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An introduction to the theory of generalized conics and their applications

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ABSTRACT

A generalized conic is a set of points with the same average distance from the pointset Γ in the Euclidean coordinate space. The measuring of the average distance is realized via integration over Γ as the set of foci. Using generalized conics we give a process for constructing convex bodies which are invariant under a fixed subgroup G of the orthogonal group in $\mathbb{R}^n$. The motivation is to present the existence of non-Euclidean Minkowski functionals with $G \subset O(n)$ in the linear isometry group provided that the closure of G is not transitive on the unit sphere. As an application, consider $\mathbb{R}^n$ as the tangent space at a point of a connected Riemannian manifold M and G as the holonomy group. If the holonomy group is not transitive on the unit sphere in the tangent space, then the Lévi-Civita connection is (re)metrizable in the sense that there is a smooth collection of non-Euclidean Minkowski functionals on the tangent spaces such that it is invariant under parallel transport with respect to the Lévi-Civita connection (according to Berger's list of possible Riemannian holonomy groups, all of them are transitive on the unit sphere in the tangent space except in the case where the manifold is a symmetric space of rank ≥ 2). We present the (re)metrizability theorem in a more general context of metrical linear connections with a torsion tensor that is not necessarily vanishing. This allows us to declare eight classes of manifolds equipped with an invariant smooth collection of Minkowski functionals on the tangent spaces. They are called Berwald manifolds in a general sense.

1. Examples and basic properties

Let Γ be a subset of the Euclidean coordinate space $\mathbb{R}^n$. The norm and distance of the elements of the space are defined with the help of the canonical inner product as usual. A generalized conic is a set of points with the same average distance from the pointset Γ. If $\Gamma = \{\gamma_1, \dots, \gamma_m\}$ is finite, then the average distance can be calculated as the arithmetic mean

$$F(x) := \frac{d(x,\gamma_1) + \cdots + d(x,\gamma_m)}{m}$$

of distances from the points γ_i. Hypersurfaces of the form $F(x) = \text{const.}$ are called polyellipses or polyellipsoids [1–3]. It is natural to take any other types of mean or their weighted versions instead of the standard arithmetic one. To include hyperbolas in the competence of the generalization we can admit a simple weighted sum of distances instead of means. Parabolas can be given as a special case if not only single points but also hyperplanes are admitted as elements of the set of foci. The pure case of such a construction is presented in the following example. If $\Gamma = \{H_1, \dots, H_m\}$ is a finite set of hyperplanes in $\mathbb{R}^n$, then the average distance can be calculated as the arithmetic mean

$$F(x) := \frac{d(x,H_1) + \cdots + d(x,H_m)}{m}$$

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of distances from the hyperplanes H_i. In particular let

$$e_1 := (1, 0 \dots, 0), \qquad e_2 := (0, 1, 0, \dots, 0), \dots, \qquad e_n := (0, \dots, 0, 1)$$

be the canonical basis and consider the hyperplanes

$$H_i := \operatorname{aff}\{e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_n\}, \quad \text{where } i = 1 \dots n.$$

Then

$$F(x) = \frac{|x^1| + \cdots + |x^n|}{n},$$

where $x = (x^1, \dots, x^n)$ and hypersurfaces of the form $F(x) = \text{const.}$ are just spheres with respect to the 1-norm. They can also be considered as a generalization of conics. In the case of a non-finite set of geometric objects we can use integration over the set of foci to calculate the average distance.

Definition 1. Let Γ be a bounded orientable submanifold in $\mathbb{R}^n$ such that vol $\Gamma < \infty$ with respect to the induced Riemannian volume form $d\gamma$. The average distance is measured as the integral

$$F(x) := \frac{1}{\operatorname{vol} \Gamma} \int_\Gamma \gamma \mapsto d(x, \gamma)\, d\gamma.$$

Hypersurfaces of the form $F(x) = \text{const.}$ are called *generalized conics* with foci $\gamma \in \Gamma$.

Remark 1. In this sense generalized conics are "limits" of sequences of polyellipses or polyellipsoids. To generalize the pure case of hyperplanes in a similar way we can use the submanifolds of Grassmannians. By taking submanifolds of the product $\mathbb{R}^n$ with Grassmannians or flag manifolds [4], mixed cases can also be presented.

Theorem 1. *F is a convex function satisfying the growth condition*

$$\liminf_{|x| \to \infty} \frac{F(x)}{|x|} > 0,$$

where $|x|$ is the Euclidean norm of x.

Proof. Convexity is clear because the integrand is a convex function of the variable x for any fixed element $\gamma \in \Gamma$. Since Γ is bounded, we can define the constant $K := \sup_{\gamma \in \Gamma} |\gamma|$. Then

$$\frac{d(x, \gamma)}{|x|} \geq 1 - \frac{K}{|x|} \geq 1 - \frac{1}{n}$$

holds on the neighbourhood $|x| > nK$ of ∞ for any $\gamma \in \Gamma$. Therefore

$$\liminf_{|x| \to \infty} \frac{F(x)}{|x|} \geq 1 > 0$$

as was to be stated. □

Corollary 1 ([5])**.** *The levels of the function F are bounded.*

Corollary 2. *F has a global minimizer.*

Proof. The statement follows from the Weierstrass's theorem [6]: if all the level sets of a continuous function defined on a non-empty, closed set in $\mathbb{R}^n$ are bounded, then it has a global minimizer. □

Let

$$\gamma \colon [0, 2\pi] \to \mathbb{R}^3, \qquad \gamma(t) := (\cos t, \sin t, 0)$$

be the unit circle in the xy-coordinate plane and

$$F(x, y, z) := \frac{1}{2\pi} \int_0^{2\pi} \sqrt{(x - \cos t)^2 + (y - \sin t)^2 + z^2}\, dt.$$

The surface of the form $F(x, y, z) = \frac{8}{2\pi}$ is a generalized conic with foci S_1 in the Euclidean space $\mathbb{R}^3$. It is obviously a revolution surface with generatrix

$$\int_0^{2\pi} \sqrt{\cos^2 t + (y - \sin t)^2 + z^2}\, dt = 8$$

in the yz-coordinate plane.

