S \wedge S_x)$$

yielding the Hamiltonian derivative (4.58).

The above derivation carries over verbatim to also verify the first Hamiltonian (7.13) for the 2 + 1 Heisenberg model (7.19). To verify the second Hamiltonian (7.14), we will separately consider the two terms in the density  $H^{(-1)} = H_1 + H_2$  given by

$$H_1 = \frac{1}{2} S_x \cdot S_y, \quad H_2 = \frac{1}{2} v_1 \xi(S) \cdot (S \wedge S_x) \quad (\text{C.14})$$

with

$$v_{1x} = S_x \cdot (S \wedge S_y) = S \cdot (S_y \wedge S_x). \quad (\text{C.15})$$

The following identity will be useful:

$$S_y \wedge S_x = v_{1x} S \quad (\text{C.16})$$

holding similarly to (C.6).

First, the variation of  $H_1$  is simply

$$\delta H_1 = \frac{1}{2} S_x \cdot D_y \delta S + \frac{1}{2} S_y \cdot D_x \delta S \equiv -\delta S \cdot S_{xy} \quad (\text{C.17})$$

modulo total  $x, y$ -derivatives. Next, the variation of  $H_2$  consists of the terms

$$\delta H_2 = \frac{1}{2} \xi(S) \cdot (S \wedge S_x) \delta v_1 + \frac{1}{2} v_1 D_x (\delta S \cdot (\xi(S) \wedge S)) + \frac{1}{2} v_1 \delta S \cdot (S \wedge S_x) \quad (\text{C.18})$$

as obtained from (C.8), (C.10), (C.12), (C.13). To evaluate the first term in (C.18), we note the variation of (C.15) is given by

$$\begin{aligned} \delta v_{1x} &= \delta S \cdot (S_y \wedge S_x) + S \cdot (S_y \wedge D_x \delta S) + S \cdot (D_y \delta S \wedge S_x) \\ &\equiv D_y (\delta S \cdot (S_x \wedge S)) + D_x (\delta S \cdot (S \wedge S_y)) + 2v_{1x} S \cdot \delta S \end{aligned}$$

through (C.16). Since the last term vanishes due to the orthogonality  $\delta S \perp S$ , this yields

$$\delta v_1 = \delta S \cdot (S \wedge S_y) - D_x^{-1} D_y (\delta S \cdot (S \wedge S_x)).$$

Hence, the first term in (C.18) becomes

$$\frac{1}{2} \xi(S) \cdot (S \wedge S_x) \delta v_1 \equiv \frac{1}{2} \delta S \cdot (S \wedge S_y) \xi(S) \cdot (S \wedge S_x) - \frac{1}{2} \delta S \cdot (S \wedge S_x) D_x^{-1} D_y (\xi(S) \cdot (S \wedge S_x)) \quad (\text{C.19})$$

after integration by parts. We simplify the second term in (C.19) by using the relations

$$D_y (\xi(S) \cdot (S \wedge S_x)) - D_x (\xi(S) \cdot (S \wedge S_y)) = S \cdot (S_x \wedge (S_y \cdot \partial_S^\perp) \xi(S) - S_y \wedge (S_x \cdot \partial_S^\perp) \xi(S)) + 2\xi(S) \cdot (S_y \wedge S_x)$$

where, similarly to (C.10) and (C.12),

$$S_x (S_y \cdot \partial_S^\perp) - S_y (S_x \cdot \partial_S^\perp) = S \cdot (S_y \wedge S_x) (S \wedge \partial_S^\perp)$$

yields

$$S \cdot (S_x \wedge (S_y \cdot \partial_S^\perp) \xi(S) - S_y \wedge (S_x \cdot \partial_S^\perp) \xi(S)) = -v_{1x} \partial_S^\perp \cdot \xi(S),$$

while

$$\xi(S) \cdot (S_y \wedge S_x) = 0$$

holds due to (C.2) and (C.16). Thus, we have

$$D_x^{-1} D_y (\xi(S) \cdot (S \wedge S_x)) = \xi(S) \cdot (S \wedge S_y) - v_1,$$

whence (C.19) simplifies to

$$\frac{1}{2} \xi(S) \cdot (S \wedge S_x) \delta v_1 \equiv \frac{1}{2} \delta S \cdot (S \wedge S_y) \xi(S) \cdot (S \wedge S_x) - \frac{1}{2} \delta S \cdot (S \wedge S_x) \xi(S) \cdot (S \wedge S_y) + \frac{1}{2} v_1 \delta S \cdot (S \wedge S_x). \quad (\text{C.20})$$

By applying vector cross-product identities to the first two terms in (C.20), we get

$$\begin{aligned} \frac{1}{2}(\delta S \wedge S) \cdot S_y(\xi(S) \wedge S) \cdot S_x - \frac{1}{2}(\delta S \wedge S) \cdot S_x(\xi(S) \wedge S) \cdot S_y &= \frac{1}{2}(\delta S \wedge S) \cdot ((S_x \wedge S_y) \wedge (\xi(S) \wedge S)) \\ &= -\frac{1}{2}(\delta S \wedge S) \cdot \xi(S)v_{1x} \end{aligned}$$

via (C.16). Hence, (C.20) reduces to

$$\frac{1}{2}\xi(S) \cdot (S \wedge S_x)\delta v_1 \equiv \frac{1}{2}\delta S \cdot (\xi(S) \wedge S)v_{1x} + \frac{1}{2}v_1\delta S \cdot (S \wedge S_x). \quad (\text{C.21})$$

Finally, combining (C.21) with the middle term in (C.18), we get a total x-derivative, so thus (C.18) becomes

$$\delta H_2 \equiv \delta S \cdot (v_1 S \wedge S_x). \quad (\text{C.22})$$

The separate variations (C.17) and (C.22) then combine to yield the Hamiltonian derivative (7.22).

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