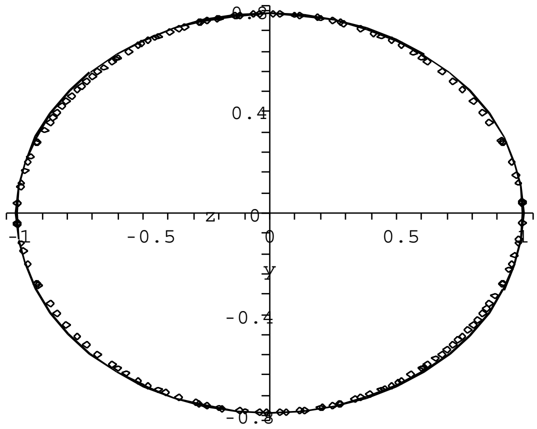

Fig. 1. The generatrix and its approximating ellipse.

Lemma 1. *The surface* $F(x, y, z) = \frac{8}{2\pi}$ *is not an ellipsoid.*

Proof. It is enough to prove that the generatrix

$$\int_0^{2\pi} \sqrt{\cos^2 t + (y - \sin t)^2 + z^2}\, dt = 8$$

is not an ellipse in the yz-coordinate plane. If $y = 0$, then we have that

$$z = \pm\sqrt{\left(\frac{8}{2\pi}\right)^2 - 1}.$$

On the other hand, if $z = 0$, then the solutions of the equation

$$\int_0^{2\pi} \sqrt{\cos^2 t + (y - \sin t)^2}\, dt = 8$$

are just $y = \pm 1$. Therefore the only possible ellipse has the parametric form

$$y(s) = \cos s \quad \text{and} \quad z(s) = \sqrt{\left(\frac{8}{2\pi}\right)^2 - 1}\,\sin s.$$

Fig. 1 shows the generatrix (pointstyle) and its approximating ellipse. Consider the auxiliary function

$$v(s) := \int_0^{2\pi} \sqrt{\cos^2 t + (y(s) - \sin t)^2 + z^2(s)}\, dt.$$

It can be written in the form

$$v(s) = 4h(s) \int_0^{\frac{\pi}{2}} \sqrt{1 - r^2(s)\sin^2 t}\, dt = 4h(s)\mathcal{E}(r(s)),$$

where

$$\mathcal{E}(r) := \int_0^{\frac{\pi}{2}} \sqrt{1 - r^2 \sin^2 t}\, dt, \quad 0 < r < 1$$

is the complete elliptic integral of the second kind,

$$h(s) := \sqrt{(1 + y(s))^2 + z^2(s)} \quad \text{and} \quad \frac{1}{4} r^2(s) := \frac{y(s)}{h^2(s)} > 0$$

provided that $-\frac{\pi}{2} < s < \frac{\pi}{2}$. In terms of the Gaussian hypergeometric function

$$\mathcal{E}(r) = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}; 1; r^2\right)$$

and Richards' result, [7] states that

$$\mathcal{E}(r) \geq \frac{\pi}{2} M_p(1, \sqrt{1-r^2}), \quad \text{where } p = \frac{3}{2} \quad \text{and} \quad M_p(x, y) := \left(\frac{x^p + y^p}{2}\right)^{\frac{1}{p}}$$

is the pth power mean of its arguments; see also [8]. Thus we have

$$v(s) \geq 2\pi h(s) M_p(1, \sqrt{1-r^2(s)}).$$

Consider the function

$$g(s) := 2\pi h(s) M_p(1, \sqrt{1-r^2(s)}), \quad 0 \leq s \leq 2\pi$$

having the properties

$$g\left(\frac{\pi}{2}\right) = 8, \qquad g'\left(\frac{\pi}{2}\right) = 0 \quad \text{and} \quad g''\left(\frac{\pi}{2}\right) > 0.$$

Here we prove that g^2 attains its local minimum at $s = \frac{\pi}{2}$. With the help of a straightforward calculation, we have

$$\frac{1}{4\pi^2}(g^2)'\left(\frac{\pi}{2}\right) = (h^2)'\left(\frac{\pi}{2}\right) - \frac{1}{2}h^2\left(\frac{\pi}{2}\right)(r^2)'\left(\frac{\pi}{2}\right) D_2 M_p^2(1, 1)$$

and

$$\begin{aligned}\frac{1}{4\pi^2}(g^2)''\left(\frac{\pi}{2}\right) &= (h^2)''\left(\frac{\pi}{2}\right) - \frac{1}{2}(h^2)'\left(\frac{\pi}{2}\right) D_2 M_p^2(1, 1)(r^2)'\left(\frac{\pi}{2}\right) \\ &\quad - \frac{1}{4}h^2\left(\frac{\pi}{2}\right) D_2 M_p^2(1, 1)\left(2(r^2)'' + (r^2)'\left(\frac{\pi}{2}\right)(r^2)'\left(\frac{\pi}{2}\right)\right) \\ &\quad + \frac{1}{4}h^2\left(\frac{\pi}{2}\right) D_2 D_2 M_p(1, 1)(r^2)'\left(\frac{\pi}{2}\right)(r^2)'\left(\frac{\pi}{2}\right),\end{aligned}$$

where

$$h^2\left(\frac{\pi}{2}\right) = \left(\frac{8}{2\pi}\right)^2, \qquad (h^2)'\left(\frac{\pi}{2}\right) = -2, \qquad (h^2)''\left(\frac{\pi}{2}\right) = 2\left(2 - \left(\frac{8}{2\pi}\right)^2\right)$$

and

$$D_2 M_p^2(1, 1) = 1, \qquad D_2 D_2 M_p^2(1, 1) = \frac{p}{2}.$$

On the other hand,

$$r^2(s) h^2(s) = 4\cos s$$

which means that

$$r^2\left(\frac{\pi}{2}\right) = 0, \qquad (r^2)'\left(\frac{\pi}{2}\right) h^2\left(\frac{\pi}{2}\right) = -4, \quad \text{and} \quad (r^2)''\left(\frac{\pi}{2}\right) h^4\left(\frac{\pi}{2}\right) = 8(h^2)'\left(\frac{\pi}{2}\right).$$

We have

$$\frac{1}{4\pi^2}(g^2)'\left(\frac{\pi}{2}\right) = 0$$

and

$$\frac{1}{4\pi^2}(g^2)''\left(\frac{\pi}{2}\right) = (h^2)''\left(\frac{\pi}{2}\right) - (4 - 2p)h^{-2}\left(\frac{\pi}{2}\right) = (h^2)''\left(\frac{\pi}{2}\right) - h^{-2}\left(\frac{\pi}{2}\right) > 0$$

as can be easily seen. Therefore the function g attains its local minimum at $s = \frac{\pi}{2}$ and there must be a parameter $0 < s_* < \frac{\pi}{2}$ such that

$$8 < g(s_*) \leq v(s_*),$$

i.e. $v(s)$ is not a constant function. □

Definition 2. Let K be a convex body containing the origin in its interior. The *Minkowski functional* L induced by K is defined as

$$L(x) := \inf\{t \mid x \in tK\}, \quad \text{where } tK := \{tk \mid k \in K\}$$

for any positive real number t. The functional L is called *smooth* if it is smooth except at the origin. *Minkowski spaces* are real vector spaces equipped with a Minkowski functional.

Remark 2. The functional L was first defined by H. Minkowski to provide a method of obtaining a norm together with a topology in very general linear spaces [9]. It is well-known that L is positive homogeneous of order 1 which means that $L(tx) = tL(x)$ for all positive real numbers t. Due to the convexity of K, it is subadditive: $L(x+y) \leq L(x)+L(y)$. Subadditivity together with homogeneity implies convexity as well. The symmetry of K with respect to the origin is equivalent to the absolute homogeneity (reversibility) $L(x) = L(-x)$. In general it is omitted as a too rigid requirement from the viewpoint of applications; see e.g. the so-called Funk metric [10] constructed by changing the base point of the vectors in the interior of K.

Definition 3. By a linear isometry with respect to the Minkowski functional L we mean a linear mapping $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $L \circ \varphi = L$.

Corollary 3. *The generalized conic $F(x, y, z) = \frac{8}{2\pi}$ induces a non-Euclidean Minkowski functional containing the Euclidean isometries leaving S_1 invariant in its linear isometry group.*

Proof. It is clear that conics together with the induced Minkowski functionals L inherit all of the symmetry properties of the set of foci. On the other hand, Lemma 1 shows that L induced by the conic $F(x, y, z) = \frac{8}{2\pi}$ cannot arise from any inner product on the Euclidean space. □

Remark 3. In general the group of linear isometries of a non-Euclidean Minkowski space is trivial. In the following sections we present results like Corollary 3 on the existence of Minkowski functionals with a given subgroup $G \subset O(n)$ in the linear isometry group. Therefore we have examples of geometric spaces with richer and richer linear isometry groups up to the Euclidean geometry.

2. The case of reducible subgroups

Let $G \subset O(n)$ be the subgroup of the orthogonal transformations in the Euclidean space $\mathbb{R}^n$. If G is reducible and $n = 2$ we can always find a finite invariant set of points $\Gamma = \{\pm x_1, \pm x_2, \}$ under the elements of G. This is clear because the invariant subspace must be of dimension 1, together with its orthogonal complement. Their Euclidean unit vectors form the set of Γ. We can choose the origin as one of the foci too. Therefore any polyellipse with foci Γ induces a non-Euclidean Minkowski functional L such that G is the subgroup of the linear isometries with respect to L. If the dimension is greater than or equal to 3, then, by the reducibility of G, we can take one of the Euclidean unit spheres

$$S_1 \subset S_2 \subset \cdots \subset S_{n-2}$$

as the invariant set under the elements of G (in the case of a one-dimensional invariant subspace, consider its orthogonal complement). S_{n-1} as the set of foci gives conics which are invariant under the whole orthogonal group because of the invariance of the set of their foci. Therefore they are spheres of dimension $n-1$. In the case of S_1 at least one of the generalized conics is different from the ellipsoids centered at the origin as Lemma 1 says, by taking $\mathbb{R}^3$ in $\mathbb{R}^n$ as a natural subspace if necessary. In what follows we are going to discuss the case of S_k in general using some recent results on the Gaussian hypergeometric function [7] and elliptic integrals [8].

Let $n \geq 4$ and $2 \leq k \leq n-2$ be fixed integers. To express $S_k \subset \mathbb{R}^n$ in a parametric form consider the mapping

$$\rho_{k-1}\colon H \to S_{k-1} \subset \mathbb{R}^k \times \mathbb{R}^{n-k}, \quad \text{where } H \subset \mathbb{R}^{k-1} \text{ and } \rho_{k-1}(u) = (\rho(u), 0)$$

which gives the points of the sphere S_{k-1} by taking $\mathbb{R}^k$ in $\mathbb{R}^n$ as a natural subspace. Then

$$\rho_k\colon H \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to S_k \subset \mathbb{R}^{k+1} \times \mathbb{R}^{n-(k+1)}, \qquad \rho_k(u, v) = (\rho(u)\cos(v), \sin(v), 0).$$

Since the determinants of the first fundamental forms of S_{k-1} and S_k are related by the formula

$$\det g_{ij}(u, v) = (\cos^2(v))^{k-1} \det h_{ij}(u),$$

we have that for all $x \in \mathbb{R}^n$,

$$F_k(x) := \frac{1}{\operatorname{Vol} S_k} \int_{S_k} \gamma \mapsto d(x, \gamma)\, d\gamma \quad \text{and}$$

$$\int_{S_k} \gamma \mapsto d(x, \gamma)\, d\gamma = \int_{S_{k-1}} \gamma \mapsto \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{D(x, \gamma, v)} \cos^{k-1}(v)\, dv \right) d\gamma,$$

where

$$D(x, \gamma, v) := \sum_{i=1}^{k} (x^i - \gamma^i \cos(v))^2 + (x^{k+1} - \sin(v))^2 + (x^{k+2})^2 + \cdots + (x^n)^2.$$

Consider the intersections of conics of the form $F_k(x) = \text{const.}$ with the plane

$$x^1 = \cdots = x^{k-1} = 0 \quad \text{and} \quad x^{k+3} = \cdots = x^n = 0.$$

They are the level sets of the function

$$f_k(y, z) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 + y^2 + z^2 - 2y \sin t} \cos^{k-1} t \, dt$$

with variables $y := x^k$ and $z := x^{k+1}$, respectively. For the sake of simplicity let $l := k - 1$; then

$$f_k(1, 0) = \frac{2^{l+2} \cdot l!}{1 \cdot 3 \cdot \cdots \cdot (2l + 1)}$$

as we can see with the help of the following calculations.

$$\begin{aligned}
f_k(1, 0) &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{2(1 - \sin t)} \cos^l t \, dt = \sqrt{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\cos \frac{t}{2} - \sin \frac{t}{2}\right) \cos^l t \, dt \\
&= \sqrt{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\cos \frac{t}{2} - \sin \frac{t}{2}\right) \left(\cos^2 \frac{t}{2} - \sin^2 \frac{t}{2}\right)^l dt \\
&= 2\sqrt{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\cos x - \sin x) \left(\cos^2 x - \sin^2 x\right)^l dx \\
&= 2\sqrt{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos x \left(\cos^2 x - \sin^2 x\right)^l dx = 2\sqrt{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos x \left(1 - 2\sin^2 x\right)^l dx
\end{aligned}$$

because $- \sin x = \sin(-x)$. Here

$$\begin{aligned}
\int \cos x \left(1 - 2\sin^2 x\right)^l dx &= \int \cos x \sum_{i=0}^{l} \binom{l}{i} (-2)^{l-i} (\sin x)^{2(l-i)} dx \\
&= \sum_{i=0}^{l} \binom{l}{i} (-2)^{l-i} \int \cos x (\sin x)^{2(l-i)} dx \\
&= \sum_{i=0}^{l} \frac{1}{2(l-i) + 1} \binom{l}{i} (-2)^{l-i} (\sin x)^{2(l-i)+1} \\
&= \sin x \sum_{i=0}^{l} \frac{1}{2l + 1 - 2i} \binom{l}{i} \left(-2\sin^2 x\right)^{l-i}
\end{aligned}$$

and thus

$$\begin{aligned}
f_k(1, 0) &= 4 \sum_{i=0}^{l} \frac{1}{2l + 1 - 2i} \binom{l}{i} (-1)^{l-i} \\
&= 4 \frac{1}{2l + 1} \sum_{i=0}^{l} \frac{2l + 1}{2l + 1 - 2i} \binom{l}{i} (-1)^{l-i} \\
&= 4 \frac{1}{2l + 1} \sum_{i=0}^{l} \binom{l}{i} (-1)^{l-i} + 4 \frac{1}{2l + 1} \sum_{i=0}^{l} \frac{2i}{2l + 1 - 2i} \binom{l}{i} (-1)^{l-i} \\
&= 4 \frac{2}{2l + 1} \sum_{i=1}^{l} \frac{i}{2l + 1 - 2i} \frac{l!}{i!(l - i)!} (-1)^{l-i} \\
&= 4 \frac{2l}{2l + 1} \sum_{i=1}^{l} \frac{1}{2l + 1 - 2i} \frac{(l - 1)!}{(i - 1)!(l - i)!} (-1)^{l-i} \\
&= 4 \frac{2l}{2l + 1} \sum_{i=0}^{l-1} \frac{1}{2l + 1 - 2i - 2} \binom{l - 1}{i} (-1)^{l-1-i}
\end{aligned}$$

$$
\begin{aligned}
&= 4\frac{2l}{2l+1}\frac{2(l-1)}{2l+1-2}\sum_{i=0}^{l-2}\frac{1}{2l+1-2i-4}\binom{l-2}{i}(-1)^{l-2-i} \\
&= \cdots \\
&= \frac{2^{l+2}\cdot l!}{1\cdot 3\cdot\cdots\cdot(2l+1)}
\end{aligned}
$$

as was to be proved.

Lemma 2. *The hypersurface* $F_k(x) = \frac{c(l)}{\mathrm{Vol}\,S_k}$ *where*

$$c(l) := \frac{2^{l+2}\cdot l!}{1\cdot 3\cdot\cdots\cdot(2l+1)}, \quad \textit{and} \quad l = k-1$$

is not an ellipsoid.

Proof. It is enough to prove that the generatrix

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1+y^2+z^2-2y\sin t}\cos^l t\,dt = c(l)$$

is not an ellipse in the yz-coordinate plane. If $y=0$ we have that

$$\sqrt{1+z^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^l t\,dt = c(l).$$

Case I. If l is odd, i.e. k is even, then

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^l t\,dt = 2\frac{(l-1)!!}{l!!},$$

where

$$
\begin{aligned}
&0!! := 1, \qquad (l-1)!! := (l-1)\cdot(l-3)\cdot(l-5)\cdot\cdots\cdot 2 \quad \text{and} \\
&l!! := l\cdot(l-2)\cdot(l-4)\cdot\cdots\cdot 1.
\end{aligned}
$$

Therefore the only possible ellipse has the parametric form

$$y(s) = \cos s \quad \text{and} \quad z(s) = b(l)\sin s$$

with

$$b(l) := \sqrt{\frac{c^2(l)l!!^2}{4(l-1)!!^2} - 1}.$$

Case II. If l is even, i.e. k is odd, then

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^l t\,dt = \pi\frac{(l-1)!!}{l!!}.$$

Therefore the only possible ellipse has the parametric form

$$y(s) = \cos s \quad \text{and} \quad z(s) = b(l)\sin s$$

with

$$b(l) := \sqrt{\frac{c^2(l)l!!^2}{\pi^2(l-1)!!^2} - 1}.$$

Consider the auxiliary function

$$v_l(s) := \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1+y^2(s)+z^2(s)-2y(s)\sin t}\cos^l t\,dt.$$

It can be written in the form

$$v_l(s) = 2^l h(s)\frac{\Gamma^2\left(\frac{l+1}{2}\right)}{\Gamma(l+1)}F\left(-\frac{1}{2}, \frac{l+1}{2}, l+1, r^2(s)\right),$$

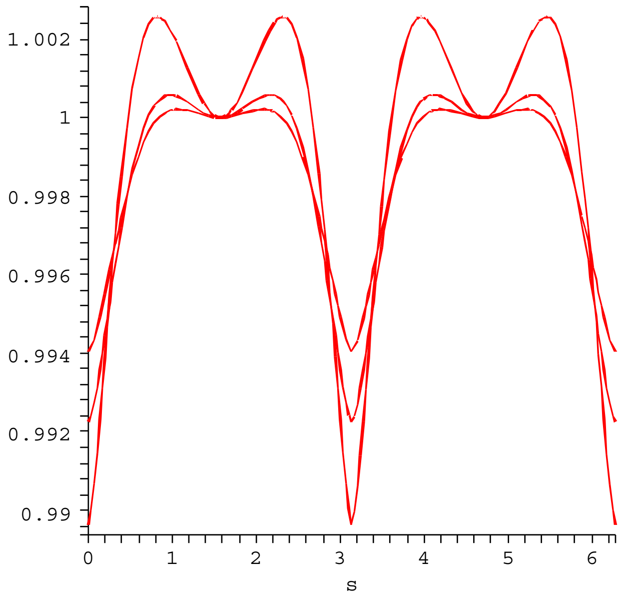

Fig. 2. The case of $l = 1, 3$ and 5.

where

$$F(a, b; c; z) := \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 (1-uz)^{-a} u^{b-1} (1-u)^{c-b-1}\, du, \quad |z| < 1$$

is the Gaussian hypergeometric function,

$$h(s) := \sqrt{(1+y(s))^2 + z^2(s)} \quad \text{and} \quad \frac{1}{4} r^2(s) := \frac{y(s)}{h^2(s)}.$$

Using the identities

$$\Gamma(m) = (m-1)!, \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad \text{and} \quad \Gamma(z+1) = z\Gamma(z)$$

we have

$$2^l \frac{\Gamma^2\left(\frac{l+1}{2}\right)}{\Gamma(l+1)} = \frac{c(l)}{\sqrt{b^2(l)+1}}.$$

On the other hand, for any parameter $-\frac{\pi}{2} < s < \frac{\pi}{2}$ Richards' result [7] states that

$$F^2\left(-\frac{1}{2}, \frac{l+1}{2}, l+1, r^2(s)\right) \geq M_p(1, \sqrt{1-r^2(s)}), \quad \text{where } p = \frac{2l+3}{l+2}.$$

Therefore

$$v_l(s) \geq h(s) \frac{c(l)}{\sqrt{b^2(l)+1}} M_p(1, \sqrt{1-r^2(s)}).$$

Fig. 2 shows the function(s)

$$g(s) := h(s) \frac{1}{\sqrt{b^2(l)+1}} M_p(1, \sqrt{1-r^2(s)}), \quad 0 \leq s \leq 2\pi$$

for $l = 1, 3$ and 5 (or $k = 2, 4$ and 6).
It has the properties

$$g\left(\frac{\pi}{2}\right) = 1, \qquad g'\left(\frac{\pi}{2}\right) = 0 \quad \text{and} \quad g''\left(\frac{\pi}{2}\right) > 0.$$

Here we prove that g^2 attains its local minimum at $s = \frac{\pi}{2}$. Using the same computation as in the proof of Lemma 1,

$$(g^2)'\left(\frac{\pi}{2}\right) = 0$$

and

$$(g^2)''\left(\frac{\pi}{2}\right) = \frac{1}{b^2(l)+1}\left((h^2)''\left(\frac{\pi}{2}\right) - (4-2p)h^{-2}\left(\frac{\pi}{2}\right)\right)$$
$$= \frac{2}{b^2(l)+1}\left(1 - b^2(l) - \frac{1}{l+2}\,\frac{1}{1+b^2(l)}\right).$$

The proof of the inequality

$$1 - b^2(l) - \frac{1}{l+2}\,\frac{1}{1+b^2(l)} > 0 \tag{1}$$

can be divided into the following steps.

First step. Inequality (1) is obviously equivalent to

$$(2-(b^2(l)+1))(b^2(l)+1)(l+2) > 1$$

which can be directly seen in the case of $l = 1, 2$. Suppose, on the contrary, that

$$(2-(b^2(l)+1))(b^2(l)+1)(l+2) \le 1$$

for some integer l and choose the smallest one. Then $l > 2$ and

$$(2-(b^2(l-2)+1))(b^2(l-2)+1)l > 1.$$

Therefore

$$\frac{(2-(b^2(l)+1))(b^2(l)+1)(l+2)}{(2-(b^2(l-2)+1))(b^2(l-2)+1)l} \le 1.$$

Second step. To present a contradiction it is enough to prove that

$$\frac{(2-(b^2(l+2)+1))(b^2(l+2)+1)(l+4)}{(2-(b^2(l)+1))(b^2(l)+1)(l+2)} > 1 \tag{2}$$

for any integer $l > 0$. Here

$$\frac{b^2(l+2)+1}{b^2(l)+1} = \frac{2^4(l+2)^2(l+2)^2}{(2l+3)^2(2l+5)^2} = \frac{(2l+4)^2(2l+4)^2}{(2l+3)^2(2l+5)^2} = \frac{(2l+4)^4}{((2l+4)^2-1)^2}$$

and inequality (2) can be written in the following equivalent forms:

$$(2-(b^2(l+2)+1))(2l+4)^4(l+4) > (2-(b^2(l)+1))((2l+4)^2-1)^2(l+2),$$

i.e.

$$2(2l+4)^4(l+4) - (b^2(l+2)+1)(2l+4)^4(l+4) > 2((2l+4)^2-1)^2(l+2) - (b^2(l)+1)((2l+4)^2-1)^2(l+2)$$

and thus

$$2(2l+4)^4(l+4) - 2((2l+4)^2-1)^2(l+2) - (b^2(l+2)+1)(2l+4)^4(l+4)$$
$$> -(b^2(l)+1)((2l+4)^2-1)^2(l+2).$$

With the help of further computations,

$$\frac{(b^2(l+2)+1)(2l+4)^4(l+4)}{(b^2(l)+1)((2l+4)^2-1)^2(l+2)} - 2\frac{(2l+4)^4(l+4) - ((2l+4)^2-1)^2(l+2)}{(b^2(l)+1)((2l+4)^2-1)^2(l+2)} < 1,$$

i.e.

$$2\frac{(2l+4)^4(l+4) - ((2l+4)^2-1)^2(l+2)}{(b^2(l)+1)((2l+4)^2-1)^2(l+2)} > \frac{(2l+4)^8(l+4) - ((2l+4)^2-1)^4(l+2)}{((2l+4)^2-1)^4(l+2)}$$

and thus

$$2((2l+4)^2-1)^2\frac{(2l+4)^4(l+4) - ((2l+4)^2-1)^2(l+2)}{(2l+4)^8(l+4) - ((2l+4)^2-1)^4(l+2)} > b^2(l)+1. \tag{3}$$

Third step. To prove inequality (3) we use an induction on l. The cases $l = 1$ and 2 can be directly seen. Suppose that inequality (3) is true up to the integer $l-2$, where $l \ge 4$, i.e.

$$2((2l)^2-1)^2\frac{(2l)^4(l+2) - ((2l)^2-1)^2 l}{(2l)^8(l+2) - ((2l)^2-1)^4 l} > b^2(l-2)+1.$$

Multiplying both sides by $\frac{(2l)^4}{((2l)^2-1)^2}$ we have

$$2(2l)^4\frac{(2l)^4(l+2)-((2l)^2-1)^2 l}{(2l)^8(l+2)-((2l)^2-1)^4 l} > b^2(l)+1$$

and it is enough to prove that for any integer $l \geq 2$,

$$2((2l+4)^2-1)^2\frac{(2l+4)^4(l+4)-((2l+4)^2-1)^2(l+2)}{(2l+4)^8(l+4)-((2l+4)^2-1)^4(l+2)} > 2(2l)^4\frac{(2l)^4(l+2)-((2l)^2-1)^2 l}{(2l)^8(l+2)-((2l)^2-1)^4 l}$$

which is equivalent to the inequality $p(l) > 0$, where

$$\begin{aligned} p(x) := {} & 327680x^{12} + 3571712x^{11} + 15024128x^{10} + 27283456x^9 + 4449280x^8 - 64296448x^7 - 101394944x^6 \\ & - 51257536x^5 + 9134912x^4 + 14982848x^3 + 2259712x^2 - 1054590x - 258300. \end{aligned}$$

Since $p(2) > 0$ and

$$p'(2) > 0, p''(2) > 0, \ldots, p^6(2) > 0 \quad \text{and} \quad p^7(2) > 0,$$

the inequality $p(l) > 0$ is satisfied for any integer $l \geq 2$ as was to be proved.
Therefore g attains its local minimum at $s = \frac{\pi}{2}$ and there must be a parameter $0 < s_* < \frac{\pi}{2}$ such that

$$1 < g(s_*) \leq \frac{v(s_*)}{c(l)},$$

i.e. $v(s)$ is not a constant function. □

Corollary 4. *The generalized conic $F_k(x) = \frac{c(l)}{\mathrm{Vol}\, S_k}$ where*

$$c(l) := \frac{2^{l+2}\cdot l!}{1\cdot 3\cdot \cdots \cdot (2l+1)}, \quad \textit{and} \quad l = k-1$$

induces a non-Euclidean Minkowski functional containing the Euclidean isometries leaving S_k invariant in its linear isometry group.

3. The case of irreducible subgroups

Surprisingly this case is almost trivial, in view of the following lemma. As we have seen above, the key step of the construction is to find an invariant set under G as the foci of a generalized conic. It is natural to consider the orbits of the points with respect to G.

Lemma 3. *For any irreducible closed subgroup G the origin is an interior point of the convex hull of non-trivial orbits.*

Proof. First of all note that the convex hulls of the orbits are invariant under G and all of them are closed and, consequently, it is a compact subsets. If G is irreducible and the origin is not a point of the convex hull of a non-trivial orbit, then we can use a simple nearest-point-type argumentation to present a contradiction as follows: taking the uniquely determined point of the convex hull nearest to the origin it can be easily seen that it must be a fixed point of any element of G. This contradicts the irreducibility. If the origin is not in the interior of the convex hull we can consider the common part H of supporting hyperplanes at the origin. It is not an empty set because the origin does not belong to any non-trivial orbit and thus, by the Krein–Milmann theorem, cannot be an extremal point of the convex hull. H is obviously an invariant linear subspace under G which contradicts the irreducibility. □

Note that if one of the convex hulls of a non-trivial orbit is an ellipsoid (as a body) centered at the origin, then it must be a ball in the Euclidean sense according to the irreducibility of G. Then G is transitive on the unit sphere, and all of the possible Minkowski functionals must be Euclidean. In any other case, Lemma 3 shows that the convex hulls of the orbits themselves induce possible Minkowski functionals (in the case of non-closed subgroups we can argue with the transitivity of the closure.)

Corollary 5. *If G is non-transitive on the unit sphere, closed and irreducible, then the convex hull of any non-trivial orbit induces a non-Euclidean Minkowski functional L such that G is the subgroup of the linear isometries with respect to L.*

Integration can be used to avoid singularities, as the following example shows. Consider the group of the symmetries of the square

$$[-1, 1] \times [-1, 1]$$

centered at the origin in the Euclidean plane. The convex hulls of all of the non-trivial orbits are polygons, i.e. the boundary of the convex hull always has singularities. The orbit

$$\Gamma = \{(-1,-1), (1,-1), (1,1), (-1,1)\}$$

induces the supremum norm

$$|(x,y)| := \frac{1}{\sqrt{2}} \max\{|x|, |y|\}.$$

To avoid the singularities at the vertices consider the function

$$F(x,y) := \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1} \sqrt{(x-t)^2+(y-s)^2}\, ds\, dt.$$

The curves of the form $F(x,y) = $ const. are just generalized conics with foci conv Γ. The following calculation shows that at least one of them is not a circle (the irreducibility implies that the invariant ellipses must be circles). According to the symmetric role of the variables x, t and y, s, respectively, we can calculate the coordinates

$$D_1F(x,y) = \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1} \frac{x-t}{\sqrt{(x-t)^2+(y-s)^2}}\, ds\, dt,$$

$$D_2F(x,y) = \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1} \frac{y-s}{\sqrt{(x-t)^2+(y-s)^2}}\, ds\, dt$$

of the gradient vector field. Here

$$\begin{aligned} D_1F(x,y) = {} & -\frac{1}{8}[(s-y)\sqrt{(x-1)^2+(y-s)^2} + (x-1)^2 \ln((s-y)+\sqrt{(x-1)^2+(y-s)^2}) \\ & + (s-y)\sqrt{(x+1)^2+(y-s)^2} + (x+1)^2 \ln((s-y) \\ & + \sqrt{(x+1)^2+(y-s)^2})]_{-1}^{1} \quad \text{and} \quad D_2F(x,y) = D_1F(y,x). \end{aligned}$$

Using these formulas consider the auxiliary function

$$v(x,y) := yD_1F(x,y) - xD_2F(x,y)$$

to measure the difference between the gradient vectors of the family of generalized conics and circles. We have

$$v(2,1) = -2\sqrt{13} + \frac{9}{2}\ln 3 - \frac{9}{2}\ln(-2+\sqrt{13}) + \frac{1}{2}\ln(-2+\sqrt{5}) - 8\ln 2 + 4\ln(-3+\sqrt{13}) + 4\ln(\sqrt{5}+1) + 8$$

which is obviously different from zero. The general case is discussed via the following theorem for the alternatives.

Definition 4. Let G be a closed subgroup; let $z \in S_{n-1}$ be a fixed point and consider its orbit Γ_z. The *minimax point* of Γ_z is such a point z_* on the sphere that the minimum

$$a := \min_{|y|=1} \max_{\gamma} d(y,\gamma)$$

is attained, where the maximum is taken on the convex hull conv Γ_z.

Consider the function

$$f: \mathbb{R} \to \mathbb{R}, \quad f(t) := \begin{cases} 0 & \text{if } t \le a \\ (t-a)e^{-\frac{1}{t-a}} & \text{if } t > a. \end{cases}$$

With the help of the standard calculus [4] it can be seen that this is a smooth convex function on the real line. Define

$$g(t) := t + f(t)$$

and take the functions

$$F(x) := \int_{\text{conv } \Gamma_z} \gamma \mapsto d(x,\gamma)\, d\gamma \quad \text{and} \quad \tilde{F}(x) := \int_{\text{conv } \Gamma_z} \gamma \mapsto g(d(x,\gamma))\, d\gamma.$$

It is clear that

$$c := F(z_*) = \tilde{F}(z_*).$$

On the other hand, one of the niveaus

$$F(x) = c \quad \text{and} \quad \tilde{F}(x) = c$$

must be different from the sphere except when the function

$$y \in S_{n-1} \mapsto \max_{\gamma} d(y,\gamma)$$

is constant, where the maximum is taken on the convex hull conv Γ_z. Since $\Gamma_z \subset S_{n-1}$, it can be easily seen that this is impossible unless conv Γ_z is the unit ball (and G is transitive). Therefore we have the following theorem for the alternatives.

Theorem 2 (*Theorem for the Alternatives*)**.** *If G is non-transitive on the unit sphere, closed and irreducible, then one of the hypersurfaces*

$$\int_{\mathrm{conv}\ \Gamma_z} \gamma \mapsto d(x,\gamma)\, d\gamma = c \quad \text{and} \quad \int_{\mathrm{conv}\ \Gamma_z} \gamma \mapsto g(d(x,\gamma))\, d\gamma = c,$$

where c is the common value of the functions F and $\tilde{F}$ at the minimax point z_*, induces a non-Euclidean Minkowski functional L such that G is the subgroup of the linear isometries with respect to L.

Remark 4. The following list [11] shows the compact connected Lie subgroups[1] which are transitive on the Euclidean unit sphere $S_{n-1} \subset \mathbb{R}^n$.

SO(n)	SO(n)	SO(n)	SO(7)	SO(8)	SO(16)
–	$U(2k+1)$	$U(2k)$	–	$U(4)$	$U(8)$
–	SU$(2k+1)$	SU$(2k)$	–	SU(4)	SU(8)
–	–	Sp(k)	–	Sp(2)	Sp(4)
–	–	Sp(k) · SO(2)	–	Sp(2) · SO(2)	Sp(4) · SO(2)
–	–	Sp(k) · Sp(1)	–	Sp(2) · Sp(1)	Sp(4) · Sp(1)
–	–	–	G_2	Spin(7)	Spin(9)
$n = 2k+1 \neq 7$	$n = 2(2k+1)$	$n = 4k \neq 8, 16$	$n = 7$	$n = 8$	$n = 16$

In the case of these groups there are no alternatives to the Euclidean geometry. For the classification see [12–14].

4. The main result

Definition 5. Let Γ be a bounded orientable submanifold in $\mathbb{R}^n$ such that vol $\Gamma < \infty$ with respect to the induced Riemannian volume form $d\gamma$. If g is a strictly monotone increasing convex function on the non-negative real numbers with initial value $g(0) = 0$, then hypersurfaces of the form

$$\frac{1}{\mathrm{vol}\ \Gamma}\int_{\Gamma} \gamma \mapsto g(d(x,\gamma))\, d\gamma = c$$

are called *generalized conics* with foci $\gamma \in \Gamma$ and g as a function of the alternative.

Theorem 3. *The function*

$$F_g(x) := \frac{1}{\mathrm{vol}\ \Gamma}\int_{\Gamma} \gamma \mapsto g(d(x,\gamma))\, d\gamma$$

is a convex function satisfying the growth condition

$$\liminf_{|x|\to\infty} \frac{F_g(x)}{|x|} > 0,$$

where $|x|$ is the Euclidean norm of x.

Proof. The convexity is trivial. Since the growth condition is equivalent to having bounded level sets for any convex function [6], we have that $\lim_{r\to\infty} \frac{g(r)}{r} > 0$ and thus

$$\liminf_{|x|\to\infty} \frac{F_g(x)}{|x|} = \frac{1}{\mathrm{vol}\ \Gamma} \liminf_{|x|\to\infty} \int_{\Gamma} \gamma \mapsto \frac{g(d(x,\gamma))}{d(\gamma,x)} \frac{d(\gamma,x)}{|x|}\, d\gamma > 0$$

as was to be proved. □

Corollary 6 (*[5]*)**.** *The levels of the function F_g are bounded.*

Corollary 7. *F_g has a global minimizer.*

Theorem 4 (*The Main Theorem*)**.** *If $G \subset O(n)$ is reducible or it is a closed irreducible subgroup which is not transitive on the Euclidean unit sphere in $\mathbb{R}^n$, then there exists an invariant subset Γ under G together with a function g of an alternative such that the generalized conic of the form*

$$\frac{1}{\mathrm{vol}\ \Gamma}\int_{\Gamma} \gamma \mapsto g(d(x,\gamma))\, d\gamma = c$$

induces a non-Euclidean Minkowski functional containing G in its linear isometry group.

[1] According to the Closed Subgroup Theorem [4], G (as a topologically closed subgroup) is a Lie subgroup. Since the orthogonal group is compact, so is G with at most finitely many components; see the list above.

5. Applications

> …a manifold carries a structure invariant under parallel transport if and only if this structure is invariant at a single point under the holonomy group.
>
> M. Berger

I. Let now (M, γ) be a connected Riemannian manifold and consider a point $p \in M$. If the closure of the holonomy group at p is not transitive on the unit sphere in T_pM, we can use Theorem 4 to construct a convex body (a generalized conic) containing the origin in its interior such that it is invariant under the element of the holonomy group at p. This induces a non-Euclidean Minkowski functional L_p in T_pM having the elements of the holonomy group as linear isometries. Extending this functional with the help of parallel transport with respect to the Riemannian structure, we have a smooth collection of functionals such that it is invariant under parallel transport with respect to the Lévi-Civita connection. In a precise terminology, manifolds having such a structure are (non-Riemannian) *Berwald manifolds* belonging to the special case of Finsler manifolds: *Finsler geometry is a non-Riemannian geometry …Instead of the Euclidean spheres in the tangent spaces, the unit vectors form the boundary of general convex sets containing the origin in their interiors.* (M. Berger).

Theorem 5. *If the closure of the holonomy group of a connected Riemannian manifold is not transitive on the unit sphere in the tangent space, then its Lévi-Civita connection is strictly Berwald metrizable by generalized conics with respect to the Riemannian structure.*

Recall that by a theorem due to Borel and Lichnerowicz the restricted holonomy group (the maximal connected subgroup containing the identity) is a closed subgroup in the orthogonal group. Since the holonomy group has at most countable components, the closedness depends on their number (finite or not). Simons [15] proved that if M is a locally irreducible Riemannian manifold such that the restricted holonomy group is not transitive on the unit sphere in the tangent space, then we have a locally symmetric space of rank ≥ 2. Then the holonomy group is contained in the group G of all orthogonal transformations leaving the curvature tensor invariant at a single point of the manifold. G is obviously a closed subgroup and, if the curvature does not vanish, then its maximal connected subgroup containing the identity coincides with the restricted holonomy group [15]. Therefore the holonomy group has at most as many components as G and it is closed; see also [16]. This means that the closure operator in Theorem 5 can be omitted in the case of both reducible Riemannian manifolds (automatically) and locally irreducible Riemannian manifolds with non-vanishing curvature tensor. To complete this panoramic view, recall that Bieberbach's theorem states that the holonomy group is finite in the case of a compact flat Riemannian manifold.

II. The next step is the generalization of Theorem 5 in the following sense. Let (M, γ) be a connected Riemannian manifold and consider a point $p \in M$. If the closure of the holonomy group at p of a metrical but not necessarily torsion free linear connection ∇ is not transitive on the unit sphere in T_pM, we can use Theorem 4 to construct a convex body (a generalized conic) containing the origin in its interior such that it is invariant under the element of the holonomy group at p. This induces a non-Euclidean Minkowski functional L_p in T_pM having the elements of the holonomy group as linear isometries. Extending this functional with the help of parallel transport with respect to the Riemannian structure, we have a smooth collection of functionals such that it is invariant under parallel transport with respect to the connection ∇. In a precise terminology, manifolds having such a structure are (non-Riemannian) *generalized Berwald manifolds*.

Theorem 6. *Let M be a connected Riemannian manifold; if the closure of the holonomy group of a metrical (but not necessarily torsion free) linear connection ∇ is not transitive on the unit sphere in the tangent space, then ∇ is strictly Berwald metrizable, in a general sense, by generalized conics with respect to the Riemannian structure.*

Conversely, if M is a generalized Berwald manifold (a manifold equipped with a smooth collection of Minkowski functionals invariant under parallel transport of a linear connection ∇ on the base manifold) with respect to the linear connection ∇, then ∇ is Riemann-metrizable [17]. The most important special cases are locally Minkowski manifolds (∇ with zero torsion and vanishing curvature tensor), Berwald manifolds (∇ with zero torsion) and (exact) Wagner manifolds [18], where the torsion is of the form

$$T = \frac{1}{2}(1 \otimes d\alpha - d\alpha \otimes 1)$$

for some globally defined smooth function $\alpha: M \to \mathbb{R}$ on the base manifold; see also [19].

III. It is well-known that metrical linear connections are uniquely determined by the torsion tensor. Taking the canonical decomposition

$$T(X, Y) := \left(T(X, Y) - \frac{1}{n-1}(\tilde{T}(X)Y - \tilde{T}(Y)X)\right) + \frac{1}{n-1}(\tilde{T}(X)Y - \tilde{T}(Y)X),$$

the traceless part

$$T(X, Y) - \frac{1}{n-1}(\tilde{T}(X)Y - \tilde{T}(Y)X)$$

is automatically zero in the case of $n = 2$. In the case of $n \geq 3$ the traceless part can be divided into two further components by separating off the axial (or totally antisymmetric) part, which means that its lowered tensor with respect to the Riemannian metric is totally antisymmetric. Following Agricola and Friedrich [20], we then have eight classes of linear connections with torsion together with eight classes of generalized Berwald manifolds depending on whether the canonical part of the torsion is identically zero or not. The most important special cases are:

i. Classical Berwald manifolds with $T = 0$.
ii. Exact Wagner manifolds with vanishing traceless part and an exact trace tensor

$$\tilde{T} = \frac{n-1}{2} d\alpha$$

in the torsion (we can speak about *closed Wagner manifolds* via the requirement of a closed trace tensor). The geometric meaning is the global (local) conformal equivalence [21] to a Berwald manifold via the exponent of the function α.

If the torsion tensor has only the pure axial (antisymmetric) component, then the linear connection has the same geodesics (as pointsets) as the Lévi-Civita connection and vice versa [20]. Therefore we have as one of the important special cases:

iii. Projectively Berwald manifolds with only the pure antisymmetric part in the torsion. The geometric meaning is obviously the projective equivalence to a Berwald manifold.

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